

ALMOST PERIODIC FACTORIZATION OF CERTAIN BLOCK TRIANGULAR MATRIX FUNCTIONS

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ABSTRACT. Let

$$G(x) = \begin{bmatrix} e^{i\lambda x} I_m & 0 \\ c_{-1} e^{-i\nu x} + c_0 + c_1 e^{i\alpha x} & e^{-i\lambda x} I_m \end{bmatrix},$$

where $c_j \in \mathbb{C}^{m \times m}$, $\alpha, \nu > 0$ and $\alpha + \nu = \lambda$. For rational α/ν such matrices G are periodic, and their Wiener-Hopf factorization with respect to the real line \mathbb{R} always exists and can be constructed explicitly. For irrational α/ν , a certain modification (called an almost periodic factorization) can be considered instead. The case of invertible c_0 and commuting $c_1 c_0^{-1}$, $c_{-1} c_0^{-1}$ was disposed of earlier—it was discovered that an almost periodic factorization of such matrices G does not always exist, and a necessary and sufficient condition for its existence was found.

This paper is devoted mostly to the situation when c_0 is not invertible but the c_j commute pairwise ($j = 0, \pm 1$). The complete description is obtained when $m \leq 3$; for an arbitrary m , certain conditions are imposed on the Jordan structure of c_j . Difficulties arising for $m = 4$ are explained, and a classification of both solved and unsolved cases is given.

The main result of the paper (existence criterion) is theoretical; however, a significant part of its proof is a constructive factorization of G in numerous particular cases. These factorizations were obtained using Maple; the code is available from the authors upon request.

1. INTRODUCTION

Let AP be the Bohr algebra of almost periodic functions, that is, the smallest C^* -algebra of $L^\infty(\mathbb{R})$ containing all the functions $e_\lambda(x) = e^{i\lambda x}$, $\lambda \in \mathbb{R}$. It is well known (the standard references for these and other properties of AP are [3, 11, 12]) that for every $f \in AP$,

1. there exists the *Bohr mean value*

$$\mathbf{M}(f) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x) dx,$$

and

2. the *Fourier coefficients* $\widehat{f}(\lambda) \stackrel{\text{def}}{=} \mathbf{M}(f e_{-\lambda})$ are different from zero for at most countably many values of $\lambda \in \mathbb{R}$.

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The set $\Omega(f) = \{\mu: \widehat{f}(\mu) \neq 0\}$ is called the *Fourier spectrum* of f , and

$$(1.1) \quad \sum_{\mu \in \Omega(f)} \widehat{f}(\mu)e_{\mu}$$

is its (formal) *Fourier series*.

We say that $f \in AP_W$ if the Fourier series (1.1) converges absolutely:

$$\sum_{\mu \in \Omega(f)} |\widehat{f}(\mu)| < \infty.$$

Finally, let

$$AP^{\pm} = \{f \in AP: \Omega(f) \subset \mathbb{R}_{\pm}\} \quad \text{and} \quad AP_W^{\pm} = AP^{\pm} \cap AP_W.$$

Here, as usual, $\mathbb{R}_{\pm} = \{x \in \mathbb{R}: \pm x \geq 0\}$.

For matrix functions f , conditions $f \in AP, AP^{\pm}, AP_W$, etc. are understood entrywise, and $\mathbf{M}(f), \widehat{f}(\mu), \Omega(f)$ are defined by exactly the same formulas as for scalar functions.

Following [6], we introduce an *AP factorization* of an $n \times n$ matrix function G as its representation in the form

$$(1.2) \quad G = G_+ \Lambda G_-,$$

where $\Lambda(x) = \text{diag}[e_{\lambda_1}, \dots, e_{\lambda_n}]$,

$$(1.3) \quad G_+^{\pm 1} \in AP^+, G_-^{\pm 1} \in AP^-,$$

and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. We say that (1.2) is an AP_W factorization of G if conditions (1.3) are replaced by the (more restrictive) conditions $G_+^{\pm 1} \in AP_W^+, G_-^{\pm 1} \in AP_W^-$.

If G is *AP factorable*, the numbers $\lambda_1, \dots, \lambda_n$ are defined uniquely; they are called the *partial AP indices* (of G). Of course, for an *AP* (AP_W) factorization (1.2) to exist it is necessary that $G^{\pm 1} \in AP$ (respectively, AP_W). However, this necessary condition is not sufficient and, except for the case of periodic matrix functions G (in which an *AP* factorization by a simple change of variable reduces to the usual Wiener-Hopf factorization), the theory of *AP* factorization is “under construction”. Its connections with integral equations, completion problems, and signal processing are discussed in [6, 7], [17, 15, 1], and [14] respectively. Explicit formulas for the factors in (1.2) for certain special types of G are obtained in [6, 9, 8]. Most of them refer to matrices G of the following block triangular form:

$$(1.4) \quad G_f = \begin{bmatrix} e_{\lambda} I_m & 0 \\ f & e_{-\lambda} I_m \end{bmatrix}$$

(so that $n = 2m$) arising in the treatment of convolution type equations on finite intervals of length λ .

In particular, the following two statements were established in [6].

Lemma 1.1. *Let f be an AP_W matrix function, and let*

$$f_0 = \sum_{\mu \in \Omega(f) \cap (-\lambda, \lambda)} \widehat{f}(\mu)e_{\mu}.$$

*Then the matrices G_f and G_{f_0} are AP (AP_W) factorable only simultaneously, and their partial *AP indices* coincide.*

Due to Lemma 1.1, for any $f \in AP_W$ in (1.4) we may suppose without loss of generality that $\Omega(f) \subset (-\lambda, \lambda)$.

Theorem 1.2. *Let $\Omega(f) \cap (-\lambda, \lambda)$ consist of at most two points, say μ and σ . Then G_f is AP_W factorable. Its partial AP indices all equal zero if and only if $\mu\sigma = 0$, $\widehat{f}(0)$ is invertible, or $\mu\sigma < 0$, $\frac{\lambda}{\mu-\sigma} \in \mathbb{Z}$ and both $\widehat{f}(\mu)$, $\widehat{f}(\sigma)$ are invertible.*

The next logical step is to consider a trinomial f with $\Omega(f) \subset (-\lambda, \lambda)$. However, with no additional restrictions on the location of $\Omega(f)$ this remains an open problem. In this paper, we concentrate on the case $\Omega(f) = \{-\nu, 0, \alpha\}$, that is,

$$(1.5) \quad f = c_{-1}e_{-\nu} + c_0 + c_1e_\alpha,$$

where $\alpha, \nu > 0$ and $\alpha + \nu = \lambda$.

If $\beta = \frac{\nu}{\alpha}$ is rational, then the matrix G_f is periodic, and its AP_W factorization exists and can be easily constructed. Thus, we suppose in what follows that β is irrational. The next result applies to the case when the matrices c_j in (1.5) commute with each other. In this case there exists a similarity T such that

$$(1.6) \quad T^{-1}c_jT = \text{diag}[c_{j1}, \dots, c_{jr}], \quad c_{jk} \in \mathbb{C}^{l_k \times l_k}, \quad k = 1, \dots, r; \quad j = 0, \pm 1,$$

and each diagonal block c_{jk} has a singleton spectrum (see [13, Section 4.4]):

$$\sigma(c_{jk}) = \{\xi_{jk}\} \quad (j = 0, \pm 1; k = 1, \dots, r).$$

As in [2], we call $\{\xi_{jk}\}_{j=-1}^1$ the *bonded eigenvalue triples* of c_j .

Theorem 1.3. *Let G_f be of the form (1.4) with f given by (1.5) and commuting coefficients c_j . Then G_f is AP factorable with zero partial AP indices if and only if, for all bonded triples $\{\xi_{-1,k}, \xi_{0,k}, \xi_{1,k}\}$,*

$$(1.7) \quad |\xi_{1,k}^\nu \xi_{-1,k}^\alpha| \neq |\xi_{0,k}|^\lambda \quad (k = 1, \dots, r).$$

In this form, Theorem 1.3 was established in [2, Theorem 7.2]; the case of invertible c_j was disposed of earlier in [9]. In fact, the result of [9] contains an additional important piece of information: if all c_j are invertible and (1.7) fails for at least one value of k , then G_f does not admit any AP factorization, even if non-zero partial AP indices are allowed. Also, it was shown in [16] that an AP factorization with zero partial AP indices of an AP_W matrix function is automatically its AP_W factorization. Hence, the following result holds.

Corollary 1.4. *Let G_f be as in Theorem 1.3 and, in addition, let c_0 be invertible. Then G_f is AP_W factorable with zero partial AP indices if condition (1.7) holds, and is not AP factorable otherwise.*

Of course, it would now be natural to consider an AP factorization of G_f with trinomial f , pairwise commuting c_j , and no restrictions imposed on the invertibility of c_0 and the values of partial AP indices. We will see, however, that this problem embraces a general setting of a trinomial f with arbitrary (not necessarily commuting) coefficients c_j and is therefore too difficult to handle at the present stage of the development. Our paper is a report on several partial results on the AP factorability of matrices (1.4), (1.5) with non-invertible c_0 .

The paper is structured as follows. Section 2 contains an auxiliary result on the factorization of block diagonal matrices. It also describes a procedure which allows us to replace a matrix of the form (1.4), (1.5) with invertible c_{-1} (and no commutativity conditions on c_j) by another matrix of the same type without changing its factorability properties. This procedure is, in fact, a variation of the one introduced in [2] for matrices (1.4) with a finite number (not limited to three) of

points μ_j in $\Omega(f) \cap (-\lambda, \lambda)$ but pairwise commuting $\widehat{f}(\mu_j)$. As a direct application of this procedure, AP_W factorability is established for matrices (1.4), (1.5) with $m = 2$, invertible c_{-1} (or c_1) and nilpotent $c_0 c_{-1}^{-1}$ (respectively, $c_0 c_1^{-1}$).

Section 3 contains necessary and sufficient factorability conditions for matrices (1.4), (1.5) with commuting c_j under certain additional restrictions on their Jordan structure. This covers, in particular, all matrices of size $m \leq 3$, invertible c_1 or c_{-1} of size $m \leq 4$, and matrices of arbitrary size, provided that each eigenvalue of at least one of the c_j corresponds to one Jordan cell. An application to difference equations is given.

In Section 4, we concentrate on 4×4 matrices c_j . An example is given explaining why this case cannot be covered in general before the AP factorability of matrices (1.4), (1.5) with arbitrary invertible non-commuting c_j is understood. All possible cases are classified, and those for which the AP factorability remains unknown are singled out.

Proofs of the results in Sections 3 and 4 are partially theoretical and partially consist in exhausting a large number of cases in which an AP_W factorization can be constructed explicitly. These cases are relegated to Section 5 the supplement at the end of this volume, where final formulas are listed. Of course, they can be checked by straightforward calculations. We emphasize, however, that a symbolic manipulation Maple program was used to obtain these formulas, and without it this paper could hardly have been completed.

2. AUXILIARY RESULTS

Suppose G is a block diagonal AP matrix: $G = \text{diag}[G_1, G_2]$. If its diagonal blocks G_1, G_2 are AP factorable, then G itself is AP factorable. Moreover, an AP factorization of G can be obtained by “pasting together” AP factorizations of G_1 and G_2 : $G_1 = G_+^{(1)} \Lambda^{(1)} G_-^{(1)}$, $G_2 = G_+^{(2)} \Lambda^{(2)} G_-^{(2)}$ imply

$$G = \text{diag}[G_+^{(1)}, G_+^{(2)}] \text{diag}[\Lambda^{(1)}, \Lambda^{(2)}] \text{diag}[G_-^{(1)}, G_-^{(2)}].$$

It is natural to ask whether the converse is true. The answer is positive provided that $G \in AP_W$ and partial AP indices of G equal zero. Indeed, a matrix $F \in AP_W$ admits an AP factorization with zero partial AP indices if and only if the corresponding Toeplitz operator T_F is invertible on L^2 [5] (see also [7]). Since T_G is a direct sum of T_{G_1} with T_{G_2} , the invertibility of T_G is equivalent to simultaneous invertibility of T_{G_1} and T_{G_2} .

We are not aware of any equivalent of AP factorability (with non-zero partial AP indices) in operator terms. Probably, the answer to the question is still positive, but we restrict our consideration to a somewhat weaker version.

Lemma 2.1. *Let $G = \text{diag}[G_1, G_2]$. If G and one of its diagonal blocks G_1, G_2 are AP factorable, then the other diagonal block is also AP factorable.*

Proof. Consider first the case when $G_1 = 1$. Then an AP factorization of G can be rewritten as

$$(2.1) \quad F_+ \begin{bmatrix} 1 & 0 \\ 0 & G_2 \end{bmatrix} = \Lambda G_-,$$

where $F_+ = G_+^{-1} \in AP^+$. Denote $F_+ = (f_{ij})_{i,j=1}^n$. From (2.1), $e_{-\lambda_j} f_{j1} \in AP^-$, so that

$$(2.2) \quad \Omega(f_{j1}) \subset [0, \lambda_j].$$

In particular, $f_{j1} = 0$ for all j (if there are any) such that $\lambda_j < 0$. Rewriting (2.1) as

$$\begin{bmatrix} 1 & 0 \\ 0 & G_2^{-1} \end{bmatrix} G_+ \Lambda = G_-^{-1},$$

we find similarly that

$$(2.3) \quad \Omega(g_{1j}) \subset [0, -\lambda_j],$$

where $G_+ = (g_{ij})_{i,j=1}^n$. Therefore, $g_{1j} = 0$ for all j (if there are any) such that $\lambda_j > 0$. Observe also that $G_+ F_+ = I$ implies that $\sum_{j=1}^n g_{1j} f_{j1} = 1$. Since for non-zero λ_j at least one of the entries g_{1j}, f_{j1} is equal to zero, the latter equality proves the existence of zero partial AP indices λ_j . Due to (2.2), (2.3), the corresponding functions g_{1j}, f_{j1} are constant, and for at least one value of j , $g_{1j} f_{j1} \neq 0$.

Applying an appropriate permutation of columns of G_+ and rows of G_- , we may suppose without loss of generality that $\lambda_1 = 0, g_{11} = c \neq 0, f_{11} = d \neq 0$. Partitioning G_+, F_+ as

$$G_+ = \begin{bmatrix} c & g_1^+ \\ g_2^+ & G_2^+ \end{bmatrix}, F_+ = \begin{bmatrix} d & f_1^+ \\ f_2^+ & F_2^+ \end{bmatrix},$$

we conclude that $c = \det F_2^+ / \det F_+ = \det F_2^+ \det G_+$. Since $c \neq 0$, the matrix F_2^+ is invertible in AP^+ simultaneously with G_+ . From (2.1) and (2.2) it follows that the left-upper entry of G_- and $H_- = G_-^{-1}$ equals d and c , respectively. Thus,

$$G_- = \begin{bmatrix} d & g_1^- \\ g_2^- & G_2^- \end{bmatrix}, H_- = \begin{bmatrix} c & h_1^- \\ h_2^- & H_2^- \end{bmatrix},$$

and $c = \det G_2^- / \det G_-$. Since $c \neq 0$, the matrix G_2^- is invertible in AP^- together with G_- . Now partition $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$. Then (2.1) yields $F_2^+ G_2 = \Lambda_2 G_2^-$, or $G_2 = (F_2^+)^{-1} \Lambda_2 G_2^-$. Since $(F_2^+)^{\pm 1} \in AP^+$ and $(G_2^-)^{\pm 1} \in AP^-$, the latter formula delivers an AP factorization of G_2 . This proves the desired statement in the case $G_1 = 1$.

If $G_1 = e_\lambda$, then the matrix $e_{-\lambda} G = \text{diag}[1, e_{-\lambda} G_2]$ is AP factorable together with G . According to the already proven particular case, $e_{-\lambda} G_2$ is AP factorable. But then G_2 is AP factorable as well.

An induction argument allows us to consider G_1 of the form $\text{diag}[e_{\lambda_1}, \dots, e_{\lambda_k}] = \Lambda_1$. Finally, for an arbitrary AP factorable $G_1 = G_+^{(1)} \Lambda_1 G_-^{(1)}$ we can write

$$G = \text{diag}[G_+^{(1)}, I] \text{diag}[\Lambda_1, G_2] \text{diag}[G_-^{(1)}, I]$$

and consider an (AP factorable) matrix $\text{diag}[\Lambda_1, G_2]$ instead of the original matrix G . □

Another technical tool we need applies to matrix functions G_f with a trinomial f containing an invertible c_{-1} coefficient.

Lemma 2.2. *Let G be of the form (1.4) with f given by (1.5). If c_{-1} is invertible, then G is AP (AP_W) factorable only simultaneously with (and has the same partial AP indices as) the matrix function*

$$(2.4) \quad G_1 = \begin{bmatrix} e_{\lambda_1} I_m & 0 \\ f_1 & e_{-\lambda_1} I_m \end{bmatrix},$$

where

$$(2.5) \quad f_1 = c_{-1}^{(1)} e_{-\nu_1} + c_0^{(1)} + c_1^{(1)} e_{\alpha_1},$$

$$c_{-1}^{(1)} = (-1)^s (c_{-1}^{-1} c_0)^{s+1}, \quad c_0^{(1)} = c_{-1}^{-1} c_1, \quad c_1^{(1)} = (-1)^{s+1} (c_{-1}^{-1} c_0)^{s+2},$$

$$\lambda_1 = \nu, \quad \nu_1 = \alpha - s\nu, \quad \alpha_1 = (s + 1)\nu - \alpha,$$

and finally, s is the integral part of $\frac{\alpha}{\nu}$: $s \in \mathbb{Z}$ and $s < \frac{\alpha}{\nu} < s + 1$.

Proof. It suffices to construct matrix functions X_+ and X_- such that $X_{\pm}^{\pm 1} \in AP_W^{\pm}$, $X_{\pm}^{\pm 1} \in AP_W^{\mp}$ and

$$(2.6) \quad X_+ G X_- = G_1.$$

To this end, let

$$(2.7) \quad \begin{aligned} X_+ &= \begin{bmatrix} c_{-1}^{-1} f e_{\nu} & -e_{\lambda+\nu} I \\ e_{-\lambda-\nu} I + (g - e_{-\lambda} I) c_{-1}^{-1} f & I - g e_{\lambda} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & c_{-1}^{-1} \end{bmatrix}, \\ X_- &= \begin{bmatrix} I & 0 \\ 0 & c_{-1} \end{bmatrix} \begin{bmatrix} e_{-\alpha} I + \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\alpha} & I \\ -I & 0 \end{bmatrix}, \end{aligned}$$

where $g = c_{-1}^{-1} c_1 - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j-1)\nu-\alpha}$. Directly from the definition of s it follows that $X_- \in AP^-$. Since $\det X_- = \det c_{-1}$ is a non-zero constant, X_-^{-1} belongs to AP^- together with X_- .

A straightforward computation shows that

$$\begin{aligned} X_+ G X_- &= \begin{bmatrix} c_{-1}^{-1} f e_{\nu} & -e_{\lambda+\nu} I \\ e_{-\lambda-\nu} I + (g - e_{-\lambda} I) c_{-1}^{-1} f & I - g e_{\lambda} \end{bmatrix} \\ &\quad \times \begin{bmatrix} e_{\lambda} I & 0 \\ c_{-1}^{-1} f & e_{-\lambda} I \end{bmatrix} \begin{bmatrix} e_{-\alpha} I + \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\alpha} & I \\ -I & 0 \end{bmatrix} \\ &= \begin{bmatrix} c_{-1}^{-1} f e_{\nu} & -e_{\lambda+\nu} I \\ e_{-\lambda-\nu} I + (g - e_{-\lambda} I) c_{-1}^{-1} f & I - g e_{\lambda} \end{bmatrix} \\ &\quad \times \begin{bmatrix} e_{\nu} I + \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+1)\nu} & e_{\lambda} I \\ c_{-1}^{-1} e_{-\alpha} f + c_{-1}^{-1} f \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\alpha} - e_{-\lambda} I & c_{-1}^{-1} f \end{bmatrix} \\ &= (y_{ij})_{i,j=1}^2, \end{aligned}$$

where

$$\begin{aligned} y_{11} &= c_{-1}^{-1} f e_{2\nu} + c_{-1}^{-1} f \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+2)\nu} - c_{-1}^{-1} f e_{2\nu} \\ &\quad - c_{-1}^{-1} f \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+2)\nu} + e_{\nu} I = e_{\nu} I, \\ y_{12} &= c_{-1}^{-1} f e_{\lambda+\nu} - c_{-1}^{-1} f e_{\lambda+\nu} = 0, \\ y_{22} &= e_{-\nu} I + (g e_{\lambda} - I) c_{-1}^{-1} f + (I - g e_{\lambda}) c_{-1}^{-1} f = e_{-\nu} I, \end{aligned}$$

and finally,

$$\begin{aligned} y_{21} &= (e_{-\lambda-\nu} I + (g - e_{-\lambda} I) c_{-1}^{-1} f) \left(e_{\nu} I + \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+1)\nu} \right) \\ &\quad + (I - g e_{\lambda}) \left(c_{-1}^{-1} e_{-\alpha} f + c_{-1}^{-1} f \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\alpha} - e_{-\lambda} I \right) \\ &= e_{-\lambda} I + \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\lambda} + (g - e_{-\lambda} I) \\ &\quad \times \left(c_{-1}^{-1} f e_{\nu} + c_{-1}^{-1} f \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+1)\nu} \right. \\ &\quad \left. - c_{-1}^{-1} f e_{\nu} - c_{-1}^{-1} f \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+1)\nu} + I \right) \\ &= e_{-\lambda} I + \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\lambda} + g - e_{-\lambda} I \\ &= \sum_{j=1}^s (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\lambda} + c_{-1}^{-1} c_1 - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j-1)\nu-\alpha} \\ &= (-1)^s (c_{-1}^{-1} c_0)^{s+1} e_{s\nu-\alpha} + c_{-1}^{-1} c_1 + (-1)^{s+1} (c_{-1}^{-1} c_0)^{s+2} e_{(s+1)\nu-\alpha} = f_1. \end{aligned}$$

This implies (2.6). Since $\det G = \det G_1 = 1$, from (2.6) it follows, in particular, that $\det X_+ = (\det X_-)^{-1}$ is a non-zero constant. It remains to show that $X_+ \in AP^+$, because then $X_+^{-1} \in AP^+$ as well. Three blocks of X_+ are obviously in AP^+ . The remaining (left-lower) block can be rewritten as

$$\begin{aligned} &e_{-\lambda-\nu} I + (g - e_{-\lambda} I) c_{-1}^{-1} f \\ &= e_{-\lambda-\nu} I + c_{-1}^{-1} c_1 e_{-\nu} + c_{-1}^{-1} c_1 c_{-1}^{-1} c_0 + (c_{-1}^{-1} c_1)^2 e_{\alpha} \\ &\quad - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j-2)\nu-\alpha} - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^{j+1} e_{(j-1)\nu-\alpha} \\ &\quad - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j-1)\nu} (c_{-1}^{-1} c_1) - e_{-\lambda-\nu} I - c_{-1}^{-1} c_0 e_{-\lambda} - c_{-1}^{-1} c_1 e_{-\nu} \\ &= c_{-1}^{-1} c_1 c_{-1}^{-1} c_0 + (c_{-1}^{-1} c_1)^2 e_{\alpha} - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j-1)\nu} (c_{-1}^{-1} c_1) \\ &\quad + (c_{-1}^{-1} c_0) e_{-\lambda} - (-1)^s (c_{-1}^{-1} c_0)^{s+3} e_{(s+1)\nu-\alpha} - (c_{-1}^{-1} c_0) e_{-\lambda}. \end{aligned}$$

Cancelling out the terms $\pm(c_{-1}^{-1}c_0)e_{-\lambda}$ in the last expression, we see that this block belongs to AP^+ as well. \square

Formula (2.6) is a particular case of the transformation introduced in [2] for an arbitrary AP polynomial (not necessarily a trinomial) f with invertible Fourier coefficient corresponding to the leftmost point in $\Omega(f) \cap (-\lambda, \lambda)$. However, in [2] only the case of commuting coefficients was considered. Also, formulas (2.7) for a trinomial case are more explicit than the general formulas of [2].

The resulting matrix G_1 in general has the same structure as the original matrix G : $\Omega(f_1) \subset \{-\nu_1, 0, \alpha_1\}$, where $\alpha_1, \nu_1 > 0$, $\alpha_1 + \nu_1 = \lambda_1$ and $\beta_1 = \nu_1/\alpha_1$ is irrational together with β . In some instances, however, G_1 may be easier to deal with. One such situation is discussed in the next theorem; other applications of Lemma 2.2 can be found in subsequent sections.

Theorem 2.3. *Let the matrix G be given by (1.4), (1.5) with c_{-1} invertible, $c_0c_{-1}^{-1}$ nilpotent and having all Jordan cells of the size at most $\lceil \frac{\alpha}{\nu} \rceil + 2$. Then 1) G is AP_W factorable, and 2) its partial AP indices equal zero if and only if c_1 is invertible.*

Proof. Due to Lemma 2.2, we may consider the matrix (2.4) instead of G . The conditions imposed on the Jordan structure of $c_0c_{-1}^{-1}$ imply that $(c_{-1}^{-1}c_0)^{s+2} = 0$. Thus, f_1 in (2.4) is in fact a binomial with $\Omega(f_1) \subset \{-\nu_1, 0\}$. According to Theorem 1.2, the matrix G_1 is AP_W factorable, and its partial AP indices equal zero if and only if the constant term $c_{-1}^{-1}c_1$ of f_1 is invertible. The latter condition is equivalent to the invertibility of c_1 . \square

Recall now the duality between an AP factorization (1.2) of G_f and that of G_{f^*} :

$$(2.8) \quad G_{f^*} = (JG_-^*)\Lambda^*(G_+^*J),$$

where $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. From (2.8) and Theorem 2.3 follows

Corollary 2.4. *Let the matrix G be given by (1.4), (1.5) with c_1 invertible, $c_0c_1^{-1}$ nilpotent and having all Jordan cells of the size at most $\lceil \frac{\nu}{\alpha} \rceil + 2$. Then G is AP_W factorable, and its partial AP indices equal zero if and only if c_{-1} is invertible.*

Observe that the condition on the size of Jordan cells is satisfied automatically if $m = 2$. Hence, the following statement holds.

Corollary 2.5. *Let the matrix G be given by (1.4), (1.5) with $m = 2$, let one of the coefficients $c_{\pm 1}$ be non-singular, and let the corresponding product $c_0c_{\pm 1}^{-1}$ be nilpotent. Then 1) G is AP_W factorable, and 2) its partial AP indices equal zero if and only if the second of the coefficients $c_{\pm 1}$ is invertible as well.*

3. MAIN RESULT

We now turn to matrices (1.4) with the off-diagonal block (1.5) having pairwise commuting coefficients $c_{\pm 1}, c_0$. The representation (1.6) is not unique, and we choose one with the maximal possible number r of diagonal blocks. Each triple $\{c_{-1,k}, c_{0k}, c_{1k}\}$ is then *irreducible*, that is, does not allow a further reduction to a block diagonal form with the help of a common similarity. Of course, the commutativity property of $\{c_{-1}, c_0, c_1\}$ is inherited by the triples $\{c_{-1,k}, c_{0k}, c_{1k}\}$.

The ambiguity of T also allows us, for each $k = 1, \dots, r$, to put one of the matrices c_{jk} (with our choice of $j = 0, \pm 1$) in its Jordan canonical form. If, for a

given k , at least one of the matrices c_{jk} is *unicellular* (that is, its canonical Jordan form consists of only one cell), then for such a T all the matrices c_{jk} with the same k automatically become upper triangular and, in addition, have a Toeplitz structure. The latter means that (p, q) -entry of each of the matrices $c_{-1,k}$, $c_{0,k}$, $c_{1,k}$ is the same as its $(p+1, q+1)$ -entry ($p, q = 1, \dots, l_k - 1$). For $l_k > 1$, the common value of the entries right above the main diagonal in c_{jk} for such k will be denoted by η_{jk} (of course, the common value of the diagonal elements of the c_{jk} in this case is ξ_{jk}).

With this notation at hand, we are ready to formulate our main result.

Theorem 3.1. *Let G be given by (1.4), (1.5) with pairwise commuting coefficients $c_{\pm 1}, c_0$. Suppose that in (1.6) for each $k = 1, \dots, r$ at least one of the following conditions holds: 1) $\xi_{0k} \neq 0$, 2) $\xi_{1,k}\xi_{-1,k} \neq 0$, 3) one of the blocks $c_{\pm 1,k}, c_{0k}$ is unicellular, 4) $l_k \leq 3$, 5) $\xi_{1,k}$ or $\xi_{-1,k}$ differs from zero and $l_k \leq 4$. Then G is not AP factorable if, for at least one value of k ,*

(3.1)

$$|\xi'_{1,k}\xi_{-1,k}^\alpha| = |\xi_{0k}|^\lambda \neq 0, \text{ or } \xi_{-1,k} = \xi_{0k} = \xi_{1,k} = 0 \text{ and } |\eta'_{1,k}\eta_{-1,k}^\alpha| = |\eta_{0k}|^\lambda \neq 0,$$

and is AP_W factorable otherwise.

Proof. Using (1.6), introduce a matrix

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & T^{-1} \end{bmatrix} G \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} = \begin{bmatrix} e_\lambda I_m & 0 \\ \text{diag}[c_{-1,k}e_{-\nu} + c_{0k} + c_{1,k}e_\alpha] & e_{-\lambda} I_m \end{bmatrix},$$

having the same factorization properties as G . By an appropriate permutation of its rows and columns, this matrix can be further rewritten as a direct sum of the blocks

$$G_k = \begin{bmatrix} e_\lambda I_{l_k} & 0 \\ c_{-1,k}e_{-\nu} + c_{0k} + c_{1,k}e_\alpha & e_{-\lambda} I_{l_k} \end{bmatrix},$$

$k = 1, \dots, r$. Let $R = \{1, \dots, r\}$ and denote by R_0 the subset of those $r \in R$ such that $\xi_{1,k} = \xi_{-1,k} = \xi_{0k} = 0$, $l_k > 1$ and (at least) one of the blocks $c_{\pm 1,k}, c_{0k}$ is unicellular. We now partition R into a disjoint union $\bigcup_{j=1}^4 R_j$, where

$$\begin{aligned} R_1 &= \{k: |\xi'_{1,k}\xi_{-1,k}^\alpha| = |\xi_{0k}|^\lambda \neq 0\}, \\ R_2 &= \{k \in R_0: |\eta'_{1,k}\eta_{-1,k}^\alpha| = |\eta_{0k}|^\lambda \neq 0\}, \\ R_3 &= R_0 \setminus R_2, \\ R_4 &= R \setminus (R_1 \cup R_0). \end{aligned}$$

For every $k \in R_0$, yet another permutation of rows and columns allows us to represent G_k as a direct sum of $\begin{bmatrix} e_\lambda & 0 \\ 0 & e_{-\lambda} \end{bmatrix}$ with

$$G'_k = \begin{bmatrix} e_\lambda I_{l_k-1} & 0 \\ c'_{-1,k}e_{-\nu} + c'_{0k} + c'_{1,k}e_\alpha & e_{-\lambda} I_{l_k-1} \end{bmatrix}.$$

Here c'_{jk} are obtained from c_{jk} by deleting its first column and last row. The Toeplitz structure of c_{jk} is inherited by c'_{jk} . In particular, the c'_{jk} pairwise commute and $\sigma(c'_{jk}) = \{\eta_{jk}\}$ ($j = 0, \pm 1$; $k \in R_0$).

Denote by $G^{(1)}$ the direct sum of all the blocks G_k , $k \in R_1$, and G'_k , $k \in R_2$. Let $G^{(2)}$ be a direct sum of all G_k ($k \in R_4$), G'_k ($k \in R_3$), and $|R_2|$ copies of

the diagonal blocks $\begin{bmatrix} e_\lambda & 0 \\ 0 & e_{-\lambda} \end{bmatrix}$. Then G can be put in the form $G^{(1)} \oplus G^{(2)}$ by an appropriate permutation of its rows and columns. In turn, $G^{(1)}$ will become a permutation of a matrix of the type (1.4) with $f = b_{-1}e_{-\nu} + b_0 + b_1e_\alpha$ and $b_j = (\bigoplus_{k \in R_1} c_{jk}) \oplus (\bigoplus_{k \in R_2} c'_{jk})$.

In terms of the sets R_j , this theorem claims that G is AP_W factorable if $R_1 \cup R_2 = \emptyset$, and is not AP factorable otherwise. This follows from Lemma 2.1, provided that $G^{(2)}$ is AP_W factorable and, for $R_1 \cup R_2 \neq \emptyset$, $G^{(1)}$ is not AP factorable. The latter statement holds due to Corollary 1.4. It remains to prove the former. We will do this by showing that each direct summand of $G^{(2)}$ is AP_W factorable. There are five types of these summands:

(i) diagonal blocks $\begin{bmatrix} e_\lambda & 0 \\ 0 & e_{-\lambda} \end{bmatrix}$,

and matrices (1.4) with f given by (1.5), pairwise commuting $c_{\pm 1}, c_0$ (slightly abusing the notation, we again denote their size by m), singleton spectra $\sigma(c_j) = \{\xi_j\}$ ($j = \pm 1, 0$) for which

- (ii) $|\xi_1^\nu \xi_{-1}^\alpha| \neq |\xi_0|^\lambda$,
- (iii) $\xi_0 = 0$, exactly one of $\xi_{\pm 1}$ differs from zero and (at least) one of the blocks $c_{\pm 1}, c_0$ is unicellular,
- (iv) $\xi_0 = 0$, exactly one of $\xi_{\pm 1}$ differs from zero, and $m \leq 4$,
- (v) $\xi_0 = \xi_1 = \xi_{-1} = 0$ and $m \leq 3$.

Indeed, the blocks G_k with $k \in R_1$ have no impact on $G^{(2)}$, $k \in R_2$ generate only summands of type (i), $k \in R_3$ yield summands of type (i) and (ii) or (iii), and $k \in R_4$ produce summands of types (ii)-(v).

The summands of type (i) are trivially AP_W factorable (with partial AP indices $\pm \lambda$). The summands of type (ii) are AP_W factorable (with zero partial AP indices) according to Theorem 1.3. It remains to consider matrices (1.4) of types (iii)-(v).

In cases (iii) and (iv) we may without loss of generality suppose that $\xi_1 = 0$, $\xi_{-1} \neq 0$; otherwise, G_{f^*} can be considered instead of G_f . If in addition, $c_0 = 0$ or $c_1 = 0$, then f is a binomial and the corresponding matrix (1.4) is AP_W factorable due to Theorem 1.2. This happens, in particular, if $m = 1$.

If all three coefficients of f differ from zero, we consider the matrix (2.4). It can happen that $c_0^{s+2} = 0$, in which case the resulting block (2.5) is a binomial. Applying Theorem 1.2 and Lemma 2.2, we conclude that (2.4), and therefore (1.4), are AP_W factorable. If $c_0^{s+2} \neq 0$, we consider cases (iii) and (iv) separately.

(iii) The matrices c_j have an upper triangular Toeplitz structure which is inherited by the coefficients $c_j^{(1)}$ of (2.5). Hence,

$$m > \text{rank } c_0^{(1)} = \text{rank } c_1$$

and

$$m > \text{rank } c_{-1}^{(1)} = \text{rank } c_0^{s+1} > \text{rank } c_1^{(1)} = \text{rank } c_0^{s+2} > 0.$$

Let $q = \max\{\text{rank } c_0^{(1)}, \text{rank } c_{-1}^{(1)}\}$, $p = m - q$. Then both p and q are strictly positive. By a permutation of its rows and columns, the matrix G_1 can be reduced to the form

$$(3.2) \quad \begin{bmatrix} e_\nu I_p & 0 \\ 0 & e_{-\nu} I_p \end{bmatrix} \oplus \begin{bmatrix} e_\nu I_q & 0 \\ f_2 & e_{-\nu} I_q \end{bmatrix},$$

where

$$(3.3) \quad f_2 = c_{-1}^{(2)}e_{-\nu_1} + c_0^{(2)} + c_1^{(2)}e_{\alpha_1}$$

and the matrices $c_j^{(2)}$ are obtained from $c_j^{(1)}$ by deleting their first p columns and last p rows. It suffices to prove now that the second direct summand in (3.2) is AP_W factorable.

If $\text{rank } c_0^{(1)} \geq \text{rank } c_{-1}^{(1)}$, this summand falls into type (ii). In the opposite case, this is again a matrix of type (iii), but its size is strictly smaller than that of the original matrix: $q < m$. By induction we now conclude that all matrices of type (iii) are AP_W factorable.

(iv) The case of unicellular c_0 is covered by (iii). Since $m \leq 4$ and $c_0^{s+2} \neq 0$, the only remaining case is $s = 0$, $m = 4$ and c_0 consisting of one 3×3 and one 1×1 Jordan cell. The same Jordan structure is possessed by the matrix $c_{-1}^{-1}c_0$. Without loss of generality we may suppose that in (2.5)

$$(3.4) \quad c_{-1}^{(1)} = c_{-1}^{-1}c_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$c_1^{(1)} = -(c_{-1}^{-1}c_0)^2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix $c_0^{(1)} = c_{-1}^{-1}c_1$ is nilpotent and commutes with (3.4). Thus,

$$c_0^{(1)} = \begin{bmatrix} 0 & z & u & b \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \end{bmatrix}.$$

If $a = b = 0$, then the matrix G_1 can be split into a direct sum of $\begin{bmatrix} e_\nu I_2 & 0 \\ 0 & e_{-\nu} I_2 \end{bmatrix}$

and $G_2 = \begin{bmatrix} e_\nu I_2 & 0 \\ f_2 & e_{-\nu} I_2 \end{bmatrix}$, where f_2 is given by (3.3) with

$$c_{-1}^{(2)} = I_2, \quad c_0^{(2)} = \begin{bmatrix} z & u \\ 0 & z \end{bmatrix}, \quad c_1^{(2)} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

The matrix G_2 is of type (ii) or (iii) (depending on whether or not z is zero), and therefore AP_W factorable. Of course, G_1 is AP_W factorable together with G_2 .

If a or b differs from zero, represent G_1 as a direct sum of $\text{diag}[e_\nu, e_{-\nu}]$ with $G_3 = \begin{bmatrix} e_\nu I_3 & 0 \\ f_3 & e_{-\nu} I_3 \end{bmatrix}$, where $f_3 = c_{-1}^{(3)}e_{-\nu_1} + c_0^{(3)} + c_1^{(3)}e_{\alpha_1}$ and

$$c_{-1}^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad c_0^{(3)} = \begin{bmatrix} z & u & b \\ 0 & z & 0 \\ 0 & a & 0 \end{bmatrix}, \quad c_1^{(3)} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The explicit AP_W factorization of G_3 is shown in Appendix A of the supplement. Hence, all matrices of type (iv) are AP_W factorable.

Finally, consider the remaining type (v). If $m \leq 2$, then each matrix c_j either is unicellular or equals zero. In both cases, an AP_W factorization exists. Therefore, we may suppose that $m = 3$. Excluding another trivial case $c_0 = 0$ (in which f is a binomial), we are left with the only possible Jordan structure of c_0 : one 2×2 and one 1×1 block. Then, without loss of generality,

$$c_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrices $c_{\pm 1}$ commute with c_0 and are nilpotent. Therefore,

$$c_{\pm 1} = \begin{bmatrix} 0 & y_{\pm} & x_{\pm} \\ 0 & 0 & 0 \\ 0 & z_{\pm} & 0 \end{bmatrix}.$$

The matrix G splits into a direct sum of $\text{diag}[e_{\lambda}, e_{-\lambda}]$ and $G_1 = \begin{bmatrix} e_{\lambda}I_2 & 0 \\ f_1 & e_{-\lambda}I_2 \end{bmatrix}$,

where $f_1 = c_{-1}^{(1)}e_{-\nu} + c_0^{(1)} + c_1^{(1)}e_{\alpha}$,

$$c_0^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c_{\pm 1}^{(1)} = \begin{bmatrix} x_{\pm} & y_{\pm} \\ 0 & z_{\pm} \end{bmatrix}.$$

From commutativity of c_1 with c_{-1} it follows that $x_+z_- = x_-z_+$; however, later on we will encounter a factorization problem for matrices G_1 with $c_{\pm 1}^{(1)}$ not satisfying this requirement. Therefore, we do not impose the condition $x_+z_- = x_-z_+$ in our consideration.

The case $x_+ = x_- = z_+ = z_- = 0$ is excluded because otherwise the triple $\{c_{-1}, c_0, c_1\}$ would be reducible. The cases $x_+z_+ \neq 0$ and $x_-z_- \neq 0$ are covered by Corollary 2.5. In all the remaining cases an AP_W factorization of G_1 also exists; it is constructed explicitly in Appendix B of the supplement. Hence, matrices G of type (v) are also AP_W factorable. □

As an application of Theorem 3.1, consider a difference equation

$$(3.5) \quad c_{-1}y(t - \nu) + c_0y(t) + c_1y(t + \alpha) = g(t) \text{ a.e. on } (0, \lambda),$$

where g is a given vector function in $L^p(0, \lambda)$, y is an unknown vector function in $L^p(\mathbb{R})$ with $\text{supp } y \subset [0, \lambda]$.

According to standard terminology, we say that (3.5) is *normally solvable* (in L^p) if the set of vector functions g for which (3.5) has a solution is closed.

Theorem 3.2. *In (3.5) let $\alpha + \nu = \lambda$, let $\frac{\alpha}{\nu} (> 0)$ be irrational, and let the coefficients $c_j \in \mathbb{C}^{m \times m}$ satisfy the conditions of Theorem 3.1. Then the equation (3.5) is normally solvable if and only if, in the notation of Theorem 3.1, condition (3.1) fails for every k .*

This result does not depend on $p \in (1, \infty)$.

Proof. As follows from [7, Section 4.1], equation (3.5) is normally solvable if and only if the Wiener-Hopf operator W_G , the symbol G of which is given by (1.4), (1.5), has closed range in $L^p(0, \infty)$.

If condition (3.1) fails for all k , then the matrix function G is AP_W factorable due to Theorem 3.1. Hence, W_G has a generalized inverse, and therefore its range is closed.

To prove the converse statement, consider first a particular case when in (1.5) each matrix c_j has a singleton spectrum $\{\xi_j\}$, and

$$|\xi_1^\nu \xi_{-1}^\alpha| = |\xi_0|^\lambda \neq 0.$$

According to Theorem 3.1, the matrix function G in this case is not AP factorable.

If $m = 1$, the homogeneous equation (3.5) takes the form

$$y(t) = \begin{cases} -\frac{\xi_{-1}}{\xi_0} y(t - \nu) & \text{if } \nu < t < \lambda, \\ -\frac{\xi_1}{\xi_0} y(t + \alpha) & \text{if } 0 < t < \nu, \end{cases}$$

and has at most one linearly independent solution (see, for example, [4]).

For $m > 1$, a similarity can be used to put the c_j simultaneously in a triangular form, with ξ_j on the diagonal. Therefore, the number of linearly independent solutions of the respective homogeneous equation (3.5) is at most m . Suppose that this equation is normally solvable. Then the corresponding Wiener-Hopf operator W_G has a closed range and a finite dimensional kernel; in other words, it is n -normal. This property, as well as the *index* $\text{ind } W_G$ of the operator W_G (the difference between the dimension of its kernel and the codimension of its range), is preserved under small perturbations. Consider such a small perturbation $W_{G_{f'}}$ with $f' = c_{-1}e_{-\nu} + (c_0 + \epsilon I) + c_1 e_\alpha$, and $0 \neq |\xi_0 + \epsilon| \neq |\xi_0|$. Then $G' = G_{f'}$ admits an AP_W factorization with zero partial AP indices (Corollary 1.4), so that $W_{G'}$ is invertible. Hence, $\text{ind } W_G = \text{ind } W_{G'} = 0$. From here it follows that $\text{codim Im } W_G$ is finite together with $\dim \text{Ker } W_G$; that is, the operator W_G is Fredholm. Since $G \in AP_W$, Theorem 2.5 of [7] implies that G is AP_W factorable. This contradiction shows that in fact the range $\text{Im } W_G$ of the operator W_G is not closed.

Finally, consider the general case when (3.1) holds for some k . Then, as was shown in the proof of Theorem 3.1, the corresponding matrix G can be split into a direct sum of summands, a non-zero number of which are of the type just considered. Hence, W_G also splits into a direct sum of operators, some of which have a non-closed range. Therefore, $\text{Im } W_G$ is not closed. \square

Remark. The above reasoning shows that for matrix functions G satisfying the conditions of Theorem 3.1 the operator W_G has a closed range if and only if G is AP factorable. This is not true in general; examples of not AP factorable 2×2 triangular matrix functions $G \in AP_W$ for which $\text{Im } W_G$ is closed can be found in [10].

4. REMARKS ON 4×4 CASES

Theorem 3.1 covers all matrices (1.4), (1.5) with commuting c_j of size $m \leq 3$. Hence, the case of reducible 4×4 triples is also covered. For irreducible $\{c_{-1}, c_0, c_1\}$, each c_j has a singleton spectrum, say $\sigma(c_j) = \{\xi_j\}$. The cases when at least one of the ξ_j differs from zero or c_j is unicellular also fall into the setting of Theorem 3.1.

This leaves us with the situation of an irreducible triple of 4×4 nilpotent matrices c_j ($j = 0, \pm 1$), none of which is unicellular. We may suppose in addition that none of them is diagonalizable (that is, has only 1×1 Jordan cells). Indeed, a diagonalizable nilpotent matrix equals zero, and the corresponding G is then AP_W factorable due to Theorem 1.2. There remain three possible Jordan structures: two 2×2 cells, one 2×2 and two 1×1 cells, and one 3×3 and one 1×1 cells.

The following example demonstrates why the case of two 2×2 Jordan cells is hard to handle.

Example. Let $c_j = \begin{bmatrix} 0 & c_j^{(0)} \\ 0 & 0 \end{bmatrix}$, where the $c_j^{(0)}$ are arbitrary (not necessarily commuting) non-singular 2×2 matrices, $j = \pm 1, 0$. Then G can be split into a direct sum of $\begin{bmatrix} e_\lambda I_2 & 0 \\ 0 & e_{-\lambda} I_2 \end{bmatrix}$ and $G_0 = \begin{bmatrix} e_\lambda I_2 & 0 \\ c_{-1}^{(0)} e_{-\nu} + c_0^{(0)} + c_1^{(0)} e_\alpha & e_{-\lambda} I_2 \end{bmatrix}$. According to Lemma 2.1, the matrices G and G_0 are AP factorable only simultaneously. Hence, the AP factorization problem for G is reduced to the corresponding problem for matrices of the form (1.4) with non-commuting coefficients of f . Since the latter problem is still open, it is not surprising that a complete description of the AP factorability for matrices (1.4), (1.5) with commuting 4×4 coefficients c_j is also missing.

We will now discuss the two remaining possibilities for the Jordan structure of c_0 . First, let c_0 consist of one 2×2 and two 1×1 Jordan cells. Without loss of generality, c_0 itself is in a Jordan form:

$$(4.1) \quad c_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the commutativity of $c_{\pm 1}$ with c_0 and their nilpotency it follows that

$$c_{\pm 1} = \begin{bmatrix} 0 & a_\pm & b_\pm & d_\pm \\ 0 & 0 & 0 & 0 \\ 0 & f_\pm & h_\pm & l_\pm \\ 0 & g_\pm & j_\pm & k_\pm \end{bmatrix},$$

where $A_\pm = \begin{bmatrix} h_\pm & l_\pm \\ j_\pm & k_\pm \end{bmatrix}$ are themselves nilpotent.

We may also use a similarity to reduce A_+ to its Jordan canonical form without disturbing c_0 and the structure of A_- . Thus, $h_+ = k_+ = j_+ = 0$ and $l_+ = 0$ or 1 .

If $l_+ = 1$, then commutativity of c_1 with c_{-1} implies that $h_- = k_- = j_- = 0$. If $l_+ = 0$ (that is, $A_+ = 0$), then we can use a similarity to reduce A_- to its Jordan canonical form without changing c_0 and A_+ . Hence, in any case it may be supposed that $h_\pm = k_\pm = j_\pm = 0$, that is,

$$(4.2) \quad c_{\pm 1} = \begin{bmatrix} 0 & a_\pm & b_\pm & d_\pm \\ 0 & 0 & 0 & 0 \\ 0 & f_\pm & 0 & l_\pm \\ 0 & g_\pm & 0 & 0 \end{bmatrix}.$$

Also, from commutativity of c_1 with c_{-1} (which is preserved under the similarities applied above),

$$(4.3) \quad l_+ g_- = l_- g_+, \quad l_+ b_- = l_- b_+, \quad b_+ f_- + d_+ g_- = b_- f_+ + d_- g_+.$$

Theorem 4.1. *Let G be given by (1.4), (1.5) with $c_0, c_{\pm 1}$ as in (4.1) and (4.2), respectively, satisfying (4.3) and forming an irreducible triple $\{c_{-1}, c_0, c_1\}$. Then G is not AP factorable if*

$$b_+ = b_- = g_+ = g_- = 0, \quad |D_-^\alpha D_+^\nu| = |l_+^\nu l_-^\alpha| \neq 0,$$

where

$$D_{\pm} = \det \begin{bmatrix} a_{\pm} & d_{\pm} \\ f_{\pm} & l_{\pm} \end{bmatrix} = a_{\pm}l_{\pm} - d_{\pm}f_{\pm},$$

and is AP_W factorable otherwise.

Proof. We need to show that G is AP_W factorable if

- i) at least one of the numbers b_{\pm}, d_{\pm} differs from zero, or
- ii) $b_+ = b_- = g_+ = g_- = l_+l_-D_+D_- = 0$

and that in the case

$$\text{iii) } b_+ = b_- = g_+ = g_- = 0, \quad l_{\pm}D_{\pm} \neq 0$$

it is AP (AP_W) factorable if and only if

$$(4.4) \quad |D_-^{\alpha}D_+^{\nu}| \neq |l_+^{\nu}l_-^{\alpha}|.$$

In case i), rewrite G as a direct sum of $\text{diag}[e_{\lambda}, e_{-\lambda}]$ and another matrix of the form (1.4), with $m = 3$ and

$$c_{\pm 1} = \begin{bmatrix} a_{\pm} & b_{\pm} & d_{\pm} \\ f_{\pm} & 0 & l_{\pm} \\ g_{\pm} & 0 & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If c_{-1} is invertible, that is, $b_-g_-l_- \neq 0$, then Lemma 2.2 can be used. A direct computation shows that

$$c_{-1}^{-1}c_0 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{b_-} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and therefore $(c_{-1}^{-1}c_0)^2 = 0$. Hence, f_1 in (2.4) is at most a binomial, and the matrix G_1 is AP_W factorable due to Theorem 1.2. The original matrix G is then also AP_W factorable.

Using (2.8) and appropriate transpositions of rows and columns, we can cover the case of invertible c_1 , that is, $b_+g_+l_+ \neq 0$. It remains to construct an AP_W factorization in the cases when, in addition to (4.3),

$$(4.5) \quad b_+g_+l_+ = b_-g_-l_- = 0.$$

This is done in Appendix C.

In cases ii) and iii), we represent G as a direct sum of $\begin{bmatrix} e_{\lambda}I_2 & 0 \\ 0 & e_{-\lambda}I_2 \end{bmatrix}$ and another matrix G_1 of the form (1.4), (1.5) with $m = 2$ and

$$c_{\pm 1}^{(1)} = \begin{bmatrix} a_{\pm} & d_{\pm} \\ f_{\pm} & l_{\pm} \end{bmatrix}, \quad c_0^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $l_+ = 0$ and $d_+f_+ \neq 0$, then the matrix G_1 is AP_W factorable due to Corollary 2.5. The same reasoning applies if $l_- = 0, d_-f_- \neq 0$. The cases $l_+ = l_- = d_+f_+ = d_-f_- = 0$ when not all of the four entries d_{\pm}, f_{\pm} equal zero are covered by Appendix B in the supplement. Observe that the case $d_{\pm} = f_{\pm} = 0$ is excluded due to the irreducibility of the original triple $\{c_{-1}, c_0, c_1\}$ given by (4.1), (4.2). Hence, the situation when $l_+ = l_- = 0$ is covered completely.

In all other cases (when at least one of l_+, l_- differs from zero) we may use the symmetry (2.8) to suppose without loss of generality that, say, $l_- \neq 0$. An obvious

similarity performed on the original 4×4 matrices $c_{\pm 1}$ (and not changing c_0) allows us to suppose in addition that $d_- = f_- = 0$. This similarity may, of course, change the values of a_{\pm} and d_+, f_+ ; however, $\det c_{\pm 1}^{(1)}$ remain the same, so that the new value of a_- is D_-/l_- . To simplify the notation, we redenote D_+ by D .

If $l_+ = 0$, then d_+, f_+ do not change under the above mentioned similarity. The only situation left uncovered by previous considerations is the case in which exactly one of d_+, f_+ differs from zero.

In case ii), we are left with only two possibilities: 1) $l_- \neq 0, l_+ = d_- = f_- = 0$, exactly one of the entries d_+, f_+ differs from zero, and 2) $l_+l_- \neq 0, d_- = f_- = 0, a_-D = 0$. Appendix D in the supplement shows that the corresponding matrix G_1 (and therefore G) is AP_W factorable.

In case iii), the additional condition $d_- = f_- = 0$ means that $a_-(= D_-/l_-) \neq 0$, and (4.4) can be rewritten as

$$(4.6) \quad |a_-^\alpha D^\nu| \neq |l_+^\nu|.$$

A straightforward calculation shows that $G_1 = X_+ G' X_-$, where

$$X_+ = \begin{bmatrix} 1 & d_+l_-e_\lambda & 0 & 0 \\ -\frac{f_+}{l_+} & a_-l_+e_\lambda - l_-(e_\nu + a_-) & -e_\nu & \frac{f_+e_\nu}{l_+} \\ 0 & d_+(a_-l_+ + a_+l_-)e_\alpha & -d_+ & a_+ \\ 0 & (a_-l_+^2 + d_+f_+l_-)e_\alpha - l_-l_+ & -l_+ & f_+ \end{bmatrix}$$

is invertible in AP_W^+ ,

$$X_- = \begin{bmatrix} 1 & -\frac{d_+l_-}{a_-l_+} & 0 & -\frac{d_+e_{-\alpha}}{a_-l_+} \\ 0 & \frac{1}{a_-l_+} & 0 & \frac{e_{-\alpha}}{a_-l_+l_-} \\ \frac{f_+(1+a_-e_{-\nu})}{D} & -\frac{a_+l_-(1+a_-e_{-\nu})}{a_-D} & \frac{f_+e_{-\lambda}}{D} & \frac{1}{l_-} - \frac{a_+(a_-e_{-\lambda}+e_{-\alpha})}{a_-D} \\ 0 & 0 & \frac{l_+}{D} & 0 \end{bmatrix}.$$

is invertible in AP_W^- , and

$$G' = \begin{bmatrix} e_\lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_-l_+e_{-\nu} + l_+ + De_\alpha & 0 & 0 & e_{-\lambda} \end{bmatrix}$$

can be split into a direct sum of I_2 with

$$G_2 = \begin{bmatrix} e_\lambda & 0 \\ a_-l_+e_{-\nu} + l_+ + De_\alpha & e_{-\lambda} \end{bmatrix}.$$

Of course, G_1 is AP (AP_W) factorable only simultaneously with G' , and in turn, G' has the same factorability properties as G_2 . The latter matrix satisfies the conditions of Corollary 1.4 with $m = 1$. In the notation of this statement, $\xi_{1,k} = D, \xi_{0k} = l_+$ and $\xi_{-1,k} = a_-l_+$ with the only value of k ($=1$), so that condition (1.7), necessary and sufficient for an AP (AP_W) factorization to exist, is equivalent to (4.6). □

Finally, let c_0 consist of one 3×3 and one 1×1 Jordan cells

$$c_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the only possible form of $c_{\pm 1}$ is

$$c_{\pm 1} = \begin{bmatrix} 0 & d_{\pm} & f_{\pm} & b_{\pm} \\ 0 & 0 & d_{\pm} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{\pm} & 0 \end{bmatrix},$$

where

$$(4.7) \quad a_+ b_- = a_- b_+.$$

The case $a_+ = a_- = b_+ = b_- = 0$ is excluded if the triple $\{c_{-1}, c_0, c_1\}$ is irreducible. Splitting G into a direct sum of $\text{diag}[e_{\lambda}, e_{-\lambda}]$ and another matrix of the form (1.4), we may suppose that $m = 3$ and

$$c_{\pm 1} = \begin{bmatrix} d_{\pm} & f_{\pm} & b_{\pm} \\ 0 & d_{\pm} & 0 \\ 0 & a_{\pm} & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the case when all four of the coefficients a_{\pm} , b_{\pm} are different from zero, an AP_W factorization exists and can be explicitly constructed (see Appendix E in the supplement). Due to the commutativity condition (4.7), the number of non-zero entries among a_{\pm} , b_{\pm} cannot equal one. However, there remain cases of exactly two or three non-zero numbers a_{\pm} , b_{\pm} , and in these cases the AP factorability of the corresponding matrices G is still unknown.

REFERENCES

- [1] M. Bakonyi, L. Rodman, I. Spitkovsky, and H. Woerdeman, *Positive extensions of matrix functions of two variables with support in an infinite band*, C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), no. 8, 859–863. MR **97i**:47023
- [2] M. A. Bastos, Yu. I. Karlovich, I. M. Spitkovsky, and P. M. Tishin, *On a new algorithm for almost periodic factorization*, Operator Theory: Advances and Applications **103** (1998), 53–74. CMP 98:16
- [3] C. Corduneanu, *Almost periodic functions*, J. Wiley & Sons, 1968. MR **58**:2006
- [4] N. K. Karapetjanc and S. G. Samko, *The functional equation $\psi(x + \alpha) - b(x)\psi(x) = g(x)$* , Izv. Akad. Nauk Armjan. SSR. Ser. Mat. **5** (1970), no. 5, 441–448. MR **44**:2101
- [5] Yu. I. Karlovich, *On the Haseman problem*, Demonstratio Math. **26** (1993), 581–595. MR **95a**:47048
- [6] Yu. I. Karlovich and I. M. Spitkovsky, *Factorization of almost periodic matrix-valued functions and the Noether theory for certain classes of equations of convolution type*, Mathematics of the USSR, Izvestiya **34** (1990), 281–316. MR **90f**:47034
- [7] ———, *(Semi)-Fredholmness of convolution operators on the spaces of Bessel potentials*, Operator Theory: Advances and Applications **71** (1994), 122–152. MR **95h**:47034
- [8] ———, *Almost periodic factorization: An analogue of Chebotarev's algorithm*, Contemporary Math. **189** (1995), 327–352. MR **96h**:47024
- [9] ———, *Factorization of almost periodic matrix functions*, J. Math. Anal. Appl. **193** (1995), 209–232. MR **96m**:47047
- [10] ———, *Semi-Fredholm properties of certain singular integral operators*, Operator Theory: Advances and Applications **90** (1996), 264–287. MR **97k**:47046
- [11] B. M. Levitan, *Almost periodic functions*, GITTL, Moscow, 1953 (in Russian). MR **15**:700a
- [12] B. M. Levitan and V. V. Zhikov, *Almost periodic functions and differential equations*, Cambridge University Press, 1982. MR **84g**:34004
- [13] G. S. Litvinchuk and I. M. Spitkovsky, *Factorization of measurable matrix functions*, Birkhäuser Verlag, Basel and Boston, 1987. MR **90g**:47030
- [14] Yu. Lyubarskii and I. Spitkovsky, *Sampling and interpolating for a lacunary spectrum*, Royal Society of Edinburgh, Proceedings **126A** (1996), 77–87. MR **97b**:41004

- [15] L. Rodman, I. M. Spitkovsky, and H. J. Woerdeman, *Carathéodory-Toeplitz and Nehari problems for matrix valued almost periodic functions*, Trans. Amer. Math. Soc. **350** (1998), 2185–2227. MR **98h**:47023
- [16] I. M. Spitkovsky, *On the factorization of almost periodic matrix functions*, Math. Notes **45** (1989), no. 5–6, 482–488. MR **90k**:47033
- [17] I. M. Spitkovsky and H. J. Woerdeman, *The Carathéodory-Toeplitz problem for almost periodic functions*, J. Functional Analysis **115** (1993), no. 2, 281–293. MR **94f**:47020

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