ALMOST PERIODIC FACTORIZATION OF CERTAIN BLOCK TRIANGULAR MATRIX FUNCTIONS

ILYA M. SPITKOVSKY AND DARRYL YONG

Abstract. Let

\[ G(x) = \begin{bmatrix} e^{i\lambda x} I_m & 0 \\ c_{-1}e^{-i\nu x} + c_0 + c_1e^{i\alpha x} & e^{-i\lambda x} I_m \end{bmatrix}, \]

where \( c_j \in \mathbb{C}^{m \times m}, \alpha, \nu > 0 \) and \( \alpha + \nu = \lambda \). For rational \( \alpha/\nu \) such matrices \( G \) are periodic, and their Wiener-Hopf factorization with respect to the real line \( \mathbb{R} \) always exists and can be constructed explicitly. For irrational \( \alpha/\nu \), a certain modification (called an almost periodic factorization) can be considered instead. The case of invertible \( c_0 \) and commuting \( c_1c_0^{-1}, c_{-1}c_0^{-1} \) was disposed of earlier—it was discovered that an almost periodic factorization of such matrices \( G \) does not always exist, and a necessary and sufficient condition for its existence was found.

This paper is devoted mostly to the situation when \( c_0 \) is not invertible but the \( c_j \) commute pairwise (\( j = 0, \pm 1 \)). The complete description is obtained when \( m \leq 3 \); for an arbitrary \( m \), certain conditions are imposed on the Jordan structure of \( c_j \). Difficulties arising for \( m = 4 \) are explained, and a classification of both solved and unsolved cases is given.

The main result of the paper (existence criterion) is theoretical; however, a significant part of its proof is a constructive factorization of \( G \) in numerous particular cases. These factorizations were obtained using Maple; the code is available from the authors upon request.

1. Introduction

Let \( AP \) be the Bohr algebra of almost periodic functions, that is, the smallest \( C^* \)-algebra of \( L^\infty(\mathbb{R}) \) containing all the functions \( e_\lambda(x) = e^{i\lambda x}, \lambda \in \mathbb{R} \). It is well known (the standard references for these and other properties of \( AP \) are [3, 11, 12]) that for every \( f \in AP \),

1. there exists the Bohr mean value

\[ M(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T f(x) \, dx, \]

and

2. the Fourier coefficients \( \hat{f}(\lambda) \) are different from zero for at most countably many values of \( \lambda \in \mathbb{R} \).
The set \( \Omega(f) = \{ \mu : \hat{f}(\mu) \neq 0 \} \) is called the Fourier spectrum of \( f \), and
\[
(1.1) \quad \sum_{\mu \in \Omega(f)} \hat{f}(\mu) e_\mu
\]
is its (formal) Fourier series.

We say that \( f \in AP_W \) if the Fourier series (1.1) converges absolutely:
\[
\sum_{\mu \in \Omega(f)} |\hat{f}(\mu)| < \infty.
\]

Finally, let
\[
AP^\pm = \{ f \in AP : \Omega(f) \subseteq \mathbb{R}_\pm \} \quad \text{and} \quad AP^\pm_W = AP^\pm \cap AP_W.
\]

Here, as usual, \( \mathbb{R}_\pm = \{ x \in \mathbb{R} : \pm x \geq 0 \} \).

For matrix functions \( f \), conditions \( f \in AP, AP^\pm, AP_W, \) etc. are understood entrywise, and \( M(f), \hat{f}(\mu), \Omega(f) \) are defined by exactly the same formulas as for scalar functions.

Following [6], we introduce an \( AP \) factorization of an \( n \times n \) matrix function \( G \) as its representation in the form
\[
(1.2) \quad G = G_+ \Lambda G_-,
\]
where \( \Lambda(x) = \text{diag}[e_{\lambda_1}, \ldots, e_{\lambda_n}] \),
\[
(1.3) \quad G_\pm^{\pm 1} \in AP^\pm, \quad G_\pm^{\pm 1} \in AP^\pm,
\]
and \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \). We say that (1.2) is an \( AP_W \) factorization of \( G \) if conditions (1.3) are replaced by the (more restrictive) conditions \( G_\pm^{\pm 1} \in AP^\pm_W, G_\pm^{\pm 1} \in AP^\pm_W \).

If \( G \) is \( AP \) factorable, the numbers \( \lambda_1, \ldots, \lambda_n \) are defined uniquely; they are called the partial \( AP \) indices (of \( G \)). Of course, for an \( AP \) (\( AP_W \)) factorization (1.2) to exist it is necessary that \( G^{\pm 1} \in AP \) (respectively, \( AP_W \)). However, this necessary condition is not sufficient and, except for the case of periodic matrix functions \( G \) (in which an \( AP \) factorization by a simple change of variable reduces to the usual Wiener-Hopf factorization), the theory of \( AP \) factorization is “under construction”. Its connections with integral equations, completion problems, and signal processing are discussed in [6, 7], [17, 13, 1], and [14] respectively. Explicit formulas for the factors in (1.2) for certain special types of \( G \) are obtained in [6, 9, 8].

Most of them refer to matrices \( G \) of the following block triangular form:
\[
(1.4) \quad G_f = \begin{bmatrix} e_\lambda I_m & 0 \\ f & e_{-\lambda} I_m \end{bmatrix}
\]
(so that \( n = 2m \)) arising in the treatment of convolution type equations on finite intervals of length \( \lambda \).

In particular, the following two statements were established in [3].

**Lemma 1.1.** Let \( f \) be an \( AP_W \) matrix function, and let
\[
f_0 = \sum_{\mu \in \Omega(f) \cap (-\lambda, \lambda)} \hat{f}(\mu) e_\mu.
\]
Then the matrices \( G_f \) and \( G_{f_0} \) are \( AP \) (\( AP_W \)) factorable only simultaneously, and their partial \( AP \) indices coincide.

Due to Lemma 1.1 for any \( f \in AP_W \) in (1.1) we may suppose without loss of generality that \( \Omega(f) \subset (-\lambda, \lambda) \).
Theorem 1.2. Let \( \Omega(f) \cap (-\lambda, \lambda) \) consist of at most two points, say \( \mu \) and \( \sigma \). Then \( G_f \) is APW factorable. Its partial AP indices all equal zero if and only if \( \mu \sigma = 0 \), \( f(0) \) is invertible, or \( \mu \sigma < 0 \), \( \frac{\lambda}{\mu \sigma} \in \mathbb{Z} \) and both \( \hat{f}(\mu) \), \( \hat{f}(\sigma) \) are invertible.

The next logical step is to consider a trinomial \( f \) with \( \Omega(f) \subset (-\lambda, \lambda) \). However, with no additional restrictions on the location of \( \Omega(f) \) this remains an open problem. In this paper, we concentrate on the case \( \Omega(f) = \{-\nu, 0, \alpha\} \), that is,

\[
(1.5) \quad f = c_{-1}e^{-\nu} + c_0 + c_1e_{\alpha},
\]

where \( \alpha, \nu > 0 \) and \( \alpha + \nu = \lambda \).

If \( \beta = \frac{\nu}{\alpha} \) is rational, then the matrix \( G_f \) is periodic, and its APW factorization exists and can be easily constructed. Thus, we suppose in what follows that \( \beta \) is irrational. The next result applies to the case when the matrices \( c_j \) in \( (1.5) \) commute with each other. In this case there exists a similarity \( T \) such that

\[
(1.6) \quad T^{-1}c_jT = \text{diag}[c_{j1}, \ldots, c_{jkr}], \quad c_{jk} \in \mathbb{C}^{l_k \times l_k}, \quad k = 1, \ldots, r; \quad j = 0, \pm1,
\]

and each diagonal block \( c_{jk} \) has a singleton spectrum (see [13] Section 4.4):

\[
\sigma(c_{jk}) = \{\xi_{jk}\} \quad (j = 0, \pm1; k = 1, \ldots, r).
\]

As in [2], we call \( \{\xi_{jk}\}_{j=-1}^1 \) the bonded eigenvalue triples of \( c_j \).

Theorem 1.3. Let \( G_f \) be of the form \( (1.4) \) with \( f \) given by \( (1.5) \) and commuting coefficients \( c_j \). Then \( G_f \) is AP factorable with zero partial AP indices if and only if, for all bonded triples \( \{\xi_{-1,k}, \xi_{0,k}, \xi_{1,k}\} \),

\[
(1.7) \quad |\xi_{0,k}^\alpha| \neq |\xi_{1,k}^\alpha| \quad (k = 1, \ldots, r).
\]

In this form, Theorem 1.3 was established in [2] Theorem 7.2]; the case of invertible \( c_j \) was disposed of earlier in [9]. In fact, the result of [9] contains an additional important piece of information: if all \( c_j \) are invertible and \( (1.7) \) fails for at least one value of \( k \), then \( G_f \) does not admit any AP factorization, even if non-zero partial AP indices are allowed. Also, it was shown in [16] that an AP factorization with zero partial AP indices of an APW matrix function is automatically its APW factorization. Hence, the following result holds.

Corollary 1.4. Let \( G_f \) be as in Theorem 1.3 and, in addition, let \( c_0 \) be invertible. Then \( G_f \) is APW factorable with zero partial AP indices if condition \( (1.7) \) holds, and is not AP factorable otherwise.

Of course, it would now be natural to consider an AP factorization of \( G_f \) with trinomial \( f \), pairwise commuting \( c_j \), and no restrictions imposed on the invertibility of \( c_0 \) and the values of partial AP indices. We will see, however, that this problem embraces a general setting of a trinomial \( f \) with arbitrary (not necessarily commuting) coefficients \( c_j \) and is therefore too difficult to handle at the present stage of the development. Our paper is a report on several partial results on the AP factorability of matrices \( (1.4), (1.5) \) with non-invertible \( c_0 \).

The paper is structured as follows. Section 2 contains an auxiliary result on the factorization of block diagonal matrices. It also describes a procedure which allows us to replace a matrix of the form \( (1.4), (1.5) \) with invertible \( c_{-1} \) (and no commutativity conditions on \( c_j \)) by another matrix of the same type without changing its factorability properties. This procedure is, in fact, a variation of the one introduced in [2] for matrices \( (1.4) \) with a finite number (not limited to three) of
points \( \mu_j \) in \( \Omega(f) \cap (-\lambda, \lambda) \) but pairwise commuting \( \hat{f}(\mu_j) \). As a direct application of this procedure, \( AP_W \) factorability is established for matrices (1.4), (1.5) with \( m = 2 \), invertible \( c_{-1} \) (or \( c_1 \)) and nilpotent \( c_0 c_{-1} \) (respectively, \( c_0 c_1^{-1} \)).

Section 3 contains necessary and sufficient factorability conditions for matrices (1.4), (1.5) with commuting \( c_j \) under certain additional restrictions on their Jordan structure. This covers, in particular, all matrices of size \( m \leq 3 \), invertible \( c_1 \) or \( c_{-1} \) of size \( m \leq 4 \), and matrices of arbitrary size, provided that each eigenvalue of at least one of the \( c_j \) corresponds to one Jordan cell. An application to difference equations is given.

In Section 4, we concentrate on \( 4 \times 4 \) matrices \( c_j \). An example is given explaining why this case cannot be covered in general before the \( AP \) factorability of matrices (1.4), (1.5) with arbitrary invertible non-commuting \( c_j \) is understood. All possible cases are classified, and those for which the \( AP \) factorability remains unknown are singled out.

Proofs of the results in Sections 3 and 4 are partially theoretical and partially consist in exhausting a large number of cases in which an \( AP_W \) factorization can be constructed explicitly. These cases are relegated to Section 5 the supplement at the end of this volume, where final formulas are listed. Of course, they can be checked by straightforward calculations. We emphasize, however, that a symbolic manipulation Maple program was used to obtain these formulas, and without it this paper could hardly have been completed.

2. Auxiliary results

Suppose \( G \) is a block diagonal \( AP \) matrix: \( G = \text{diag}[G_1, G_2] \). If its diagonal blocks \( G_1, G_2 \) are \( AP \) factorable, then \( G \) itself is \( AP \) factorable. Moreover, an \( AP \) factorization of \( G \) can be obtained by “pasting together” \( AP \) factorizations of \( G_1 \) and \( G_2 \): \( G_1 = G_+^{(1)} \Lambda_+^{(1)} G_-^{(1)}, \quad G_2 = G_+^{(2)} \Lambda_+^{(2)} G_-^{(2)}, \) imply

\[
G = \text{diag}[G_+^{(1)}, G_+^{(2)}] \text{diag}[\Lambda_+^{(1)}, \Lambda_+^{(2)}] \text{diag}[G_-^{(1)}, G_-^{(2)}].
\]

It is natural to ask whether the converse is true. The answer is positive provided that \( G \in AP_W \) and partial \( AP \) indices of \( G \) equal zero. Indeed, a matrix \( F \in AP_W \) admits an \( AP \) factorization with zero partial \( AP \) indices if and only if the corresponding Toeplitz operator \( T_F \) is invertible on \( L^2 \) (see also [7]). Since \( T_G \) is a direct sum of \( T_{G_1} \) with \( T_{G_2} \), the invertibility of \( T_G \) is equivalent to simultaneous invertibility of \( T_{G_1} \) and \( T_{G_2} \).

We are not aware of any equivalent of \( AP \) factorability (with non-zero partial \( AP \) indices) in operator terms. Probably, the answer to the question is still positive, but we restrict our consideration to a somewhat weaker version.

Lemma 2.1. Let \( G = \text{diag}[G_1, G_2] \). If \( G \) and one of its diagonal blocks \( G_1, G_2 \) are \( AP \) factorable, then the other diagonal block is also \( AP \) factorable.

Proof. Consider first the case when \( G_1 = 1 \). Then an \( AP \) factorization of \( G \) can be rewritten as

\[
F_+ \begin{bmatrix} 1 & 0 \\ 0 & G_2 \end{bmatrix} = \Lambda G_-,
\]
where \( F_+ = G_+^{-1} \in AP^+ \). Denote \( F_+ = (f_{ij})_{i,j=1}^n \). From (2.1), \( e^{-\lambda_j} f_{j1} \in AP^- \), so that

\[
\Omega(f_{j1}) \subset [0, \lambda_j].
\]

In particular, \( f_{j1} = 0 \) for all \( j \) (if there are any) such that \( \lambda_j < 0 \). Rewriting (2.1) as

\[
\begin{bmatrix} 1 & 0 \\ 0 & G_2^{-1} \end{bmatrix} G_+ \Lambda = G_+^{-1},
\]

we find similarly that

\[
\Omega(g_{ij}) \subset [0, -\lambda_j],
\]

where \( G_+ = (g_{ij})_{i,j=1}^n \). Therefore, \( g_{ij} = 0 \) for all \( j \) (if there are any) such that \( \lambda_j > 0 \). Observe also that \( G_+ F_+ = I \) implies that \( \sum_{j=1}^n g_{ij} f_{j1} = 1 \). Since for non-zero \( \lambda_j \) at least one of the entries \( g_{1j}, f_{j1} \) is equal to zero, the latter equality proves the existence of zero partial \( AP \) indices \( \lambda_j \). Due to (2.2), (2.3), the corresponding functions \( g_{1j}, f_{j1} \) are constant, and for at least one value of \( j \), \( g_{1j} f_{j1} \neq 0 \).

Applying an appropriate permutation of columns of \( G_+ \) and rows of \( G_- \), we may suppose without loss of generality that \( \lambda_1 = 0 \), \( g_{11} = c \neq 0 \), \( f_{11} = d \neq 0 \).

Partitioning \( G_+, F_+ \) as

\[ G_+ = \begin{bmatrix} c & g_1^+ \\ g_2^+ & G_2^+ \end{bmatrix}, \quad F_+ = \begin{bmatrix} d & f_1^+ \\ f_2^+ & F_2^+ \end{bmatrix}, \]

we conclude that \( c = \det F_2^+ / \det F_+ = \det F_2^+ \det G_+ \). Since \( c \neq 0 \), the matrix \( F_2^+ \) is invertible in \( AP^+ \) simultaneously with \( G_+ \). From (2.1) and (2.2) it follows that the left-upper entry of \( G_- \) and \( H_- = G_-^{-1} \) equals \( d \) and \( c \), respectively. Thus,

\[ G_- = \begin{bmatrix} d & g_1^- \\ g_2^- & G_2^- \end{bmatrix}, \quad H_- = \begin{bmatrix} c & h_1^- \\ h_2^- & H_2^- \end{bmatrix}, \]

and \( c = \det G_2^- / \det G_- \). Since \( c \neq 0 \), the matrix \( G_2^- \) is invertible in \( AP^- \) together with \( G_- \). Now partition \( \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \). Then (2.1) yields \( F_2^+ G_2 = \Lambda_2 G_2^+ \), or \( G_2 = (F_2^+)^{-1} \Lambda_2 G_2 \). Since \( (F_2^+)^{\pm 1} \in AP^+ \) and \( (G_2^-)^{\pm 1} \in AP^- \), the latter formula delivers an \( AP \) factorization of \( G_2 \). This proves the desired statement in the case \( G_1 = 1 \).

If \( G_1 = e_{\lambda} \), then the matrix \( e^{-\lambda} G = \text{diag}[1, e^{-\lambda} G_2] \) is \( AP \) factorable together with \( G \). According to the already proven particular case, \( e^{-\lambda} G_2 \) is \( AP \) factorable. But then \( G_2 \) is \( AP \) factorable as well.

An induction argument allows us to consider \( G_1 \) of the form \( \text{diag}[e_{\lambda_1}, \ldots, e_{\lambda_k}] = \Lambda_1 \). Finally, for an arbitrary \( AP \) factorable \( G_1 = G_1(1) \Lambda_1 G_1(1) \) we can write

\[ G = \text{diag}[G_1(1), I] \text{diag}[\Lambda_1, G_2] \text{diag}[G_1(1), I] \]

and consider an \( (AP \) factorable) matrix \( \text{diag}[\Lambda_1, G_2] \) instead of the original matrix \( G \).

Another technical tool we need applies to matrix functions \( G_f \) with a trinomial \( f \) containing an invertible \( c_{-1} \) coefficient.
Lemma 2.2. Let $G$ be of the form (1.4) with $f$ given by (1.5). If $c_{-1}$ is invertible, then $G$ is $AP$ ($AP^W$) factorable only simultaneously with (and has the same partial $AP$ indices as) the matrix function

\begin{equation}
G_1 = \begin{bmatrix} e_{\lambda_1} I_m & 0 \\ f_1 & e_{-\lambda_1} I_m \end{bmatrix},
\end{equation}

where

\begin{equation}
f_1 = c_{-1}^{(1)} e_{-\nu} + c_0^{(1)} + c_1^{(1)} e_{\alpha_1},
\end{equation}

\begin{align*}
c_{-1}^{(1)} &= (-1)^s (c_{-1}^{-1} c_0)^{s+1}, \\
c_0^{(1)} &= c_{-1}^{-1} c_1, \\
c_1^{(1)} &= (-1)^{s+1} (c_{-1}^{-1} c_0)^{s+2},
\end{align*}

\[\lambda_1 = \nu, \quad \nu_1 = \alpha - s\nu, \quad \alpha_1 = (s + 1)\nu - \alpha,\]

and finally, $s$ is the integral part of $\frac{\nu}{\nu^*}; \ s \in \mathbb{Z}$ and $s < \frac{\nu}{\nu^*} < s + 1$.

Proof. It suffices to construct matrix functions $X_+$ and $X_-$ such that $X_+ \in AP^W$, $X_- \in AP^W$ and

\begin{equation}
X_+ G X_- = G_1.
\end{equation}

To this end, let

\begin{equation}
X_+ = \begin{bmatrix} c_{-1}^{-1} f e_{\nu} \\ e_{-\lambda_1} - \nu I + (g - e_{-\lambda_1})e_{-1}^{-1} f I - ge_{\lambda} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & c_{-1}^{-1} \end{bmatrix},
\end{equation}

\begin{equation}
X_- = \begin{bmatrix} I & 0 \\ 0 & c_{-1} \\ e_{-\alpha} I + \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{j-1} I - I - I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},
\end{equation}

with $g = c_{-1}^{-1} c_1 - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{j-1} - \alpha$. Directly from the definition of $s$ it follows that $X_- \in AP^-$. Since $\det X_- = \det c_{-1}$ is a non-zero constant, $X_-^{-1}$ belongs to $AP^-$ together with $X_-$. A straightforward computation shows that

\begin{align*}
X_+ G X_- &= \begin{bmatrix} c_{-1}^{-1} f e_{\nu} \\ e_{-\lambda_1} - \nu I + (g - e_{-\lambda_1})e_{-1}^{-1} f I - ge_{\lambda} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & c_{-1}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} e_{\nu} I + \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{j+1} \\ c_{-1}^{-1} e_{-\alpha} f + c_{-1}^{-1} f \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{j-1} - e_{-\lambda} I - c_{-1}^{-1} f \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\
&= (y_{ij})_{i,j=1}^{s},
\end{align*}
where
\[
y_{11} = c^{-1}_1 f e_{2\nu} + c^{-1}_1 f \sum_{j=1}^{s} (-1)^j (c^{-1}_1 c_0)^j e_{(j+2)\nu} - c^{-1}_1 f e_{2\nu} \\
- c^{-1}_1 f \sum_{j=1}^{s} (-1)^j (c^{-1}_1 c_0)^j e_{(j+2)\nu} + e_{\nu} I = e_{\nu} I,
\]
\[
y_{12} = c^{-1}_1 f e_{\lambda +\nu} - c^{-1}_1 f e_{\lambda +\nu} = 0,
\]
\[
y_{22} = e_{-\nu} I + (ge_{\lambda} - I)e^{-1}_1 f + (I - ge_{\lambda})c^{-1}_1 f = e_{-\nu} I,
\]
and finally,
\[
y_{21} = (e_{-\lambda -\nu} I + (g - e_{-\lambda} I)c^{-1}_1 f) \left( e_{\nu} I + \sum_{j=1}^{s} (-1)^j (c^{-1}_1 c_0)^j e_{(j+1)\nu} \right) \\
+ (I - ge_{\lambda}) \left( c^{-1}_1 e_{-\alpha} + (I - c^{-1}_1 f \sum_{j=1}^{s} (-1)^j (c^{-1}_1 c_0)^j e_{j\nu -\alpha} - e_{-\lambda} I) \right) \\
= e_{-\lambda} I + \sum_{j=1}^{s} (-1)^j (c^{-1}_1 c_0)^j e_{j\nu -\lambda} + (g - e_{-\lambda} I) \\
\times \left( c^{-1}_1 f e_{\nu} + c^{-1}_1 f \sum_{j=1}^{s} (-1)^j (c^{-1}_1 c_0)^j e_{(j+1)\nu} \\
- c^{-1}_1 f e_{\nu} - c^{-1}_1 f \sum_{j=1}^{s} (-1)^j (c^{-1}_1 c_0)^j e_{(j+1)\nu} + I \right) \\
= e_{-\lambda} I + \sum_{j=1}^{s} (-1)^j (c^{-1}_1 c_0)^j e_{j\nu -\lambda} + g - e_{-\lambda} I
\]
\[
= \sum_{j=1}^{s} (-1)^j (c^{-1}_1 c_0)^j e_{j\nu -\lambda} + c^{-1}_1 c_1 - \sum_{j=1}^{s+2} (-1)^j (c^{-1}_1 c_0)^j e_{(j-1)\nu -\alpha} \\
= (-1)^s (c^{-1}_1 c_0)^{s+1} e_{\nu -\alpha} + c^{-1}_1 c_1 + (-1)^{s+1} (c^{-1}_1 c_0)^{s+2} e_{(s+1)\nu -\alpha} = f_1.
\]

This implies (2.10). Since \( \det G = \det G_1 = 1 \), from (2.10) it follows, in particular, that \( \det X_+ = (\det X_-)^{-1} \) is a non-zero constant. It remains to show that \( X_+ \in AP^+ \), because then \( X_+^{-1} \in AP^+ \) as well. Three blocks of \( X_+ \) are obviously in \( AP^+ \).

The remaining (left-lower) block can be rewritten as
\[
e_{-\lambda -\nu} I + (g - e_{-\lambda} I)c^{-1}_1 f \\
= e_{-\lambda -\nu} I + c^{-1}_1 c_1 e_{-\nu} + c^{-1}_1 c_1 c^{-1}_1 c_0 + (c^{-1}_1 c_1)^2 e_{\alpha} \\
- \sum_{j=1}^{s+2} (-1)^j (c^{-1}_1 c_0)^j e_{(j-2)\nu -\alpha} - \sum_{j=1}^{s+2} (-1)^j (c^{-1}_1 c_0)^j e_{(j-1)\nu -\alpha} \\
- \sum_{j=1}^{s+2} (-1)^j (c^{-1}_1 c_0)^j e_{(j-1)\nu -\alpha} - e_{-\lambda -\nu} I - c^{-1}_1 c_0 e_{-\lambda} - c^{-1}_1 c_1 e_{-\nu} \\
= c^{-1}_1 c_1 c^{-1}_1 c_0 + (c^{-1}_1 c_1)^2 e_{\alpha} - \sum_{j=1}^{s+2} (-1)^j (c^{-1}_1 c_0)^j e_{(j-1)\nu -\alpha} \\
+ (c^{-1}_1 c_0) e_{-\lambda} - (1)^s (c^{-1}_1 c_0)^{s+3} e_{(s+1)\nu -\alpha} - (c^{-1}_1 c_0) e_{-\lambda}.
\]
Cancelling out the terms $\pm(c_0^{-1}c_0)e^{-\lambda}$ in the last expression, we see that this block belongs to $AP^+$ as well.

Formula (2.6) is a particular case of the transformation introduced in [2] for an arbitrary $AP$ polynomial (not necessarily a trinomial) $f$ with invertible Fourier coefficient corresponding to the leftmost point in $\Omega(f) \cap (\lambda, \lambda)$. However, in [2] only the case of commuting coefficients was considered. Also, formulas (2.7) for a trinomial case are more explicit than the general formulas of [2].

The resulting matrix $G_1$ in general has the same structure as the original matrix $G$: $\Omega(f_1) \subset \{-\nu_1, 0, \alpha_1\}$, where $\alpha_1, \nu_1 > 0$, $\alpha_1 + \nu_1 = \lambda_1$ and $\beta_1 = \nu_1/\alpha_1$ is irrational together with $\beta$. In some instances, however, $G_1$ may be easier to deal with. One such situation is discussed in the next theorem; other applications of Lemma 2.2 can be found in subsequent sections.

**Theorem 2.3.** Let the matrix $G$ be given by (1.4), (1.5) with $c_{-1}$ invertible, $c_0c_{-1}^{-1}$ nilpotent and having all Jordan cells of the size at most $\left\lceil \frac{\alpha_1}{\nu_1} \right\rceil + 2$. Then 1) $G$ is $AP_W$ factorable, and 2) its partial $AP$ indices equal zero if and only if $c_1$ is invertible.

**Proof.** Due to Lemma 2.2, we may consider the matrix (2.4) instead of $G$. The conditions imposed on the Jordan structure of $c_0c_{-1}^{-1}$ imply that $(c_{-1}^{-1}c_0)^{\nu_1+2} = 0$. Thus, $f_1$ in (2.4) is in fact a binomial with $\Omega(f_1) \subset \{-\nu_1, 0\}$. According to Theorem 1.2, the matrix $G_1$ is $AP_W$ factorable, and its partial $AP$ indices equal zero if and only if the constant term $c_{-1}^{-1}c_1$ of $f_1$ is invertible. The latter condition is equivalent to the invertibility of $c_1$.

Recall now the duality between an $AP$ factorization (1.2) of $G_f$ and that of $G_{f^*}$:

\begin{equation}
G_{f^*} = (JG_f^*)^*(G_f^* J),
\end{equation}

where $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. From (2.8) and Theorem 2.3 follows

**Corollary 2.4.** Let the matrix $G$ be given by (1.4), (1.5) with $c_1$ invertible, $c_0c_1^{-1}$ nilpotent and having all Jordan cells of the size at most $\left\lceil \frac{\alpha_1}{\nu_1} \right\rceil + 2$. Then $G$ is $AP_W$ factorable, and its partial $AP$ indices equal zero if and only if $c_{-1}$ is invertible.

Observe that the condition on the size of Jordan cells is satisfied automatically if $m = 2$. Hence, the following statement holds.

**Corollary 2.5.** Let the matrix $G$ be given by (1.4), (1.5) with $m = 2$, let one of the coefficients $c_{\pm 1}$ be non-singular, and let the corresponding product $c_0c_{\pm 1}^{-1}$ be nilpotent. Then 1) $G$ is $AP_W$ factorable, and 2) its partial $AP$ indices equal zero if and only if the second of the coefficients $c_{\pm 1}$ is invertible as well.

### 3. Main result

We now turn to matrices (1.4) with the off-diagonal block (1.5) having pairwise commuting coefficients $c_{1,1}$, $c_0$. The representation (1.6) is not unique, and we choose one with the maximal possible number $r$ of diagonal blocks. Each triple $\{c_{-1, k}, c_{0, k}, c_{1, k}\}$ is then irreducible, that is, does not allow a further reduction to a block diagonal form with the help of a common similarity. Of course, the commutativity property of $\{c_{-1}, c_0, c_1\}$ is inherited by the triples $\{c_{-1, k}, c_{0, k}, c_{1, k}\}$.

The ambiguity of $T$ also allows us, for each $k = 1, \ldots, r$, to put one of the matrices $c_{j, k}$ (with our choice of $j = 0, \pm 1$) in its Jordan canonical form. If, for a
given \( k \), at least one of the matrices \( c_{jk} \) is \textit{unicellular} (that is, its canonical Jordan form consists of only one cell), then for such a \( T \) all the matrices \( c_{jk} \) with the same \( k \) automatically become upper triangular and, in addition, have a Toeplitz structure. The latter means that \((p,q)\)-entry of each of the matrices \( c_{-1,k}, c_{0,k}, c_{1,k} \) is the same as its \((p+1,q+1)\)-entry \((p,q = 1, \ldots, l_k - 1)\). For \( l_k > 1 \), the common value of the entries right above the main diagonal in \( c_{jk} \) for such \( k \) will be denoted by \( \eta_{jk} \) (of course, the common value of the diagonal elements of the \( c_{jk} \) in this case is \( \xi_{jk} \)).

With this notation at hand, we are ready to formulate our main result.

**Theorem 3.1.** Let \( G \) be given by (1.3), (1.4) with pairwise commuting coefficients \( c_{\pm k}, c_0 \). Suppose that in (1.3) for each \( k = 1, \ldots, r \) at least one of the following conditions holds: 1) \( \xi_{0k} \neq 0 \), 2) \( \xi_{1,k}\xi_{-1,k} \neq 0 \), 3) one of the blocks \( c_{\pm k}, c_0 \) is unicellular, 4) \( l_k \leq 3 \), 5) \( \xi_{1,k} \) or \( \xi_{-1,k} \) differs from zero and \( l_k \leq 4 \). Then \( G \) is not \( AP \) factorable if, for at least one value of \( k \),

\[
|\xi_{1,k}^\nu \xi_{-1,k}^\alpha| = |\xi_{0k}|^\lambda \neq 0, \quad \text{or} \quad \xi_{-1,k} = \xi_{0k} = \xi_{1,k} = 0 \quad \text{and} \quad |\eta_{1,k}^\nu \eta_{-1,k}^\alpha| = |\eta_{0k}|^\lambda \neq 0,
\]

and is \( AP_W \) factorable otherwise.

**Proof.** Using (1.6), introduce a matrix

\[
\begin{pmatrix}
T^{-1} & 0 & T^{-1} \\
0 & T & 0 \\
T^{-1} & 0 & T
\end{pmatrix}
\begin{pmatrix}
e_{\lambda}I_m & 0 \\
0 & e_{-\lambda}I_m
\end{pmatrix}
\begin{pmatrix}
diag(c_{-1,k}e^{\nu} + c_{0,k} + c_{1,k}e^{\alpha}) & e_{-\lambda}I_m \\
e_{\lambda}I_m & 0
\end{pmatrix},
\]

having the same factorization properties as \( G \). By an appropriate permutation of its rows and columns, this matrix can be further rewritten as a direct sum of the blocks

\[
G_k = \begin{pmatrix}
e_{\lambda}I_{l_k} & 0 \\
c_{-1,k}e^{\nu} + c_{0,k} + c_{1,k}e^{\alpha} & e_{-\lambda}I_{l_k}
\end{pmatrix},
\]

\( k = 1, \ldots, r \). Let \( R = \{1, \ldots, r\} \) and denote by \( R_0 \) the subset of those \( r \in R \) such that \( \xi_{1,k} = \xi_{-1,k} = \xi_{0k} = 0 \), \( l_k > 1 \) and (at least) one of the blocks \( c_{\pm k}, c_0 \) is unicellular. We now partition \( R \) into a disjoint union \( \bigcup_{j=1}^{4} R_j \), where

\[
R_1 = \{ k : |\xi_{1,k}^\nu \xi_{-1,k}^\alpha| = |\xi_{0k}|^\lambda \neq 0 \},
\]

\[
R_2 = \{ k \in R_0 : |\eta_{1,k}^\nu \eta_{-1,k}^\alpha| = |\eta_{0k}|^\lambda \neq 0 \},
\]

\[
R_3 = R_0 \setminus R_2,
\]

\[
R_4 = R \setminus (R_1 \cup R_0).
\]

For every \( k \in R_0 \), yet another permutation of rows and columns allows us to represent \( G_k \) as a direct sum of \( \begin{pmatrix} e_{\lambda} & 0 \\ 0 & e_{-\lambda} \end{pmatrix} \) with

\[
G'_k = \begin{pmatrix}
e_{\lambda}I_{l_k-1} & 0 \\
c_{-1,k}e^{\nu} + c_{0,k} + c_{1,k}e^{\alpha} & e_{-\lambda}I_{l_k-1}
\end{pmatrix}.
\]

Here \( c'_{jk} \) are obtained from \( c_{jk} \) by deleting its first column and last row. The Toeplitz structure of \( c_{jk} \) is inherited by \( c'_{jk} \). In particular, the \( c'_{jk} \) pairwise commute and \( \sigma(c'_{jk}) = \{ \eta_{jk} \} (j = 0, \pm 1; k \in R_0) \).

Denote by \( G^{(1)} \) the direct sum of all the blocks \( G_k, k \in R_1 \), and \( G'_k, k \in R_3 \).

Let \( G^{(2)} \) be a direct sum of all \( G_k (k \in R_4) \), \( G'_k (k \in R_3) \), and \( |R_2| \) copies of
the diagonal blocks \[
\begin{bmatrix}
e_\lambda & 0 \\
0 & e_{-\lambda}
\end{bmatrix}
\]. Then \(G\) can be put in the form \(G^{(1)} \oplus G^{(2)}\) by an appropriate permutation of its rows and columns. In turn, \(G^{(1)}\) will become a permutation of a matrix of the type \((1.4)\) with \(f = b_{-1}e_{-\nu} + b_0 + b_1e_\alpha\) and \(b_j = (\bigoplus_{k \in R_1} c_{jk}) \oplus (\bigoplus_{k \in R_2} c'_{jk})\).

In terms of the sets \(R_j\), this theorem claims that \(G\) is \(AP_W\) factorable if \(R_1 \cup R_2 = \emptyset\), and is not \(AP\) factorable otherwise. This follows from Lemma\(2.1\) provided that \(G^{(2)}\) is \(AP_W\) factorable and, for \(R_1 \cup R_2 \neq \emptyset\), \(G^{(1)}\) is not \(AP\) factorable. The latter statement holds due to Corollary \(1.4\). It remains to prove the former. We will do this by showing that each direct summand of \(G^{(2)}\) is \(AP_W\) factorable. There are five types of these summands:

(i) diagonal blocks \[
\begin{bmatrix}
e_\lambda & 0 \\
0 & e_{-\lambda}
\end{bmatrix},
\]

and matrices \((1.4)\) with \(f\) given by \((1.5)\), pairwise commuting \(c_{\pm 1},\ c_0\) (slightly abusing the notation, we again denote their size by \(m\)), singleton spectra \(\sigma(c_j) = \{\xi_j\}\) \((j = \pm 1, 0)\) for which

(ii) \(|\xi^7_1 - \xi^7_0| \neq |\xi_0|^3\),

(iii) \(\xi_0 = 0\), exactly one of \(\xi_{\pm 1}\) differs from zero and (at least) one of the blocks \(c_{\pm 1}, c_0\) is unicellular,

(iv) \(\xi_0 = 0\), exactly one of \(\xi_{\pm 1}\) differs from zero, and \(m \leq 4\),

(v) \(\xi_0 = \xi_1 = \xi_{-1} = 0\) and \(m \leq 3\).

Indeed, the blocks \(G_k\) with \(k \in R_1\) have no impact on \(G^{(2)}\), \(k \in R_2\) generate only summands of type (i), \(k \in R_3\) yield summands of type (i) and (ii) or (iii), and \(k \in R_4\) produce summands of types (ii)-(v).

The summands of type (i) are trivially \(AP_W\) factorable (with partial \(AP\) indices \(\pm \lambda\)). The summands of type (ii) are \(AP_W\) factorable (with zero partial \(AP\) indices) according to Theorem \(1.3\). It remains to consider matrices \((1.4)\) of types (iii)-(v).

In cases (iii) and (iv) we may without loss of generality suppose that \(\xi_1 = 0\), \(\xi_{-1} \neq 0\); otherwise, \(G_f\) can be considered instead of \(G_f\). If in addition, \(c_0 = 0\) or \(c_1 = 0\), then \(f\) is a binomial and the corresponding matrix \((1.4)\) is \(AP_W\) factorable due to Theorem \(1.2\). This happens, in particular, if \(m = 1\).

If all three coefficients of \(f\) differ from zero, we consider the matrix \((2.4)\). It can happen that \(c_0^{s+2} = 0\), in which case the resulting block \((2.5)\) is a binomial. Applying Theorem \(1.2\) and Lemma\(2.2\) we conclude that \((2.4)\), and therefore \((1.4)\), are \(AP_W\) factorable. If \(c_0^{s+2} \neq 0\), we consider cases (iii) and (iv) separately.

(iii) The matrices \(c_j\) have an upper triangular Toeplitz structure which is inherited by the coefficients \(c_j^{(1)}\) of \((2.5)\). Hence,

\[
m > \text{rank} \ c_0^{(1)} = \text{rank} \ c_1
\]

and

\[
m > \text{rank} \ c_{-1}^{(1)} = \text{rank} \ c_0^{s+1} > \text{rank} \ c_1^{(1)} = \text{rank} \ c_0^{s+2} > 0.
\]

Let \(q = \max\{\text{rank} \ c_0^{(1)}, \text{rank} \ c_{-1}^{(1)}\}, \ p = m - q\). Then both \(p\) and \(q\) are strictly positive. By a permutation of its rows and columns, the matrix \(G_1\) can be reduced to the form

\[
\begin{bmatrix}
e_\nu I_p & 0 \\
0 & e_{-\nu} I_p
\end{bmatrix} \oplus \begin{bmatrix}
e_\nu I_q & 0 \\
0 & e_{-\nu} I_q
\end{bmatrix},
\]

\((3.2)\)
where

\( f_2 = c_1^{(2)} e_{-\alpha_1} + c_0^{(2)} + c_1^{(2)} e_{\alpha_1} \)

and the matrices \( c_j^{(2)} \) are obtained from \( c_j^{(1)} \) by deleting their first \( p \) columns and last \( p \) rows. It suffices to prove now that the second direct summand in (3.2) is \( AP_W \) factorable.

If \( \text{rank } c_1^{(1)} \geq \text{rank } c_{-1}^{(1)} \), this summand falls into type (ii). In the opposite case, this is again a matrix of type (iii), but its size is strictly smaller than that of the original matrix: \( q < m \). By induction we now conclude that all matrices of type (iii) are \( AP_W \) factorable.

(iv) The case of unicellular \( c_0 \) is covered by (iii). Since \( m \leq 4 \) and \( c_0^{a+2} \neq 0 \), the only remaining case is \( s = 0, m = 4 \) and \( c_0 \) consisting of one \( 3 \times 3 \) and one \( 1 \times 1 \) Jordan cell. The same Jordan structure is possessed by the matrix \( c_{-1}^{(1)} c_0 \). Without loss of generality we may suppose that in (2.5)

\[
(3.4) \quad c_{-1}^{(1)} = c_{-1}^{-1} c_0 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Then

\[
(3.4) \quad c_1^{(1)} = -(c_{-1}^{-1} c_0)^2 = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

The matrix \( c_0^{(1)} = c_{-1}^{-1} c_1 \) is nilpotent and commutes with (3.4). Thus,

\[
(3.4) \quad c_0^{(1)} = \begin{bmatrix}
0 & z & u & b \\
0 & 0 & z & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & a & 0
\end{bmatrix}.
\]

If \( a = b = 0 \), then the matrix \( G_1 \) can be split into a direct sum of

\[
\begin{bmatrix}
e_\nu I_2 & 0 \\
f_2 & e_{-\nu} I_2
\end{bmatrix},
\]

where \( f_2 \) is given by (3.3) with

\[
(3.4) \quad c_1^{(2)} = I_2, \quad c_0^{(2)} = \begin{bmatrix}
z & u \\
0 & z
\end{bmatrix}, \quad c_1^{(2)} = \begin{bmatrix}
0 & -1 \\
0 & 0
\end{bmatrix}.
\]

The matrix \( G_2 \) is of type (ii) or (iii) (depending on whether or not \( z \) is zero), and therefore \( AP_W \) factorable. Of course, \( G_1 \) is \( AP_W \) factorable together with \( G_2 \).

If \( a \) or \( b \) differs from zero, represent \( G_1 \) as a direct sum of \( \text{diag} \{ e_\nu, e_{-\nu} \} \) with

\[
G_3 = \begin{bmatrix}
e_\nu I_2 & 0 \\
f_3 & e_{-\nu} I_3
\end{bmatrix}, \quad \text{where } f_3 = c_{-1}^{(3)} e_{-\nu_1} + c_0^{(3)} + c_1^{(3)} e_{\alpha_1}
\]

and

\[
(3.4) \quad c_{-1}^{(3)} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad c_0^{(3)} = \begin{bmatrix}
z & u & b \\
0 & z & 0 \\
0 & a & 0
\end{bmatrix}, \quad c_1^{(3)} = \begin{bmatrix}
0 & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

The explicit \( AP_W \) factorization of \( G_3 \) is shown in Appendix A of the supplement. Hence, all matrices of type (iv) are \( AP_W \) factorable.
Finally, consider the remaining type (v). If \( m \leq 2 \), then each matrix \( c_j \) either is unicellular or equals zero. In both cases, an \( AP_W \) factorization exists. Therefore, we may suppose that \( m = 3 \). Excluding another trivial case \( c_0 = 0 \) (in which \( f \) is a binomial), we are left with the only possible Jordan structure of \( c_0 \): one \( 2 \times 2 \) and one \( 1 \times 1 \) block. Then, without loss of generality,

\[
c_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

The matrices \( c_{\pm 1} \) commute with \( c_0 \) and are nilpotent. Therefore,

\[
c_{\pm 1} = \begin{bmatrix} 0 & y_{\pm} & x_{\pm} \\ 0 & 0 & 0 \\ 0 & 0 & z_{\pm} \end{bmatrix}.
\]

The matrix \( G \) splits into a direct sum of \( \text{diag}[e_\lambda, e_{-\lambda}] \) and

\[
G_1 = \begin{bmatrix} e_\lambda I_2 & 0 \\ f_1 & e_{-\lambda} I_2 \end{bmatrix},
\]

where \( f_1 = c_0^{(1)} e_\nu + c_0^{(1)} + c_1^{(1)} e_\alpha \),

\[
c_0^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c_1^{(1)} = \begin{bmatrix} x_{\pm} & y_{\pm} \\ 0 & z_{\pm} \end{bmatrix}.
\]

From commutativity of \( c_1 \) with \( c_{-1} \) it follows that \( x_{+} z_{-} = x_{-} z_{+} \); however, later on we will encounter a factorization problem for matrices \( G_1 \) with \( c_{1}^{(1)} \) not satisfying this requirement. Therefore, we do not impose the condition \( x_{+} z_{-} = x_{-} z_{+} \) in our consideration.

The case \( x_{+} = x_{-} = z_{+} = z_{-} = 0 \) is excluded because otherwise the triple \( \{ c_{-1}, c_0, c_1 \} \) would be reducible. The cases \( x_{+} z_{+} \neq 0 \) and \( x_{-} z_{-} \neq 0 \) are covered by Corollary 2.5. In all the remaining cases an \( AP_W \) factorization of \( G_1 \) also exists; it is constructed explicitly in Appendix B of the supplement. Hence, matrices \( G \) of type (v) are also \( AP_W \) factorable.

As an application of Theorem 3.1, consider a difference equation

\[
cy_1 y(t - \nu) + c_0 y(t) + c_1 y(t + \alpha) = g(t) \quad \text{a.e. on } (0, \lambda),
\]

where \( g \) is a given vector function in \( L^p(0, \lambda) \), \( y \) is an unknown vector function in \( L^p(\mathbb{R}) \) with \( \text{supp } y \subset [0, \lambda] \).

According to standard terminology, we say that (3.5) is normally solvable (in \( L^p \)) if the set of vector functions \( g \) for which (3.5) has a solution is closed.

**Theorem 3.2.** In (3.5) let \( \alpha + \nu = \lambda \), let \( \frac{\nu}{\lambda} \) (\( > 0 \)) be irrational, and let the coefficients \( c_j \in \mathbb{C}^{m \times m} \) satisfy the conditions of Theorem 3.1. Then the equation (3.5) is normally solvable if and only if, in the notation of Theorem 3.1, condition (3.1) fails for every \( k \).

This result does not depend on \( p \in (1, \infty) \).

**Proof.** As follows from [7, Section 4.1], equation (3.5) is normally solvable if and only if the Wiener-Hopf operator \( W_G \), the symbol \( G \) of which is given by (1.4), (1.5), has closed range in \( L^p(0, \infty) \).

If condition (3.1) fails for all \( k \), then the matrix function \( G \) is \( AP_W \) factorable due to Theorem 3.1. Hence, \( W_G \) has a generalized inverse, and therefore its range is closed.
To prove the converse statement, consider first a particular case when in (1.5) each matrix $c_j$ has a singleton spectrum $\{\xi_j\}$, and
\[ |\xi_1^{e_1} - \xi_0^{e_0}| = |\xi_0|^\lambda \neq 0. \]
According to Theorem 5.1 the matrix function $G$ in this case is not AP factorable. If $m = 1$, the homogeneous equation (5.5) takes the form
\[ y(t) = \begin{cases} -\frac{\xi_1 - s_0}{\xi_0} y(t - \nu) & \text{if } \nu < t < \lambda, \\ -\frac{\xi_0 + \alpha}{\xi_0} y(t + \alpha) & \text{if } 0 < t < \nu, \end{cases} \]
and has at most one linearly independent solution (see, for example, [4]).

For $m > 1$, a similarity can be used to put the $c_j$ simultaneously in a triangular form, with $\xi_j$ on the diagonal. Therefore, the number of linearly independent solutions of the respective homogeneous equation (5.5) is at most $m$. Suppose that this equation is normally solvable. Then the corresponding Wiener-Hopf operator $W_G$ has a closed range and a finite dimensional kernel; in other words, it is $m$-normal. This property, as well as the index $\text{ind} W_G$ of the operator $W_G$ (the difference between the dimension of its kernel and the codimension of its range), is preserved under small perturbations. Consider such a small perturbation $W_{G'}$, with $f' = c_{-1} e_{-\nu} + (c_0 + eI) + c_1 e_{\alpha}$, and $0 \neq |\xi_0 + \epsilon| \neq |\xi_0|$. Then $G' = G f'$ admits an AP factorization with zero partial AP indices (Corollary 4.4), so that $W_{G'}$ is invertible. Hence, $\text{ind} W_G = \text{ind} W_{G'} = 0$. From here it follows that codim $\text{Im} W_G$ is finite together with $\text{dim Ker} W_G$; that is, the operator $W_G$ is Fredholm. Since $G \in AP_W$, Theorem 2.5 of [7] implies that $G$ is $AP_W$ factorable. This contradiction shows that in fact the range $\text{Im} W_G$ of the operator $W_G$ is not closed.

Finally, consider the general case when (3.1) holds for some $k$. Then, as was shown in the proof of Theorem 5.1 the corresponding matrix $G$ can be split into a direct sum of summands, a non-zero number of which are of the type just considered. Hence, $W_G$ also splits into a direct sum of operators, some of which have a non-closed range. Therefore, $\text{Im} W_G$ is not closed.

\textbf{Remark.} The above reasoning shows that for matrix functions $G$ satisfying the conditions of Theorem 5.1 the operator $W_G$ has a closed range if and only if $G$ is AP factorable. This is not true in general; examples of not AP factorable $2 \times 2$ triangular matrix functions $G \in AP_W$ for which $\text{Im} W_G$ is closed can be found in [10].

4. REMARKS ON $4 \times 4$ CASES

Theorem 5.1 covers all matrices (1.4), (1.5) with commuting $c_j$ of size $m \leq 3$. Hence, the case of reducible $4 \times 4$ triples is also covered. For irreducible $\{c_{-1}, c_0, c_1\}$, each $c_j$ has a singleton spectrum, say $\delta(c_j) = \{\xi_j\}$. The cases when at least one of the $\xi_j$ differs from zero or $c_j$ is unicellular also fall into the setting of Theorem 5.1.

This leaves us with the situation of an irreducible triple of $4 \times 4$ nilpotent matrices $c_j$ ($j = 0, \pm 1$), none of which is unicellular. We may suppose in addition that none of them is diagonalizable (that is, has only $1 \times 1$ Jordan cells). Indeed, a diagonalizable nilpotent matrix equals zero, and the corresponding $G$ is then $AP_W$ factorable due to Theorem 4.2. There remain three possible Jordan structures: two $2 \times 2$ cells, one $2 \times 2$ and two $1 \times 1$ cells, and one $3 \times 3$ and one $1 \times 1$ cells.

The following example demonstrates why the case of two $2 \times 2$ Jordan cells is hard to handle.
Example. Let \( c_j = \begin{bmatrix} 0 & c_j^{(0)}_j \\ 0 & 0 \end{bmatrix} \), where the \( c_j^{(0)} \) are arbitrary (not necessarily commuting) non-singular 2 \( \times \) 2 matrices, \( j = \pm 1, 0 \). Then \( G \) can be split into a direct sum of \( e^\lambda I_2 \) and \( G_0 = \begin{bmatrix} e^\lambda I_2 & 0 \\ 0 & e^-\lambda I_2 \end{bmatrix} \). According to Lemma 2.1, the matrices \( G \) and \( G_0 \) are AP factorable only simultaneously. Hence, the AP factorization problem for \( G \) is reduced to the corresponding problem for matrices of the form (1.4) with non-commuting coefficients of \( f \). Since the latter problem is still open, it is not surprising that a complete description of the AP factorability for matrices (1.4), (1.5) with commuting 4 \( \times \) 4 coefficients \( c_j \) is also missing.

We will now discuss the two remaining possibilities for the Jordan structure of \( c_0 \). First, let \( c_0 \) consist of one 2 \( \times \) 2 and two 1 \( \times \) 1 Jordan cells. Without loss of generality, \( c_0 \) itself is in a Jordan form:

\[
(4.1) \quad c_0 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix}.
\]

From the commutativity of \( c_{\pm 1} \) with \( c_0 \) and their nilpotency it follows that

\[
(4.2) \quad c_{\pm 1} = \begin{bmatrix}
a_{\pm} & b_{\pm} & d_{\pm} \\
0 & 0 & 0 \\
0 & f_{\pm} & h_{\pm} & l_{\pm} \\
0 & g_{\pm} & j_{\pm} & k_{\pm} 
\end{bmatrix},
\]

where \( A_{\pm} = \begin{bmatrix} h_{\pm} & l_{\pm} \\ j_{\pm} & k_{\pm} \end{bmatrix} \) are themselves nilpotent.

We may also use a similarity to reduce \( A_+ \) to its Jordan canonical form without disturbing \( c_0 \) and the structure of \( A_- \). Thus, \( h_+ = k_+ = j_+ = 0 \) and \( l_+ = 0 \) or 1.

If \( l_+ = 1 \), then commutativity of \( c_1 \) with \( c_{-1} \) implies that \( h_- = k_- = j_- = 0 \). If \( l_+ = 0 \) (that is, \( A_+ = 0 \)), then we can use a similarity to reduce \( A_- \) to its Jordan canonical form without changing \( c_0 \) and \( A_+ \). Hence, in any case it may be supposed that \( h_\pm = k_\pm = j_\pm = 0 \), that is,

\[
(4.3) \quad l_+ g_- = l_- g_+, \quad l_+ b_- = l_- b_+, \quad b_+ f_- + d_+ g_- = b_- f_+ + d_- g_+.
\]

Theorem 4.1. Let \( G \) be given by (1.4), (1.5) with \( c_0, c_{\pm 1} \) as in (4.1) and (4.2), respectively, satisfying (1.3) and forming an irreducible triple \( \{c_{-1}, c_0, c_1\} \). Then \( G \) is not AP factorable if

\[
b_+ = b_- = g_+ = g_- = 0, \quad |D^n_- D^n_+| = |l^n_+ l^n_-| \neq 0,
\]
where
\[ D_\pm = \det \begin{bmatrix} a_\pm & d_\pm \\ f_\pm & l_\pm \end{bmatrix} = a_\pm l_\pm - d_\pm f_\pm, \]
and is \( AP_W \) factorable otherwise.

**Proof.** We need to show that \( G \) is \( AP_W \) factorable if

i) at least one of the numbers \( b_\pm, d_\pm \) differs from zero, or

ii) \( b_+ = b_- = g_+ = g_- = l_+ l_- D_+ D_- = 0 \)

and that in the case

iii) \( b_+ = b_- = g_+ = g_- = 0, \quad l_+ D_+ \neq 0 \)

it is \( AP \) (\( AP_W \)) factorable if and only if

\[ |D^0 D'_+| \neq |P'_+ P_0|. \]

In case i), rewrite \( G \) as a direct sum of \( \text{diag}[e_\lambda, e_{-\lambda}] \) and another matrix of the form (1.4), with \( m = 3 \) and

\[ c_{\pm 1} = \begin{bmatrix} a_\pm & b_\pm & d_\pm \\ f_\pm & 0 & l_\pm \\ g_\pm & 0 & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

If \( c_{-1} \) is invertible, that is, \( b_- g_- l_- \neq 0 \), then Lemma 2.2 can be used. A direct computation shows that

\[ c_{-1}^{-1} c_0 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{b_-} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]

and therefore \( (c_{-1}^{-1} c_0)^2 = 0 \). Hence, \( f_1 \) in (2.4) is at most a binomial, and the matrix \( G_1 \) is \( AP_W \) factorable due to Theorem 1.2. The original matrix \( G \) is then also \( AP_W \) factorable.

Using (2.8) and appropriate transpositions of rows and columns, we can cover the case of invertible \( c_1 \), that is, \( b_+ g_+ l_+ \neq 0 \). It remains to construct an \( AP_W \) factorization in the cases when, in addition to (4.3),

\[ b_+ g_+ l_+ = b_- g_- l_- = 0. \]

(4.5)

This is done in Appendix C.

In cases ii) and iii), we represent \( G \) as a direct sum of \( \begin{bmatrix} e_\lambda I_2 & 0 \\ 0 & e_{-\lambda} I_2 \end{bmatrix} \) and another matrix \( G_1 \) of the form (1.4), (1.5) with \( m = 2 \) and

\[ c_{\pm 1}^{(1)} = \begin{bmatrix} a_\pm & d_\pm \\ f_\pm & l_\pm \end{bmatrix}, \quad c_0^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \]

If \( l_+ = 0 \) and \( d_+ f_+ \neq 0 \), then the matrix \( G_1 \) is \( AP_W \) factorable due to Corollary 2.5. The same reasoning applies if \( l_- = 0, \quad d_- f_- \neq 0 \). The cases \( l_+ = l_- = d_+ f_+ = d_- f_- = 0 \) when not all of the four entries \( d_\pm, f_\pm \) equal zero are covered by Appendix B in the supplement. Observe that the case \( d_\pm = f_\pm = 0 \) is excluded due to the irreducibility of the original triple \( \{c_{-1}, c_0, c_1\} \) given by (4.1), (4.2). Hence, the situation when \( l_+ = l_- = 0 \) is covered completely.

In all other cases (when at least one of \( l_+, l_- \) differs from zero) we may use the symmetry (2.8) to suppose without loss of generality that, say, \( l_- \neq 0 \). An obvious
similarity performed on the original $4 \times 4$ matrices $c_{\pm 1}$ (and not changing $c_0$) allows us to suppose in addition that $d_- = f_- = 0$. This similarity may, of course, change the values of $a_\pm$ and $d_\pm$, $f_\pm$; however, $\det c_{\pm 1}$ remains the same, so that the new value of $a_-$ is $D_-/l_-$. To simplify the notation, we redefine $D_\pm$ by $D$.

If $l_+ = 0$, then $d_+, f_+$ do not change under the above mentioned similarity. The only situation left uncovered by previous considerations is the case in which exactly one of $d_+, f_+$ differs from zero.

In case ii), we are left with only two possibilities: 1) $l_- \neq 0$, $l_+ = d_- = f_- = 0$, exactly one of the entries $d_+$, $f_+$ differs from zero, and 2) $l_+l_- \neq 0$, $d_- = f_- = 0$, $a_-D = 0$. Appendix D in the supplement shows that the corresponding matrix $G_1$ (and therefore $G$) is $AP_W$ factorable.

In case iii), the additional condition $d_- = f_- = 0$ means that $a_- (= D_-/l_-) \neq 0$, and (4.4) can be rewritten as

$$\det \rho = |a_- D'| \neq |l'_-|.$$  

A straightforward calculation shows that $G_1 = X_+G'X_-$, where

$$X_+ = \begin{pmatrix} 1 & d_+l_-e_\lambda & 0 & 0 \\ -\frac{f_+}{l_+} & a_-l_+e_\lambda - l_-(e_\nu + a_-) & -e_\nu & \frac{l'_{+}e_\mu}{l_+} \\ 0 & d_+(a_-l_+ + a_+l_-)e_\alpha & -d_+ & a_+ \\ 0 & (a_-l'_+ + d_+f_+l_-)e_\alpha - l_-l_+ & -l_+ & f_+ \end{pmatrix}$$

is invertible in $AP_W^+$, and

$$X_- = \begin{pmatrix} 1 & -\frac{d_+}{a_-l_+} & 0 & -\frac{d_+e_\alpha}{a_-l_+} \\ 0 & \frac{a_-l_+}{a_-D} & f_+e_\alpha & \frac{l'_{-}e_\mu}{l_-} \\ f_+(1+a_-e_\nu) & a_-l_+(1+a_-e_\nu) & \frac{f_+e_\alpha}{D} & \frac{a_-l'_+ + d_+f_+l_-}{a_-D} \\ \frac{D}{D} & 0 & \frac{D}{D} & 0 \end{pmatrix}$$

is invertible in $AP_W^-$, and

$$G' = \begin{pmatrix} e_\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_-l_+e_\nu + l_+ + De_\alpha & 0 & 0 & e_-\lambda \end{pmatrix}$$

can be split into a direct sum of $I_2$ with

$$G_2 = \begin{pmatrix} e_\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_-l_+e_\nu + l_+ + De_\alpha & 0 & 0 & e_-\lambda \end{pmatrix}.$$  

Of course, $G_2$ is $AP$ ($AP_W$) factorable only simultaneously with $G'$, and in turn, $G'$ has the same factorability properties as $G_2$. The latter matrix satisfies the conditions of Corollary 4.4 with $m = 1$. In the notation of this statement, $\xi_{1,k} = D$, $\xi_{0k} = l_+$ and $\xi_{-1,k} = a_-l_+$ with the only value of $k (=1)$, so that condition (4.7), necessary and sufficient for an $AP$ ($AP_W$) factorization to exist, is equivalent to (4.6).

Finally, let $c_0$ consist of one $3 \times 3$ and one $1 \times 1$ Jordan cells

$$c_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
Then the only possible form of $c \pm_1$ is

$$c \pm_1 = \begin{bmatrix}
0 & d_\pm & f_\pm & b_\pm \\
0 & 0 & d_\pm & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & a_\pm & 0
\end{bmatrix},$$

where

(4.7) \quad a_+ b_- = a_- b_+.

The case $a_+ = a_- = b_+ = b_- = 0$ is excluded if the triple $\{c_{-1}, c_0, c_1\}$ is irreducible. Splitting $G$ into a direct sum of $\text{diag}[e_\lambda, e_{-\lambda}]$ and another matrix of the form (1.4), we may suppose that $m = 3$ and

$$c \pm_1 = \begin{bmatrix}
d_\pm & f_\pm & b_\pm \\
0 & d_\pm & 0 \\
0 & a_\pm & 0
\end{bmatrix}, \quad c_0 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}.$$

In the case when all four of the coefficients $a_\pm, b_\pm$ are different from zero, an $AP_W$ factorization exists and can be explicitly constructed (see Appendix E in the supplement). Due to the commutativity condition (4.7), the number of non-zero entries among $a_\pm, b_\pm$ cannot equal one. However, there remain cases of exactly two or three non-zero numbers $a_\pm, b_\pm$, and in these cases the $AP$ factorability of the corresponding matrices $G$ is still unknown.

References


Department of Mathematics, The College of William and Mary, Williamsburg, VA 23187-8795
E-mail address: ilya@math.wm.edu

Department of Applied Mathematics, University of Washington, Seattle, WA 98195
E-mail address: dyong@u.washington.edu