MATHEMATICS OF COMPUTATION
Volume 69, Number 231, Pages 1053–1070
S 0025-5718(99)01161-8
Article electronically published on August 25, 1999

ALMOST PERIODIC FACTORIZATION OF CERTAIN BLOCK TRIANGULAR MATRIX FUNCTIONS

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Abstract. Let

$$G(x) = \begin{bmatrix} e^{i\lambda x} I_m & 0 \\ c_{-1}e^{-i\nu x} + c_0 + c_1 e^{i\alpha x} & e^{-i\lambda x} I_m \end{bmatrix},$$

where $c_j \in \mathbb{C}^{m\times m}$, $\alpha, \nu > 0$ and $\alpha + \nu = \lambda$. For rational α/ν such matrices G are periodic, and their Wiener-Hopf factorization with respect to the real line $\mathbb{R}$ always exists and can be constructed explicitly. For irrational α/ν, a certain modification (called an almost periodic factorization) can be considered instead. The case of invertible c_0 and commuting $c_1 c_0^{-1}$, $c_{-1} c_0^{-1}$ was disposed of earlier—it was discovered that an almost periodic factorization of such matrices G does not always exist, and a necessary and sufficient condition for its existence was found.

This paper is devoted mostly to the situation when c_0 is not invertible but the c_j commute pairwise ($j = 0, \pm 1$). The complete description is obtained when $m \leq 3$; for an arbitrary m, certain conditions are imposed on the Jordan structure of c_j. Difficulties arising for $m = 4$ are explained, and a classification of both solved and unsolved cases is given.

The main result of the paper (existence criterion) is theoretical; however, a significant part of its proof is a constructive factorization of G in numerous particular cases. These factorizations were obtained using Maple; the code is available from the authors upon request.

1. Introduction

Let AP be the Bohr algebra of almost periodic functions, that is, the smallest C^*-algebra of $L^\infty(\mathbb{R})$ containing all the functions $e_\lambda(x) = e^{i\lambda x}$, $\lambda \in \mathbb{R}$. It is well known (the standard references for these and other properties of AP are [3, 11, 12]) that for every $f \in AP$,

1. there exists the *Bohr mean value*

$$\mathbf{M}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\,dx,$$

and

2. the *Fourier coefficients* $\widehat{f}(\lambda) \overset{\text{def}}{=} \mathbf{M}(f e_{-\lambda})$ are different from zero for at most countably many values of $\lambda \in \mathbb{R}$.

Received by the editor March 12, 1997 and, in revised form, September 18, 1998.
1991 *Mathematics Subject Classification.* Primary 47A68, 47-04, 42A75.
Key words and phrases. Almost periodic matrix functions, factorization, explicit computation.
The first author's research was partially supported by NSF Grant DMS-9800704.
The second author's research was started during a Research Experience for Undergraduates sponsored by the NSF at the College of William and Mary during the summer of 1995.

The set $\Omega(f) = \{\mu\colon \widehat{f}(\mu) \neq 0\}$ is called the *Fourier spectrum* of f, and

$$\sum_{\mu\in\Omega(f)} \widehat{f}(\mu)e_\mu \tag{1.1}$$

is its (formal) *Fourier series.*

We say that $f \in AP_W$ if the Fourier series (1.1) converges absolutely:

$$\sum_{\mu\in\Omega(f)} |\widehat{f}(\mu)| < \infty.$$

Finally, let

$$AP^\pm = \{f \in AP\colon \Omega(f) \subset \mathbb{R}_\pm\} \quad \text{and} \quad AP_W^\pm = AP^\pm \cap AP_W.$$

Here, as usual, $\mathbb{R}_\pm = \{x \in \mathbb{R}\colon \pm x \geq 0\}$.

For matrix functions f, conditions $f \in AP, AP^\pm,\ AP_W$, etc. are understood entrywise, and $\mathbf{M}(f)$, $\widehat{f}(\mu)$, $\Omega(f)$ are defined by exactly the same formulas as for scalar functions.

Following [6], we introduce an *AP factorization* of an $n \times n$ matrix function G as its representation in the form

$$G = G_+\Lambda G_-, \tag{1.2}$$

where $\Lambda(x) = \operatorname{diag}[e_{\lambda_1}, \dots , e_{\lambda_n}]$,

$$G_+^{\pm 1} \in AP^+,\ G_-^{\pm 1} \in AP^-, \tag{1.3}$$

and $\lambda_1, \dots , \lambda_n \in \mathbb{R}$. We say that (1.2) is an AP_W factorization of G if conditions (1.3) are replaced by the (more restrictive) conditions $G_+^{\pm 1} \in AP_W^+,\ G_-^{\pm 1} \in AP_W^-$.

If G is AP factorable, the numbers $\lambda_1, \dots , \lambda_n$ are defined uniquely; they are called the *partial AP indices* (of G). Of course, for an AP (AP_W) factorization (1.2) to exist it is necessary that $G^{\pm 1} \in AP$ (respectively, AP_W). However, this necessary condition is not sufficient and, except for the case of periodic matrix functions G (in which an AP factorization by a simple change of variable reduces to the usual Wiener-Hopf factorization), the theory of AP factorization is "under construction". Its connections with integral equations, completion problems, and signal processing are discussed in [6, 7], [17, 15, 1], and [14] respectively. Explicit formulas for the factors in (1.2) for certain special types of G are obtained in [6, 9, 8]. Most of them refer to matrices G of the following block triangular form:

$$G_f = \begin{bmatrix} e_\lambda I_m & 0 \\ f & e_{-\lambda} I_m \end{bmatrix} \tag{1.4}$$

(so that $n = 2m$) arising in the treatment of convolution type equations on finite intervals of length λ.

In particular, the following two statements were established in [6].

Lemma 1.1. *Let f be an AP_W matrix function, and let*

$$f_0 = \sum_{\mu\in\Omega(f)\cap(-\lambda,\lambda)} \widehat{f}(\mu)e_\mu.$$

Then the matrices G_f and G_{f_0} are AP (AP_W) factorable only simultaneously, and their partial AP indices coincide.

Due to Lemma 1.1, for any $f \in AP_W$ in (1.4) we may suppose without loss of generality that $\Omega(f) \subset (-\lambda, \lambda)$.

Theorem 1.2. *Let $\Omega(f)\cap(-\lambda,\lambda)$ consist of at most two points, say μ and σ. Then G_f is AP_W factorable. Its partial AP indices all equal zero if and only if $\mu\sigma = 0$, $\widehat{f}(0)$ is invertible, or $\mu\sigma < 0$, $\frac{\lambda}{\mu-\sigma} \in \mathbb{Z}$ and both $\widehat{f}(\mu)$, $\widehat{f}(\sigma)$ are invertible.*

The next logical step is to consider a trinomial f with $\Omega(f) \subset (-\lambda,\lambda)$. However, with no additional restrictions on the location of $\Omega(f)$ this remains an open problem. In this paper, we concentrate on the case $\Omega(f) = \{-\nu, 0, \alpha\}$, that is,

$$f = c_{-1}e_{-\nu} + c_0 + c_1 e_\alpha, \tag{1.5}$$

where $\alpha, \nu > 0$ and $\alpha + \nu = \lambda$.

If $\beta = \frac{\nu}{\alpha}$ is rational, then the matrix G_f is periodic, and its AP_W factorization exists and can be easily constructed. Thus, we suppose in what follows that β is irrational. The next result applies to the case when the matrices c_j in (1.5) commute with each other. In this case there exists a similarity T such that

$$T^{-1}c_jT = \mathrm{diag}[c_{j1},\dots,c_{jr}], \quad c_{jk} \in \mathbb{C}^{l_k\times l_k},\ k = 1,\dots,r;\ j = 0,\pm1, \tag{1.6}$$

and each diagonal block c_{jk} has a singleton spectrum (see [13, Section 4.4]):

$$\sigma(c_{jk}) = \{\xi_{jk}\} \quad (j = 0,\pm1; k = 1,\dots r).$$

As in [2], we call $\{\xi_{jk}\}_{j=-1}^{1}$ the *bonded* eigenvalue triples of c_j.

Theorem 1.3. *Let G_f be of the form* (1.4) *with f given by* (1.5) *and commuting coefficients c_j. Then G_f is AP factorable with zero partial AP indices if and only if, for all bonded triples $\{\xi_{-1,k},\, \xi_{0,k},\, \xi_{1,k}\}$,*

$$|\xi_{1,k}^{\nu}\xi_{-1,k}^{\alpha}| \neq |\xi_{0,k}|^{\lambda} \quad (k = 1,\dots,r). \tag{1.7}$$

In this form, Theorem 1.3 was established in [2, Theorem 7.2]; the case of invertible c_j was disposed of earlier in [9]. In fact, the result of [9] contains an additional important piece of information: if all c_j are invertible and (1.7) fails for at least one value of k, then G_f does not admit *a*ny AP factorization, even if non-zero partial AP indices are allowed. Also, it was shown in [16] that an AP factorization with zero partial AP indices of an AP_W matrix function is automatically its AP_W factorization. Hence, the following result holds.

Corollary 1.4. *Let G_f be as in Theorem* 1.3 *and, in addition, let c_0 be invertible. Then G_f is AP_W factorable with zero partial AP indices if condition* (1.7) *holds, and is not AP factorable otherwise.*

Of course, it would now be natural to consider an AP factorization of G_f with trinomial f, pairwise commuting c_j, and no restrictions imposed on the invertibility of c_0 and the values of partial AP indices. We will see, however, that this problem embraces a general setting of a trinomial f with arbitrary (not necessarily commuting) coefficients c_j and is therefore too difficult to handle at the present stage of the development. Our paper is a report on several partial results on the AP factorability of matrices (1.4), (1.5) with non-invertible c_0.

The paper is structured as follows. Section 2 contains an auxiliary result on the factorization of block diagonal matrices. It also describes a procedure which allows us to replace a matrix of the form (1.4), (1.5) with invertible c_{-1} (and no commutativity conditions on c_j) by another matrix of the same type without changing its factorability properties. This procedure is, in fact, a variation of the one introduced in [2] for matrices (1.4) with a finite number (not limited to three) of points μ_j in $\Omega(f) \cap (-\lambda, \lambda)$ but pairwise commuting $\widehat{f}(\mu_j)$. As a direct application of this procedure, AP_W factorability is established for matrices (1.4), (1.5) with $m = 2$, invertible c_{-1} (or c_1) and nilpotent $c_0 c_{-1}^{-1}$ (respectively, $c_0 c_1^{-1}$).

Section 3 contains necessary and sufficient factorability conditions for matrices (1.4), (1.5) with commuting c_j under certain additional restrictions on their Jordan structure. This covers, in particular, all matrices of size $m \leq 3$, invertible c_1 or c_{-1} of size $m \leq 4$, and matrices of arbitrary size, provided that each eigenvalue of at least one of the c_j corresponds to one Jordan cell. An application to difference equations is given.

In Section 4, we concentrate on 4×4 matrices c_j. An example is given explaining why this case cannot be covered in general before the AP factorability of matrices (1.4), (1.5) with arbitrary invertible non-commuting c_j is understood. All possible cases are classified, and those for which the AP factorability remains unknown are singled out.

Proofs of the results in Sections 3 and 4 are partially theoretical and partially consist in exhausting a large number of cases in which an AP_W factorization can be constructed explicitly. These cases are relegated to Section 5 the supplement at the end of this volume, where final formulas are listed. Of course, they can be checked by straightforward calculations. We emphasize, however, that a symbolic manipulation Maple program was used to obtain these formulas, and without it this paper could hardly have been completed.

2. Auxiliary results

Suppose G is a block diagonal AP matrix: $G = \operatorname{diag}[G_1, G_2]$. If its diagonal blocks G_1, G_2 are AP factorable, then G itself is AP factorable. Moreover, an AP factorization of G can be obtained by "pasting together" AP factorizations of G_1 and G_2: $G_1 = G_+^{(1)} \Lambda^{(1)} G_-^{(1)}$, $G_2 = G_+^{(2)} \Lambda^{(2)} G_-^{(2)}$ imply

$$G = \operatorname{diag}[G_+^{(1)}, G_+^{(2)}] \operatorname{diag}[\Lambda^{(1)}, \Lambda^{(2)}] \operatorname{diag}[G_-^{(1)}, G_-^{(2)}].$$

It is natural to ask whether the converse is true. The answer is positive provided that $G \in AP_W$ and partial AP indices of G equal zero. Indeed, a matrix $F \in AP_W$ admits an AP factorization with zero partial AP indices if and only if the corresponding Toeplitz operator T_F is invertible on L^2 [5] (see also [7]). Since T_G is a direct sum of T_{G_1} with T_{G_2}, the invertibility of T_G is equivalent to simultaneous invertibility of T_{G_1} and T_{G_2}.

We are not aware of any equivalent of AP factorability (with non-zero partial AP indices) in operator terms. Probably, the answer to the question is still positive, but we restrict our consideration to a somewhat weaker version.

Lemma 2.1. *Let $G = \operatorname{diag}[G_1, G_2]$. If G and one of its diagonal blocks G_1, G_2 are AP factorable, then the other diagonal block is also AP factorable.*

Proof. Consider first the case when $G_1 = 1$. Then an AP factorization of G can be rewritten as

$$F_+ \begin{bmatrix} 1 & 0 \\ 0 & G_2 \end{bmatrix} = \Lambda G_-, \tag{2.1}$$

where $F_+ = G_+^{-1} \in AP^+$. Denote $F_+ = (f_{ij})_{i,j=1}^n$. From (2.1), $e_{-\lambda_j} f_{j1} \in AP^-$, so that

$$\Omega(f_{j1}) \subset [0, \lambda_j]. \tag{2.2}$$

In particular, $f_{j1} = 0$ for all j (if there are any) such that $\lambda_j < 0$. Rewriting (2.1) as

$$\begin{bmatrix} 1 & 0 \\ 0 & G_2^{-1} \end{bmatrix} G_+ \Lambda = G_-^{-1},$$

we find similarly that

$$\Omega(g_{1j}) \subset [0, -\lambda_j], \tag{2.3}$$

where $G_+ = (g_{ij})_{i,j=1}^n$. Therefore, $g_{1j} = 0$ for all j (if there are any) such that $\lambda_j > 0$. Observe also that $G_+ F_+ = I$ implies that $\sum_{j=1}^n g_{1j} f_{j1} = 1$. Since for nonzero λ_j at least one of the entries g_{1j}, f_{j1} is equal to zero, the latter equality proves the existence of zero partial AP indices λ_j. Due to (2.2), (2.3), the corresponding functions g_{1j}, f_{j1} are constant, and for at least one value of j, $g_{1j} f_{j1} \neq 0$.

Applying an appropriate permutation of columns of G_+ and rows of G_-, we may suppose without loss of generality that $\lambda_1 = 0$, $g_{11} = c \neq 0$, $f_{11} = d \neq 0$. Partitioning G_+, F_+ as

$$G_+ = \begin{bmatrix} c & g_1^+ \\ g_2^+ & G_2^+ \end{bmatrix}, \; F_+ = \begin{bmatrix} d & f_1^+ \\ f_2^+ & F_2^+ \end{bmatrix},$$

we conclude that $c = \det F_2^+ / \det F_+ = \det F_2^+ \det G_+$. Since $c \neq 0$, the matrix F_2^+ is invertible in AP^+ simultaneously with G_+. From (2.1) and (2.2) it follows that the left-upper entry of G_- and $H_- = G_-^{-1}$ equals d and c, respectively. Thus,

$$G_- = \begin{bmatrix} d & g_1^- \\ g_2^- & G_2^- \end{bmatrix}, \; H_- = \begin{bmatrix} c & h_1^- \\ h_2^- & H_2^- \end{bmatrix},$$

and $c = \det G_2^- / \det G_-$. Since $c \neq 0$, the matrix G_2^- is invertible in AP^- together with G_-. Now partition $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$. Then (2.1) yields $F_2^+ G_2 = \Lambda_2 G_2^-$, or $G_2 = (F_2^+)^{-1} \Lambda_2 G_2^-$. Since $(F_2^+)^{\pm 1} \in AP^+$ and $(G_2^-)^{\pm 1} \in AP^-$, the latter formula delivers an AP factorization of G_2. This proves the desired statement in the case $G_1 = 1$.

If $G_1 = e_\lambda$, then the matrix $e_{-\lambda} G = \mathrm{diag}[1, e_{-\lambda} G_2]$ is AP factorable together with G. According to the already proven particular case, $e_{-\lambda} G_2$ is AP factorable. But then G_2 is AP factorable as well.

An induction argument allows us to consider G_1 of the form $\mathrm{diag}[e_{\lambda_1}, \dots, e_{\lambda_k}] = \Lambda_1$. Finally, for an arbitrary AP factorable $G_1 = G_+^{(1)} \Lambda_1 G_-^{(1)}$ we can write

$$G = \mathrm{diag}[G_+^{(1)}, I] \, \mathrm{diag}[\Lambda_1, G_2] \, \mathrm{diag}[G_-^{(1)}, I]$$

and consider an (AP factorable) matrix $\mathrm{diag}[\Lambda_1, G_2]$ instead of the original matrix G. $\square$

Another technical tool we need applies to matrix functions G_f with a trinomial f containing an invertible c_{-1} coefficient.

Lemma 2.2. *Let G be of the form (1.4) with f given by (1.5). If c_{-1} is invertible, then G is AP (AP_W) factorable only simultaneously with (and has the same partial AP indices as) the matrix function*

$$G_1 = \begin{bmatrix} e_{\lambda_1} I_m & 0 \\ f_1 & e_{-\lambda_1} I_m \end{bmatrix}, \tag{2.4}$$

where

$$f_1 = c_{-1}^{(1)} e_{-\nu_1} + c_0^{(1)} + c_1^{(1)} e_{\alpha_1}, \tag{2.5}$$

$$c_{-1}^{(1)} = (-1)^s (c_{-1}^{-1} c_0)^{s+1},\ c_0^{(1)} = c_{-1}^{-1} c_1,\ c_1^{(1)} = (-1)^{s+1} (c_{-1}^{-1} c_0)^{s+2},$$

$$\lambda_1 = \nu,\ \nu_1 = \alpha - s\nu,\ \alpha_1 = (s+1)\nu - \alpha,$$

and finally, s is the integral part of $\frac{\alpha}{\nu}$: $s \in \mathbb{Z}$ and $s < \frac{\alpha}{\nu} < s+1$.

Proof. It suffices to construct matrix functions X_+ and X_- such that $X_+^{\pm 1} \in AP_W^+$, $X_-^{\pm 1} \in AP_W^-$ and

$$X_+ G X_- = G_1. \tag{2.6}$$

To this end, let

$$\begin{aligned} X_+ &= \begin{bmatrix} c_{-1}^{-1} f e_\nu & -e_{\lambda+\nu} I \\ e_{-\lambda-\nu} I + (g - e_{-\lambda} I) c_{-1}^{-1} f & I - g e_\lambda \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & c_{-1}^{-1} \end{bmatrix}, \\ X_- &= \begin{bmatrix} I & 0 \\ 0 & c_{-1} \end{bmatrix} \begin{bmatrix} e_{-\alpha} I + \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\alpha} & I \\ -I & 0 \end{bmatrix}, \end{aligned} \tag{2.7}$$

where $g = c_{-1}^{-1} c_1 - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j-1)\nu-\alpha}$. Directly from the definition of s it follows that $X_- \in AP^-$. Since $\det X_- = \det c_{-1}$ is a non-zero constant, X_-^{-1} belongs to AP^- together with X_-.

A straightforward computation shows that

$$\begin{aligned} X_+ G X_- &= \begin{bmatrix} c_{-1}^{-1} f e_\nu & -e_{\lambda+\nu} I \\ e_{-\lambda-\nu} I + (g - e_{-\lambda} I) c_{-1}^{-1} f & I - g e_\lambda \end{bmatrix} \\ &\quad \times \begin{bmatrix} e_\lambda I & 0 \\ c_{-1}^{-1} f & e_{-\lambda} I \end{bmatrix} \begin{bmatrix} e_{-\alpha} I + \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\alpha} & I \\ -I & 0 \end{bmatrix} \\ &= \begin{bmatrix} c_{-1}^{-1} f e_\nu & -e_{\lambda+\nu} I \\ e_{-\lambda-\nu} I + (g - e_{-\lambda} I) c_{-1}^{-1} f & I - g e_\lambda \end{bmatrix} \\ &\quad \times \begin{bmatrix} e_\nu I + \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+1)\nu} & e_\lambda I \\ c_{-1}^{-1} e_{-\alpha} f + c_{-1}^{-1} f \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\alpha} - e_{-\lambda} I & c_{-1}^{-1} f \end{bmatrix} \\ &= (y_{ij})_{i,j=1}^2, \end{aligned}$$

where

$$
\begin{aligned}
y_{11} &= c_{-1}^{-1} f e_{2\nu} + c_{-1}^{-1} f \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+2)\nu} - c_{-1}^{-1} f e_{2\nu} \\
&\quad - c_{-1}^{-1} f \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+2)\nu} + e_\nu I = e_\nu I, \\
y_{12} &= c_{-1}^{-1} f e_{\lambda+\nu} - c_{-1}^{-1} f e_{\lambda+\nu} = 0, \\
y_{22} &= e_{-\nu} I + (g e_\lambda - I) c_{-1}^{-1} f + (I - g e_\lambda) c_{-1}^{-1} f = e_{-\nu} I,
\end{aligned}
$$

and finally,

$$
\begin{aligned}
y_{21} &= \big(e_{-\lambda-\nu} I + (g - e_{-\lambda} I) c_{-1}^{-1} f\big) \left(e_\nu I + \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+1)\nu} \right) \\
&\quad + (I - g e_\lambda) \left(c_{-1}^{-1} e_{-\alpha} f + c_{-1}^{-1} f \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\alpha} - e_{-\lambda} I \right) \\
&= e_{-\lambda} I + \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\lambda} + (g - e_{-\lambda} I) \\
&\quad \times \Bigg(c_{-1}^{-1} f e_\nu + c_{-1}^{-1} f \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+1)\nu} \\
&\qquad - c_{-1}^{-1} f e_\nu - c_{-1}^{-1} f \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j+1)\nu} + I \Bigg) \\
&= e_{-\lambda} I + \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\lambda} + g - e_{-\lambda} I \\
&= \sum_{j=1}^{s} (-1)^j (c_{-1}^{-1} c_0)^j e_{j\nu-\lambda} + c_{-1}^{-1} c_1 - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j-1)\nu-\alpha} \\
&= (-1)^s (c_{-1}^{-1} c_0)^{s+1} e_{s\nu-\alpha} + c_{-1}^{-1} c_1 + (-1)^{s+1} (c_{-1}^{-1} c_0)^{s+2} e_{(s+1)\nu-\alpha} = f_1.
\end{aligned}
$$

This implies (2.6). Since $\det G = \det G_1 = 1$, from (2.6) it follows, in particular, that $\det X_+ = (\det X_-)^{-1}$ is a non-zero constant. It remains to show that $X_+ \in AP^+$, because then $X_+^{-1} \in AP^+$ as well. Three blocks of X_+ are obviously in AP^+. The remaining (left-lower) block can be rewritten as

$$
\begin{aligned}
& e_{-\lambda-\nu} I + (g - e_{-\lambda} I) c_{-1}^{-1} f \\
&\quad = e_{-\lambda-\nu} I + c_{-1}^{-1} c_1 e_{-\nu} + c_{-1}^{-1} c_1 c_{-1}^{-1} c_0 + (c_{-1}^{-1} c_1)^2 e_\alpha \\
&\qquad - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j-2)\nu-\alpha} - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^{j+1} e_{(j-1)\nu-\alpha} \\
&\qquad - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j-1)\nu} (c_{-1}^{-1} c_1) - e_{-\lambda-\nu} I - c_{-1}^{-1} c_0 e_{-\lambda} - c_{-1}^{-1} c_1 e_{-\nu} \\
&\quad = c_{-1}^{-1} c_1 c_{-1}^{-1} c_0 + (c_{-1}^{-1} c_1)^2 e_\alpha - \sum_{j=1}^{s+2} (-1)^j (c_{-1}^{-1} c_0)^j e_{(j-1)\nu} (c_{-1}^{-1} c_1) \\
&\qquad + (c_{-1}^{-1} c_0) e_{-\lambda} - (-1)^s (c_{-1}^{-1} c_0)^{s+3} e_{(s+1)\nu-\alpha} - (c_{-1}^{-1} c_0) e_{-\lambda}.
\end{aligned}
$$

Cancelling out the terms $\pm(c_{-1}^{-1}c_0)e_{-\lambda}$ in the last expression, we see that this block belongs to AP^+ as well. $\square$

Formula (2.6) is a particular case of the transformation introduced in [2] for an arbitrary AP polynomial (not necessarily a trinomial) f with invertible Fourier coefficient corresponding to the leftmost point in $\Omega(f)\cap(-\lambda,\lambda)$. However, in [2] only the case of commuting coefficients was considered. Also, formulas (2.7) for a trinomial case are more explicit than the general formulas of [2].

The resulting matrix G_1 in general has the same structure as the original matrix G: $\Omega(f_1)\subset\{-\nu_1,0,\alpha_1\}$, where $\alpha_1,\nu_1>0$, $\alpha_1+\nu_1=\lambda_1$ and $\beta_1=\nu_1/\alpha_1$ is irrational together with β. In some instances, however, G_1 may be easier to deal with. One such situation is discussed in the next theorem; other applications of Lemma 2.2 can be found in subsequent sections.

Theorem 2.3. *Let the matrix G be given by* (1.4), (1.5) *with c_{-1} invertible, $c_0c_{-1}^{-1}$ nilpotent and having all Jordan cells of the size at most $\left[\frac{\alpha}{\nu}\right]+2$. Then* 1) *$G$ is AP_W factorable, and* 2) *its partial AP indices equal zero if and only if c_1 is invertible.*

Proof. Due to Lemma 2.2, we may consider the matrix (2.4) instead of G. The conditions imposed on the Jordan structure of $c_0c_{-1}^{-1}$ imply that $(c_{-1}^{-1}c_0)^{s+2}=0$. Thus, f_1 in (2.4) is in fact a binomial with $\Omega(f_1)\subset\{-\nu_1,0\}$. According to Theorem 1.2, the matrix G_1 is AP_W factorable, and its partial AP indices equal zero if and only if the constant term $c_{-1}^{-1}c_1$ of f_1 is invertible. The latter condition is equivalent to the invertibility of c_1. $\square$

Recall now the duality between an AP factorization (1.2) of G_f and that of G_{f^*}:

$$G_{f^*}=(JG_-^*)\Lambda^*(G_+^*J), \tag{2.8}$$

where $J=\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$. From (2.8) and Theorem 2.3 follows

Corollary 2.4. *Let the matrix G be given by* (1.4), (1.5) *with c_1 invertible, $c_0c_1^{-1}$ nilpotent and having all Jordan cells of the size at most $\left[\frac{\nu}{\alpha}\right]+2$. Then G is AP_W factorable, and its partial AP indices equal zero if and only if c_{-1} is invertible.*

Observe that the condition on the size of Jordan cells is satisfied automatically if $m=2$. Hence, the following statement holds.

Corollary 2.5. *Let the matrix G be given by* (1.4), (1.5) *with $m=2$, let one of the coefficients $c_{\pm1}$ be non-singular, and let the corresponding product $c_0c_{\pm1}^{-1}$ be nilpotent. Then* 1) *G is AP_W factorable, and* 2) *its partial AP indices equal zero if and only if the second of the coefficients $c_{\pm1}$ is invertible as well.*

3. Main result

We now turn to matrices (1.4) with the off-diagonal block (1.5) having pairwise commuting coefficients $c_{\pm1}$, c_0. The representation (1.6) is not unique, and we choose one with the maximal possible number r of diagonal blocks. Each triple $\{c_{-1,k},c_{0k},c_{1k}\}$ is then *irreducible*, that is, does not allow a further reduction to a block diagonal form with the help of a common similarity. Of course, the commutativity property of $\{c_{-1},c_0,c_1\}$ is inherited by the triples $\{c_{-1,k},c_{0k},c_{1k}\}$.

The ambiguity of T also allows us, for each $k=1,\dots,r$, to put one of the matrices c_{jk} (with our choice of $j=0,\pm1$) in its Jordan canonical form. If, for a given k, at least one of the matrices c_{jk} is *unicellular* (that is, its canonical Jordan form consists of only one cell), then for such a T all the matrices c_{jk} with the same k automatically become upper triangular and, in addition, have a Toeplitz structure. The latter means that (p,q)-entry of each of the matrices $c_{-1,k}$, $c_{0,k}$, $c_{1,k}$ is the same as its $(p+1,q+1)$-entry $(p,q=1,\dots,l_k-1)$. For $l_k>1$, the common value of the entries right above the main diagonal in c_{jk} for such k will be denoted by η_{jk} (of course, the common value of the diagonal elements of the c_{jk} in this case is ξ_{jk}).

With this notation at hand, we are ready to formulate our main result.

Theorem 3.1. *Let G be given by* (1.4), (1.5) *with pairwise commuting coefficients $c_{\pm1}$, c_0. Suppose that in* (1.6) *for each $k=1,\dots,r$ at least one of the following conditions holds:* 1) $\xi_{0k}\neq 0$, 2) $\xi_{1,k}\xi_{-1,k}\neq 0$, 3) *one of the blocks $c_{\pm1,k}$, c_{0k} is unicellular,* 4) $l_k\leq 3$, 5) *$\xi_{1,k}$ or $\xi_{-1,k}$ differs from zero and $l_k\leq 4$. Then G is not AP factorable if, for at least one value of k,*

(3.1)
$$|\xi_{1,k}^{\nu}\xi_{-1,k}^{\alpha}|=|\xi_{0k}|^{\lambda}\neq 0,\ \text{ or } \xi_{-1,k}=\xi_{0k}=\xi_{1,k}=0 \text{ and } |\eta_{1,k}^{\nu}\eta_{-1,k}^{\alpha}|=|\eta_{0k}|^{\lambda}\neq 0,$$

and is AP_W factorable otherwise.

Proof. Using (1.6), introduce a matrix

$$\begin{bmatrix} T^{-1} & 0\\ 0 & T^{-1}\end{bmatrix} G \begin{bmatrix} T & 0\\ 0 & T\end{bmatrix} = \begin{bmatrix} e_\lambda I_m & 0\\ \operatorname{diag}[c_{-1,k}e_{-\nu}+c_{0k}+c_{1,k}e_\alpha] & e_{-\lambda}I_m\end{bmatrix},$$

having the same factorization properties as G. By an appropriate permutation of its rows and columns, this matrix can be further rewritten as a direct sum of the blocks

$$G_k=\begin{bmatrix} e_\lambda I_{l_k} & 0\\ c_{-1,k}e_{-\nu}+c_{0k}+c_{1,k}e_\alpha & e_{-\lambda}I_{l_k}\end{bmatrix},$$

$k=1,\dots r$. Let $R=\{1,\dots,r\}$ and denote by R_0 the subset of those $r\in R$ such that $\xi_{1,k}=\xi_{-1,k}=\xi_{0k}=0$, $l_k>1$ and (at least) one of the blocks $c_{\pm1,k}$, c_{0k} is unicellular. We now partition R into a disjoint union $\bigcup_{j=1}^4 R_j$, where

$$\begin{aligned} R_1&=\{k\colon |\xi_{1,k}^{\nu}\xi_{-1,k}^{\alpha}|=|\xi_{0k}|^{\lambda}\neq 0\},\\ R_2&=\{k\in R_0\colon |\eta_{1,k}^{\nu}\eta_{-1,k}^{\alpha}|=|\eta_{0k}|^{\lambda}\neq 0\},\\ R_3&=R_0\setminus R_2,\\ R_4&=R\setminus(R_1\cup R_0).\end{aligned}$$

For every $k\in R_0$, yet another permutation of rows and columns allows us to represent G_k as a direct sum of $\begin{bmatrix} e_\lambda & 0\\ 0 & e_{-\lambda}\end{bmatrix}$ with

$$G_k'=\begin{bmatrix} e_\lambda I_{l_k-1} & 0\\ c_{-1,k}'e_{-\nu}+c_{0k}'+c_{1,k}'e_\alpha & e_{-\lambda}I_{l_k-1}\end{bmatrix}.$$

Here c_{jk}' are obtained from c_{jk} by deleting its first column and last row. The Toeplitz structure of c_{jk} is inherited by c_{jk}'. In particular, the c_{jk}' pairwise commute and $\sigma(c_{jk}')=\{\eta_{jk}\}$ $(j=0,\pm1;\ k\in R_0)$.

Denote by $G^{(1)}$ the direct sum of all the blocks G_k, $k\in R_1$, and G_k', $k\in R_2$. Let $G^{(2)}$ be a direct sum of all G_k $(k\in R_4)$, G_k' $(k\in R_3)$, and $|R_2|$ copies of the diagonal blocks $\begin{bmatrix} e_\lambda & 0 \\ 0 & e_{-\lambda} \end{bmatrix}$. Then G can be put in the form $G^{(1)} \oplus G^{(2)}$ by an appropriate permutation of its rows and columns. In turn, $G^{(1)}$ will become a permutation of a matrix of the type (1.4) with $f = b_{-1}e_{-\nu} + b_0 + b_1 e_\alpha$ and $b_j = \left(\bigoplus_{k\in R_1} c_{jk}\right) \oplus \left(\bigoplus_{k\in R_2} c'_{jk}\right)$.

In terms of the sets R_j, this theorem claims that G is AP_W factorable if $R_1 \cup R_2 = \emptyset$, and is not AP factorable otherwise. This follows from Lemma 2.1, provided that $G^{(2)}$ is AP_W factorable and, for $R_1 \cup R_2 \neq \emptyset$, $G^{(1)}$ is not AP factorable. The latter statement holds due to Corollary 1.4. It remains to prove the former. We will do this by showing that each direct summand of $G^{(2)}$ is AP_W factorable. There are five types of these summands:

(i) diagonal blocks $\begin{bmatrix} e_\lambda & 0 \\ 0 & e_{-\lambda} \end{bmatrix}$,

and matrices (1.4) with f given by (1.5), pairwise commuting $c_{\pm 1}$, c_0 (slightly abusing the notation, we again denote their size by m), singleton spectra $\sigma(c_j) = \{\xi_j\}$ $(j = \pm 1, 0)$ for which

(ii) $|\xi_1^\nu \xi_{-1}^\alpha| \neq |\xi_0|^\lambda$,
(iii) $\xi_0 = 0$, exactly one of $\xi_{\pm 1}$ differs from zero and (at least) one of the blocks $c_{\pm 1}, c_0$ is unicellular,
(iv) $\xi_0 = 0$, exactly one of $\xi_{\pm 1}$ differs from zero, and $m \leq 4$,
(v) $\xi_0 = \xi_1 = \xi_{-1} = 0$ and $m \leq 3$.

Indeed, the blocks G_k with $k \in R_1$ have no impact on $G^{(2)}$, $k \in R_2$ generate only summands of type (i), $k \in R_3$ yield summands of type (i) and (ii) or (iii), and $k \in R_4$ produce summands of types (ii)-(v).

The summands of type (i) are trivially AP_W factorable (with partial AP indices $\pm\lambda$). The summands of type (ii) are AP_W factorable (with zero partial AP indices) according to Theorem 1.3. It remains to consider matrices (1.4) of types (iii)-(v).

In cases (iii) and (iv) we may without loss of generality suppose that $\xi_1 = 0$, $\xi_{-1} \neq 0$; otherwise, G_{f^*} can be considered instead of G_f. If in addition, $c_0 = 0$ or $c_1 = 0$, then f is a binomial and the corresponding matrix (1.4) is AP_W factorable due to Theorem 1.2. This happens, in particular, if $m = 1$.

If all three coefficients of f differ from zero, we consider the matrix (2.4). It can happen that $c_0^{s+2} = 0$, in which case the resulting block (2.5) is a binomial. Applying Theorem 1.2 and Lemma 2.2, we conclude that (2.4), and therefore (1.4), are AP_W factorable. If $c_0^{s+2} \neq 0$, we consider cases (iii) and (iv) separately.

(iii) The matrices c_j have an upper triangular Toeplitz structure which is inherited by the coefficients $c_j^{(1)}$ of (2.5). Hence,

$$m > \operatorname{rank} c_0^{(1)} = \operatorname{rank} c_1$$

and

$$m > \operatorname{rank} c_{-1}^{(1)} = \operatorname{rank} c_0^{s+1} > \operatorname{rank} c_1^{(1)} = \operatorname{rank} c_0^{s+2} > 0.$$

Let $q = \max\{\operatorname{rank} c_0^{(1)}, \operatorname{rank} c_{-1}^{(1)}\}$, $p = m - q$. Then both p and q are strictly positive. By a permutation of its rows and columns, the matrix G_1 can be reduced to the form

$$\begin{bmatrix} e_\nu I_p & 0 \\ 0 & e_{-\nu} I_p \end{bmatrix} \oplus \begin{bmatrix} e_\nu I_q & 0 \\ f_2 & e_{-\nu} I_q \end{bmatrix}, \tag{3.2}$$

where

$$f_2 = c_{-1}^{(2)} e_{-\nu_1} + c_0^{(2)} + c_1^{(2)} e_{\alpha_1} \tag{3.3}$$

and the matrices $c_j^{(2)}$ are obtained from $c_j^{(1)}$ by deleting their first p columns and last p rows. It suffices to prove now that the second direct summand in (3.2) is AP_W factorable.

If $\operatorname{rank} c_0^{(1)} \geq \operatorname{rank} c_{-1}^{(1)}$, this summand falls into type (ii). In the opposite case, this is again a matrix of type (iii), but its size is strictly smaller than that of the original matrix: $q < m$. By induction we now conclude that all matrices of type (iii) are AP_W factorable.

(iv) The case of unicellular c_0 is covered by (iii). Since $m \leq 4$ and $c_0^{s+2} \neq 0$, the only remaining case is $s = 0$, $m = 4$ and c_0 consisting of one 3×3 and one 1×1 Jordan cell. The same Jordan structure is possessed by the matrix $c_{-1}^{-1} c_0$. Without loss of generality we may suppose that in (2.5)

$$c_{-1}^{(1)} = c_{-1}^{-1} c_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.4}$$

Then

$$c_1^{(1)} = -(c_{-1}^{-1} c_0)^2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix $c_0^{(1)} = c_{-1}^{-1} c_1$ is nilpotent and commutes with (3.4). Thus,

$$c_0^{(1)} = \begin{bmatrix} 0 & z & u & b \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \end{bmatrix}.$$

If $a = b = 0$, then the matrix G_1 can be split into a direct sum of $\begin{bmatrix} e_\nu I_2 & 0 \\ 0 & e_{-\nu} I_2 \end{bmatrix}$ and $G_2 = \begin{bmatrix} e_\nu I_2 & 0 \\ f_2 & e_{-\nu} I_2 \end{bmatrix}$, where f_2 is given by (3.3) with

$$c_{-1}^{(2)} = I_2,\ c_0^{(2)} = \begin{bmatrix} z & u \\ 0 & z \end{bmatrix},\ c_1^{(2)} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}.$$

The matrix G_2 is of type (ii) or (iii) (depending on whether or not z is zero), and therefore AP_W factorable. Of course, G_1 is AP_W factorable together with G_2.

If a or b differs from zero, represent G_1 as a direct sum of $\operatorname{diag}[e_\nu, e_{-\nu}]$ with $G_3 = \begin{bmatrix} e_\nu I_3 & 0 \\ f_3 & e_{-\nu} I_3 \end{bmatrix}$, where $f_3 = c_{-1}^{(3)} e_{-\nu_1} + c_0^{(3)} + c_1^{(3)} e_{\alpha_1}$ and

$$c_{-1}^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ c_0^{(3)} = \begin{bmatrix} z & u & b \\ 0 & z & 0 \\ 0 & a & 0 \end{bmatrix},\ c_1^{(3)} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The explicit AP_W factorization of G_3 is shown in Appendix A of the supplement. Hence, all matrices of type (iv) are AP_W factorable.

Finally, consider the remaining type (v). If $m \le 2$, then each matrix c_j either is unicellular or equals zero. In both cases, an AP_W factorization exists. Therefore, we may suppose that $m = 3$. Excluding another trivial case $c_0 = 0$ (in which f is a binomial), we are left with the only possible Jordan structure of c_0: one 2×2 and one 1×1 block. Then, without loss of generality,

$$c_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrices $c_{\pm 1}$ commute with c_0 and are nilpotent. Therefore,

$$c_{\pm 1} = \begin{bmatrix} 0 & y_\pm & x_\pm \\ 0 & 0 & 0 \\ 0 & z_\pm & 0 \end{bmatrix}.$$

The matrix G splits into a direct sum of $\operatorname{diag}[e_\lambda, e_{-\lambda}]$ and $G_1 = \begin{bmatrix} e_\lambda I_2 & 0 \\ f_1 & e_{-\lambda} I_2 \end{bmatrix}$, where $f_1 = c_{-1}^{(1)} e_{-\nu} + c_0^{(1)} + c_1^{(1)} e_\alpha$,

$$c_0^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad c_{\pm 1}^{(1)} = \begin{bmatrix} x_\pm & y_\pm \\ 0 & z_\pm \end{bmatrix}.$$

From commutativity of c_1 with c_{-1} it follows that $x_+ z_- = x_- z_+$; however, later on we will encounter a factorization problem for matrices G_1 with $c_{\pm 1}^{(1)}$ not satisfying this requirement. Therefore, we do not impose the condition $x_+ z_- = x_- z_+$ in our consideration.

The case $x_+ = x_- = z_+ = z_- = 0$ is excluded because otherwise the triple $\{c_{-1}, c_0, c_1\}$ would be reducible. The cases $x_+ z_+ \neq 0$ and $x_- z_- \neq 0$ are covered by Corollary 2.5. In all the remaining cases an AP_W factorization of G_1 also exists; it is constructed explicitly in Appendix B of the supplement. Hence, matrices G of type (v) are also AP_W factorable. □

As an application of Theorem 3.1, consider a difference equation

$$c_{-1} y(t-\nu) + c_0 y(t) + c_1 y(t+\alpha) = g(t) \text{ a.e. on } (0, \lambda), \tag{3.5}$$

where g is a given vector function in $L^p(0, \lambda)$, y is an unknown vector function in $L^p(\mathbb{R})$ with $\operatorname{supp} y \subset [0, \lambda]$.

According to standard terminology, we say that (3.5) is *normally solvable* (in L^p) if the set of vector functions g for which (3.5) has a solution is closed.

Theorem 3.2. *In* (3.5) *let* $\alpha + \nu = \lambda$, *let* $\frac{\alpha}{\nu}$ (> 0) *be irrational, and let the coefficients* $c_j \in \mathbb{C}^{m \times m}$ *satisfy the conditions of Theorem* 3.1. *Then the equation* (3.5) *is normally solvable if and only if, in the notation of Theorem* 3.1, *condition* (3.1) *fails for every* k.

This result does not depend on $p \in (1, \infty)$.

Proof. As follows from [7, Section 4.1], equation (3.5) is normally solvable if and only if the Wiener-Hopf operator W_G, the symbol G of which is given by (1.4), (1.5), has closed range in $L^p(0, \infty)$.

If condition (3.1) fails for all k, then the matrix function G is AP_W factorable due to Theorem 3.1. Hence, W_G has a generalized inverse, and therefore its range is closed.

To prove the converse statement, consider first a particular case when in (1.5) each matrix c_j has a singleton spectrum $\{\xi_j\}$, and

$$|\xi_1^\nu \xi_{-1}^\alpha| = |\xi_0|^\lambda \neq 0.$$

According to Theorem 3.1, the matrix function G in this case is not AP factorable.

If $m = 1$, the homogeneous equation (3.5) takes the form

$$y(t) = \begin{cases} -\frac{\xi_{-1}}{\xi_0} y(t-\nu) & \text{if } \nu < t < \lambda, \\ -\frac{\xi_1}{\xi_0} y(t+\alpha) & \text{if } 0 < t < \nu, \end{cases}$$

and has at most one linearly independent solution (see, for example, [4]).

For $m > 1$, a similarity can be used to put the c_j simultaneously in a triangular form, with ξ_j on the diagonal. Therefore, the number of linearly independent solutions of the respective homogeneous equation (3.5) is at most m. Suppose that this equation is normally solvable. Then the corresponding Wiener-Hopf operator W_G has a closed range and a finite dimensional kernel; in other words, it is n-normal. This property, as well as the *index* $\operatorname{ind} W_G$ of the operator W_G (the difference between the dimension of its kernel and the codimension of its range), is preserved under small perturbations. Consider such a small perturbation $W_{G_{f'}}$ with $f' = c_{-1}e_{-\nu} + (c_0 + \epsilon I) + c_1 e_\alpha$, and $0 \neq |\xi_0 + \epsilon| \neq |\xi_0|$. Then $G' = G_{f'}$ admits an AP_W factorization with zero partial AP indices (Corollary 1.4), so that $W_{G'}$ is invertible. Hence, $\operatorname{ind} W_G = \operatorname{ind} W_{G'} = 0$. From here it follows that $\operatorname{codim} \operatorname{Im} W_G$ is finite together with $\dim \operatorname{Ker} W_G$; that is, the operator W_G is Fredholm. Since $G \in AP_W$, Theorem 2.5 of [7] implies that G is AP_W factorable. This contradiction shows that in fact the range $\operatorname{Im} W_G$ of the operator W_G is not closed.

Finally, consider the general case when (3.1) holds for some k. Then, as was shown in the proof of Theorem 3.1, the corresponding matrix G can be split into a direct sum of summands, a non-zero number of which are of the type just considered. Hence, W_G also splits into a direct sum of operators, some of which have a non-closed range. Therefore, $\operatorname{Im} W_G$ is not closed. □

Remark. The above reasoning shows that for matrix functions G satisfying the conditions of Theorem 3.1 the operator W_G has a closed range if and only if G is AP factorable. This is not true in general; examples of not AP factorable 2×2 triangular matrix functions $G \in AP_W$ for which $\operatorname{Im} W_G$ is closed can be found in [10].

4. Remarks on 4×4 cases

Theorem 3.1 covers all matrices (1.4), (1.5) with commuting c_j of size $m \leq 3$. Hence, the case of reducible 4×4 triples is also covered. For irreducible $\{c_{-1}, c_0, c_1\}$, each c_j has a singleton spectrum, say $\sigma(c_j) = \{\xi_j\}$. The cases when at least one of the ξ_j differs from zero or c_j is unicellular also fall into the setting of Theorem 3.1.

This leaves us with the situation of an irreducible triple of 4×4 nilpotent matrices c_j ($j = 0, \pm 1$), none of which is unicellular. We may suppose in addition that none of them is diagonalizable (that is, has only 1×1 Jordan cells). Indeed, a diagonalizable nilpotent matrix equals zero, and the corresponding G is then AP_W factorable due to Theorem 1.2. There remain three possible Jordan structures: two 2×2 cells, one 2×2 and two 1×1 cells, and one 3×3 and one 1×1 cells.

The following example demonstrates why the case of two 2×2 Jordan cells is hard to handle.

Example. Let $c_j = \begin{bmatrix} 0 & c_j^{(0)} \\ 0 & 0 \end{bmatrix}$, where the $c_j^{(0)}$ are arbitrary (not necessarily commuting) non-singular 2×2 matrices, $j = \pm 1, 0$. Then G can be split into a direct sum of $\begin{bmatrix} e_\lambda I_2 & 0 \\ 0 & e_{-\lambda} I_2 \end{bmatrix}$ and $G_0 = \begin{bmatrix} e_\lambda I_2 & 0 \\ c_{-1}^{(0)} e_{-\nu} + c_0^{(0)} + c_1^{(0)} e_\alpha & e_{-\lambda} I_2 \end{bmatrix}$. According to Lemma 2.1, the matrices G and G_0 are AP factorable only simultaneously. Hence, the AP factorization problem for G is reduced to the corresponding problem for matrices of the form (1.4) with non-commuting coefficients of f. Since the latter problem is still open, it is not surprising that a complete description of the AP factorability for matrices (1.4), (1.5) with commuting 4×4 coefficients c_j is also missing.

We will now discuss the two remaining possibilities for the Jordan structure of c_0. First, let c_0 consist of one 2×2 and two 1×1 Jordan cells. Without loss of generality, c_0 itself is in a Jordan form:

$$c_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.1}$$

From the commutativity of $c_{\pm 1}$ with c_0 and their nilpotency it follows that

$$c_{\pm 1} = \begin{bmatrix} 0 & a_\pm & b_\pm & d_\pm \\ 0 & 0 & 0 & 0 \\ 0 & f_\pm & h_\pm & l_\pm \\ 0 & g_\pm & j_\pm & k_\pm \end{bmatrix},$$

where $A_\pm = \begin{bmatrix} h_\pm & l_\pm \\ j_\pm & k_\pm \end{bmatrix}$ are themselves nilpotent.

We may also use a similarity to reduce A_+ to its Jordan canonical form without disturbing c_0 and the structure of A_-. Thus, $h_+ = k_+ = j_+ = 0$ and $l_+ = 0$ or 1.

If $l_+ = 1$, then commutativity of c_1 with c_{-1} implies that $h_- = k_- = j_- = 0$. If $l_+ = 0$ (that is, $A_+ = 0$), then we can use a similarity to reduce A_- to its Jordan canonical form without changing c_0 and A_+. Hence, in any case it may be supposed that $h_\pm = k_\pm = j_\pm = 0$, that is,

$$c_{\pm 1} = \begin{bmatrix} 0 & a_\pm & b_\pm & d_\pm \\ 0 & 0 & 0 & 0 \\ 0 & f_\pm & 0 & l_\pm \\ 0 & g_\pm & 0 & 0 \end{bmatrix}. \tag{4.2}$$

Also, from commutativity of c_1 with c_{-1} (which is preserved under the similarities applied above),

$$l_+ g_- = l_- g_+, \; l_+ b_- = l_- b_+, \; b_+ f_- + d_+ g_- = b_- f_+ + d_- g_+. \tag{4.3}$$

Theorem 4.1. *Let G be given by* (1.4), (1.5) *with c_0, $c_{\pm 1}$ as in* (4.1) *and* (4.2), *respectively, satisfying* (4.3) *and forming an irreducible triple $\{c_{-1}, c_0, c_1\}$. Then G is not AP factorable if*

$$b_+ = b_- = g_+ = g_- = 0, \quad |D_-^\alpha D_+^\nu| = |l_+^\nu l_-^\alpha| \neq 0,$$

where

$$D_\pm = \det \begin{bmatrix} a_\pm & d_\pm \\ f_\pm & l_\pm \end{bmatrix} = a_\pm l_\pm - d_\pm f_\pm,$$

and is AP_W factorable otherwise.

Proof. We need to show that G is AP_W factorable if

i) at least one of the numbers $b_\pm, d_\pm$ differs from zero, or

ii) $b_+ = b_- = g_+ = g_- = l_+ l_- D_+ D_- = 0$

and that in the case

iii) $b_+ = b_- = g_+ = g_- = 0,\ l_\pm D_\pm \neq 0$

it is AP (AP_W) factorable if and only if

$$|D_-^\alpha D_+^\nu| \neq |l_+^\nu l_-^\alpha|. \tag{4.4}$$

In case i), rewrite G as a direct sum of $\mathrm{diag}[e_\lambda, e_{-\lambda}]$ and another matrix of the form (1.4), with $m = 3$ and

$$c_{\pm 1} = \begin{bmatrix} a_\pm & b_\pm & d_\pm \\ f_\pm & 0 & l_\pm \\ g_\pm & 0 & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If c_{-1} is invertible, that is, $b_- g_- l_- \neq 0$, then Lemma 2.2 can be used. A direct computation shows that

$$c_{-1}^{-1} c_0 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{b_-} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and therefore $(c_{-1}^{-1} c_0)^2 = 0$. Hence, f_1 in (2.4) is at most a binomial, and the matrix G_1 is AP_W factorable due to Theorem 1.2. The original matrix G is then also AP_W factorable.

Using (2.8) and appropriate transpositions of rows and columns, we can cover the case of invertible c_1, that is, $b_+ g_+ l_+ \neq 0$. It remains to construct an AP_W factorization in the cases when, in addition to (4.3),

$$b_+ g_+ l_+ = b_- g_- l_- = 0. \tag{4.5}$$

This is done in Appendix C.

In cases ii) and iii), we represent G as a direct sum of $\begin{bmatrix} e_\lambda I_2 & 0 \\ 0 & e_{-\lambda} I_2 \end{bmatrix}$ and another matrix G_1 of the form (1.4), (1.5) with $m = 2$ and

$$c_{\pm 1}^{(1)} = \begin{bmatrix} a_\pm & d_\pm \\ f_\pm & l_\pm \end{bmatrix}, \quad c_0^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $l_+ = 0$ and $d_+ f_+ \neq 0$, then the matrix G_1 is AP_W factorable due to Corollary 2.5. The same reasoning applies if $l_- = 0$, $d_- f_- \neq 0$. The cases $l_+ = l_- = d_+ f_+ = d_- f_- = 0$ when not all of the four entries $d_\pm$, $f_\pm$ equal zero are covered by Appendix B in the supplement. Observe that the case $d_\pm = f_\pm = 0$ is excluded due to the irreducibility of the original triple $\{c_{-1}, c_0, c_1\}$ given by (4.1), (4.2). Hence, the situation when $l_+ = l_- = 0$ is covered completely.

In all other cases (when at least one of l_+, l_- differs from zero) we may use the symmetry (2.8) to suppose without loss of generality that, say, $l_- \neq 0$. An obvious similarity performed on the original 4×4 matrices $c_{\pm 1}$ (and not changing c_0) allows us to suppose in addition that $d_- = f_- = 0$. This similarity may, of course, change the values of $a_\pm$ and d_+, f_+; however, $\det c_{\pm 1}^{(1)}$ remain the same, so that the new value of a_- is D_-/l_-. To simplify the notation, we redenote D_+ by D.

If $l_+ = 0$, then d_+, f_+ do not change under the above mentioned similarity. The only situation left uncovered by previous considerations is the case in which exactly one of d_+, f_+ differs from zero.

In case ii), we are left with only two possibilities: 1) $l_- \neq 0$, $l_+ = d_- = f_- = 0$, exactly one of the entries d_+, f_+ differs from zero, and 2) $l_+ l_- \neq 0$, $d_- = f_- = 0$, $a_- D = 0$. Appendix D in the supplement shows that the corresponding matrix G_1 (and therefore G) is AP_W factorable.

In case iii), the additional condition $d_- = f_- = 0$ means that $a_- (= D_-/l_-) \neq 0$, and (4.4) can be rewritten as

$$|a_-^\alpha D^\nu| \neq |l_+^\nu|. \tag{4.6}$$

A straightforward calculation shows that $G_1 = X_+ G' X_-$, where

$$X_+ = \begin{bmatrix} 1 & d_+ l_- e_\lambda & 0 & 0 \\ -\frac{f_+}{l_+} & a_- l_+ e_\lambda - l_-(e_\nu + a_-) & -e_\nu & \frac{f_+ e_\nu}{l_+} \\ 0 & d_+(a_- l_+ + a_+ l_-) e_\alpha & -d_+ & a_+ \\ 0 & (a_- l_+^2 + d_+ f_+ l_-) e_\alpha - l_- l_+ & -l_+ & f_+ \end{bmatrix}$$

is invertible in AP_W^+,

$$X_- = \begin{bmatrix} 1 & -\frac{d_+ l_-}{a_- l_+} & 0 & -\frac{d_+ e_{-\alpha}}{a_- l_+} \\ 0 & \frac{1}{a_- l_+} & 0 & \frac{e_{-\alpha}}{a_- l_+ l_-} \\ \frac{f_+(1 + a_- e_{-\nu})}{D} & -\frac{a_+ l_-(1 + a_- e_{-\nu})}{a_- D} & \frac{f_+ e_{-\lambda}}{D} & \frac{1}{l_-} - \frac{a_+(a_- e_{-\lambda} + e_{-\alpha})}{a_- D} \\ 0 & 0 & \frac{l_+}{D} & 0 \end{bmatrix}.$$

is invertible in AP_W^-, and

$$G' = \begin{bmatrix} e_\lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_- l_+ e_{-\nu} + l_+ + D e_\alpha & 0 & 0 & e_{-\lambda} \end{bmatrix}$$

can be split into a direct sum of I_2 with

$$G_2 = \begin{bmatrix} e_\lambda & 0 \\ a_- l_+ e_{-\nu} + l_+ + D e_\alpha & e_{-\lambda} \end{bmatrix}.$$

Of course, G_1 is AP (AP_W) factorable only simultaneously with G', and in turn, G' has the same factorability properties as G_2. The latter matrix satisfies the conditions of Corollary 1.4 with $m = 1$. In the notation of this statement, $\xi_{1,k} = D$, $\xi_{0k} = l_+$ and $\xi_{-1,k} = a_- l_+$ with the only value of k (=1), so that condition (1.7), necessary and sufficient for an AP (AP_W) factorization to exist, is equivalent to (4.6). □

Finally, let c_0 consist of one 3×3 and one 1×1 Jordan cells

$$c_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the only possible form of $c_{\pm 1}$ is

$$c_{\pm 1} = \begin{bmatrix} 0 & d_\pm & f_\pm & b_\pm \\ 0 & 0 & d_\pm & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_\pm & 0 \end{bmatrix},$$

where

$$a_+ b_- = a_- b_+. \tag{4.7}$$

The case $a_+ = a_- = b_+ = b_- = 0$ is excluded if the triple $\{c_{-1}, c_0, c_1\}$ is irreducible. Splitting G into a direct sum of $\operatorname{diag}[e_\lambda, e_{-\lambda}]$ and another matrix of the form (1.4), we may suppose that $m = 3$ and

$$c_{\pm 1} = \begin{bmatrix} d_\pm & f_\pm & b_\pm \\ 0 & d_\pm & 0 \\ 0 & a_\pm & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the case when all four of the coefficients $a_\pm$, $b_\pm$ are different from zero, an AP_W factorization exists and can be explicitly constructed (see Appendix E in the supplement). Due to the commutativity condition (4.7), the number of non-zero entries among $a_\pm$, $b_\pm$ cannot equal one. However, there remain cases of exactly two or three non-zero numbers $a_\pm$, $b_\pm$, and in these cases the AP factorability of the corresponding matrices G is still unknown.

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