

SOME EXAMPLES RELATED TO THE *abc*-CONJECTURE FOR ALGEBRAIC NUMBER FIELDS

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ABSTRACT. We present a numerical method for finding extreme examples of identities related to the uniform *abc*-conjecture for algebraic number fields.

1. INTRODUCTION

Let K be an algebraic number field and let V_K denote the set of primes on K , that is, $v \in V_K$ is an equivalence class of non-trivial norms on K (finite or infinite). Let $\|x\|_v = N_{K/\mathbb{Q}}(\mathfrak{p})^{-v_{\mathfrak{p}}(x)}$ if v is a prime defined by a prime ideal \mathfrak{p} of the ring of integers \mathfrak{O}_K in K and $v_{\mathfrak{p}}$ is the corresponding valuation. Let $\|x\|_v = |\varphi(x)|^e$ for all distinct (non-conjugate) embeddings $\varphi : K \rightarrow \mathbb{C}$, with $e = 1$ if $\varphi(K) \subset \mathbb{R}$ and $e = 2$ otherwise. We define the *height* of $(a, b, c) \in (K^*)^3$ to be

$$H_K(a, b, c) = \prod_{v \in V_K} \max(\|a\|_v, \|b\|_v, \|c\|_v),$$

and the *conductor* of (a, b, c) to be

$$N_K(a, b, c) = \prod_{\mathfrak{p} \in I_K(a, b, c)} N_{K/\mathbb{Q}}(\mathfrak{p}),$$

where $I_K(a, b, c)$ is the set of all prime ideals \mathfrak{p} of \mathfrak{O}_K for which $\|a\|_{\mathfrak{p}}, \|b\|_{\mathfrak{p}}, \|c\|_{\mathfrak{p}}$ are not all equal. Let $\Delta_{K/\mathbb{Q}}$ denote the discriminant of K .

The uniform *abc*-conjecture. For every $\varepsilon > 0$ there exists a constant C_{ε} , depending only on ε , such that

$$H_K(a, b, c) < C_{\varepsilon}^{[K:\mathbb{Q}]} (|\Delta_{K/\mathbb{Q}}| N_K(a, b, c))^{1+\varepsilon},$$

for all $a, b, c \in K^*$ satisfying $a + b + c = 0$.

Remark. In [4], $\Delta_{K/\mathbb{Q}}^{1+\varepsilon}$ is replaced by $\Delta_{K/\mathbb{Q}}^A$ for some constant A . The choice $A = 1 + \varepsilon$ is suggested by a theorem in [1].

We define a real valued function on $K \setminus \{0, 1\}$ by

$$l_K(x) = \frac{\log H_K(x, 1-x, 1)}{\log |\Delta_{K/\mathbb{Q}}| + \log N_K(x, 1-x, 1)}.$$

The uniform *abc*-conjecture is equivalent to the statement that $l_K(x)$ is bounded and its biggest limit point equals 1. Examples of $x \in K \setminus \{0, 1\}$ for which $l_K(x)$ is big may therefore be of interest. The definition of $l_K(x)$ suggests defining a function

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on the algebraic numbers (excluding 0 and 1) by $l(x) = l_{\mathbb{Q}(x)}(x)$. It is not hard to show that the conjecture implies that $l(x)$ is bounded, and one could expect that the biggest limit point of $l(x)$ also equals 1.

2. EXAMPLES

We are looking for algebraic numbers x for which $l_K(x)$ is large, that is, numbers x for which $H_K(x, 1 - x, 1)$ is relatively large and $N_K(x, 1 - x, 1)$ relatively small. One method is to approximate a number $\sqrt[n]{k}$, $k \in K$, by an element y in K and then hope that $l(k/y^n)$ is large. We will try to do so in a few norm–Euclidean quadratic fields.

Let $K = \mathbb{Q}(\sqrt{d})$, for a square free integer d . An integral basis for K over \mathbb{Q} is $\{1, \alpha\}$, where

$$\alpha = \begin{cases} (1 + \sqrt{d})/2 & \text{if } d \equiv 1 \pmod{4}, \\ \sqrt{d} & \text{otherwise.} \end{cases}$$

Consider $\varphi : K \rightarrow \mathbb{R}^2$, where $\varphi(x + y\alpha) = (x, y)$, and define multiplication on \mathbb{R}^2 by

$$(x_1, y_1)(x_2, y_2) = \begin{cases} (x_1 x_2 + y_1 y_2 d, x_1 y_2 + y_1 x_2) & \text{if } \alpha = \sqrt{d}, \\ (x_1 x_2 + y_1 y_2 \frac{d-1}{4}, x_1 y_2 + y_1 x_2 + y_1 y_2) & \text{if } \alpha = \frac{1+\sqrt{d}}{2}. \end{cases}$$

Then $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in K$ and $\varphi(\mathfrak{O}_K) = \mathbb{Z}^2$. We extend the norm from the image of K to \mathbb{R}^2 ,

$$N : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad N(a, b) = |(a + b\alpha)(a - b\alpha)|,$$

so $N(a, b) = |N_{K/\mathbb{Q}}(a + b\alpha)|$ for all $a, b \in \mathbb{Q}$. For any $x \in \mathbb{R}^2$ we define the subset T_x of \mathfrak{O}_K to be $\{r \in \mathfrak{O}_K : N(x - \varphi(r)) < 1\}$. Note that if T_x is non–empty for all $x \in \mathbb{R}^2$, then K is a norm–Euclidean domain. The following theorem from [3] gives a non–empty subset of T_x in some special cases:

Theorem. *Let d be 2, 3, 6, or 7. For any $(x, y) \in \mathbb{R}^2$ let*

$$S_{(x,y)} = \{[x] + [y]\alpha + a + b\alpha : a = -1, 0, 1, 2, b = 0, 1\} \subset \mathfrak{O}_K,$$

where $[a]$ denotes the largest integer less than a . Then $T_{(x,y)} \cap S_{(x,y)} \neq \emptyset$ for all $(x, y) \in \mathbb{R}^2$. For $d = -11, -7, -3, -2, -1$, the statement is true with

$$S_{(x,y)} = \{[x] + [y]\alpha + a + b\alpha : a = 0, 1, b = 0, 1\}.$$

Now select an element $a \in \varphi(\mathfrak{O}_K)$ such that the equation $x^n - a = 0$ has a solution $x \in \mathbb{R}^2 \setminus \varphi(K)$, where n is a positive integer. We want to expand x in a continued fraction p_i/q_i , where $p_i, q_i \in \varphi(\mathfrak{O}_K)$. Set $x_0 = x$ and construct the sequences $\{x_i\} \subset \mathbb{R}^2$ and $\{a_i\} \subset \varphi(\mathfrak{O}_K)$ by

$$x_{i+1} = \frac{(1, 0)}{x_i - a_i}, \quad \text{where } a_i \in \varphi(T_{x_i}),$$

i.e. x_{i+1} is the inverse of $x_i - a_i$ with respect to the multiplication in \mathbb{R}^2 defined above. To get uniqueness, one needs a rule for selecting a particular $a_i \in \varphi(T_{x_i})$. To do this, choose an ordering of the S_{x_i} of the theorem and let a_i be the first

element in S_{x_i} satisfying $N(x_i - a_i) \leq N(x_i - a)$ for all $a \in S_{x_i}$. Let p_i/q_i be the continued fraction given by

$$\begin{aligned} p_i &= a_i p_{i-1} + p_{i-2}, & p_{-1} &= (1, 0), & p_0 &= a_0, \\ q_i &= a_i q_{i-1} + q_{i-2}, & q_{-1} &= (0, 0), & q_0 &= (1, 0). \end{aligned}$$

Then one can check that

$$x q_i - p_i = -\frac{x q_{i-1} - p_{i-1}}{x_{i+1}} = \frac{(-1, 0)^i}{x_1 x_2 \cdots x_{i+1}},$$

and, if we take the norm on both sides,

$$N(x q_i - p_i) = \frac{1}{N(x_1) \cdots N(x_{i+1})} = N(x_0 - a_0) \cdots N(x_i - a_i) < 1.$$

Note that $N(x - p_i/q_i) \rightarrow 0$ does not have to imply $|\varphi^{-1}(x) - \varphi^{-1}(p_i/q_i)| \rightarrow 0$, where $\varphi^{-1} : \mathbb{R}^2 \rightarrow \mathbb{C} : (x, y) \mapsto x + y\alpha$ and $|\cdot|$ is the usual absolute-value on \mathbb{C} .

Now for some examples of identities $a + b = c$ for which $l(a/c)$ are large. The examples are computed using the method described above, for $n = 2, 3, 4$ in the real cases and $n = 2, 3, 4, 5$ in the complex cases. We only searched among equations $x^n - a = 0$ with $N(a) \leq 10000$. The rational examples are well known and are included here for completeness. There is a table of extremal (rational) *abc*-examples to be found at URL: <http://www.math.chalmers.se/~jub/abc>.

l	identity	author
1.768124	$(\sqrt{2})^{17} + (1 - \sqrt{2})^5 (3 - \sqrt{2}) = (1 + \sqrt{2})^5 (3 + \sqrt{2})$	N.B.
1.707221	$(1 - 2\frac{1+\sqrt{-7}}{2}) + (1 - \frac{1+\sqrt{-7}}{2})^{13} = (\frac{1+\sqrt{-7}}{2})^{13}$	N.B.
1.629912	$2 + 3^{10} 109 = 23^5$	E.R.
1.625991	$11^2 + 3^2 5^6 7^3 = 2^{21} 23$	B.W.
1.623490	$19 1307 + 7 29^2 31^8 = 2^8 3^{22} 5^4$	J.B.-J.B.
1.580756	$283 + 5^{11} 13^2 = 2^8 3^8 17^3$	J.B.-J.B., A.N.
1.567887	$1 + 2 3^7 = 5^4 7$	B. W.
1.561437	$(1 + \sqrt{2})^{14} + 1 = (1 + \sqrt{2})^7 (\sqrt{2})^3 13^2$	N.B.
1.547075	$7^3 + 3^{10} = 2^{11} 29$	B.W.
1.544434	$7^2 41^2 311^3 + 11^{16} 13^2 79 = 2 3^3 5^{23} 953$	A.N.
1.536714	$5^3 + 2^9 3^{17} 13^2 = 11^5 17 31^3 137$	H.R.-P.M.
1.528940	$(8 - 3\sqrt{7})^2 (5 - 2\sqrt{7}) + (8 - 3\sqrt{7})^7 (3 - \sqrt{7})^3 (5 + 2\sqrt{7})^{12} = (4 - 3\sqrt{7})^4$	N.B.
1.526999	$13 19^6 + 2^{30} 5 = 3^{13} 11^2 31$	A.N.
1.522160	$3^{18} 23 2269 + 17^3 29 31^8 = 2^{10} 5^2 7^{15}$	A.N.
1.518102	$(5 + 2\sqrt{6})^9 (2 - \sqrt{6})^9 (3 - \sqrt{6}) (1 + \sqrt{6}) (1 - \sqrt{6}) 7^2 + 1 = (5 + 2\sqrt{6})^8$	N.B.
1.502839	$239 + 5^8 17^3 = 2^{10} 37^4$	J.B.-J.B., A.N.

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