SOME EXAMPLES RELATED TO THE abc–CONJECTURE
FOR ALGEBRAIC NUMBER FIELDS

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Abstract. We present a numerical method for finding extreme examples of identities related to the uniform abc–conjecture for algebraic number fields.

1. Introduction

Let $K$ be an algebraic number field and let $V_k$ denote the set of primes on $K$, that is, $v \in V_k$ is an equivalence class of non-trivial norms on $K$ (finite or infinite). Let $\|x\|_v = N_{K/Q}(p)^{-\nu_p(x)}$ if $v$ is a prime defined by a prime ideal $p$ of the ring of integers $\mathcal{O}_K$ in $K$ and $\nu_p$ is the corresponding valuation. Let $\|x\|_v = |\varphi(x)|^e$ for all distinct (non-conjugate) embeddings $\varphi : K \to \mathbb{C}$, with $e = 1$ if $\varphi(K) \subset \mathbb{R}$ and $e = 2$ otherwise. We define the height of $(a, b, c) \in (K^*)^3$ to be

$$H_K(a, b, c) = \prod_{v \in V_k} \max (\|a\|_v, \|b\|_v, \|c\|_v),$$

and the conductor of $(a, b, c)$ to be

$$N_K(a, b, c) = \prod_{p \in I_k(a, b, c)} N_{K/Q}(p),$$

where $I_k(a, b, c)$ is the set of all prime ideals $p$ of $\mathcal{O}_K$ for which $\|a\|_p, \|b\|_p, \|c\|_p$ are not all equal. Let $\Delta_{K/Q}$ denote the discriminant of $K$.

The uniform abc–conjecture. For every $\varepsilon > 0$ there exists a constant $C_\varepsilon$, depending only on $\varepsilon$, such that

$$H_K(a, b, c) < C_\varepsilon^{\frac{\varepsilon}{K/Q}} \left(\|\Delta_{K/Q}\| N_K(a, b, c)\right)^{1+\varepsilon},$$

for all $a, b, c \in K^*$ satisfying $a + b + c = 0$.

Remark. In [4], $\Delta_{K/Q}^{1+\varepsilon}$ is replaced by $\Delta_{K/Q}^A$ for some constant $A$. The choice $A = 1 + \varepsilon$ is suggested by a theorem in [1].

We define a real valued function on $K \setminus \{0, 1\}$ by

$$l_K(x) = \frac{\log H_K(x, 1 - x, 1)}{\log |\Delta_{K/Q}| + \log N_K(x, 1 - x, 1)}.$$

The uniform abc–conjecture is equivalent to the statement that $l_K(x)$ is bounded and its biggest limit point equals 1. Examples of $x \in K \setminus \{0, 1\}$ for which $l_K(x)$ is big may therefore be of interest. The definition of $l_K(x)$ suggests defining a function

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on the algebraic numbers (excluding 0 and 1) by \( l(x) = l_{\mathbb{Q}(x)}(x) \). It is not hard to show that the conjecture implies that \( l(x) \) is bounded, and one could expect that the biggest limit point of \( l(x) \) also equals 1.

2. Examples

We are looking for algebraic numbers \( x \) for which \( l_K(x) \) is large, that is, numbers \( x \) for which \( H_K(x, 1 - x, 1) \) is relatively large and \( N_K(x, 1 - x, 1) \) relatively small.

One method is to approximate a number \( \sqrt[4]{k} \), \( k \in \mathbb{N} \), by an element \( y \) in \( K \) and then hope that \( l(k/y^n) \) is large. We will try to do so in a few norm–Euclidean quadratic fields.

Let \( K = \mathbb{Q}(\sqrt{d}) \), for a square free integer \( d \). An integral basis for \( K \) over \( \mathbb{Q} \) is \( \{1, \alpha\} \), where

\[
\alpha = \begin{cases} 
(1 + \sqrt{d})/2 & \text{if } d \equiv 1 \pmod{4}, \\
\sqrt{d} & \text{otherwise}.
\end{cases}
\]

Consider \( \varphi : K \to \mathbb{R}^2 \), where \( \varphi(x + y \alpha) = (x, y) \), and define multiplication on \( \mathbb{R}^2 \) by

\[
(x_1, y_1) (x_2, y_2) = \begin{cases} 
(x_1 x_2 + y_1 y_2 d, x_1 y_2 + y_1 x_2) & \text{if } \alpha = \sqrt{d}, \\
(x_1 x_2 + y_1 y_2 \frac{d - 1}{2}, x_1 y_2 + y_1 x_2 + y_1 \alpha) & \text{if } \alpha = \frac{1 + \sqrt{d}}{2}.
\end{cases}
\]

Then \( \varphi(x y) = \varphi(x) \varphi(y) \) for all \( x, y \in K \) and \( \varphi(\mathcal{O}_K) = \mathbb{Z}^2 \). We extend the norm from the image of \( K \) to \( \mathbb{R}^2 \),

\[
N : \mathbb{R}^2 \to \mathbb{R}, \quad N(a, b) = |(a + b \alpha) (a - b \alpha)|,
\]

so \( N(a, b) = |N_{K/\mathbb{Q}}(a + b \alpha)| \) for all \( a, b \in \mathbb{Q} \). For any \( x \in \mathbb{R}^2 \) we define the subset \( T_x \) of \( \mathcal{O}_K \) to be \( \{r \in \mathcal{O}_K : N(x - \varphi(r)) < 1\} \). Note that if \( T_x \) is non-empty for all \( x \in \mathbb{R}^2 \), then \( K \) is a norm–Euclidean domain. The following theorem from [3] gives a non-empty subset of \( T_x \) in some special cases:

**Theorem.** Let \( d \) be 2, 3, 6, or 7. For any \((x, y) \in \mathbb{R}^2\) let

\[
S_{(x, y)} = \{[a] + [y] \alpha + a + b \alpha : a = -1, 0, 1, 2, b = 0, 1\} \subset \mathcal{O}_K,
\]

where \([a]\) denotes the largest integer less than \( a \). Then \( T_{(x, y)} \cap S_{(x, y)} \neq \emptyset \) for all \((x, y) \in \mathbb{R}^2\). For \( d = -11, -7, -3, -2, -1 \), the statement is true with

\[
S_{(x, y)} = \{[x] + [y] \alpha + a + b \alpha : a = 0, 1, b = 0, 1\}.
\]

Now select an element \( a \in \varphi(\mathcal{O}_K) \) such that the equation \( x^n - a = 0 \) has a solution \( x \in \mathbb{R}^2 \setminus \varphi(K) \), where \( n \) is a positive integer. We want to expand \( x \) in a continued fraction \( p_i/q_i \), where \( p_i, q_i \in \varphi(\mathcal{O}_K) \). Set \( x_0 = x \) and construct the sequences \( \{x_i\} \subset \mathbb{R}^2 \) and \( \{a_i\} \subset \varphi(\mathcal{O}_K) \) by

\[
x_{i+1} = \frac{(1, 0)}{x_i - a_i}, \quad \text{where } a_i \in \varphi(T_{x_i}).
\]

i.e. \( x_{i+1} \) is the inverse of \( x_i - a_i \) with respect to the multiplication in \( \mathbb{R}^2 \) defined above. To get uniqueness, one needs a rule for selecting a particular \( a_i \in \varphi(T_{x_i}) \). To do this, choose an ordering of the \( S_{x_i} \) of the theorem and let \( a_i \) be the first
element in $S_x$, satisfying $N(x_i - a_i) \leq N(x_i - a)$ for all $a \in S_x$. Let $p_i/q_i$ be the continued fraction given by

\[
p_i = a_i p_{i-1} + p_{i-2}, \quad p_{-1} = (1, 0), \quad p_0 = a_0, \\
q_i = a_i q_{i-1} + q_{i-2}, \quad q_{-1} = (0, 0), \quad q_0 = (1, 0).
\]

Then one can check that

\[
x q_i - p_i = \frac{x q_{i-1} - p_{i-1}}{x_{i+1}} = \frac{(-1)^i}{x_1 x_2 \cdots x_{i+1}},
\]

and, if we take the norm on both sides,

\[
N(x q_i - p_i) = \frac{1}{N(x_1) \cdots N(x_{i+1})} = N(x_0 - a_0) \cdots N(x_i - a_i) < 1.
\]

Note that $N(x - p_i/q_i) \to 0$ does not have to imply $|\varphi^{-1}(x) - \varphi^{-1}(p_i/q_i)| \to 0$, where $\varphi^{-1} : \mathbb{R}^2 \to \mathbb{C} : (x, y) \to x + y \alpha$ and $\|\cdot\|$ is the usual absolute-value on $\mathbb{C}$.

Now for some examples of identities $a + b = c$ for which $l(a/c)$ are large. The examples are computed using the method described above, for $n = 2, 3, 4, 5$ in the real cases and $n = 2, 3, 4, 5$ in the complex cases. We only searched among equations $x^n - a = 0$ with $N(a) \leq 10000$. The rational examples are well known and are included here for completeness. There is a table of extremal (rational) $abc$-examples to be found at URL: http://www.math.chalmers.se/~jub/abc.

<table>
<thead>
<tr>
<th>$l$</th>
<th>identity</th>
<th>author</th>
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<tbody>
<tr>
<td>1.768124</td>
<td>$(\sqrt{2})^{17} + (1 - \sqrt{2})^5 (3 - \sqrt{2}) = (1 + \sqrt{2})^5 (3 + \sqrt{2})$</td>
<td>N.B.</td>
</tr>
<tr>
<td>1.707221</td>
<td>$(1 - 2 + \sqrt{2})^{13} + (1 - 2 - \sqrt{2})^{13} = (1 + \sqrt{2})^{13}$</td>
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<tr>
<td>1.629912</td>
<td>$2 + 3^{10} 109 = 23^5$</td>
<td>E.R.</td>
</tr>
<tr>
<td>1.625991</td>
<td>$11^2 + 32^5 5^7 7 = 2^{21} 23$</td>
<td>B.W.</td>
</tr>
<tr>
<td>1.623490</td>
<td>$19 1307 + 7 29^2 31^8 = 2^8 3^{22} 5^4$</td>
<td>J.B.-J.B.</td>
</tr>
<tr>
<td>1.560756</td>
<td>$283 + 5^{11} 13^2 = 2^{8} 3^{8} 17^3$</td>
<td>J.B.-J.B., A.N.</td>
</tr>
<tr>
<td>1.567887</td>
<td>$1 + 2 3^7 = 5^7$</td>
<td>B. W.</td>
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<tr>
<td>1.561437</td>
<td>$(1 + \sqrt{2})^{14} + 1 = (1 + \sqrt{2})^7 (\sqrt{2})^3 13^2$</td>
<td>N.B.</td>
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<tr>
<td>1.547075</td>
<td>$7^3 + 3^{10} = 2^{11} 29$</td>
<td>B.W.</td>
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<tr>
<td>1.544434</td>
<td>$7^2 41^2 311^3 + 11^{16} 13^2 79 = 2^{33} 5^{23} 953$</td>
<td>A.N.</td>
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<tr>
<td>1.536714</td>
<td>$5^7 + 2^9 3^{17} 13^2 = 11^5 17 31^3 137$</td>
<td>H.R.-P.M.</td>
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<tr>
<td>1.528940</td>
<td>$(8 - 3\sqrt{7})^2 (5 - 2\sqrt{7}) + (8 - 3\sqrt{7})^7 (3 - \sqrt{7})^{12} (5 + 2\sqrt{7})^{12} = (4 - 3\sqrt{7})^3$</td>
<td>N.B.</td>
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<tr>
<td>1.526999</td>
<td>$13 19^6 + 3^{20} 5 = 3^{13} 11^2 31$</td>
<td>A.N.</td>
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<td>1.522160</td>
<td>$3^{18} 23 2269 + 17^3 29 31^8 = 2^{10} 5^2 7^{15}$</td>
<td>A.N.</td>
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<tr>
<td>1.518102</td>
<td>$(5 + 2\sqrt{5})^8 (2 - \sqrt{5})^3 (3 - \sqrt{5}) (1 + \sqrt{5}) (1 - \sqrt{5}) 17^2 + 1 = (5 + 2\sqrt{5})^8$</td>
<td>N.B.</td>
</tr>
<tr>
<td>1.502839</td>
<td>$239 + 5^8 17^3 = 2^{10} 37^4$</td>
<td>J.B.-J.B., A.N.</td>
</tr>
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REFERENCES


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