

## BIVARIATE COMPOSITE VECTOR VALUED RATIONAL INTERPOLATION

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ABSTRACT. In this paper we point out that bivariate vector valued rational interpolants (BVRI) have much to do with the vector-grid to be interpolated. When a vector-grid is well-defined, one can directly design an algorithm to compute the BVRI. However, the algorithm no longer works if a vector-grid is ill-defined. Taking the policy of “divide and conquer”, we define a kind of bivariate composite vector valued rational interpolant and establish the corresponding algorithm. A numerical example shows our algorithm still works even if a vector-grid is ill-defined.

### 1. MOTIVATION

Let  $\{(x_i, y_j) | i, j = 0, 1, \dots, n\}$  be a set of points in  $\mathbb{R}^2$  and let these points be reordered into the following array

$$(1.1) \quad \begin{array}{ccc} (x_0, y_0) & \cdots & (x_0, y_n) \\ \vdots & \ddots & \vdots \\ (x_n, y_0) & \cdots & (x_n, y_n) \end{array}$$

where  $x_i > x_{i+1}$  and  $y_i < y_{i+1}$  for  $i = 0, 1, \dots, n - 1$ . We call this array a *square point-grid*, and denote it by  $\Pi^n$ . Let  $\vec{v}_{i,j} \in \mathbb{C}^d$  be a  $d$ -dimensional vector associated with the point  $(x_i, y_j) \in \Pi^n$ . Similarly we arrange these  $\vec{v}_{i,j}$  into the following array

$$(1.2) \quad \begin{array}{cccc} \vec{v}_{0,0} & \vec{v}_{0,1} & \cdots & \vec{v}_{0,n} \\ \vec{v}_{1,0} & \vec{v}_{1,1} & \cdots & \vec{v}_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_{n,0} & \vec{v}_{n,1} & \cdots & \vec{v}_{n,n} \end{array}$$

and call it a vector-grid, denoted by  $\vec{V}^n$ .

For a  $d$ -dimensional vector  $\vec{v} = (v_1, v_2, \dots, v_d) \in \mathbb{C}^d$ , its generalized inverse (or the Samelson inverse) is defined as (see [3])

$$(1.3) \quad \vec{v}^{-1} = \frac{(v_1^*, v_2^*, \dots, v_d^*)}{\sum_{i=1}^d v_i v_i^*},$$

where  $v_i^*$  denotes the complex conjugate of  $v_i$ .

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**Definition 1.1.** A  $d$ -dimensional vector valued polynomial

$$\vec{N}(x, y) = (N_1(x, y), N_2(x, y), \dots, N_d(x, y))$$

is said to be of degree  $n$  and we write  $\partial\vec{N} = n$ , if  $\partial N_i(x, y) \leq n$  for  $i = 1, 2, \dots, d$  and  $\partial N_j(x, y) = n$  for some  $j$  ( $1 \leq j \leq d$ ).

**Definition 1.2.** Denote by  $H_n$  the collection of all bivariate polynomials with total degree not exceeding  $n$  and by  $\vec{H}_n$  the collection of  $d$  dimensional bivariate vector valued polynomials of degree  $n$ . Then the set

$$\vec{H}_{n,m} = \{\vec{N}(x, y)/M(x, y) | \vec{N}(x, y) \in \vec{H}_n, M(x, y) \in H_m\}$$

is called the collection of bivariate vector valued rational functions of type  $(n/m)$ .

All vectors in this paper are  $d$ -dimensional unless otherwise specified.

Making use of the Samelson inverse and reciprocal difference, one of the authors constructed the following Thiele-type branched continued fraction (see [5]):

$$(1.4) \quad \vec{r}_n(x, y) = \vec{t}_0(y) + \frac{x - x_0}{\vec{t}_1(y)} + \dots + \frac{x - x_{n-1}}{\vec{t}_n(y)},$$

where

$$(1.5) \quad \begin{aligned} \vec{t}_l(y) &= \vec{c}_{l,0}(\overline{x_0, \dots, x_l}; y_0) \\ &+ \frac{y - y_0}{\vec{c}_{l,1}(\overline{x_0, \dots, x_l}; y_0, y_1)} + \dots + \frac{y - y_{n-1}}{\vec{c}_{l,n}(\overline{x_0, \dots, x_l}; y_0, \dots, y_n)}, \end{aligned}$$

and  $\vec{c}_{i,j}(\overline{x_0, \dots, x_i}; y_0, \dots, y_j)$  are computed through the following recursive process

$$(1.6) \quad \vec{c}_{0,0}(\overline{x_i}; y_j) = \vec{v}_{i,j} \quad (i = 0, 1, \dots, n, j = 0, 1, \dots, n),$$

$$(1.7)$$

$$\vec{c}_{0,j}(\overline{x_i}; y_0, \dots, y_j) = \frac{y_j - y_{j-1}}{\vec{c}_{0,j-1}(\overline{x_i}; y_0, \dots, y_{j-2}, y_j) - \vec{c}_{0,j-1}(\overline{x_i}; y_0, \dots, y_{j-2}, y_{j-1})},$$

$$(1.8)$$

$$\vec{c}_{i,0}(\overline{x_0, \dots, x_i}; y_j) = \frac{x_i - x_{i-1}}{\vec{c}_{i-1,0}(\overline{x_0, \dots, x_{i-2}, x_i}; y_j) - \vec{c}_{i-1,0}(\overline{x_0, \dots, x_{i-2}, x_{i-1}}; y_j)},$$

$$(1.9)$$

$$\begin{aligned} &\vec{c}_{i,j}(\overline{x_0, \dots, x_i}; y_0, \dots, y_j) \\ &= \frac{y_j - y_{j-1}}{\vec{c}_{i,j-1}(\overline{x_0, \dots, x_i}; y_0, \dots, y_{j-2}, y_j) - \vec{c}_{i,j-1}(\overline{x_0, \dots, x_i}; y_0, \dots, y_{j-2}, y_{j-1})}. \end{aligned}$$

It is not difficult to prove (see [5]) that  $\vec{r}_n(x, y) \in \vec{H}_{n^2+2n, 2[(n^2+2n)/2]}$  (here  $[x]$  denotes the greatest integer not exceeding  $x$ ) and

$$(1.10) \quad \vec{r}_n(x_i, y_j) = \vec{v}_{i,j} \quad (i = 0, 1, \dots, n, j = 0, 1, \dots, n).$$

If the roles of  $x$  and  $y$  are interchanged, one will obtain a so-called dual Thiele-type branched continued fraction (see [5])

$$(1.11) \quad \vec{r}_n^*(x, y) = \vec{t}_0^*(x) + \frac{y - y_0}{\vec{t}_1^*(x)} + \dots + \frac{y - y_{n-1}}{\vec{t}_n^*(x)},$$

where

$$(1.12) \quad \begin{aligned} \vec{t}_l^*(x) &= \vec{c}_{0,l}^*(x_0; \overline{y_0, \dots, y_l}) \\ &+ \frac{x - x_0}{\vec{c}_{1,l}^*(x_0, x_1; \overline{y_0, \dots, y_l})} + \dots + \frac{x - x_{n-1}}{\vec{c}_{n,l}^*(x_0, \dots, x_n; \overline{y_0, \dots, y_l})} \end{aligned}$$

and  $\vec{c}_{i,j}^*(x_0, \dots, x_i; \overline{y_0, \dots, y_j})$  are computed according to the following recursive process

$$(1.13) \quad \vec{c}_{0,0}^*(x_i, \overline{y_j}) = \vec{v}_{i,j} \quad (i = 0, 1, \dots, n, j = 0, 1, \dots, n),$$

$$(1.14)$$

$$(1.15) \quad \begin{aligned} &\vec{c}_{0,j}^*(x_i; \overline{y_0, \dots, y_j}) \\ &= \frac{y_j - y_{j-1}}{\vec{c}_{0,j-1}^*(x_i; \overline{y_0, \dots, y_{j-2}, y_j}) - \vec{c}_{0,j-1}^*(x_i; \overline{y_0, \dots, y_{j-2}, y_{j-1}})}, \end{aligned}$$

$$(1.16) \quad \vec{c}_{i,0}^*(x_0, \dots, x_i; \overline{y_j})$$

$$= \frac{x_i - x_{i-1}}{\vec{c}_{i-1,0}^*(x_0, \dots, x_{i-2}, x_i; \overline{y_j}) - \vec{c}_{i-1,0}^*(x_0, \dots, x_{i-2}, x_{i-1}; \overline{y_j})},$$

$$(1.16)$$

$$\vec{c}_{i,j}^*(x_0, \dots, x_i; \overline{y_0, \dots, y_j})$$

$$= \frac{x_i - x_{i-1}}{\vec{c}_{i-1,j}^*(x_0, \dots, x_{i-2}, x_i; \overline{y_0, \dots, y_j}) - \vec{c}_{i-1,j}^*(x_0, \dots, x_{i-2}, x_{i-1}; \overline{y_0, \dots, y_j})}.$$

To distinguish  $\vec{r}_n(x, y)$  from  $\vec{r}_n^*(x, y)$ , we might as well call  $\vec{r}_n(x, y)$  defined in (1.4)–(1.9) an *x/y-type* and  $\vec{r}_n^*(x, y)$  defined in (1.11)–(1.16) a *y/x-type*. It can be proved that  $\vec{r}_n^*(x, y) \in \vec{H}_{n^2+2n, 2[(n^2+2n)/2]}$  and

$$(1.17) \quad \vec{r}_n^*(x_i, y_j) = \vec{v}_{i,j} \quad (i = 0, 1, \dots, n, j = 0, 1, \dots, n).$$

Although both  $\vec{r}_n(x, y)$  and  $\vec{r}_n^*(x, y)$  are of the same rational type and have the same interpolation properties, one can by no means assert that  $\vec{r}_n(x, y) \equiv \vec{r}_n^*(x, y)$ , as is shown by a numerical example in [5]. However, if the square point-grid  $\Pi^n$  is symmetric, by which we mean  $x_i = y_i$  for  $i = 0, 1, \dots, n$ , and the vector-grid  $\vec{V}^n$  is symmetric, by which we mean  $\vec{v}_{i,j} = \vec{v}_{j,i}$  for  $i, j = 0, 1, \dots, n$ , then we can conclude  $\vec{r}_n(x, y) \equiv \vec{r}_n^*(y, x)$  (see [5]).

In what follows, we only restrict ourselves to the discussion of *x/y-type* bivariate vector valued rational interpolants (BVRI), because the results in *x/y-type* BVRI can easily be transplanted into *y/x-type*.

For convenience, let us simply set

$$(1.18) \quad \vec{c}_{i,j}^{(i,j)} = \vec{c}_{i,j}^*(x_0, \dots, x_i; \overline{y_0, \dots, y_j}) \quad (i = 0, 1, \dots, n, j = 0, 1, \dots, n).$$

Then we have the following algorithm to compute  $\vec{r}_n(x, y)$ .

**Algorithm 1.1.** *This algorithm is carried out according to the following three steps.*

a) For  $i = 0, 1, \dots, n$  and  $j = 0, 1, \dots, n$ , let

$$\vec{c}_{i,j}^{(0,0)} = \vec{v}_{i,j}.$$

b) For  $j = 0, 1, \dots, n$ ,  $p = 1, 2, \dots, n$ , and  $i = p, p + 1, \dots, n$ , let

$$\vec{c}_{i,j}^{(p,0)} = \frac{x_i - x_{p-1}}{\vec{c}_{i,j}^{(p-1,0)} - \vec{c}_{p-1,j}^{(p-1,0)}}.$$

c) For  $i = 0, 1, \dots, n$ ,  $q = 1, 2, \dots, n$ , and  $j = q, q + 1, \dots, n$ , let

$$\vec{c}_{i,j}^{(i,q)} = \frac{y_j - y_{q-1}}{\vec{c}_{i,j}^{(i,q-1)} - \vec{c}_{i,q-1}^{(i,q-1)}}.$$

It is easy to verify that

$$\vec{r}_n(x_i, y_j) = \vec{v}_{i,j}, \quad \forall (x_i, y_j) \in \Pi^n.$$

**Definition 1.3.** A vector-grid  $\vec{V}^n$  is said to be *well-defined* if the  $\vec{c}_{i,j}^{(p,q)}$  as defined in Algorithm 1.1 satisfy  $\vec{c}_{i,j}^{(p-1,0)} \neq \vec{c}_{p-1,j}^{(p-1,0)}$  for  $j = 0, 1, \dots, n$ ,  $p = 1, 2, \dots, n$ , and  $i = p, p + 1, \dots, n$ , and  $\vec{c}_{i,j}^{(i,q-1)} \neq \vec{c}_{i,q-1}^{(i,q-1)}$  for  $i = 0, 1, \dots, n$ ,  $q = 1, 2, \dots, n$ , and  $j = q, q + 1, \dots, n$ . Otherwise the grid  $\vec{V}^n$  is said to be *ill-defined*.

It is clear that if a vector-grid  $\vec{V}^n$  is ill-defined, then Algorithm 1.1 does not work any more.

**Example 1.1.** Let  $\Pi^1$  and a two-dimensional vector-grid  $\vec{V}^1$  be given as follows:

$$\Pi^1 : \begin{pmatrix} (1, 0) & (1, 1) \\ (0, 0) & (0, 1), \end{pmatrix}$$

$$\vec{V}^1 : \begin{pmatrix} (1, 0) & (1, 1) \\ (0, 0) & (1, 0). \end{pmatrix}$$

Proceeding by Algorithm 1.1, we obtain

$$\begin{array}{cc} \vec{c}_{0,0}^{(0,0)} = (1, 0) & \vec{c}_{0,1}^{(0,0)} = (1, 1) \\ \vec{c}_{1,0}^{(0,0)} = (0, 0) & \vec{c}_{1,1}^{(0,0)} = (1, 0) \\ & \downarrow \\ \vec{c}_{0,0}^{(0,0)} = (1, 0) & \vec{c}_{0,1}^{(0,0)} = (1, 1) \\ \vec{c}_{1,0}^{(1,0)} = (1, 0) & \vec{c}_{1,1}^{(1,0)} = (0, 1) \\ & \downarrow \\ \vec{c}_{0,0}^{(0,0)} = (1, 0) & \vec{c}_{0,1}^{(0,1)} = (0, 1) \\ \vec{c}_{1,0}^{(1,0)} = (1, 0) & \vec{c}_{1,1}^{(1,1)} = (-\frac{1}{2}, \frac{1}{2}). \end{array}$$

Obviously  $\vec{V}^1$  is well-defined. As a result,

$$\vec{t}_0(y) = \vec{c}_{0,0} + \frac{y - y_0}{\vec{c}_{0,1}} = (1, 0) + \frac{y}{(0, 1)} = (1, y),$$

$$\vec{t}_1(y) = \vec{c}_{1,0} + \frac{y - y_0}{\vec{c}_{1,1}} = (1, 0) + \frac{y}{(-\frac{1}{2}, \frac{1}{2})} = (1 - y, y).$$

Consequently we get

$$\begin{aligned} \vec{r}_1(x, y) &= \vec{t}_0(y) + \frac{x - x_0}{\vec{t}_1(y)} = (1, y) + \frac{x - 1}{(1 - y, y)} \\ &= \frac{(y^2 + (1 - y)^2 + (x - 1)(1 - y), y(y^2 + (1 - y)^2 + x - 1))}{(1 - y)^2 + y^2}. \end{aligned}$$

**Example 1.2.** Let  $\Pi^2$  and the two-dimensional vector-grid  $\vec{V}^2$  be given as follows:

$$\begin{aligned} \Pi^2 : \quad & \begin{pmatrix} (0, 0) & (0, 1) & (0, 2) \\ (-1, 0) & (-1, 1) & (-1, 2) \\ (-2, 0) & (-2, 1) & (-2, 2), \end{pmatrix} \\ \vec{V}^2 : \quad & \begin{pmatrix} (2, 2) & (6, 0) & (24, 24) \\ (12, 6) & (6, 0) & (12, 6) \\ (0, 0) & (6, 0) & (-2, 2). \end{pmatrix} \end{aligned}$$

We see  $\vec{v}_{0,1} = \vec{v}_{1,1} = \vec{v}_{2,1} = (6, 0)$ , which leads to  $\vec{c}_{0,1}^{(0,0)} = \vec{c}_{1,1}^{(0,0)} = \vec{c}_{2,1}^{(0,0)}$ ; therefore  $\vec{V}^2$  is ill-defined and we cannot use Algorithm 1.1 to construct a vector-valued rational function  $\vec{r}_2(x, y)$  that interpolates  $\vec{V}^2$  over  $\Pi^2$ .

In the next section, we define a new interpolant with a corresponding algorithm more reliable than Algorithm 1.1.

2. THE DEFINITION AND COMPUTATION OF BCVRI

Let us decompose the grid  $\Pi^n$  into the following two triangular grids:

$$(2.1) \quad \begin{pmatrix} (x_0, y_0) \\ (x_1, y_0) & (x_1, y_1) \\ \vdots & \vdots & \ddots \\ (x_n, y_0) & (x_n, y_1) & \cdots & (x_n, y_n) \end{pmatrix}$$

and

$$(2.2) \quad \begin{pmatrix} (x_0, y_1) & (x_0, y_2) & \cdots & (x_0, y_n) \\ & (x_1, y_2) & \cdots & (x_1, y_n) \\ & & \ddots & \vdots \\ & & & (x_{n-1}, y_n), \end{pmatrix}$$

denoted by LB and RU, respectively. We hope to use the policy of “divide and conquer” to construct a kind of composite vector valued rational interpolant. In what follows we abbreviate the term *bivariate composite vector valued rational interpolant* as BCVRI.

Let

$$(2.3) \quad \vec{R}_n(LB; x, y) = \vec{S}_0(LB; y) + \frac{x - x_n}{\vec{S}_1(LB; y)} + \cdots + \frac{x - x_1}{\vec{S}_n(LB; y)},$$

$$(2.4) \quad \vec{R}_n(RU; x, y) = \vec{S}_0(RU; y) + \frac{x - x_0}{\vec{S}_1(RU; y)} + \cdots + \frac{x - x_{n-2}}{\vec{S}_{n-1}(RU; y)},$$

where

$$(2.5) \quad \vec{S}_k(LB; y) = \vec{a}_{k,0} + \frac{y - y_0}{\vec{a}_{k,1}} + \cdots + \frac{y - y_{n-k-1}}{\vec{a}_{k,n-k}}, \quad k = 0, 1, \dots, n,$$

$$(2.6) \quad \vec{S}_k(RU; y) = \vec{b}_{k,k+1} + \frac{y - y_{k+1}}{\vec{b}_{k,k+2}} + \cdots + \frac{y - y_{n-1}}{\vec{b}_{k,n}}, \quad k = 0, 1, \dots, n - 1.$$

Suppose  $\Pi^n$  is uniform, i.e.,

$$(2.7) \quad x_{i-1} - x_i = x_i - x_{i+1} = y_{i+1} - y_i = y_i - y_{i-1}, \quad i = 1, 2, \dots, n - 1,$$

and let

$$(2.8) \quad P(x, y) = \prod_{i=0}^n (x + y - x_n - y_i),$$

$$(2.9) \quad Q(x, y) = \prod_{i=0}^{n-1} (x + y - x_i - y_n).$$

It is clear that  $P(x, y)$  and  $Q(x, y)$  are polynomials of degree  $n + 1$  and  $n$ , respectively, and

$$(2.10) \quad \begin{aligned} P(x_i, y_j) &= 0, & Q(x_i, y_j) &\neq 0 & \text{if } (x_i, y_j) \in \text{LB}, \\ P(x_i, y_j) &\neq 0, & Q(x_i, y_j) &= 0 & \text{if } (x_i, y_j) \in \text{RU}. \end{aligned}$$

When  $\Pi^n$  is not uniform, by which we mean that the conditions (2.7) are not satisfied, one can also construct polynomials  $P(x, y)$  and  $Q(x, y)$  such that (2.10) holds. In general, however, the degrees of the polynomials will be much higher. For example,

$$\begin{aligned} P(x, y) &= \prod_{(x_i, y_j) \in \text{LB}} [(x - x_i)^2 + (y - y_j)^2] \\ Q(x, y) &= \prod_{(x_i, y_j) \in \text{RU}} [(x - x_i)^2 + (y - y_j)^2] \end{aligned}$$

are the polynomials satisfying (2.10) with degree  $(n + 1)(n + 2)$  and  $n(n + 1)$ , respectively.

Now we define a BCVRI over  $\Pi^n$  as follows:

$$(2.11) \quad \vec{R}_n(x, y) = Q(x, y)\vec{R}_n(\text{LB}; x, y) + P(x, y)\vec{R}_n(\text{RU}; x, y).$$

The following algorithm aims at computing the coefficients  $\vec{a}_{k,l}$  and  $\vec{b}_{k,l}$  in branched continued fractions  $\vec{R}_n(\text{LB}; x, y)$  and  $\vec{R}_n(\text{RU}; x, y)$  simultaneously.

**Algorithm 2.1.** *This algorithm proceeds as follows.*

a) For  $i = 0, 1, \dots, n$  and  $j = 0, 1, \dots, i$ , let

$$(2.12) \quad \vec{A}_{i,j}^{(0,0)} = \vec{v}_{i,j} / Q(x_i, y_j).$$

For  $i = 0, 1, \dots, n - 1$  and  $j = i + 1, i + 2, \dots, n$ , let

$$(2.13) \quad \vec{B}_{i,j}^{(0,0)} = \vec{v}_{i,j} / P(x_i, y_j).$$

b) For  $j = 0, 1, \dots, n$ ,  $p = 1, 2, \dots, n - j$ , and  $i = j, j + 1, \dots, n - p$ , let

$$(2.14) \quad \vec{A}_{i,j}^{(p,0)} = \frac{x_i - x_{n-p+1}}{\vec{A}_{i,j}^{(p-1,0)} - \vec{A}_{n-p+1,j}^{(p-1,0)}}.$$

c) For  $i = 0, 1, \dots, n$ ,  $q = 1, 2, \dots, i$ , and  $j = q, q + 1, \dots, i$ , let

$$(2.15) \quad \vec{A}_{i,j}^{(n-i,q)} = \frac{y_j - y_{q-1}}{\vec{A}_{i,j}^{(n-i,q-1)} - \vec{A}_{i,q-1}^{(n-i,q-1)}}.$$

d) For  $j = 1, 2, \dots, n$ ,  $p = 1, 2, \dots, j - 1$ , and  $i = p, p + 1, \dots, j - 1$ , let

$$(2.16) \quad \vec{B}_{i,j}^{(p,0)} = \frac{x_i - x_{p-1}}{\vec{B}_{i,j}^{(p-1,0)} - \vec{B}_{p-1,j}^{(p-1,0)}}.$$

e) For  $i = 0, 1, \dots, n - 1$  and  $j = i + 1, i + 2, \dots, n$ , let

$$(2.17) \quad \vec{B}_{i,j}^{(i,i+1)} = \vec{B}_{i,j}^{(i,0)}.$$

f) For  $i = 0, 1, \dots, n - 2$ ,  $j = i + 2, i + 3, \dots, n$ , and  $q = i + 2, i + 3, \dots, j$ , let

$$(2.18) \quad \vec{B}_{i,j}^{(i,q)} = \frac{y_j - y_{q-1}}{\vec{B}_{i,j}^{(i,q-1)} - \vec{B}_{i,q-1}^{(i,q-1)}}.$$

**Theorem 2.1.** *Let*

$$(2.19) \quad \vec{a}_{k,l} = \vec{A}_{n-k,l}^{(k,l)} \quad (k = 0, 1, \dots, n, l = 0, 1, \dots, n - k),$$

$$(2.20) \quad \vec{b}_{k,l} = \vec{B}_{k,l}^{(k,l)} \quad (k = 0, 1, \dots, n - 1, l = k + 1, \dots, n).$$

Then

$$\vec{R}_n(x, y) \in \vec{H}_{\max(\partial Q+n^2+n-2, \partial P+n^2+n-1), n^2+n-2} \quad \text{for even } n,$$

$$\vec{R}_n(x, y) \in \vec{H}_{\max(\partial Q+n^2+n-1, \partial P+n^2+n-2), n^2+n-2} \quad \text{for odd } n,$$

where  $\partial Q$  and  $\partial P$  denote the total degrees of polynomials  $Q(x, y)$  and  $P(x, y)$ , respectively. (In particular, if  $\Pi^n$  is uniform, then  $\partial Q = n$  and  $\partial P = n + 1$ . In this case,  $\vec{R}_n(x, y) \in \vec{H}_{n^2+2n, n^2+n-2}$  for even  $n$  and  $\vec{R}_n(x, y) \in \vec{H}_{n^2+2n-1, n^2+n-2}$  for odd  $n$ .) Moreover,

$$\vec{R}_n(x_i, y_j) = \vec{v}_{i,j} \quad \forall (x_i, y_j) \in \Pi^n.$$

*Proof.* It is not difficult to show by induction that

$$\vec{R}_n(LB; x, y) \in \vec{H}_{(n^2+2n)/2, (n^2+2n)/2} \quad \text{for even } n,$$

$$\vec{R}_n(LB; x, y) \in \vec{H}_{(n^2+2n-1)/2, (n^2+2n-3)/2} \quad \text{for odd } n,$$

$$\vec{R}_n(RU; x, y) \in \vec{H}_{(n^2-2)/2, (n^2-4)/2} \quad \text{for even } n,$$

$$\vec{R}_n(RU; x, y) \in \vec{H}_{(n^2-1)/2, (n^2-1)/2} \quad \text{for odd } n.$$

Therefore

$$\vec{R}_n(x, y) \in \vec{H}_{\max(\partial Q+n^2+n-2, \partial P+n^2+n-1), n^2+n-2} \quad \text{for even } n$$

$$\vec{R}_n(x, y) \in \vec{H}_{\max(\partial Q+n^2+n-1, \partial P+n^2+n-2), n^2+n-2} \quad \text{for odd } n.$$

Since  $\Pi^n = LB \cup RU$ ,  $(x_i, y_j) \in \Pi^n$  implies  $(x_i, y_j) \in LB$  or  $(x_i, y_j) \in RU$ . If  $(x_i, y_j) \in LB$ , then from (2.5), (2.19) and (2.15) it follows that

$$\begin{aligned} \vec{S}_k(LB; y_j) &= \vec{A}_{n-k,0}^{(k,0)} + \frac{y_j - y_0}{\vec{A}_{n-k,1}^{(k,1)}} + \dots + \frac{y_j - y_{j-1}}{\vec{A}_{n-k,j}^{(k,j)}} \\ &= \vec{A}_{n-k,0}^{(k,0)} + \frac{y_j - y_0}{\vec{A}_{n-k,1}^{(k,1)}} + \dots + \frac{y_j - y_{j-2}}{\vec{A}_{n-k,j}^{(k,j-1)}} \\ &= \dots = \vec{A}_{n-k,j}^{(k,0)}. \end{aligned}$$

By (2.3), (2.14) and (2.12) one has

$$\begin{aligned} \vec{R}_n(\text{LB}; x_i, y_j) &= \vec{A}_{n,j}^{(0,0)} + \sqrt{\frac{x_i - x_n}{\vec{A}_{n-1,j}^{(1,0)}}} + \cdots + \sqrt{\frac{x_i - x_{i+1}}{\vec{A}_{i,j}^{(n-i,0)}}} \\ &= \vec{A}_{n,j}^{(0,0)} + \sqrt{\frac{x_i - x_n}{\vec{A}_{n-1,j}^{(1,0)}}} + \cdots + \sqrt{\frac{x_i - x_{i+2}}{\vec{A}_{i,j}^{(n-i-1,0)}}} \\ &= \cdots = \vec{A}_{i,j}^{(0,0)} = \vec{v}_{i,j}/Q(x_i, y_j). \end{aligned}$$

Therefore, by (2.11) one finally gets

$$\vec{R}_n(x_i, y_j) = Q(x_i, y_j)\vec{R}_n(\text{LB}; x_i, y_j) = \vec{v}_{i,j}.$$

If  $(x_i, y_j) \in \text{RU}$ , then from (2.6), (2.20) and (2.18) it follows that

$$\begin{aligned} \vec{S}_k(\text{RU}; y_j) &= \vec{B}_{k,k+1}^{(k,k+1)} + \sqrt{\frac{y_j - y_{k+1}}{\vec{B}_{k,k+2}^{(k,k+2)}}} + \cdots + \sqrt{\frac{y_j - y_{j-1}}{\vec{B}_{k,j}^{(k,j)}}} \\ &= \vec{B}_{k,k+1}^{(k,k+1)} + \sqrt{\frac{y_j - y_{k+1}}{\vec{B}_{k,k+2}^{(k,k+2)}}} + \cdots + \sqrt{\frac{y_j - y_{j-2}}{\vec{B}_{k,j}^{(k,j-1)}}} \\ &= \cdots = \vec{B}_{k,j}^{(k,k+1)}. \end{aligned}$$

Thus, from (2.4), (2.17), (2.16) and (2.13) we get

$$\begin{aligned} \vec{R}_n(\text{RU}; x_i, y_j) &= \vec{B}_{0,j}^{(0,1)} + \sqrt{\frac{x_i - x_0}{\vec{B}_{1,j}^{(1,2)}}} + \cdots + \sqrt{\frac{x_i - x_{i-1}}{\vec{B}_{i,j}^{(i,i+1)}}} \\ &= \vec{B}_{0,j}^{(0,0)} + \sqrt{\frac{x_i - x_0}{\vec{B}_{1,j}^{(1,0)}}} + \cdots + \sqrt{\frac{x_i - x_{i-1}}{\vec{B}_{i,j}^{(i,0)}}} \\ &= \vec{B}_{0,j}^{(0,0)} + \sqrt{\frac{x_i - x_0}{\vec{B}_{1,j}^{(1,0)}}} + \cdots + \sqrt{\frac{x_i - x_{i-2}}{\vec{B}_{i,j}^{(i-1,0)}}} \\ &= \cdots = \vec{B}_{i,j}^{(0,0)} = \vec{v}_{i,j}/P(x_i, y_j). \end{aligned}$$

Hence, by (2.11) we have

$$\vec{R}_n(x_i, y_j) = P(x_i, y_j)\vec{R}_n(\text{RU}; x_i, y_j) = \vec{v}_{i,j}.$$

The proof is completed. □

### 3. THE COMPLEXITY OF ALGORITHMS

Instead of (2.1) and (2.2) one can also carry out other triangular decompositions of the square grid, for instance, the decomposition

$$\begin{array}{ccccccc} (x_1, y_0) & & & & (x_0, y_0) & (x_0, y_1) & \cdots & (x_0, y_n) \\ (x_2, y_0) & (x_2, y_1) & & & & (x_1, y_1) & \cdots & (x_1, y_n) \\ \vdots & \vdots & \ddots & & & & \ddots & \vdots \\ (x_n, y_0) & (x_n, y_1) & \cdots & (x_n, y_{n-1}) & & & & (x_n, y_n) \end{array}$$

and the decomposition, along another diagonal,

$$\begin{array}{ccccccc}
 (x_0, y_0) & \cdots & (x_0, y_{n-1}) & (x_0, y_n) & & & (x_1, y_n) \\
 (x_1, y_0) & \cdots & (x_1, y_{n-1}) & & & (x_2, y_{n-1}) & (x_2, y_n) \\
 \vdots & & & & & \vdots & \vdots \\
 (x_n, y_0) & & & (x_n, y_1) & \cdots & (x_n, y_{n-1}) & (x_n, y_n).
 \end{array}$$

It is not difficult to define the corresponding BCVRIs based on the above decompositions which interpolate  $\vec{V}^n$  over  $\Pi^n$ .

For a vector valued continued fraction, the complexity is obviously related to the computation of the Samelson inverses. From (1.3) we know that carrying out a Samelson inversion for a  $d$ -dimensional vector demands at least  $2d$  operations of multiplications or divisions. Therefore we take the number of Samelson inverses in an algorithm as the criterion for judging whether the algorithm is complicated or not.

Suppose  $N_1$  and  $N_2$  are the total numbers of Samelson inverses to be computed for the vector valued rational interpolants of form (2.11) and (1.4)–(1.5), respectively. Then

$$\begin{aligned}
 N_1 &= \sum_{j=0}^n \frac{(n-j)(n-j+1)}{2} + \sum_{i=0}^n \frac{i(i+1)}{2} + \sum_{j=1}^n \frac{j(j-1)}{2} + \sum_{i=0}^{n-2} \frac{(n-i+1)(n-i)}{2} \\
 &= \frac{n(n+1)(n+2)}{6} + \frac{n(n+1)(n+2)}{6} + \frac{(n-1)n(n+1)}{6} + \frac{(n-1)n(n+1)}{6} \\
 &= \frac{n(n+1)(2n+1)}{3},
 \end{aligned}$$

$$N_2 = n(n+1)^2,$$

which shows that it is  $n(n+1)(n+2)/3$  times more economical to compute the BCVRI  $\vec{R}_n(x, y)$  in (2.11) yielded by the decomposition (2.1) and (2.2) of  $\Pi^n$  than to compute (1.4) and (1.5) directly. Therefore at least  $2n(n+1)(n+2)d/3$  multiplications are saved through our decomposition method.

#### 4. NUMERICAL EXAMPLE

Let us consider again the grid  $\Pi^2$  and the corresponding vector-grid  $\vec{V}^2$  in Example 1.2, i.e.,

$$\begin{array}{ccc}
 \Pi^2 : & \begin{pmatrix} (0, 0) & (0, 1) & (0, 2) \\ (-1, 0) & (-1, 1) & (-1, 2) \\ (-2, 0) & (-2, 1) & (-2, 2) \end{pmatrix}, \\
 \\
 \vec{V}^2 : & \begin{pmatrix} (2, 2) & (6, 0) & (24, 24) \\ (12, 6) & (6, 0) & (12, 6) \\ (0, 0) & (6, 0) & (-2, 2) \end{pmatrix}.
 \end{array}$$

In this case,  $\Pi^2$  is uniform and  $\vec{V}^2$  is ill-defined. We mentioned in Example 1.2 that the computational procedure in Algorithm 1.1 breaks down. In fact,  $\vec{r}_2(x, y)$  does not exist at all in this case. Otherwise,  $\vec{r}_2(x, y)$  can be written as

$$\vec{r}_2(x, y) = \vec{t}_0(y) + \frac{x - x_0}{t_1(y)} + \frac{x - x_1}{t_2(y)}.$$

Whatever a reordering of the square point-grid  $\Pi^2$  and vector-grid  $\vec{V}^2$  is made, we always have a whole column in  $\vec{V}^2$ , entries of which are all equal to  $(6, 0)$ , i.e.,  $\vec{v}_{0,j} = \vec{v}_{1,j} = \vec{v}_{2,j} = (6, 0)$  with some  $j$  in  $\{0, 1, 2\}$ . Therefore we have

$$\vec{r}_2(x_0, y_j) = \vec{r}_2(x_1, y_j) = \vec{r}_2(x_2, y_j),$$

which leads to

$$\vec{t}_0(y_j) + \frac{x_1 - x_0}{\vec{t}_1(y_j)} = \vec{t}_0(y_j).$$

The above relations imply  $(x_1 - x_0)/\vec{t}_1(y_j) = 0$  which is impossible because  $x_1 \neq x_0$ .

Next, we turn to the construction of a BCVRI defined in (2.11). By (2.8) and (2.9),

$$\begin{aligned} P(x, y) &= (x + y + 2)(x + y + 1)(x + y), \\ Q(x, y) &= (x + y - 2)(x + y - 1). \end{aligned}$$

By (2.12) and (2.13), one gets

$$\begin{aligned} \vec{A}_{0,0}^{(0,0)} &= (1, 1), \\ \vec{A}_{1,0}^{(0,0)} &= (2, 1), \quad \vec{A}_{1,1}^{(0,0)} = (3, 0), \\ \vec{A}_{2,0}^{(0,0)} &= (0, 0), \quad \vec{A}_{2,1}^{(0,0)} = (1, 0), \quad \vec{A}_{2,2}^{(0,0)} = (-1, 1) \end{aligned}$$

and

$$\begin{aligned} \vec{B}_{0,1}^{(0,0)} &= (1, 0), \quad \vec{B}_{0,2}^{(0,0)} = (1, 1), \\ \vec{B}_{1,2}^{(0,0)} &= (2, 1). \end{aligned}$$

According to (2.14) and (2.15), one obtains in order

$$\begin{aligned} \vec{A}_{0,0}^{(1,0)} &= (1, 1), \\ \vec{A}_{1,0}^{(1,0)} &= (2/5, 1/5), \quad \vec{A}_{1,1}^{(1,0)} = (1/2, 0), \\ \vec{A}_{2,0}^{(0,0)} &= (0, 0), \quad \vec{A}_{2,1}^{(0,0)} = (1, 0), \quad \vec{A}_{2,2}^{(0,0)} = (-1, 1), \\ \vec{A}_{0,0}^{(2,0)} &= (3/5, 4/5), \\ \vec{A}_{1,0}^{(1,0)} &= (2/5, 1/5), \quad \vec{A}_{1,1}^{(1,0)} = (1/2, 0), \\ \vec{A}_{2,0}^{(0,0)} &= (0, 0), \quad \vec{A}_{2,1}^{(0,0)} = (1, 0), \quad \vec{A}_{2,2}^{(0,0)} = (-1, 1), \\ \vec{A}_{0,0}^{(2,0)} &= (3/5, 4/5), \\ \vec{A}_{1,0}^{(1,0)} &= (2/5, 1/5), \quad \vec{A}_{1,1}^{(1,1)} = (2, -4), \\ \vec{A}_{2,0}^{(0,0)} &= (0, 0), \quad \vec{A}_{2,1}^{(0,1)} = (1, 0), \quad \vec{A}_{2,2}^{(0,1)} = (-1, 1), \\ \vec{A}_{0,0}^{(2,0)} &= (3/5, 4/5), \\ \vec{A}_{1,0}^{(1,0)} &= (2/5, 1/5), \quad \vec{A}_{1,1}^{(1,1)} = (2, -4), \\ \vec{A}_{2,0}^{(0,0)} &= (0, 0), \quad \vec{A}_{2,1}^{(0,1)} = (1, 0), \quad \vec{A}_{2,2}^{(0,2)} = (-2/5, 1/5). \end{aligned}$$

By Theorem 2.1

$$\begin{aligned} \vec{S}_0(LB; y) &= \vec{A}_{2,0}^{(0,0)} + \sqrt{\frac{y - y_0}{\vec{A}_{2,1}^{(0,1)}}} + \sqrt{\frac{y - y_1}{\vec{A}_{2,2}^{(0,2)}}} \\ &= (0, 0) + \sqrt{\frac{y}{(1, 0)}} + \sqrt{\frac{y - 1}{(-2/5, 1/5)}} \\ &= \frac{(3y - 2y^2, y^2 - y)}{(3 - 2y)^2 + (y - 1)^2}, \\ \vec{S}_1(LB; y) &= \vec{A}_{1,0}^{(1,0)} + \sqrt{\frac{y - y_0}{\vec{A}_{1,1}^{(1,1)}}} = (2/5, 1/5) + \sqrt{\frac{y}{(2, -4)}} \\ &= \left(\frac{4 + y}{10}, \frac{1 - y}{5}\right), \\ \vec{S}_2(LB; y) &= \vec{A}_{0,0}^{(2,0)} = (3/5, 4/5). \end{aligned}$$

This leads to

$$\begin{aligned} \vec{R}_2(LB; x, y) &= \vec{S}_0(LB; y) + \sqrt{\frac{x - x_2}{\vec{S}_1(LB; y)}} + \sqrt{\frac{x - x_1}{\vec{S}_2(LB; y)}} \\ &= \frac{(3y - 2y^2, y^2 - y)}{(3 - 2y)^2 + (y - 1)^2} + \sqrt{\frac{x + 2}{(\frac{4+y}{10}, \frac{1-y}{5})}} + \sqrt{\frac{x + 1}{(3/5, 4/5)}} \\ &= \frac{(3y - 2y^2, y^2 - y)}{(3 - 2y)^2 + (y - 1)^2} + \frac{(10x + 20)(6x + y + 10, 8x - 2y + 10)}{(6x + y + 10)^2 + (8x - 2y + 10)^2}. \end{aligned}$$

According to (2.16)–(2.18), one derives

$$\begin{aligned} \vec{B}_{0,1}^{(0,0)} &= (1, 0), & \vec{B}_{0,2}^{(0,0)} &= (1, 1), \\ & & \vec{B}_{1,2}^{(1,0)} &= (-1, 0), \\ \vec{B}_{0,1}^{(0,1)} &= (1, 0), & \vec{B}_{0,2}^{(0,1)} &= (1, 1), \\ & & \vec{B}_{1,2}^{(1,2)} &= (-1, 0), \\ \vec{B}_{0,1}^{(0,1)} &= (1, 0), & \vec{B}_{0,2}^{(0,2)} &= (0, 1), \\ & & \vec{B}_{1,2}^{(1,2)} &= (-1, 0). \end{aligned}$$

By Theorem 2.1

$$\begin{aligned} \vec{S}_0(RU; y) &= \vec{B}_{0,1}^{(0,1)} + \sqrt{\frac{y - y_1}{\vec{B}_{0,2}^{(0,2)}}} = (1, 0) + \sqrt{\frac{y - 1}{(0, 1)}} = (1, y - 1), \\ \vec{S}_1(RU; y) &= \vec{B}_{1,2}^{(1,2)} = (-1, 0), \end{aligned}$$

which results in

$$\begin{aligned}\vec{R}_2(RU; x, y) &= \vec{S}_0(RU; y) + \frac{x - x_0}{\vec{S}_1(RU; y)} \\ &= (1, y - 1) + \frac{x}{(-1, 0)} = (1 - x, y - 1).\end{aligned}$$

Hence we finally obtain

$$\begin{aligned}\vec{R}_2(x, y) &= Q(x, y)\vec{R}_2(LB; x, y) + P(x, y)\vec{R}_2(RU; x, y) \\ &= (x + y - 2)(x + y - 1) \left[ \frac{(3y - 2y^2, y^2 - y)}{(3 - 2y)^2 + (y - 1)^2} \right. \\ &\quad \left. + \frac{(10x + 20)(6x + y + 10, 8x - 2y + 10)}{(6x + y + 10)^2 + (8x - 2y + 10)^2} \right] \\ &\quad + (x + y + 2)(x + y + 1)(x + y)(1 - x, y - 1).\end{aligned}$$

It is easy to verify that  $\vec{R}_2(x, y)$  interpolates  $\vec{V}^2$  over  $\Pi^2$ . In our example, the vector-grid  $\vec{V}^2$  is ill-defined, and, what is more, as mentioned at the beginning of this section, in this case one fails to find a rational interpolant  $\vec{r}_2(x, y)$  of the form (1.4). However,  $\vec{R}_2(x, y)$ , as a BCVRI defined in (2.11), still exists. Hence, compared with Algorithm 1.1, our new algorithm for BCVRI is more reliable in the sense that it can overcome the nonexistence of some  $\vec{r}_n(x, y)$ , and more economical in the sense that it involves fewer Samelson inverses.

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