

## ENUMERATING SOLUTIONS TO $p(a) + q(b) = r(c) + s(d)$

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ABSTRACT. Let  $p, q, r, s$  be polynomials with integer coefficients. This paper presents a fast method, using very little temporary storage, to find all small integers  $(a, b, c, d)$  satisfying  $p(a) + q(b) = r(c) + s(d)$ . Numerical results include all small solutions to  $a^4 + b^4 + c^4 = d^4$ ; all small solutions to  $a^4 + b^4 = c^4 + d^4$ ; and the smallest positive integer that can be written in 5 ways as a sum of two coprime cubes.

### 1. INTRODUCTION

Let  $H$  be a positive integer. How can one find all positive integers  $a, b, c, d \leq H$  satisfying  $a^3 + 2b^3 + 3c^3 = 4d^3$ ?

The following method is standard. Sort the set  $\{(a^3 + 2b^3, a, b) : a, b \leq H\}$  into increasing order in the first component. Similarly sort  $\{(4d^3 - 3c^3, c, d) : c, d \leq H\}$ . Now merge the sorted lists, looking for collisions. The sorting takes time  $H^{2+o(1)}$  and space  $H^{2+o(1)}$ .

It does not seem to be widely known that one can save a factor of  $H$  in space. Section 3 explains how to enumerate  $\{(a^3 + 2b^3, a, b)\}$  and  $\{(4d^3 - 3c^3, c, d)\}$  in order, using  $O(H^2)$  heap operations on two heaps of size  $H$ . Heaps are reviewed in section 2. The remaining sections of this paper give several numerical examples. See <http://pobox.com/~djb/sortedsums.html> for a UNIX implementation of most of the algorithms discussed here.

A standard improvement is to split the range of  $a^3 + 2b^3$  and  $4d^3 - 3c^3$  into several (0-adic or  $p$ -adic) intervals. For example, one can separately consider each possibility for  $4d^3 - 3c^3 \pmod 7$ , and skip pairs  $(a, b)$  with  $a^3 + 2b^3 \pmod 7 \in \{2, 5\}$ .

**Notes.** Lander and Parkin in [11] enumerated solutions to  $a^4 + b^4 = c^4 + d^4$  using time  $H^{3+o(1)}$  and space  $H^{1+o(1)}$ .

Ekl in [2] pointed out that the time of the Lander-Parkin method could be reduced to  $H^{2+o(1)}$ . I made the same observation independently in April 1997, when Yuri Tschinkel asked me about the example described in section 4 below. David W. Wilson made the same observation independently in October 1997, for the example described in section 5 below. The difference between my method, Ekl's method, and the Lander-Parkin method is the difference between a heap, a balanced tree, and an unstructured array.

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The use of heaps to enumerate sums in sorted order actually appeared much earlier in another context, namely William S. Brown's algorithm for multiplication of sparse power series. See [9, exercise 5.2.3–29]; compare [9, exercise 5–18].

## 2. HEAPS

A *heap* is a sequence  $x_1, x_2, \dots, x_n$  satisfying  $x_{\lfloor k/2 \rfloor} \leq x_k$  for  $2 \leq k \leq n$ : i.e.,  $x_1 \leq x_2$ ,  $x_1 \leq x_3$ ,  $x_2 \leq x_4$ ,  $x_2 \leq x_5$ ,  $x_3 \leq x_6$ ,  $x_3 \leq x_7$ , etc.

The smallest element of a heap  $x_1, x_2, \dots, x_n$  is  $x_1$ . Given  $y$ , one can permute  $y, x_2, \dots, x_n$  into a new heap by the following algorithm. First set  $j \leftarrow 1$ . Then perform the following steps repeatedly: set  $k \leftarrow 2j$ ; stop if  $k > n$ ; set  $k \leftarrow k + 1$  if  $k < n$  and  $x_{k+1} < x_k$ ; stop if  $y \leq x_k$ ; exchange  $y$ , which is now in the  $j$ th position, with  $x_k$ ; set  $j \leftarrow k$ . The total number of operations here is  $O(\log n)$ .

In particular, using  $O(\log n)$  operations, one can permute  $x_n, x_2, \dots, x_{n-1}$  into a new heap. By a similar algorithm, also using  $O(\log n)$  operations, one can permute  $x_1, x_2, \dots, x_n, y$  into a new heap.

**Notes.** Heaps were published by Williams in [22]. Floyd in [5] pointed out an algorithm using  $O(n)$  operations to permute any sequence of length  $n$  into a new heap.

For some practical improvements in heap performance see [9, exercise 5.2.3–18] and [9, exercise 5.2.3–28]. The bottom-up algorithm in [9, exercise 5.2.3–18] is due to Floyd; the “new” algorithms announced many years later in [1] and [21] are the same as Floyd's.

There are many other data structures that support insertion of new elements and removal of the smallest element. Any such structure is called a *priority queue*. Examples include *leftist trees*, as discussed in [9, section 5.2.3]; *loser selection trees*, as discussed in [9, section 5.4.1]; *balanced trees*, as discussed in [9, section 6.2.3]; and *B-trees*, as discussed in [9, section 6.2.4]. See also [10, page 152]. The reader can replace the heap in section 3 with any priority queue. Beware, however, that some “fast” priority queues are several times bigger and slower than heaps; see, for example, section 10 below.

## 3. ENUMERATING SUMS

Fix  $p, q \in \mathbf{Z}[x]$ . This section explains how to print  $\{(p(a) + q(b), a, b) : a, b \leq H\}$  in increasing order in the first component, using space  $H^{1+o(1)}$ .

First build a table of  $\{(p(a), a) : a \leq H\}$ . Sort the table into increasing order in the first component; say  $p(a_1) \leq p(a_2) \leq \dots$ .

Next build a heap containing  $\{(p(a_1) + q(b), 1, b) : b \leq H\}$ . Perform the following operations repeatedly until the heap is empty:

1. Find and remove the smallest element  $(y, n, b)$  in the heap.
2. Print  $(y, a_n, b)$ ; by induction  $y = p(a_n) + q(b)$  at this point.
3. Insert  $(p(a_{n+1}) - p(a_n) + y, n + 1, b)$  into the heap if  $a_{n+1}$  exists.

Step 1 and step 3 can be combined into a single heap operation.

This algorithm takes time  $H^{1+o(1)}$  for initializations, plus  $H^{o(1)}$  for each of the  $H^2$  outputs, for a total of  $H^{2+o(1)}$ . There are at most  $H$  elements in the heap at any moment.

**Refinements.** One can easily save half the heap operations if  $p = q$ : start with  $\{(p(a_n) + p(a_n), n, a_n)\}$ ; print  $(y, b, a_n)$  along with  $(y, a_n, b)$  if  $a_n \neq b$ .

One can speed up the manipulation of  $y$ , and in some cases save space, by storing  $p(a_2) - p(a_1), p(a_3) - p(a_2), \dots$  instead of  $p(a_2), p(a_3), \dots$ .

One need not bother building tables of  $n \mapsto a_n$  and  $n \mapsto p(a_n)$  if  $p$  is a sufficiently dull function.

**Generalizations.** Given functions  $p, q, r, s$  from finite sets  $A, B, C, D$  to an ordered group, one can enumerate  $\{(a, b, c, d) \in A \times B \times C \times D : p(a) + q(b) = r(c) + s(d)\}$  by the same algorithm. For example, one can enumerate small solutions  $(a, b, c, d)$  to  $a^3 + 2b^3 = 3c^3 + 4d^3$  with  $a, b, c, d \in \mathbf{Z}[w]/(w^2 + w + 1)$ , using lexicographic order on  $\mathbf{Z}[w]/(w^2 + w + 1)$ . See section 10 for another example.

One can restrict attention to a subset of  $A \times B$ , simply by skipping to the next suitable  $a$  for each  $b$ . See sections 9 and 10 for examples.

There are many functions that are not of the form  $a, b \mapsto p(a) + q(b)$  but that are nevertheless amenable to sorted enumeration. For example, one can apply the method here to any function  $f$  such that  $a \mapsto f(a, b)$  is monotone for each  $b$ . See section 6 for an example.

4. EXAMPLE:  $a^3 + b^3 = c^3 + d^3$

There are 12137664 solutions  $(a, b, c, d)$  to  $a^3 + b^3 = c^3 + d^3 > 0$  with  $a \neq c, a \neq d, -10^5 \leq a, b, c, d \leq 10^5$ , and  $a\mathbf{Z} + b\mathbf{Z} + c\mathbf{Z} + d\mathbf{Z} = \mathbf{Z}$ . In other words, there are 12137664 rational points of height at most  $10^5$  on the surface  $x^3 + y^3 + z^3 = 1$  away from the lines on the surface.

This computation took  $1.4 \cdot 10^{13}$  cycles on a Pentium II-350. It takes roughly twice as long to do a similar computation for  $pa^3 + qb^3 = pc^3 + qd^3$ ; roughly three times as long for  $pa^3 + pb^3 = rc^3 + sd^3$ ; and roughly four times as long for  $pa^3 + qb^3 = rc^3 + sd^3$ .

**Notes.** Peyre and Tschinkel have checked some of my numerical results and some of their theoretical computations against the best available conjecture. See [16]. Heath-Brown in [8] had previously enumerated solutions to  $a^3 + b^3 = c^3 + 2d^3$  and  $a^3 + b^3 = c^3 + 3d^3$  with  $-10^3 \leq a, b, c \leq 10^3$  by a cubic-time method.

In some cases one can save time by using [8, Theorem 1].

5. EXAMPLE: MANY EQUAL SUMS OF TWO POSITIVE CUBES

The smallest integer that can be written in  $k$  ways as a sum of two cubes of positive integers is 1729 for  $k = 2$ ; 87539319 for  $k = 3$ ; 6963472309248 for  $k = 4$ ; and 48988659276962496 for  $k = 5$ . There are no 6-way integers below  $10^{18}$ . (There are two other 5-way integers below  $10^{18}$ :  $391909274215699968 = 8 \cdot 48988659276962496$  and  $490593422681271000$ .)

This computation took  $7.9 \cdot 10^{14}$  cycles on an UltraSPARC II-296.

**Notes.** The answer for  $k = 3$  was found by Leech in [14]. The answer for  $k = 4$  was found by Rosenstiel, Dardis, and Rosenstiel in [17]. The answer for  $k = 5$  was found by David W. Wilson in 1997 and independently by me in 1998. There is an answer for every  $k$ ; see [19] for the best known bounds.

## 6. EXAMPLE: MANY EQUAL SUMS OF TWO CUBES

The smallest positive integer that can be written in  $k$  ways as a sum of two cubes is 91 for  $k = 2$ ; 728 for  $k = 3$ ; 2741256 for  $k = 4$ ; 6017193 for  $k = 5$ ; 1412774811 for  $k = 6$ ; 11302198488 for  $k = 7$ ; and 137513849003496 for  $k = 8$ . There are no 9-way integers below  $2.5 \cdot 10^{17}$ . (There are 37 other 8-way integers below  $2.5 \cdot 10^{17}$ .)

This computation took  $9.2 \cdot 10^{14}$  cycles on an UltraSPARC II-296. To keep the heap small, I enumerated pairs  $(a, b)$  with  $a \geq b/2$  and  $1 \leq a^3 + (b-a)^3 \leq 2.5 \cdot 10^{17}$ , in order of  $a^3 + (b-a)^3$ ; these conditions imply  $1 \leq b \leq 10^6$ .

**Notes.** The answers for  $k \in \{5, 6, 7\}$  were found by Randall Rathbun, according to [7, page 141]. The answer for  $k = 8$  appears to be new.

## 7. EXAMPLE: MANY EQUAL SUMS OF TWO COPRIME CUBES

The smallest positive integer that can be written in  $k$  ways as a sum of two cubes of coprime integers is 91 for  $k = 2$ ; 3367 for  $k = 3$ ; 16776487 for  $k = 4$ ; and 506433677359393 for  $k = 5$ . Each of these integers is squarefree. There are no 6-way integers below  $2.5 \cdot 10^{17}$ . (There is one other 5-way integer, namely 137904678696613339.)

I found these results during the computation described in section 6. A separate computation, skipping pairs  $(a, b)$  with a common factor, would have been somewhat faster.

**Notes.** The answer for  $k = 4$  was found by Rathbun, according to [7, page 141]. The answer for  $k = 5$  appears to be new.

Silverman proved in [18] that the number of pairs of integers  $(a, b)$  satisfying  $a^3 + b^3 = n$  is bounded by a particular function of the rank over  $\mathbf{Q}$  of the elliptic curve  $x^3 + y^3 = n$ , if  $n$  is cubefree. It is not known how tight Silverman's bound is.

Paul Vojta found that 15170835645 can be written in 3 ways as a sum of two cubes of coprime *positive* integers.

8. EXAMPLE:  $a^4 + b^4 = c^4 + d^4$ 

There are 516 solutions  $(a, b, c, d)$  to  $a^4 + b^4 = c^4 + d^4$  with  $0 < b \leq a$ ,  $0 < d \leq c$ ,  $c < a \leq 10^6$ , and  $a\mathbf{Z} + b\mathbf{Z} + c\mathbf{Z} + d\mathbf{Z} = \mathbf{Z}$ . This computation took roughly  $10^{15}$  cycles on an UltraSPARC II-296.

The fourth power of  $10^6$  does not fit into a 64-bit integer. I actually enumerated values of  $(a^4 \bmod m) + (b^4 \bmod m) + (0 \bmod m)$  greater than or equal to  $m$ , where  $m = 2^{60} - 93$ . Then I checked each collision  $a^4 + b^4 \equiv c^4 + d^4 \pmod{m}$  to see whether  $a^4 + b^4 = c^4 + d^4$ .

**Notes.** 218 of the 516 solutions were already known: Lander and Parkin in [11] exhaustively found all solutions with  $a^4 + b^4 < 7.885 \cdot 10^{15}$ ; Lander, Parkin, and Selfridge in [13] exhaustively found all solutions with  $a^4 + b^4 \leq 5.3 \cdot 10^{16}$ ; Zajta in [23] found many solutions with  $a \leq 10^6$  by various ad-hoc techniques.

9. EXAMPLE:  $a^4 + b^4 + c^4 = d^4$

The only positive solutions  $(a, b, c, d)$  to  $a^4 + b^4 + c^4 = d^4$  with  $d \leq 2.1 \cdot 10^7$  and  $a\mathbf{Z} + b\mathbf{Z} + c\mathbf{Z} + d\mathbf{Z} = \mathbf{Z}$  are permutations of the solutions

- (95800, 414560, 217519, 422481),
- (1390400, 2767624, 673865, 2813001),
- (5507880, 8332208, 1705575, 8707481),
- (5870000, 11289040, 8282543, 12197457),
- (12552200, 14173720, 4479031, 16003017),
- (3642840, 7028600, 16281009, 16430513),
- (2682440, 18796760, 15365639, 20615673).

This computation took  $4.5 \cdot 10^{15}$  cycles on a Pentium II-350.

I used several  $p$ -adic restrictions here. One can permute  $a, b, c$  so that  $a \in 2\mathbf{Z}$  and  $b \in 10\mathbf{Z}$ . Then  $a \in 8\mathbf{Z}$ ,  $b \in 40\mathbf{Z}$ ,  $d - 1 \in 8\mathbf{Z}$ , and  $c \equiv \pm d \pmod{1024}$  by [20, Theorem 1]; also  $d \notin 5\mathbf{Z}$ . There are roughly  $H^2/320$  possibilities for  $(a, b)$  and  $H^2/10240$  possibilities for  $(c, d)$  if  $d \leq H$ . I enumerated sums modulo  $2^{60} - 93$  as in section 8.

**Notes.** Euler conjectured that  $a^4 + b^4 + c^4 = d^4$  had no positive integer solutions. Ward in [20] proved that there are no solutions with  $d \leq 10^4$ . Lander, Parkin, and Selfridge in [13] proved that there are no solutions with  $d \leq 2.2 \cdot 10^5$ . Elkies in [4] proved that there are infinitely many solutions with  $a\mathbf{Z} + b\mathbf{Z} + c\mathbf{Z} + d\mathbf{Z} = \mathbf{Z}$ , and exhibited two examples. Elkies commented that the smaller example, with  $d = 20615673$ , “seems beyond the range of reasonable exhaustive computer search.” Frye in [6] subsequently found the solutions with  $d = 422481$ , and proved that there are no other solutions with  $d \leq 2 \cdot 10^6$ . Allan MacLeod subsequently found the solutions with  $d = 2813001$  by Elkies’s method. The solutions with  $d \in \{8707481, 12197457, 16003017, 16430513\}$  appear to be new.

For each  $(c, d)$  satisfying various  $p$ -adic restrictions, Ward factored  $d^4 - c^4$  into primes and then found all representations of  $d^4 - c^4$  as a sum of squares; the total time of Ward’s algorithm is  $H^{2+o(1)}$  with modern factoring methods, but the  $o(1)$  is fairly large. Lander, Parkin, Selfridge, and Frye instead enumerated possibilities for  $b$ , and checked for each  $b$  whether  $d^4 - c^4 - b^4$  was a fourth power; Frye estimated that his program used about  $H^3/490000$  fourth-power tests to find all solutions with  $d \leq H$ .

10. EXAMPLE:  $a^7 + b^7 + c^7 + d^7 = e^7 + f^7 + g^7 + h^7$

The five smallest integers that can be written in 2 ways as sums of four positive seventh powers are 2056364173794800, 12191487610289536, 263214614245734400, 696885239160606459, and 1560510414117060608. There are no other examples below  $420^7$ .

I began this computation by generating a sorted table of  $\{a^7 + b^7 : a \geq b\}$ . Then I enumerated sums  $(a^7 + b^7) + (c^7 + d^7)$  in order, skipping inputs  $((a, b), (c, d))$  with  $b < c$ . Searching up to  $155^7$ , to verify the smallest example, took  $1.4 \cdot 10^{10}$  cycles (and roughly 340 kilobytes of memory) on an UltraSPARC I-167. Searching up to  $420^7$  took  $1.4 \cdot 10^{12}$  cycles.

**Notes.** All the examples here were found by Ekl in [2] and [3]. However, Ekl needed  $1.6 \cdot 10^{11}$  cycles on an HP PRISM-50 (and roughly 8900 kilobytes of memory) to find the first example. Presumably the main reason is that the priority queue in [2] and [3] was a balanced tree, whereas the priority queue here is a heap.

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