MATHEMATICS OF COMPUTATION
Volume 70, Number 233, Pages 431–447
S 0025-5718(00)01280-1
Article electronically published on June 12, 2000

EXPLICIT UPPER BOUNDS FOR EXPONENTIAL SUMS OVER PRIMES

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Dedicated to the memory of Chen Jing Run

ABSTRACT. We give explicit upper bounds for linear trigonometric sums over primes.

1. INTRODUCTION

In 1937 I.M. Vinogradov [12] proved that every sufficiently large odd number is the sum of three prime numbers. Later Chen and Wang [2] gave a lower bound for the result of Vinogradov, which is very large, around 10^{43000}. The method used is the Hardy-Littlewood circle method, and the following sums play an important role in the proof:

$$\sum_{p\le x} \mathrm{e}(\alpha p), \qquad S(x,\alpha) = \sum_{n\le x} \Lambda(n)\ \mathrm{e}(n\alpha),$$

where Λ is the function of Von Mangoldt and $\mathrm{e}(\alpha) = e^{2i\pi\alpha}$.

In [1] Chen proved that if $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \le 1$, $q \le x$, then

$$\left|\sum_{p\le x} \mathrm{e}(\alpha p)\right| \le 1.2\ x\,(\log x)^{3/4} \log\log x \left(\sqrt{\frac{5}{q} + \frac{q\log q}{x}} + \sqrt{\log q}\ \exp -\frac{1}{2}\sqrt{\log x}\right).$$

More recently in [3] Chen and Wang proved that

$$|S(x,\alpha)| \le 0.177\ \frac{x}{\sqrt{q}}\,(\log x)^3 + 3.8\ x^{4/5}(\log x)^{2.2} + 0.08\ \sqrt{xq}\,(\log x)^{3.5}.$$

Our purpose is to improve on these two estimates. By a classical elementary transformation it suffices to consider $S(x,\alpha)$.

In order to estimate this sum, a useful identity has been proved by R.C. Vaughan [11]. Recently Daboussi [4] gave another identity, which has the advantage of involving nice coefficients. This permits us to give a new explicit upper bound for $|S(x,\alpha)|$.

In this paper we will need sharp versions of some classical inequalities which have their own independent interest. We will prove

Received by the editor November 3, 1998.
2000 *Mathematics Subject Classification.* Primary 11L07, 11L20.
Key words and phrases. Prime numbers, exponential sums, sieves.

TABLE 1. Upper bounds for $x^{-1}|S(x,\alpha)|$ when $q = (\log x)^3$

x	$x^{-1}\|S(x,\alpha)\|$
10^{200}	0.385
10^{300}	0.293
10^{400}	0.241
10^{500}	0.207
10^{1000}	0.129
10^{2000}	0.080
10^{5000}	0.042
10^{10000}	0.026
10^{20000}	0.016
10^{43000}	0.010

Theorem 1. *For $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \le 1$, $q \le x$, we have*

$$\begin{aligned}|S(x,\alpha)| \quad &\le \quad 14.86\,\sqrt{\log\log x + 0.5}\; x\,(\log x)^{3/4}\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}}\\ &\quad + 6.45\,\sqrt{\log\log x + 0.5}\; x\,(\log x)^{5/4}\,\exp\left(-\frac{1}{2}\sqrt{\log x}\right).\end{aligned}$$

From this theorem we can compute numerical upper bounds for $x^{-1}|S(x,\alpha)|$ (cf. Table 1) with the choice $q = (\log x)^3$ for which the result of Chen and Wang is not even as good as the trivial upper bound.

Definitions and notations. For x real we will denote by $\lfloor x\rfloor$ the greatest integer $\le x$, $\{x\}$ the fractional part of x, $\lceil x\rceil$ the smallest integer $\ge x$, $\|x\|$ the distance from x to the nearest integer, $\lfloor x\rceil$ the smallest integer n such that $|x-n| \le 1/2$ (n is unique if $\{x\} \ne 1/2$). The letter p denotes always a prime number, $\pi(x)$ denotes the number of primes $\le x$. We denote by μ and φ the Möbius and Euler functions, respectively. The functions $\Omega(n)$ and $\omega(n)$ count the number of prime factors of n, respectively, with and without multiplicity. We define the functions u_z and v_z by $u_z(m) = 1$ if $(\forall p,\ p\,|\,m \Rightarrow p > z)$ and $u_z(m) = 0$ otherwise, and $v_z(m) = 1$ if $(\forall p,\ p\,|\,m \Rightarrow p \le z)$ and $v_z(m) = 0$ otherwise.

2. Vinogradov type lemmas

In [1] Chen improved Vinogradov Lemmas 8a and 8b [13]. In this section, we further improve the results of Chen.

Lemma 1. *Let $x \in \mathbb{R}$, $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \le 1$, $(a,q) = 1$, $U > 0$. We then have*

$$\sum_{x<n\le x+q} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) \le 2\,U + \frac{2}{\pi}\,q\log 4q.$$

Remark 1. This is the analog of Vinogradov Lemma 8a. Chen obtained $5\,U + q\log q$. The factor 2 instead of 5 is obtained by using $t = \lfloor t\rceil + \delta$ with $|\delta| \le 1/2$ which is more precise than the classical $t = \lfloor t\rfloor + \{t\}$. The factor $2/\pi$ has been obtained by dealing directly with $(\sin t)^{-1}$ without using the classical inequality $\sin t \ge 2t/\pi$ for

$0 \le t \le \pi/2$. Indeed we simply used the fact that $(\log\tan(t/2))' = (\sin t)^{-1}$. We acknowledge the referee's improvement of this lemma (see below).

Proof. The result is trivial for $q \le 2$. We therefore suppose $q \ge 3$.

Let $m_0 = \lfloor x \rfloor + \lfloor (q+1)/2 \rfloor$. We have

$$\sum_{x<n\le x+q} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) = \sum_{-q/2<m\le q/2} \min\left(U, \frac{1}{|\sin(\pi\alpha(m_0+m))|}\right).$$

Now writing

$$b = \left\lfloor am_0 + \frac{\beta m_0}{q} \right\rceil \quad \text{and} \quad b + \delta = am_0 + \frac{\beta m_0}{q}$$

(hence $|\delta| \le 1/2$), we obtain

$$\alpha(m_0+m) = \frac{1}{q}\left(am + am_0 + \frac{\beta m_0}{q} + \frac{\beta m}{q}\right) = \frac{1}{q}(am+b) + \frac{1}{q}\left(\delta + \frac{\beta m}{q}\right).$$

When m runs through the integers in the interval $-q/2 < m \le q/2$, $am+b$ runs through a complete set of residue classes modulo q. We introduce r such that $am + b \equiv r \bmod q$ and $-q/2 < r \le q/2$. Using $\left|\delta + \frac{\beta m}{q}\right| \le 1$, we get for $|r| \ge 2$

$$\min\left(U, \frac{1}{|\sin(\pi\alpha(m_0+m))|}\right) \le \sin\left(\frac{\pi}{q}(|r|-1)\right).$$

For $r = \pm 1$, the referee observed that

$$\|\alpha(m_0+m)\| = \left\|\frac{f(r)}{q}\right\|,$$

where $f(r) = r + \delta + \theta(r)$, with $|\theta(r)| \le \frac{1}{2}$. It follows that $f(1) - f(-1) \ge 1$ so that

$$\max\left\{\left\|\frac{f(1)}{q}\right\|, \left\|\frac{f(-1)}{q}\right\|\right\} \ge \frac{1}{2q}.$$

Thus one of the two terms for $r = \pm 1$ can be bounded by $|\sin(\pi/2q)|^{-1}$. Hence we obtain

$$\sum_{x<n\le x+q} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) \le 2\,U + \frac{1}{\sin\left(\frac{\pi}{2q}\right)} + 2\sum_{2\le r\le q/2} \frac{1}{\sin\left(\frac{\pi}{q}(r-1)\right)}$$

(the sum on the right hand side is empty for $q = 3$).

Using the convexity of the function $t \longmapsto 1/\sin(\pi t/q)$ for $0 < t \le q/2$, we obtain

$$\sum_{1\le r\le \frac{q}{2}-1} \frac{1}{\sin\frac{\pi}{q}r} \le \int_{\frac{1}{2}}^{\frac{q-1}{2}} \frac{dt}{\sin\frac{\pi t}{q}} \le \frac{q}{\pi}\log\cot\frac{\pi}{4q} \le \frac{q}{\pi}\log\frac{4q}{\pi},$$

and we have for $q \ge 3$

$$\frac{1}{\sin\left(\frac{\pi}{2q}\right)} + \frac{2q}{\pi}\log\frac{4q}{\pi} \le \frac{2q}{\pi}\log 4q,$$

which completes the proof of Lemma 1. □

Lemma 2. *Let $N \ge 1$, $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \le 1$, $(a,q) = 1$, $U > 0$. We then have*

$$\sum_{1\le n\le N} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) \le \left\lceil\frac{N}{q}\right\rceil\left(2\,U + \frac{2}{\pi}\,q\log 4q\right).$$

Proof. We divide the interval $1 \le n \le N$ into subintervals $kq+1 \le n \le (k+1)q$, for which we apply Lemma 1. There are at most $\left\lceil \frac{N}{q} \right\rceil$ such subintervals. □

Lemma 3. *Let $N \ge 1$, $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \le 1$, $(a,q)=1$, $x > 0$. We then have*

$$\sum_{1\le n\le N} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|}\right) \le 2\ \frac{x}{q} \log\left(\frac{8N}{q} + 4\right) + \frac{2}{\pi}\ N \log 4q + \frac{3}{\pi}\ q \log 5q.$$

Remark 2. This is the analog of Vinogradov Lemma 8b.

Proof. We can assume without loss of generality that N is an integer.

Using the convexity of $t \longmapsto \frac{1}{t}$ for $t > 0$, we obtain for $N \ge 1$

$$\sum_{1\le n\le N} \frac{1}{n} \le \int_{\frac{1}{2}}^{N+\frac{1}{2}} \frac{dt}{t} = \log(2N+1).$$

This proves the result for $q \le 2$. We can now suppose $q \ge 3$.

Writing $K = \left\lceil \frac{N}{q} - \frac{1}{2} \right\rceil$, we have $K \ge \frac{N}{q} - \frac{1}{2}$ and $Kq + \frac{q}{2} \ge N$. Hence

$$\sum_{1\le n\le N} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|}\right) \le \sum_{k=0}^{K} S_k,$$

where

$$S_0 = \sum_{1\le n\le q/2} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|}\right),$$

and for $k \ge 1$

$$S_k = \sum_{kq-q/2<n\le kq+q/2} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|}\right).$$

For $1 \le n \le q/2$, we have $an \not\equiv 0 \bmod q$ and $\alpha n = \frac{an}{q} + \frac{1}{q}\frac{\beta n}{q}$ with $\left|\frac{\beta n}{q}\right| \le \frac{1}{2}$. Hence for $an \equiv r \bmod q$ with $-q/2 \le r \le q/2$ and $r \ne 0$, we have

$$|\sin(\pi\alpha n)| \ge \left|\sin\left(\frac{\pi}{q}\left(|r| - \frac{1}{2}\right)\right)\right|$$

and

$$\begin{aligned} S_0 &\le 2 \sum_{1\le r\le q/2} \left|\sin\left(\frac{\pi}{q}\left(r - \frac{1}{2}\right)\right)\right|^{-1} \\ &\le 2\left|\sin\left(\frac{\pi}{2q}\right)\right|^{-1} + 2\int_{\frac{3}{2}}^{\frac{q+1}{2}} \frac{dt}{\sin\left(\frac{\pi}{q}\left(t - \frac{1}{2}\right)\right)} \\ &\le \frac{2}{\sin\left(\frac{\pi}{2q}\right)} + \frac{2q}{\pi} \log\cot\left(\frac{\pi}{2q}\right) \\ &\le \frac{2q}{\pi}\left(2\frac{\frac{\pi}{2q}}{\sin\left(\frac{\pi}{2q}\right)} + \log\left(\frac{2q}{\pi}\right)\right) \\ &\le \frac{2q}{\pi} \log 5q \qquad \text{for } q \ge 4. \end{aligned}$$

For $q = 3$ we also have $S_0 \le 2|\sin \frac{\pi}{2q}|^{-1} = 4 \le \frac{2q}{\pi} \log 5q$. This proves the result for $K < 1$, so from now we assume that $K \ge 1$.

By Lemma 1 we have

$$\begin{aligned}
\sum_{1\le k\le K} S_k &\le \sum_{1\le k\le K} \sum_{kq-q/2<n\le kq+q/2} \min\left(\frac{x}{q(k-\frac{1}{2})}, \frac{1}{|\sin(\pi\alpha n)|}\right) \\
&\le \sum_{1\le k\le K} \left(2\,\frac{x}{q(k-\frac{1}{2})} + \frac{2}{\pi} q\log 4q\right) \\
&\le \frac{2}{\pi} Kq\log 4q + \frac{2x}{q} \sum_{1\le k\le K} \frac{1}{(k-\frac{1}{2})}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{1\le k\le K} S_k &\le \frac{2}{\pi} Kq\log 4q + \frac{2x}{q}\left(2 + \int_{\frac{3}{2}}^{K+\frac{1}{2}} \frac{dt}{(t-\frac{1}{2})}\right) \\
&\le \frac{2}{\pi} Kq\log 4q + \frac{2x}{q}(2 + \log K) \\
&\le \frac{2}{\pi}\left(\frac{N}{q} + \frac{1}{2}\right) q\log 4q + \frac{2x}{q}\log\left(e^2 K\right) \\
&\le \frac{2}{\pi} N\log 4q + \frac{q}{\pi}\log 4q + \frac{2x}{q}\log\left(e^2 K\right).
\end{aligned}$$

Finally,

$$\begin{aligned}
&\sum_{1\le n\le N} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|}\right) \\
&\le \frac{2}{\pi} N\log 4q + \frac{2q}{\pi}\left(\log q + \log 5 + \frac{\log q}{2} + \frac{\log 4}{2}\right) + \frac{2x}{q}\log\left(e^2 K\right) \\
&\le \frac{2}{\pi} N\log 4q + \frac{3q}{\pi}\left(\log q + \frac{2\log 5}{3} + \frac{\log 4}{3}\right) + \frac{2x}{q}\log\left(e^2 K\right) \\
&\le \frac{2}{\pi} N\log 4q + \frac{3q}{\pi}\log 5q + \frac{2x}{q}\log\left(e^2 K\right) \\
&\le \frac{2}{\pi} N\log 4q + \frac{3q}{\pi}\log 5q + \frac{2x}{q}\log\left(\frac{8N}{q} + 4\right),
\end{aligned}$$

which completes the proof. □

3. Rankin's method

Elliott [6, pages 81-83] has given an effective version of Rankin's method. In this section we generalize and improve his results numerically.

Lemma 4. *Let $z \ge 2$, f a multiplicative function with $f \ge 0$, and*

$$S = \sum_{p\le z} \frac{f(p)}{1+f(p)} \log p.$$

We assume $S > 0$ and write $K(t) = \log t - 1 + \frac{1}{t}$ for $t \ge 1$.

For any y with $\log y \ge S$ we have

$$\begin{aligned}\sum_{n>y} v_z(n)\mu^2(n)f(n) &\le \left(\prod_{p\le z}(1+f(p))\right)\exp\left(-\frac{\log y}{\log z}K\left(\frac{\log y}{S}\right)\right),\\ \sum_{n\le y} v_z(n)\mu^2(n)f(n) &\ge \left(\prod_{p\le z}(1+f(p))\right)\left(1-\exp\left(-\frac{\log y}{\log z}K\left(\frac{\log y}{S}\right)\right)\right).\end{aligned}$$

In particular for any y with $\log y \ge 7S$ we have

$$\begin{aligned}\sum_{n>y} v_z(n)\mu^2(n)f(n) &\le \left(\prod_{p\le z}(1+f(p))\right)\exp\left(-\frac{\log y}{\log z}\right),\\ \sum_{n\le y} v_z(n)\mu^2(n)f(n) &\ge \left(\prod_{p\le z}(1+f(p))\right)\left(1-\exp\left(-\frac{\log y}{\log z}\right)\right).\end{aligned}$$

Proof. The special case for $\log y \ge 7S$ is a direct consequence of the general case $\log y \ge S$, as for all $t \ge 7$ we have $K(t) \ge K(7) \ge 1$.

We note that

$$\sum_{n\le y} v_z(n)\mu^2(n)f(n) + \sum_{n>y} v_z(n)\mu^2(n)f(n) = \prod_{p\le z}(1+f(p)),$$

which shows that the required lower bound for the first sum will follow from the required upper bound for the second sum.

For all $\eta \ge 0$ we have

$$\begin{aligned}\sum_{n>y} v_z(n)\mu^2(n)f(n) &\le \sum_{n=1}^{\infty} v_z(n)\mu^2(n)f(n)\left(\frac{n}{y}\right)^{\eta}\\ &\le y^{-\eta}\prod_{p\le z}\left(1+f(p)\,p^{\eta}\right).\end{aligned}$$

Now

$$\prod_{p\le z}\left(1+f(p)\,p^{\eta}\right) = \left(\prod_{p\le z}(1+f(p))\right)\left(\prod_{p\le z}\left(1+\frac{f(p)}{1+f(p)}\,(p^{\eta}-1)\right)\right).$$

Using $\log(1+u) \le u$ for $u \ge 0$ we get

$$\prod_{p\le z}\left(1+\frac{f(p)}{1+f(p)}\,(p^{\eta}-1)\right) \le \exp\left(\sum_{p\le z}\frac{f(p)}{1+f(p)}\,(p^{\eta}-1)\right),$$

$$\begin{aligned}
\sum_{p\le z}\frac{f(p)}{1+f(p)}\,(p^\eta-1) &\le \sum_{p\le z}\frac{f(p)}{1+f(p)}\,(\exp(\eta\log p)-1)\\
&\le \sum_{p\le z}\frac{f(p)}{1+f(p)}\,\sum_{k=1}^{\infty}\frac{(\eta\log p)^k}{k!}\\
&\le \sum_{k=1}^{\infty}\frac{\eta^k(\log z)^{k-1}}{k!}\sum_{p\le z}\frac{f(p)}{1+f(p)}\,\log p\\
&\le \frac{S}{\log z}\sum_{k=1}^{\infty}\frac{\eta^k(\log z)^k}{k!}\\
&\le \frac{S}{\log z}\,(\exp(\eta\log z)-1)\,.
\end{aligned}$$

Writing $\nu=\eta\log z$ we get

$$y^{-\eta}\prod_{p\le z}\left(1+\frac{f(p)}{1+f(p)}\,(p^\eta-1)\right)\le\exp\left(\frac{S}{\log z}\left(\exp(\nu)-1-\nu\frac{\log y}{S}\right)\right).$$

The last inequality is valid for any $\nu\ge 0$, in particular for $\nu=\log\left(\frac{\log y}{S}\right)$. Hence

$$\begin{aligned}
&\sum_{n>y}v_z(n)\mu^2(n)f(n)\\
&\le \left(\prod_{p\le z}(1+f(p))\right)\exp\left(\frac{S}{\log z}\left(\frac{\log y}{S}-1-\frac{\log y}{S}\log\left(\frac{\log y}{S}\right)\right)\right)\\
&\le \left(\prod_{p\le z}(1+f(p))\right)\exp\left(-\frac{\log y}{\log z}K\left(\frac{\log y}{S}\right)\right).
\end{aligned}$$

□

4. Effective inequalities

Lemma 5. *For all $x>1$ we have*

$$\pi(2x)-\pi(x)<\frac{x}{\log x}.$$

Proof. P. Dusart [5] improved some results of [9] and proved for $x\ge 60184$ that

$$\frac{x}{\log(x)-1}<\pi(x)<\frac{x}{\log(x)-1.1}.$$

This implies for $x\ge 60184$ that

$$\pi(2x)-\pi(x)\le\frac{2x}{\log x-0.41}-\frac{x}{\log x-1}\le\frac{x}{\log x}\left(1-\frac{0.016}{\log x}\right)<\frac{x}{\log x}$$

using the inequalities $1+u<\frac{1}{1-u}<1+\frac{6}{5}u$ (valid for $0<u<1/6$). The result can be easily extented for all $x>1$ by computer evidence. □

Remark 3. We note that the result of this lemma is sharp for $x=113/2$ for which

$$\pi(113)-\pi(113/2)=14<\frac{113/2}{\log(113/2)}=14.0051\ldots.$$

Lemma 6. *For $z \geq 2$ we have*

$$\sum_{q \leq z} \left(1 + \frac{q}{z}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)} \geq \log z.$$

Proof. By Lemma 8 of Montgomery-Vaughan [7] we have for $z \geq 100$

$$\sum_{q \leq z} \left(1 + \frac{q}{z}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)} \geq \log z + 0.361$$

and the result follows by a direct computation for $2 \leq z < 100$. □

Lemma 7. *For $2 \leq z \leq x$ we have*

$$\sum_{x < m \leq 2x} u_z(m) \leq \frac{x}{\log z}.$$

Proof. Suppose first that $\sqrt{2x} \leq z \leq x$. For $x < m \leq 2x$, we have $u_z(m) = 1$ if and only if m is prime. Using Lemma 5 we obtain

$$\sum_{x < m \leq 2x} u_z(m) = \pi(2x) - \pi(x) \leq \frac{x}{\log x} \leq \frac{x}{\log z}.$$

Hence we can suppose $z < \sqrt{2x}$.

By Corollary 1 of Montgomery-Vaughan [7] we have for any positive number z,

$$\sum_{x < m \leq 2x} u_z(m) \leq x \left(\sum_{q \leq z} \left(1 + \frac{3qz}{2x}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)} \right)^{-1},$$

and using Lemma 6 we obtain for $z \leq \sqrt{\frac{2}{3}x}$

$$\sum_{x < m \leq 2x} u_z(m) \leq x \left(\sum_{q \leq z} \left(1 + \frac{q}{z}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)} \right)^{-1} \leq \frac{x}{\log z}.$$

Thus we can suppose $\sqrt{\frac{2}{3}x} < z < \sqrt{2x}$.

For $x > 15$ we have $(2x)^{1/3} < \sqrt{\frac{2}{3}x} < z < \sqrt{2x}$ and

$$\sum_{x < m \leq 2x} u_z(m) = \pi(2x) - \pi(x) + \pi_2(2x, z) - \pi_2(x, z),$$

where

$$\pi_2(x, z) = \#\{n \leq x,\ \Omega(n) = 2,\ p \,|\, n \Longrightarrow p > z\}.$$

We have using Lemma 5

$$\pi_2(2x, z) - \pi_2(x, z) \leq \sum_{z < p \leq \sqrt{2x}} \left(\pi\left(\frac{2x}{p}\right) - \pi\left(\frac{x}{p}\right)\right) \leq \sum_{z < p \leq \sqrt{2x}} \frac{x/p}{\log(x/p)}.$$

For $x > 15$ we have $x/z > e$ and the function $t \longmapsto t/\log t$ is increasing for $t > e$. Hence

$$\pi_2(2x, z) - \pi_2(x, z) \leq \frac{x/z}{\log(x/z)} \left(\pi(\sqrt{2x}) - \pi(z)\right) \leq \frac{x/z}{\log(x/z)} \left(\pi(2z) - \pi(z)\right),$$

and using Lemma 5 we obtain

$$\pi_2(2x,z)-\pi_2(x,z)\le \frac{x}{\log(x/z)\log z}.$$

Therefore we have for $x>15$

$$\sum_{x<m\le 2x} u_z(m)\le \frac{x}{\log x}+\frac{x}{\log(x/z)\log z},$$

and using the inequality $z<\sqrt{2x}$ we obtain for $x>200$

$$\sum_{x<m\le 2x} u_z(m)\le \frac{x}{\log z}\left(\frac{\log\sqrt{2x}}{\log x}+\frac{1}{\log\sqrt{x/2}}\right)\le \frac{x}{\log z}$$

and it suffices to show the result for $x\le 200$ and $z<\sqrt{2x}$, which can be verified easily by computer. This completes the proof of Lemma 7. □

Corollary 1. *For $2\le z\le x$ we have*

$$\sum_{x\le m<2x} u_z(m)\le \frac{x}{\log z}.$$

Proof. If x is not an integer or if x is an integer and $z=x$ (in this case $u_z(x)=0$), we have

$$\sum_{x\le d<2x} u_z(d)\le \sum_{x<d\le 2x} u_z(d)\le \frac{x}{\log z}.$$

If x is an integer and $z<x$, we have

$$\sum_{x\le d<2x} u_z(d)=\sum_{x^-<d\le 2x^-} u_z(d)\le \frac{x}{\log z}.$$

□

Lemma 8. *For $x\ge 2$ and $1\le h\le x$ we have*

$$\sum_{n\le x}\Lambda(n)\Lambda(n+h)\le 15\ x\ (\log\log x+0.5).$$

Proof. If h is odd, we have $\Lambda(n)\Lambda(n+h)=0$ if n is not a power of 2. Hence, when h is odd,

$$\sum_{n\le x}\Lambda(n)\Lambda(n+h)\le \sum_{r\le\frac{\log x}{\log 2}}\log 2\ \log 2x\le \log x\ \log 2x\le 2x.$$

We can suppose that h is even, and $\Lambda(n)\Lambda(n+h)\ne 0$ implies that n is odd; therefore $n\ge 3$ and $x\ge 3$.

The contribution of the terms for which n and $n+h$ are not both primes is at most

$$2\log 2x\sum_{\substack{p^r\le 2x\\ r\ge 2}}\log p\le 2\ \pi(\sqrt{2x})\ \log^2 2x.$$

By inequality 3.6 of Rosser and Schoenfeld [8] we have

$$\forall x>1,\quad \pi(x)<1.25506\ \frac{x}{\log x};$$

therefore the contribution of the terms for which n and $n+h$ are not both primes is at most

$$7.1\ \sqrt{x}\ \log 2x.$$

By the theorem of Siebert [10] the number of primes $p \le x$ such that $p+h$ is prime is at most

$$16 \prod_{p \ge 3} \left(1 - \frac{1}{(p-1)^2}\right) \frac{x}{\log^2 x} \prod_{\substack{p \mid h \\ p \ge 3}} \frac{p-1}{p-2}.$$

We remark that

$$\frac{p-1}{p-2} = \left(1 - \frac{1}{p}\right)^{-1} \left(1 - \frac{1}{(p-1)^2}\right)^{-1},$$

and when h is even

$$\prod_{\substack{p \mid h \\ p \ge 3}} \left(1 - \frac{1}{p}\right) = 2\ \frac{\varphi(h)}{h}$$

so that Siebert's expression can be written as

$$8 \prod_{\substack{p \\ (p,h)=1}} \left(1 - \frac{1}{(p-1)^2}\right) \frac{x}{\log^2 x} \cdot \frac{h}{\varphi(h)} \le \frac{8\,x}{\log^2 x} \cdot \frac{h}{\varphi(h)}.$$

By inequality 3.41 and 3.42 of Rosser and Schoenfeld [8] we have for $h \ge 3$

$$\frac{h}{\varphi(h)} \le e^{\gamma} \log\log h + \frac{2.50637}{\log\log h};$$

hence for $x \ge 3$ we have

$$\sum_{n \le x} \Lambda(n)\Lambda(n+h) \le 8\ \frac{\log 2x}{\log x} x \left(e^{\gamma} \log\log x + \frac{2.50637}{\log\log x}\right) + 7.1\ \sqrt{x}\ \log 2x,$$

and for $x \ge 10^8$ we obtain

$$\sum_{n \le x} \Lambda(n)\Lambda(n+h) \le 15\ x\ (\log\log x + 0.5).$$

For $x < 10^8$ we have

$$\sum_{n \le x} \Lambda(n)\Lambda(n+h) \le \log 2x \sum_{n \le x} \Lambda(n),$$

and we use the inequality 3.35 of Rosser and Schoenfeld [8]

$$\sum_{n \le x} \Lambda(n) < 1.03883\ x \quad \text{for all } x > 0,$$

which gives for $10 \le x < 10^8$

$$\sum_{n \le x} \Lambda(n)\Lambda(n+h) \le 1.03883\ \log(2.10^8)\ x \le 20\ x \le 15\ x\ (\log\log x + 0.5).$$

For $x < 10$ the inequality is verified by direct computation. □

5. Sums of type I and II

For $z \geq 3$ depending on x only we can split $S(x,\alpha)$ as follows:

$$S(x,\alpha) = \sum_{n \leq x} \Lambda(n)\ \mathrm{e}(n\alpha) = S_1(x,\alpha) + S_2(x,\alpha),$$

where

$$\begin{aligned} S_1(x,\alpha) &= \sum_{n \leq x} v_z(n)\Lambda(n)\ \mathrm{e}(n\alpha), \\ S_2(x,\alpha) &= \sum_{n \leq x} u_z(n)\Lambda(n)\ \mathrm{e}(n\alpha). \end{aligned}$$

We can estimate $S_1(x,\alpha)$ trivially:

$$|S_1(x,\alpha)| \leq \sum_{\substack{p^r \leq x \\ p \leq z}} \log p = \sum_{p \leq z} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor \leq \pi(z) \log x \leq z \log x.$$

Now we split $S_2(x,\alpha)$ into $B_1(x,\alpha) - B_2(x,\alpha)$ (see [4] for details) where

$$\begin{aligned} B_1(x,\alpha) &= \sum_{n \leq x} u_z(n)\ \log(n)\ \mathrm{e}(n\alpha), \\ B_2(x,\alpha) &= \sum_{z \leq d \leq x/z} u_z(d) \sum_{z \leq m \leq x/d} u_z(m)\ \Lambda(m)\ \mathrm{e}(md\alpha). \end{aligned}$$

6. Sums of type I

Lemma 9. *For $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \leq 1$, $(a,q) = 1$, $3^7 \leq z^7 \leq y \leq x$, we have*

$$\begin{aligned} |B_1(x,\alpha)| \leq\ & \frac{2}{3}\ e^{\gamma} x \log x \log 3z\ \ \exp\left(-\frac{\log y}{\log z}\right) \\ & +\ 2\ \frac{x}{q} \log x \log 3y + \frac{2}{\pi}\ y \log x \log 4q + \frac{3}{\pi}\ q \log x \log 5q. \end{aligned}$$

Proof. We write

$$B_1(x,\alpha) = \sum_{n \leq x} u_z(n)\ \mathrm{e}(n\alpha) \int_1^n \frac{dt}{t} = \int_1^x \frac{dt}{t} \sum_{t \leq n \leq x} u_z(n)\ \mathrm{e}(n\alpha).$$

Introducing $T_1(t,x,\alpha) = \sum_{t \leq n \leq x} u_z(n)\ \mathrm{e}(n\alpha)$, we see that

$$|B_1(x,\alpha)| \leq \log x \sup_{1 \leq t \leq x} |T_1(t,x,\alpha)|.$$

By the Möbius inversion formula

$$T_1(t,x,\alpha) = \sum_{\substack{n,d \\ t \leq nd \leq x}} v_z(n)\ \mu(n)\ \mathrm{e}(nd\alpha).$$

Let y such that $z^7 \leq y \leq x$. We have

$$\begin{aligned} T_{1,1}(t,x,\alpha) &= \sum_{n \leq y} \sum_{t \leq nd \leq x} v_z(n)\ \mu(n)\ \mathrm{e}(nd\alpha), \\ T_{1,2}(t,x,\alpha) &= \sum_{y < n \leq x} \sum_{t \leq nd \leq x} v_z(n)\ \mu(n)\ \mathrm{e}(nd\alpha). \end{aligned}$$

Clearly

$$|T_{1,1}(t,x,\alpha)| \le \sum_{n\le y} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi n\alpha)|}\right).$$

Hence by Lemma 3

$$|T_{1,1}(t,x,\alpha)| \le 2\ \frac{x}{q}\log 3y + \frac{2}{\pi}\ y\log 4q + \frac{3}{\pi}\ q\log 5q.$$

We have

$$|T_{1,2}(t,x,\alpha)| \le \sum_{n>y} v_z(n)\ |\mu(n)| \sum_{d\le x/n} 1 \le x\sum_{n>y}\frac{v_z(n)}{n}\mu^2(n).$$

In [8], Rosser and Schoenfeld proved (inequality 3.24) that

$$\sum_{p\le x}\frac{\log p}{p} < \log x \quad \text{for all } x>1.$$

Using this inequality we get

$$0 < S = \sum_{p\le z}\frac{\frac{1}{p}}{1+\frac{1}{p}}\log p \le \sum_{p\le z}\frac{\log p}{p} \le \log z \le \frac{\log y}{7}.$$

Hence by Rankin's method (Lemma 4) we get

$$|T_{1,2}(t,x,\alpha)| \le x\left(\prod_{p\le z}\left(1+\frac{1}{p}\right)\right)\exp\left(-\frac{\log y}{\log z}\right).$$

Now for $z\ge 3$ we have

$$\prod_{p\le z}\left(1+\frac{1}{p}\right)\prod_{p\le z}\left(1-\frac{1}{p}\right) = \prod_{p\le z}\left(1-\frac{1}{p^2}\right) \le \frac{3}{4}\cdot\frac{8}{9} = \frac{2}{3}.$$

In [8], Rosser and Schoenfeld proved (inequality 3.31) that

$$\prod_{p\le x}\frac{p}{p-1} < e^{\gamma}\sum_{1\le n\le x}\frac{1}{n} \quad \text{for all } x\ge 1.$$

Using these inequalities we obtain

$$|T_{1,2}(t,x,\alpha)| \le \frac{2}{3}\ e^{\gamma}x\left(\sum_{1\le n\le z}\frac{1}{n}\right)\exp\left(-\frac{\log y}{\log z}\right),$$

and finally

$$|T_{1,2}(t,x,\alpha)| \le \frac{2}{3}\ e^{\gamma}x\log 3z\exp\left(-\frac{\log y}{\log z}\right),$$

which completes the proof. □

7. Sums of type II

Let J satisfy $2^J \lceil z \rceil \le x/z < 2^{J+1} \lceil z \rceil$. We have

$$|B_2(x,\alpha)| \le \sum_{0\le j\le J} \sum_{2^j\lceil z\rceil \le d < 2^{j+1}\lceil z\rceil} u_z(d) \left| \sum_{z\le m\le x/d} u_z(m)\ \Lambda(m)\ \mathrm{e}(md\alpha) \right|.$$

We observe that $J\log 2 \le \log x - 2\log z \le \log x - \log 2$, and we define

$$T_2(M) = \sum_{M\le d<2M} u_z(d) \left| \sum_{z\le m\le x/d} u_z(m)\ \Lambda(m)\ \mathrm{e}(md\alpha) \right|.$$

We get

$$|B_2(x,\alpha)| \le \frac{\log x}{\log 2} \sup_{\substack{z\le M\le x/z\\ M\in\mathbb{N}}} |T_2(M)|.$$

By the Cauchy-Schwarz inequality

$$|T_2(M)|^2 \le \left(\sum_{M\le d<2M} u_z^2(d) \right) \sum_{M\le d<2M} \left| \sum_{z\le m\le x/d} u_z(m)\ \Lambda(m)\ \mathrm{e}(md\alpha) \right|^2.$$

By Corollary 1, we have

$$\sum_{M\le d<2M} u_z^2(d) \le \frac{M}{\log z}.$$

Expanding the square and summing first over the d's we obtain

$$|T_2(M)|^2 \le \frac{M}{\log z} \sum_{z\le m\le x/M} \Lambda(m) \sum_{z\le m'\le x/M} \Lambda(m') \left| \sum_{d\in I(m,m')} \mathrm{e}((m-m')d\alpha) \right|,$$

where $I(m,m')$ is the interval of d's such that $M \le d \le \min(2M-1, \frac{x}{m}, \frac{x}{m'})$.

We distinguish $m = m'$ and $m \ne m'$ and obtain

$$|T_2(M)|^2 \le |T_{2,1}(M)|^2 + |T_{2,2}(M)|^2,$$

where

$$|T_{2,1}(M)|^2 = \frac{M^2}{\log z} \sum_{z\le m\le x/M} \Lambda^2(m)$$

and

$$|T_{2,2}(M)|^2 = 2\ \frac{M}{\log z} \sum_{1\le h\le x/M} \sum_{z\le m\le x/M} \Lambda(m)\ \Lambda(m+h) \left| \sum_{d\in I(m,m+h)} \mathrm{e}(hd\alpha) \right|.$$

We have

$$\left| \sum_{d\in I(m,m+h)} \mathrm{e}(hd\alpha) \right| \le \min\left(M, \frac{1}{|\sin(\pi h\alpha)|} \right).$$

By Lemma 8

$$\sum_{z\le m\le x/M} \Lambda(m)\ \Lambda(m+h) \le 15\ (\log\log x + 0.5)\ \frac{x}{M}.$$

So

$$|T_{2,2}(M)|^2 \le 30\ (\log\log x + 0.5)\frac{x}{\log z} \sum_{1\le h\le x/M} \min\left(M, \frac{1}{|\sin(\pi h\alpha)|}\right).$$

Using Lemma 2 and $z \le M \le x/z$ we get

$$\begin{aligned}
\sum_{1\le h\le x/M} \min\left(M, \frac{1}{|\sin(\pi h\alpha)|}\right) &\le \left(\frac{x}{Mq}+1\right)\left(2M + \frac{2}{\pi} q\log 4q\right) \\
&\le \frac{2x}{q} + 2M + \frac{2x\log 4q}{\pi M} + \frac{2}{\pi} q\log 4q \\
&\le \frac{2x}{q} + \frac{2x}{z} + \frac{2x\log 4q}{\pi z} + \frac{2}{\pi} q\log 4q \\
&\le 2\,x\left(\frac{1}{q} + \frac{\pi+\log 4q}{\pi z} + \frac{q\log 4q}{\pi x}\right).
\end{aligned}$$

We obtain

$$|T_{2,2}(M)|^2 \le 60\ (\log\log x + 0.5)\ \frac{x^2}{\log z}\left(\frac{1}{q} + \frac{\log 93q}{\pi z} + \frac{q\log 4q}{\pi x}\right),$$

$$\sum_{n\le x} \Lambda(n) < 1.03883\ x \quad \text{for all } x > 0.$$

For $z \ge 3$, the function $M \longmapsto M\log(x/M)$ is increasing on $[z, x/z]$. Hence by Rosser and Schoenfeld [8] inequality 3.35 we have

$$|T_{2,1}(M)|^2 \le 1.03883\ xM\frac{\log(x/M)}{\log z} \le 1.03883\ \frac{x^2}{z}$$

and

$$\begin{aligned}
|B_2(x,\alpha)| \quad &\le \quad \frac{\sqrt{1.03883}}{\log 2}\frac{x}{\sqrt{z}}\log x \\
&\quad + \frac{1}{\log 2}\sqrt{60\ (\log\log x + 0.5)}\frac{x\log x}{\sqrt{\log z}}\left(\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}} + \sqrt{\frac{\log 93q}{\pi z}}\right).
\end{aligned}$$

Finally

$$\begin{aligned}
|B_2(x,\alpha)| \quad &\le \quad 1.48\,\frac{x}{\sqrt{z}}\log x \\
&\quad + 11.18\,\sqrt{\log\log x + 0.5}\frac{x\log x}{\sqrt{\log z}}\left(\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}} + \sqrt{\frac{\log 93q}{\pi z}}\right).
\end{aligned}$$

8. Proof of Theorem 1

We can suppose $x \geq 10^{184}$; otherwise

$$\sqrt{\log\log x + 0.5}\; x\,(\log x)^{5/4}\, \exp\left(-\frac{1}{2}\sqrt{\log x}\right) > 0.166$$

and the result is trivial using Rosser and Schoenfeld [8] inequality 3.35. Furthermore we can suppose

$$(\log x)^{3/2}\log\log x \leq q \leq \frac{x}{(\log x)^{5/2}\log\log x};$$

otherwise the result is trivial.

We choose $\log z = \sqrt{\log x}$ and we obtain

$$\begin{aligned}
|B_2(x,\alpha)| \quad &\leq \quad 11.18\,\sqrt{\log\log x + 0.5}\; x\,(\log x)^{3/4}\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}} \\
&\quad + 1.48\, x\,\log x\, \exp\left(-\frac{1}{2}\sqrt{\log x}\right) \\
&\quad + 6.31\,\sqrt{\log\log x + 0.5}\; x\,(\log x)^{5/4}\, \exp\left(-\frac{1}{2}\sqrt{\log x}\right) \\
&\leq \quad 11.18\,\sqrt{\log\log x + 0.5}\; x\,(\log x)^{3/4}\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}} \\
&\quad + 6.44\,\sqrt{\log\log x + 0.5}\; x\,(\log x)^{5/4}\, \exp\left(-\frac{1}{2}\sqrt{\log x}\right).
\end{aligned}$$

Let us first suppose that

$$(\log x)^{3/2}\log\log x \leq q \leq (\log x)^3.$$

Let $\log y = \sqrt{\log x}\log q$. We then have for $x \geq 10^{184}$

$$\begin{aligned}
|B_1(x,\alpha)| \quad &\leq \quad 1.19\,\frac{x}{q}\,\log x(\log z + \log 3) + 2\,\frac{x}{q}\,\log x(\log y + \log 3) \\
&\quad + 0.64\,\exp(\sqrt{\log x}\log q)\log x\log 4q + 0.96\; q\log x\log 5q \\
&\leq \quad 3.68\,\sqrt{\log\log x + 0.5}\; x\,(\log x)^{3/4}\,\sqrt{\frac{1}{q}}.
\end{aligned}$$

Now let us suppose that

$$(\log x)^3 < q \leq \frac{x}{(\log x)^{5/2}\log\log x}.$$

We choose

$$y = x\,(\log x)^{-1/2}\,\exp(-\sqrt{\log x}),$$

$$
\begin{aligned}
|B_1(x,\alpha)| &\le 3.95\,x\,(\log x)^{3/2}\,\exp(-\sqrt{\log x}) + 2\,\frac{x}{q}\,(\log x)^2 \\
&\quad + \frac{2}{\pi}\,x\,(\log x)^{3/2}\exp(-\sqrt{\log x}) + \frac{3}{\pi}\,q\,\log x\log 5q \\
&\le 5.59\,x\,(\log x)^{3/2}\,\exp(-\sqrt{\log x}) + 0.45\,x\,(\log x)^{3/4}\sqrt{\frac{1}{q}} \\
&\quad + 0.11\,x\,(\log x)^{3/4}\sqrt{\frac{q\log 4q}{\pi x}} \\
&\le 0.56\,x\,(\log x)^{3/4}\sqrt{\frac{1}{q}+\frac{2}{3\pi}\cdot\frac{q\log 4q}{x}} \\
&\quad + 4.59\,x\,(\log x)^{3/2}\,\exp(-\sqrt{\log x}).
\end{aligned}
$$

Hence for all q we have

$$
\begin{aligned}
|B_1(x,\alpha)| &\le 3.68\,\sqrt{\log\log x + 0.5}\;x\,(\log x)^{3/4}\sqrt{\frac{1}{q}+\frac{2}{3\pi}\cdot\frac{q\log 4q}{x}} \\
&\quad + 0.01\,\sqrt{\log\log x + 0.5}\;x\,(\log x)^{5/4}\,\exp\left(-\frac{1}{2}\sqrt{\log x}\right).
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
|S(x,\alpha)| &\le |S_1(x,\alpha)| + |B_1(x,\alpha)| + |B_2(x,\alpha)| \\
&\le 14.86\,\sqrt{\log\log x + 0.5}\;x\,(\log x)^{3/4}\sqrt{\frac{1}{q}+\frac{2}{3\pi}\cdot\frac{q\log 4q}{x}} \\
&\quad + 6.45\,\sqrt{\log\log x + 0.5}\;x\,(\log x)^{5/4}\,\exp\left(-\frac{1}{2}\sqrt{\log x}\right).
\end{aligned}
$$

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