EXPLICIT UPPER BOUNDS
FOR EXPONENTIAL SUMS OVER PRIMES

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Dedicated to the memory of Chen Jing Run

ABSTRACT. We give explicit upper bounds for linear trigonometric sums over primes.

1. INTRODUCTION

In 1937 I.M. Vinogradov [12] proved that every sufficiently large odd number is the sum of three prime numbers. Later Chen and Wang [2] gave a lower bound for the result of Vinogradov, which is very large, around 10^{43000}. The method used is the Hardy-Littlewood circle method, and the following sums play an important role in the proof:

$$\sum_{p \leq x} \mathrm{e}(\alpha p), \qquad S(x, \alpha) = \sum_{n \leq x} \Lambda(n) \, \mathrm{e}(n\alpha),$$

where Λ is the function of Von Mangoldt and $\mathrm{e}(\alpha) = e^{2i\pi\alpha}$.

In [1] Chen proved that if $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \leq 1$, $q \leq x$, then

$$\left| \sum_{p \leq x} \mathrm{e}(\alpha p) \right| \leq 1.2 \; x \, (\log x)^{3/4} \log \log x \left(\sqrt{\frac{5}{q} + \frac{q \log q}{x}} + \sqrt{\log q} \; \exp{-\frac{1}{2}\sqrt{\log x}} \right).$$

More recently in [3] Chen and Wang proved that

$$|S(x, \alpha)| \leq 0.177 \, \frac{x}{\sqrt{q}} \, (\log x)^3 + 3.8 \; x^{4/5}(\log x)^{2.2} + 0.08 \; \sqrt{xq} \, (\log x)^{3.5}.$$

Our purpose is to improve on these two estimates. By a classical elementary transformation it suffices to consider $S(x, \alpha)$.

In order to estimate this sum, a useful identity has been proved by R.C. Vaughan [11]. Recently Daboussi [4] gave another identity, which has the advantage of involving nice coefficients. This permits us to give a new explicit upper bound for $|S(x, \alpha)|$.

In this paper we will need sharp versions of some classical inequalities which have their own independent interest. We will prove

TABLE 1. Upper bounds for $x^{-1}|S(x,\alpha)|$ when $q = (\log x)^3$

x	$x^{-1}|S(x,\alpha)|$
10^{200}	0.385
10^{300}	0.293
10^{400}	0.241
10^{500}	0.207
10^{1000}	0.129
10^{2000}	0.080
10^{5000}	0.042
10^{10000}	0.026
10^{20000}	0.016
10^{43000}	0.010

Theorem 1. *For $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \leq 1$, $q \leq x$, we have*

$$
\begin{aligned}
|S(x,\alpha)| \quad \leq \quad & 14.86 \sqrt{\log\log x + 0.5}\; x\, (\log x)^{3/4} \sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}} \\
& + 6.45 \sqrt{\log\log x + 0.5}\; x\, (\log x)^{5/4} \exp\left(-\frac{1}{2}\sqrt{\log x}\right).
\end{aligned}
$$

From this theorem we can compute numerical upper bounds for $x^{-1}|S(x,\alpha)|$ (cf. Table 1) with the choice $q = (\log x)^3$ for which the result of Chen and Wang is not even as good as the trivial upper bound.

Definitions and notations. For x real we will denote by $\lfloor x \rfloor$ the greatest integer $\leq x$, $\{x\}$ the fractional part of x, $\lceil x \rceil$ the smallest integer $\geq x$, $\|x\|$ the distance from x to the nearest integer, $\lfloor x \rceil$ the smallest integer n such that $|x - n| \leq 1/2$ (n is unique if $\{x\} \neq 1/2$). The letter p denotes always a prime number, $\pi(x)$ denotes the number of primes $\leq x$. We denote by μ and φ the Möbius and Euler functions, respectively. The functions $\Omega(n)$ and $\omega(n)$ count the number of prime factors of n, respectively, with and without multiplicity. We define the functions u_z and v_z by $u_z(m) = 1$ if $(\forall p,\ p\,|\,m \Rightarrow p > z)$ and $u_z(m) = 0$ otherwise, and $v_z(m) = 1$ if $(\forall p,\ p\,|\,m \Rightarrow p \leq z)$ and $v_z(m) = 0$ otherwise.

2. VINOGRADOV TYPE LEMMAS

In [1] Chen improved Vinogradov Lemmas 8a and 8b [13]. In this section, we further improve the results of Chen.

Lemma 1. *Let $x \in \mathbb{R}$, $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \leq 1$, $(a,q) = 1$, $U > 0$. We then have*

$$
\sum_{x < n \leq x+q} \min\left(U, \frac{1}{|\sin(\pi\alpha n)|}\right) \leq 2\, U + \frac{2}{\pi}\, q\log 4q.
$$

Remark 1. This is the analog of Vinogradov Lemma 8a. Chen obtained $5\,U + q\log q$. The factor 2 instead of 5 is obtained by using $t = \lfloor t \rfloor + \delta$ with $|\delta| \leq 1/2$ which is more precise than the classical $t = \lfloor t \rfloor + \{t\}$. The factor $2/\pi$ has been obtained by dealing directly with $(\sin t)^{-1}$ without using the classical inequality $\sin t \geq 2t/\pi$ for

$0 \leq t \leq \pi/2$. Indeed we simply used the fact that $(\log \tan(t/2))' = (\sin t)^{-1}$. We acknowledge the referee's improvement of this lemma (see below).

Proof. The result is trivial for $q \leq 2$. We therefore suppose $q \geq 3$.

Let $m_0 = \lfloor x \rfloor + \lfloor (q+1)/2 \rfloor$. We have

$$\sum_{x < n \leq x+q} \min\left(U, \frac{1}{|\sin(\pi \alpha n)|}\right) = \sum_{-q/2 < m \leq q/2} \min\left(U, \frac{1}{|\sin(\pi \alpha (m_0 + m))|}\right).$$

Now writing

$$b = \left\lfloor am_0 + \frac{\beta m_0}{q} \right\rceil \quad \text{and} \quad b + \delta = am_0 + \frac{\beta m_0}{q}$$

(hence $|\delta| \leq 1/2$), we obtain

$$\alpha(m_0 + m) = \frac{1}{q}\left(am + am_0 + \frac{\beta m_0}{q} + \frac{\beta m}{q}\right) = \frac{1}{q}(am + b) + \frac{1}{q}\left(\delta + \frac{\beta m}{q}\right).$$

When m runs through the integers in the interval $-q/2 < m \leq q/2$, $am + b$ runs through a complete set of residue classes modulo q. We introduce r such that $am + b \equiv r \bmod q$ and $-q/2 < r \leq q/2$. Using $\left|\delta + \frac{\beta m}{q}\right| \leq 1$, we get for $|r| \geq 2$

$$\min\left(U, \frac{1}{|\sin(\pi \alpha(m_0 + m))|}\right) \leq \sin\left(\frac{\pi}{q}(|r| - 1)\right).$$

For $r = \pm 1$, the referee observed that

$$\|\alpha(m_0 + m)\| = \left\|\frac{f(r)}{q}\right\|,$$

where $f(r) = r + \delta + \theta(r)$, with $|\theta(r)| \leq \frac{1}{2}$. It follows that $f(1) - f(-1) \geq 1$ so that

$$\max\left\{\left\|\frac{f(1)}{q}\right\|, \left\|\frac{f(-1)}{q}\right\|\right\} \geq \frac{1}{2q}.$$

Thus one of the two terms for $r = \pm 1$ can be bounded by $|\sin(\pi/2q)|^{-1}$. Hence we obtain

$$\sum_{x < n \leq x+q} \min\left(U, \frac{1}{|\sin(\pi \alpha n)|}\right) \leq 2\,U + \frac{1}{\sin\left(\frac{\pi}{2q}\right)} + 2 \sum_{2 \leq r \leq q/2} \frac{1}{\sin\left(\frac{\pi}{q}(r-1)\right)}$$

(the sum on the right hand side is empty for $q = 3$).

Using the convexity of the function $t \longmapsto 1/\sin(\pi t/q)$ for $0 < t \leq q/2$, we obtain

$$\sum_{1 \leq r \leq \frac{q}{2} - 1} \frac{1}{\sin\frac{\pi}{q}r} \leq \int_{\frac{1}{2}}^{\frac{q-1}{2}} \frac{dt}{\sin\frac{\pi t}{q}} \leq \frac{q}{\pi} \log \cot \frac{\pi}{4q} \leq \frac{q}{\pi} \log \frac{4q}{\pi},$$

and we have for $q \geq 3$

$$\frac{1}{\sin\left(\frac{\pi}{2q}\right)} + \frac{2q}{\pi} \log \frac{4q}{\pi} \leq \frac{2q}{\pi} \log 4q,$$

which completes the proof of Lemma 1. $\square$

Lemma 2. *Let $N \geq 1$, $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \leq 1$, $(a, q) = 1$, $U > 0$. We then have*

$$\sum_{1 \leq n \leq N} \min\left(U, \frac{1}{|\sin(\pi \alpha n)|}\right) \leq \left\lceil \frac{N}{q} \right\rceil \left(2\,U + \frac{2}{\pi}\,q \log 4q\right).$$

Proof. We divide the interval $1 \leq n \leq N$ into subintervals $kq + 1 \leq n \leq (k+1)q$, for which we apply Lemma 1. There are at most $\left\lceil \frac{N}{q} \right\rceil$ such subintervals. $\qquad\square$

Lemma 3. *Let $N \geq 1$, $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \leq 1$, $(a, q) = 1$, $x > 0$. We then have*

$$\sum_{1 \leq n \leq N} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|} \right) \leq 2\,\frac{x}{q} \log\left(\frac{8N}{q} + 4 \right) + \frac{2}{\pi}\, N \log 4q + \frac{3}{\pi}\, q \log 5q.$$

Remark 2. This is the analog of Vinogradov Lemma 8b.

Proof. We can assume without loss of generality that N is an integer.

Using the convexity of $t \longmapsto \frac{1}{t}$ for $t > 0$, we obtain for $N \geq 1$

$$\sum_{1 \leq n \leq N} \frac{1}{n} \leq \int_{\frac{1}{2}}^{N + \frac{1}{2}} \frac{dt}{t} = \log(2N + 1).$$

This proves the result for $q \leq 2$. We can now suppose $q \geq 3$.

Writing $K = \left\lceil \frac{N}{q} - \frac{1}{2} \right\rceil$, we have $K \geq \frac{N}{q} - \frac{1}{2}$ and $Kq + \frac{q}{2} \geq N$. Hence

$$\sum_{1 \leq n \leq N} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|} \right) \leq \sum_{k=0}^{K} S_k,$$

where

$$S_0 = \sum_{1 \leq n \leq q/2} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|} \right),$$

and for $k \geq 1$

$$S_k = \sum_{kq - q/2 < n \leq kq + q/2} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi\alpha n)|} \right).$$

For $1 \leq n \leq q/2$, we have $an \not\equiv 0 \bmod q$ and $\alpha n = \frac{an}{q} + \frac{1}{q}\frac{\beta n}{q}$ with $\left| \frac{\beta n}{q} \right| \leq \frac{1}{2}$. Hence for $an \equiv r \bmod q$ with $-q/2 \leq r \leq q/2$ and $r \neq 0$, we have

$$|\sin(\pi\alpha n)| \geq \left| \sin\left(\frac{\pi}{q}\left(|r| - \frac{1}{2} \right) \right) \right|$$

and

$$
\begin{aligned}
S_0 &\leq 2 \sum_{1 \leq r \leq q/2} \left| \sin\left(\frac{\pi}{q}\left(r - \frac{1}{2} \right) \right) \right|^{-1} \\
&\leq 2 \left| \sin\left(\frac{\pi}{2q} \right) \right|^{-1} + 2 \int_{\frac{3}{2}}^{\frac{q+1}{2}} \frac{dt}{\sin\left(\frac{\pi}{q}\left(t - \frac{1}{2} \right) \right)} \\
&\leq \frac{2}{\sin\left(\frac{\pi}{2q} \right)} + \frac{2q}{\pi} \log \cot\left(\frac{\pi}{2q} \right) \\
&\leq \frac{2q}{\pi} \left(2 \frac{\frac{\pi}{2q}}{\sin\left(\frac{\pi}{2q} \right)} + \log\left(\frac{2q}{\pi} \right) \right) \\
&\leq \frac{2q}{\pi} \log 5q \quad \text{for } q \geq 4.
\end{aligned}
$$

For $q = 3$ we also have $S_0 \leq 2|\sin \frac{\pi}{2q}|^{-1} = 4 \leq \frac{2q}{\pi} \log 5q$. This proves the result for $K < 1$, so from now on we assume that $K \geq 1$.

By Lemma 1 we have

$$
\begin{aligned}
\sum_{1 \leq k \leq K} S_k &\leq \sum_{1 \leq k \leq K} \sum_{kq-q/2 < n \leq kq+q/2} \min \left(\frac{x}{q(k-\frac{1}{2})}, \frac{1}{|\sin(\pi \alpha n)|} \right) \\
&\leq \sum_{1 \leq k \leq K} \left(2 \frac{x}{q(k-\frac{1}{2})} + \frac{2}{\pi} q \log 4q \right) \\
&\leq \frac{2}{\pi} Kq \log 4q + \frac{2x}{q} \sum_{1 \leq k \leq K} \frac{1}{(k-\frac{1}{2})}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{1 \leq k \leq K} S_k &\leq \frac{2}{\pi} Kq \log 4q + \frac{2x}{q} \left(2 + \int_{\frac{3}{2}}^{K+\frac{1}{2}} \frac{dt}{(t-\frac{1}{2})} \right) \\
&\leq \frac{2}{\pi} Kq \log 4q + \frac{2x}{q} (2 + \log K) \\
&\leq \frac{2}{\pi} \left(\frac{N}{q} + \frac{1}{2} \right) q \log 4q + \frac{2x}{q} \log \left(e^2 K \right) \\
&\leq \frac{2}{\pi} N \log 4q + \frac{q}{\pi} \log 4q + \frac{2x}{q} \log \left(e^2 K \right).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{1 \leq n \leq N} &\min \left(\frac{x}{n}, \frac{1}{|\sin(\pi \alpha n)|} \right) \\
&\leq \frac{2}{\pi} N \log 4q + \frac{2q}{\pi} \left(\log q + \log 5 + \frac{\log q}{2} + \frac{\log 4}{2} \right) + \frac{2x}{q} \log \left(e^2 K \right) \\
&\leq \frac{2}{\pi} N \log 4q + \frac{3q}{\pi} \left(\log q + \frac{2 \log 5}{3} + \frac{\log 4}{3} \right) + \frac{2x}{q} \log \left(e^2 K \right) \\
&\leq \frac{2}{\pi} N \log 4q + \frac{3q}{\pi} \log 5q + \frac{2x}{q} \log \left(e^2 K \right) \\
&\leq \frac{2}{\pi} N \log 4q + \frac{3q}{\pi} \log 5q + \frac{2x}{q} \log \left(\frac{8N}{q} + 4 \right),
\end{aligned}
$$

which completes the proof.   □

3. RANKIN'S METHOD

Elliott [6, pages 81-83] has given an effective version of Rankin's method. In this section we generalize and improve his results numerically.

Lemma 4. *Let $z \geq 2$, f a multiplicative function with $f \geq 0$, and*

$$
S = \sum_{p \leq z} \frac{f(p)}{1 + f(p)} \log p.
$$

We assume $S > 0$ and write $K(t) = \log t - 1 + \frac{1}{t}$ for $t \geq 1$.

For any y with $\log y \geq S$ we have

$$\sum_{n>y} v_z(n)\mu^2(n)f(n) \leq \left(\prod_{p\leq z}(1+f(p))\right)\exp\left(-\frac{\log y}{\log z}K\left(\frac{\log y}{S}\right)\right),$$

$$\sum_{n\leq y} v_z(n)\mu^2(n)f(n) \geq \left(\prod_{p\leq z}(1+f(p))\right)\left(1-\exp\left(-\frac{\log y}{\log z}K\left(\frac{\log y}{S}\right)\right)\right).$$

In particular for any y with $\log y \geq 7S$ we have

$$\sum_{n>y} v_z(n)\mu^2(n)f(n) \leq \left(\prod_{p\leq z}(1+f(p))\right)\exp\left(-\frac{\log y}{\log z}\right),$$

$$\sum_{n\leq y} v_z(n)\mu^2(n)f(n) \geq \left(\prod_{p\leq z}(1+f(p))\right)\left(1-\exp\left(-\frac{\log y}{\log z}\right)\right).$$

Proof. The special case for $\log y \geq 7S$ is a direct consequence of the general case $\log y \geq S$, as for all $t \geq 7$ we have $K(t) \geq K(7) \geq 1$.

We note that

$$\sum_{n\leq y} v_z(n)\mu^2(n)f(n) + \sum_{n>y} v_z(n)\mu^2(n)f(n) = \prod_{p\leq z}(1+f(p)),$$

which shows that the required lower bound for the first sum will follow from the required upper bound for the second sum.

For all $\eta \geq 0$ we have

$$\sum_{n>y} v_z(n)\mu^2(n)f(n) \leq \sum_{n=1}^{\infty} v_z(n)\mu^2(n)f(n)\left(\frac{n}{y}\right)^\eta$$

$$\leq y^{-\eta}\prod_{p\leq z}\left(1+f(p)\,p^\eta\right).$$

Now

$$\prod_{p\leq z}\left(1+f(p)\,p^\eta\right) = \left(\prod_{p\leq z}(1+f(p))\right)\left(\prod_{p\leq z}\left(1+\frac{f(p)}{1+f(p)}\,(p^\eta-1)\right)\right).$$

Using $\log(1+u) \leq u$ for $u \geq 0$ we get

$$\prod_{p\leq z}\left(1+\frac{f(p)}{1+f(p)}\,(p^\eta-1)\right) \leq \exp\left(\sum_{p\leq z}\frac{f(p)}{1+f(p)}\,(p^\eta-1)\right),$$

$$\sum_{p \leq z} \frac{f(p)}{1 + f(p)} \, (p^\eta - 1) \;\leq\; \sum_{p \leq z} \frac{f(p)}{1 + f(p)} \, (\exp(\eta \log p) - 1)$$

$$\leq \;\; \sum_{p \leq z} \frac{f(p)}{1 + f(p)} \, \sum_{k=1}^{\infty} \frac{(\eta \log p)^k}{k!}$$

$$\leq \;\; \sum_{k=1}^{\infty} \frac{\eta^k (\log z)^{k-1}}{k!} \sum_{p \leq z} \frac{f(p)}{1 + f(p)} \, \log p$$

$$\leq \;\; \frac{S}{\log z} \sum_{k=1}^{\infty} \frac{\eta^k (\log z)^k}{k!}$$

$$\leq \;\; \frac{S}{\log z} \, (\exp(\eta \log z) - 1) \,.$$

Writing $\nu = \eta \log z$ we get

$$y^{-\eta} \prod_{p \leq z} \left(1 + \frac{f(p)}{1 + f(p)} \, (p^\eta - 1) \right) \leq \exp \left(\frac{S}{\log z} \left(\exp(\nu) - 1 - \nu \frac{\log y}{S} \right) \right).$$

The last inequality is valid for any $\nu \geq 0$, in particular for $\nu = \log \left(\frac{\log y}{S} \right)$. Hence

$$\sum_{n > y} v_z(n) \mu^2(n) f(n)$$

$$\leq \;\; \left(\prod_{p \leq z} (1 + f(p)) \right) \exp \left(\frac{S}{\log z} \left(\frac{\log y}{S} - 1 - \frac{\log y}{S} \log \left(\frac{\log y}{S} \right) \right) \right)$$

$$\leq \;\; \left(\prod_{p \leq z} (1 + f(p)) \right) \exp \left(-\frac{\log y}{\log z} K \left(\frac{\log y}{S} \right) \right).$$

$\square$

4. Effective inequalities

Lemma 5. *For all $x > 1$ we have*

$$\pi(2x) - \pi(x) < \frac{x}{\log x}.$$

Proof. P. Dusart [5] improved some results of [9] and proved for $x \geq 60184$ that

$$\frac{x}{\log(x) - 1} < \pi(x) < \frac{x}{\log(x) - 1.1}.$$

This implies for $x \geq 60184$ that

$$\pi(2x) - \pi(x) \leq \frac{2x}{\log x - 0.41} - \frac{x}{\log x - 1} \leq \frac{x}{\log x} \left(1 - \frac{0.016}{\log x} \right) < \frac{x}{\log x}$$

using the inequalities $1 + u < \frac{1}{1-u} < 1 + \frac{6}{5} u$ (valid for $0 < u < 1/6$). The result can be easily extented for all $x > 1$ by computer evidence. $\square$

Remark 3. We note that the result of this lemma is sharp for $x = 113/2$ for which

$$\pi(113) - \pi(113/2) = 14 < \frac{113/2}{\log(113/2)} = 14.0051 \dots .$$

Lemma 6. *For $z \geq 2$ we have*

$$\sum_{q \leq z} \left(1 + \frac{q}{z}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)} \geq \log z.$$

Proof. By Lemma 8 of Montgomery-Vaughan [7] we have for $z \geq 100$

$$\sum_{q \leq z} \left(1 + \frac{q}{z}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)} \geq \log z + 0.361$$

and the result follows by a direct computation for $2 \leq z < 100$. □

Lemma 7. *For $2 \leq z \leq x$ we have*

$$\sum_{x < m \leq 2x} u_z(m) \leq \frac{x}{\log z}.$$

Proof. Suppose first that $\sqrt{2x} \leq z \leq x$. For $x < m \leq 2x$, we have $u_z(m) = 1$ if and only if m is prime. Using Lemma 5 we obtain

$$\sum_{x < m \leq 2x} u_z(m) = \pi(2x) - \pi(x) \leq \frac{x}{\log x} \leq \frac{x}{\log z}.$$

Hence we can suppose $z < \sqrt{2x}$.

By Corollary 1 of Montgomery-Vaughan [7] we have for any positive number z,

$$\sum_{x < m \leq 2x} u_z(m) \leq x \left(\sum_{q \leq z} \left(1 + \frac{3qz}{2x}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)}\right)^{-1},$$

and using Lemma 6 we obtain for $z \leq \sqrt{\frac{2}{3}x}$

$$\sum_{x < m \leq 2x} u_z(m) \leq x \left(\sum_{q \leq z} \left(1 + \frac{q}{z}\right)^{-1} \frac{\mu^2(q)}{\varphi(q)}\right)^{-1} \leq \frac{x}{\log z}.$$

Thus we can suppose $\sqrt{\frac{2}{3}x} < z < \sqrt{2x}$.

For $x > 15$ we have $(2x)^{1/3} < \sqrt{\frac{2}{3}x} < z < \sqrt{2x}$ and

$$\sum_{x < m \leq 2x} u_z(m) = \pi(2x) - \pi(x) + \pi_2(2x, z) - \pi_2(x, z),$$

where

$$\pi_2(x, z) = \#\{n \leq x, \ \Omega(n) = 2, \ p \,|\, n \Longrightarrow p > z\}.$$

We have using Lemma 5

$$\pi_2(2x, z) - \pi_2(x, z) \leq \sum_{z < p \leq \sqrt{2x}} \left(\pi\left(\frac{2x}{p}\right) - \pi\left(\frac{x}{p}\right)\right) \leq \sum_{z < p \leq \sqrt{2x}} \frac{x/p}{\log(x/p)}.$$

For $x > 15$ we have $x/z > e$ and the function $t \longmapsto t/\log t$ is increasing for $t > e$. Hence

$$\pi_2(2x, z) - \pi_2(x, z) \leq \frac{x/z}{\log(x/z)} \left(\pi(\sqrt{2x}) - \pi(z)\right) \leq \frac{x/z}{\log(x/z)} \left(\pi(2z) - \pi(z)\right),$$

and using Lemma 5 we obtain

$$\pi_2(2x, z) - \pi_2(x, z) \leq \frac{x}{\log(x/z) \log z}.$$

Therefore we have for $x > 15$

$$\sum_{x < m \leq 2x} u_z(m) \leq \frac{x}{\log x} + \frac{x}{\log(x/z) \log z},$$

and using the inequality $z < \sqrt{2x}$ we obtain for $x > 200$

$$\sum_{x < m \leq 2x} u_z(m) \leq \frac{x}{\log z} \left(\frac{\log \sqrt{2x}}{\log x} + \frac{1}{\log \sqrt{x/2}} \right) \leq \frac{x}{\log z}$$

and it suffices to show the result for $x \leq 200$ and $z < \sqrt{2x}$, which can be verified easily by computer. This completes the proof of Lemma 7. $\qquad\square$

Corollary 1. *For $2 \leq z \leq x$ we have*

$$\sum_{x \leq m < 2x} u_z(m) \leq \frac{x}{\log z}.$$

Proof. If x is not an integer or if x is an integer and $z = x$ (in this case $u_z(x) = 0$), we have

$$\sum_{x \leq d < 2x} u_z(d) \leq \sum_{x < d \leq 2x} u_z(d) \leq \frac{x}{\log z}.$$

If x is an integer and $z < x$, we have

$$\sum_{x \leq d < 2x} u_z(d) = \sum_{x^- < d \leq 2x^-} u_z(d) \leq \frac{x}{\log z}.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 8. *For $x \geq 2$ and $1 \leq h \leq x$ we have*

$$\sum_{n \leq x} \Lambda(n) \Lambda(n + h) \leq 15 \, x \, (\log \log x + 0.5).$$

Proof. If h is odd, we have $\Lambda(n)\Lambda(n+h) = 0$ if n is not a power of 2. Hence, when h is odd,

$$\sum_{n \leq x} \Lambda(n) \Lambda(n + h) \leq \sum_{r \leq \frac{\log x}{\log 2}} \log 2 \, \log 2x \leq \log x \, \log 2x \leq 2x.$$

We can suppose that h is even, and $\Lambda(n)\Lambda(n + h) \neq 0$ implies that n is odd; therefore $n \geq 3$ and $x \geq 3$.

The contribution of the terms for which n and $n + h$ are not both primes is at most

$$2 \log 2x \sum_{\substack{p^r \leq 2x \\ r \geq 2}} \log p \leq 2 \, \pi(\sqrt{2x}) \, \log^2 2x.$$

By inequality 3.6 of Rosser and Schoenfeld [8] we have

$$\forall x > 1, \quad \pi(x) < 1.25506 \, \frac{x}{\log x};$$

therefore the contribution of the terms for which n and $n + h$ are not both primes is at most

$$7.1 \sqrt{x} \, \log 2x.$$

By the theorem of Siebert [10] the number of primes $p \leq x$ such that $p + h$ is prime is at most

$$16 \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2} \right) \frac{x}{\log^2 x} \prod_{\substack{p \mid h \\ p \geq 3}} \frac{p-1}{p-2}.$$

We remark that

$$\frac{p-1}{p-2} = \left(1 - \frac{1}{p} \right)^{-1} \left(1 - \frac{1}{(p-1)^2} \right)^{-1},$$

and when h is even

$$\prod_{\substack{p \mid h \\ p \geq 3}} \left(1 - \frac{1}{p} \right) = 2 \, \frac{\varphi(h)}{h}$$

so that Siebert's expression can be written as

$$8 \prod_{\substack{p \\ (p,h)=1}} \left(1 - \frac{1}{(p-1)^2} \right) \frac{x}{\log^2 x} \cdot \frac{h}{\varphi(h)} \leq \frac{8 \, x}{\log^2 x} \cdot \frac{h}{\varphi(h)}.$$

By inequality 3.41 and 3.42 of Rosser and Schoenfeld [8] we have for $h \geq 3$

$$\frac{h}{\varphi(h)} \leq e^\gamma \log \log h + \frac{2.50637}{\log \log h};$$

hence for $x \geq 3$ we have

$$\sum_{n \leq x} \Lambda(n)\Lambda(n+h) \leq 8 \, \frac{\log 2x}{\log x} x \left(e^\gamma \log \log x + \frac{2.50637}{\log \log x} \right) + 7.1 \sqrt{x} \, \log 2x,$$

and for $x \geq 10^8$ we obtain

$$\sum_{n \leq x} \Lambda(n)\Lambda(n+h) \leq 15 \, x \, (\log \log x + 0.5).$$

For $x < 10^8$ we have

$$\sum_{n \leq x} \Lambda(n)\Lambda(n+h) \leq \log 2x \sum_{n \leq x} \Lambda(n),$$

and we use the inequality 3.35 of Rosser and Schoenfeld [8]

$$\sum_{n \leq x} \Lambda(n) < 1.03883 \, x \quad \text{for all } x > 0,$$

which gives for $10 \leq x < 10^8$

$$\sum_{n \leq x} \Lambda(n)\Lambda(n+h) \leq 1.03883 \, \log(2.10^8) \, x \leq 20 \, x \leq 15 \, x \, (\log \log x + 0.5).$$

For $x < 10$ the inequality is verified by direct computation. □

5. Sums of type I and II

For $z \geq 3$ depending on x only we can split $S(x, \alpha)$ as follows:

$$S(x, \alpha) = \sum_{n \leq x} \Lambda(n) \, \mathrm{e}(n\alpha) = S_1(x, \alpha) + S_2(x, \alpha),$$

where

$$S_1(x, \alpha) = \sum_{n \leq x} v_z(n) \Lambda(n) \, \mathrm{e}(n\alpha),$$

$$S_2(x, \alpha) = \sum_{n \leq x} u_z(n) \Lambda(n) \, \mathrm{e}(n\alpha).$$

We can estimate $S_1(x, \alpha)$ trivially:

$$|S_1(x, \alpha)| \leq \sum_{\substack{p^r \leq x \\ p \leq z}} \log p = \sum_{p \leq z} \log p \left\lfloor \frac{\log x}{\log p} \right\rfloor \leq \pi(z) \log x \leq z \log x.$$

Now we split $S_2(x, \alpha)$ into $B_1(x, \alpha) - B_2(x, \alpha)$ (see [4] for details) where

$$B_1(x, \alpha) = \sum_{n \leq x} u_z(n) \, \log(n) \, \mathrm{e}(n\alpha),$$

$$B_2(x, \alpha) = \sum_{z \leq d \leq x/z} u_z(d) \sum_{z \leq m \leq x/d} u_z(m) \, \Lambda(m) \, \mathrm{e}(md\alpha).$$

6. Sums of type I

Lemma 9. *For* $\alpha = \frac{a}{q} + \frac{\beta}{q^2}$, $|\beta| \leq 1$, $(a, q) = 1$, $3^7 \leq z^7 \leq y \leq x$, *we have*

$$
\begin{aligned}
|B_1(x, \alpha)| \quad \leq \quad & \frac{2}{3} \, e^\gamma x \log x \log 3z \, \exp\left(-\frac{\log y}{\log z} \right) \\
& + 2 \, \frac{x}{q} \log x \log 3y + \frac{2}{\pi} \, y \log x \log 4q + \frac{3}{\pi} \, q \log x \log 5q.
\end{aligned}
$$

Proof. We write

$$B_1(x, \alpha) = \sum_{n \leq x} u_z(n) \, \mathrm{e}(n\alpha) \int_1^n \frac{dt}{t} = \int_1^x \frac{dt}{t} \sum_{t \leq n \leq x} u_z(n) \, \mathrm{e}(n\alpha).$$

Introducing $T_1(t, x, \alpha) = \sum_{t \leq n \leq x} u_z(n) \, \mathrm{e}(n\alpha)$, we see that

$$|B_1(x, \alpha)| \leq \log x \sup_{1 \leq t \leq x} |T_1(t, x, \alpha)|.$$

By the Möbius inversion formula

$$T_1(t, x, \alpha) = \sum_{\substack{n, d \\ t \leq nd \leq x}} v_z(n) \, \mu(n) \, \mathrm{e}(nd\alpha).$$

Let y such that $z^7 \leq y \leq x$. We have

$$T_{1,1}(t, x, \alpha) = \sum_{n \leq y} \sum_{t \leq nd \leq x} v_z(n) \, \mu(n) \, \mathrm{e}(nd\alpha),$$

$$T_{1,2}(t, x, \alpha) = \sum_{y < n \leq x} \sum_{t \leq nd \leq x} v_z(n) \, \mu(n) \, \mathrm{e}(nd\alpha).$$

Clearly

$$|T_{1,1}(t,x,\alpha)| \leq \sum_{n \leq y} \min\left(\frac{x}{n}, \frac{1}{|\sin(\pi n\alpha)|}\right).$$

Hence by Lemma 3

$$|T_{1,1}(t,x,\alpha)| \leq 2\,\frac{x}{q}\log 3y + \frac{2}{\pi}\,y\log 4q + \frac{3}{\pi}\,q\log 5q.$$

We have

$$|T_{1,2}(t,x,\alpha)| \leq \sum_{n>y} v_z(n)\,|\mu(n)| \sum_{d \leq x/n} 1 \leq x \sum_{n>y} \frac{v_z(n)}{n}\mu^2(n).$$

In [8], Rosser and Schoenfeld proved (inequality 3.24) that

$$\sum_{p \leq x} \frac{\log p}{p} < \log x \quad \text{for all } x > 1.$$

Using this inequality we get

$$0 < S = \sum_{p \leq z} \frac{\frac{1}{p}}{1+\frac{1}{p}}\log p \leq \sum_{p \leq z} \frac{\log p}{p} \leq \log z \leq \frac{\log y}{7}.$$

Hence by Rankin's method (Lemma 4) we get

$$|T_{1,2}(t,x,\alpha)| \leq x \left(\prod_{p \leq z}\left(1 + \frac{1}{p}\right)\right)\exp\left(-\frac{\log y}{\log z}\right).$$

Now for $z \geq 3$ we have

$$\prod_{p \leq z}\left(1 + \frac{1}{p}\right)\prod_{p \leq z}\left(1 - \frac{1}{p}\right) = \prod_{p \leq z}\left(1 - \frac{1}{p^2}\right) \leq \frac{3}{4}\cdot\frac{8}{9} = \frac{2}{3}.$$

In [8], Rosser and Schoenfeld proved (inequality 3.31) that

$$\prod_{p \leq x}\frac{p}{p-1} < e^\gamma \sum_{1 \leq n \leq x}\frac{1}{n} \quad \text{for all } x \geq 1.$$

Using these inequalities we obtain

$$|T_{1,2}(t,x,\alpha)| \leq \frac{2}{3}\,e^\gamma x \left(\sum_{1 \leq n \leq z}\frac{1}{n}\right)\exp\left(-\frac{\log y}{\log z}\right),$$

and finally

$$|T_{1,2}(t,x,\alpha)| \leq \frac{2}{3}\,e^\gamma x \log 3z \exp\left(-\frac{\log y}{\log z}\right),$$

which completes the proof. $\square$

7. Sums of type II

Let J satisfy $2^J \lceil z \rceil \leq x/z < 2^{J+1} \lceil z \rceil$. We have

$$|B_2(x, \alpha)| \leq \sum_{0 \leq j \leq J} \sum_{2^j \lceil z \rceil \leq d < 2^{j+1} \lceil z \rceil} u_z(d) \left| \sum_{z \leq m \leq x/d} u_z(m) \, \Lambda(m) \, \mathrm{e}(md\alpha) \right|.$$

We observe that $J \log 2 \leq \log x - 2 \log z \leq \log x - \log 2$, and we define

$$T_2(M) = \sum_{M \leq d < 2M} u_z(d) \left| \sum_{z \leq m \leq x/d} u_z(m) \, \Lambda(m) \, \mathrm{e}(md\alpha) \right|.$$

We get

$$|B_2(x, \alpha)| \leq \frac{\log x}{\log 2} \sup_{\substack{z \leq M \leq x/z \\ M \in \mathbb{N}}} |T_2(M)|.$$

By the Cauchy-Schwarz inequality

$$|T_2(M)|^2 \leq \left(\sum_{M \leq d < 2M} u_z^2(d) \right) \sum_{M \leq d < 2M} \left| \sum_{z \leq m \leq x/d} u_z(m) \, \Lambda(m) \, \mathrm{e}(md\alpha) \right|^2.$$

By Corollary 1, we have

$$\sum_{M \leq d < 2M} u_z^2(d) \leq \frac{M}{\log z}.$$

Expanding the square and summing first over the d's we obtain

$$|T_2(M)|^2 \leq \frac{M}{\log z} \sum_{z \leq m \leq x/M} \Lambda(m) \sum_{z \leq m' \leq x/M} \Lambda(m') \left| \sum_{d \in I(m,m')} \mathrm{e}((m - m')d\alpha) \right|,$$

where $I(m, m')$ is the interval of d's such that $M \leq d \leq \min(2M-1, \frac{x}{m}, \frac{x}{m'})$.

We distinguish $m = m'$ and $m \neq m'$ and obtain

$$|T_2(M)|^2 \leq |T_{2,1}(M)|^2 + |T_{2,2}(M)|^2,$$

where

$$|T_{2,1}(M)|^2 = \frac{M^2}{\log z} \sum_{z \leq m \leq x/M} \Lambda^2(m)$$

and

$$|T_{2,2}(M)|^2 = 2 \frac{M}{\log z} \sum_{1 \leq h \leq x/M} \sum_{z \leq m \leq x/M} \Lambda(m) \, \Lambda(m + h) \left| \sum_{d \in I(m,m+h)} \mathrm{e}(hd\alpha) \right|.$$

We have

$$\left| \sum_{d \in I(m,m+h)} \mathrm{e}(hd\alpha) \right| \leq \min \left(M, \frac{1}{|\sin(\pi h\alpha)|} \right).$$

By Lemma 8

$$\sum_{z \le m \le x/M} \Lambda(m) \, \Lambda(m+h) \le 15 \, (\log\log x + 0.5) \, \frac{x}{M}.$$

So

$$|T_{2,2}(M)|^2 \le 30 \, (\log\log x + 0.5) \frac{x}{\log z} \sum_{1 \le h \le x/M} \min\left(M, \frac{1}{|\sin(\pi h\alpha)|}\right).$$

Using Lemma 2 and $z \le M \le x/z$ we get

$$
\begin{aligned}
\sum_{1 \le h \le x/M} \min\left(M, \frac{1}{|\sin(\pi h\alpha)|}\right) &\le \left(\frac{x}{Mq}+1\right)\left(2M + \frac{2}{\pi}q\log 4q\right) \\
&\le \frac{2x}{q} + 2M + \frac{2x\log 4q}{\pi M} + \frac{2}{\pi}q\log 4q \\
&\le \frac{2x}{q} + \frac{2x}{z} + \frac{2x\log 4q}{\pi z} + \frac{2}{\pi}q\log 4q \\
&\le 2\,x\left(\frac{1}{q} + \frac{\pi + \log 4q}{\pi z} + \frac{q\log 4q}{\pi x}\right).
\end{aligned}
$$

We obtain

$$|T_{2,2}(M)|^2 \le 60 \, (\log\log x + 0.5) \, \frac{x^2}{\log z}\left(\frac{1}{q} + \frac{\log 93q}{\pi z} + \frac{q\log 4q}{\pi x}\right),$$

$$\sum_{n \le x} \Lambda(n) < 1.03883 \, x \quad \text{for all } x > 0.$$

For $z \ge 3$, the function $M \longmapsto M\log(x/M)$ is increasing on $[z, x/z]$. Hence by Rosser and Schoenfeld [8] inequality 3.35 we have

$$|T_{2,1}(M)|^2 \le 1.03883 \, xM\frac{\log(x/M)}{\log z} \le 1.03883 \, \frac{x^2}{z}$$

and

$$
\begin{aligned}
|B_2(x,\alpha)| &\le \frac{\sqrt{1.03883}}{\log 2}\frac{x}{\sqrt{z}}\log x \\
&\quad + \frac{1}{\log 2}\sqrt{60\,(\log\log x + 0.5)}\frac{x\log x}{\sqrt{\log z}}\left(\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}} + \sqrt{\frac{\log 93q}{\pi z}}\right).
\end{aligned}
$$

Finally

$$
\begin{aligned}
|B_2(x,\alpha)| &\le 1.48 \, \frac{x}{\sqrt{z}}\log x \\
&\quad + 11.18 \, \sqrt{\log\log x + 0.5}\frac{x\log x}{\sqrt{\log z}}\left(\sqrt{\frac{1}{q} + \frac{q\log 4q}{\pi x}} + \sqrt{\frac{\log 93q}{\pi z}}\right).
\end{aligned}
$$

8. Proof of Theorem 1

We can suppose $x \geq 10^{184}$; otherwise

$$\sqrt{\log \log x + 0.5} \, x \, (\log x)^{5/4} \exp\left(-\frac{1}{2}\sqrt{\log x}\right) > 0.166$$

and the result is trivial using Rosser and Schoenfeld [8] inequality 3.35. Furthermore we can suppose

$$(\log x)^{3/2} \log \log x \leq q \leq \frac{x}{(\log x)^{5/2} \log \log x};$$

otherwise the result is trivial.

We choose $\log z = \sqrt{\log x}$ and we obtain

$$
\begin{aligned}
|B_2(x, \alpha)| \quad \leq \quad & 11.18 \sqrt{\log \log x + 0.5} \, x \, (\log x)^{3/4} \sqrt{\frac{1}{q} + \frac{q \log 4q}{\pi x}} \\
& + 1.48 \, x \, \log x \, \exp\left(-\frac{1}{2}\sqrt{\log x}\right) \\
& + 6.31 \sqrt{\log \log x + 0.5} \, x \, (\log x)^{5/4} \exp\left(-\frac{1}{2}\sqrt{\log x}\right) \\
\leq \quad & 11.18 \sqrt{\log \log x + 0.5} \, x \, (\log x)^{3/4} \sqrt{\frac{1}{q} + \frac{q \log 4q}{\pi x}} \\
& + 6.44 \sqrt{\log \log x + 0.5} \, x \, (\log x)^{5/4} \exp\left(-\frac{1}{2}\sqrt{\log x}\right).
\end{aligned}
$$

Let us first suppose that

$$(\log x)^{3/2} \log \log x \leq q \leq (\log x)^3.$$

Let $\log y = \sqrt{\log x} \log q$. We then have for $x \geq 10^{184}$

$$
\begin{aligned}
|B_1(x, \alpha)| \quad \leq \quad & 1.19 \frac{x}{q} \log x (\log z + \log 3) + 2 \frac{x}{q} \log x (\log y + \log 3) \\
& + 0.64 \exp(\sqrt{\log x} \log q) \log x \log 4q + 0.96 \, q \log x \log 5q \\
\leq \quad & 3.68 \sqrt{\log \log x + 0.5} \, x \, (\log x)^{3/4} \sqrt{\frac{1}{q}}.
\end{aligned}
$$

Now let us suppose that

$$(\log x)^3 < q \leq \frac{x}{(\log x)^{5/2} \log \log x}.$$

We choose

$$y = x \, (\log x)^{-1/2} \exp(-\sqrt{\log x}),$$

$$
\begin{aligned}
|B_1(x,\alpha)| \;\leq\; & 3.95\, x\, (\log x)^{3/2}\, \exp(-\sqrt{\log x}) + 2\,\frac{x}{q}\,(\log x)^2 \\
& + \frac{2}{\pi}\, x\, (\log x)^{3/2}\, \exp(-\sqrt{\log x}) + \frac{3}{\pi}\, q\, \log x \log 5q \\
\;\leq\; & 5.59\, x\, (\log x)^{3/2}\, \exp(-\sqrt{\log x}) + 0.45\, x\, (\log x)^{3/4}\, \sqrt{\frac{1}{q}} \\
& + 0.11\, x\, (\log x)^{3/4}\, \sqrt{\frac{q \log 4q}{\pi x}} \\
\;\leq\; & \;\, 0.56\, x\, (\log x)^{3/4}\, \sqrt{\frac{1}{q} + \frac{2}{3\pi}\cdot\frac{q \log 4q}{x}} \\
& + 4.59\, x\, (\log x)^{3/2}\, \exp(-\sqrt{\log x}).
\end{aligned}
$$

Hence for all q we have

$$
\begin{aligned}
|B_1(x,\alpha)| \;\leq\; & 3.68\, \sqrt{\log\log x + 0.5}\; x\, (\log x)^{3/4}\, \sqrt{\frac{1}{q} + \frac{2}{3\pi}\cdot\frac{q \log 4q}{x}} \\
& + 0.01\, \sqrt{\log\log x + 0.5}\; x\, (\log x)^{5/4}\, \exp\left(-\frac{1}{2}\sqrt{\log x}\right).
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
|S(x,\alpha)| \;\leq\; & |S_1(x,\alpha)| + |B_1(x,\alpha)| + |B_2(x,\alpha)| \\
\;\leq\; & 14.86\, \sqrt{\log\log x + 0.5}\; x\, (\log x)^{3/4}\, \sqrt{\frac{1}{q} + \frac{2}{3\pi}\cdot\frac{q \log 4q}{x}} \\
& + 6.45\, \sqrt{\log\log x + 0.5}\; x\, (\log x)^{5/4}\, \exp\left(-\frac{1}{2}\sqrt{\log x}\right).
\end{aligned}
$$

REFERENCES

[1] J. Chen, *On the estimation of some trigonometrical sums and their application (Chinese)*, Scientia Sinica, 28 (1985), pp. 449–458. MR **87h:**11078

[2] J. Chen and T. Wang, *On the Goldbach problem (Chinese)*, Acta Mathematica Sinica, 32 (1989), No. 5, pp. 702–718. MR **91e:**11108

[3] ———, *Estimation of linear trigonometric sums with primes (Chinese)*, Acta Mathematica Sinica, 37 (1994), No. 1, pp. 25–31. MR **95c:**11102

[4] H. Daboussi, *Effective estimates of exponential sums over primes*, Analytic Number Theory, Vol. 1, Progr. Math., 138, Birkhäuser, Boston, 1996, pp. 231–244. MR **97i:**11088

[5] P. Dusart, *Autour de la fonction qui compte le nombre de nombres premiers*, PhD thesis, Université de Limoges, 1998.

[6] P. D. T. A. Elliott, *Probabilistic Number Theory I*, vol. 239 of Grundlehren der mathematichen Wissenschaften, Springer-Verlag, 1979. MR **82h:**10002a

[7] H. L. Montgomery and R. C. Vaughan, *The large sieve*, Mathematika, 20 (1973), pp. 119–134. MR **51:**10260

[8] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois Journal of Mathematics, 6 (1962), pp. 64–94. MR **25:**1139

[9] ———, *Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$*, Mathematics of Computation, 29 (1975), pp. 243–269. MR **56:**15581a

[10] H. Siebert, *Montgomery's weighted sieve for dimension two*, Monatshefte für Mathematik, 82 (1976), pp. 327–336. MR **54:**12690

[11] R. C. Vaughan, *An elementary method in prime number theory*, Acta Arithmetica, 37 (1980), pp. 111–115. MR **82c:**10055

[12] I. M. Vinogradov, *Representation of an odd number as the sum of three primes*, Dokl. Akad. Nauk SSSR, 15 (1937), pp. 291–294.

[13] ⸺, *The method of Trigonometrical Sums in the Theory of Numbers*, translated from the Russian, revised and annotated by K.F. Roth and A. Davenport, Interscience, London, 1954. MR **15:**941b

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