ON A CONJECTURE OF ERDÖS AND STEWART

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Abstract. For any \( k \geq 1 \), let \( p_k \) be the \( k \)th prime number. In this paper, we confirm a conjecture of Erdös and Stewart concerning all the solutions of the diophantine equation \( n! + 1 = p_k^a p_{k+1}^b \), when \( p_{k-1} \leq n < p_k \).

1. Introduction

For any \( k \geq 1 \) let \( p_k \) be the \( k \)th prime number. From [3], we found out that Erdös and Stewart conjectured that the only solutions of the equation

\[
n! + 1 = p_k^a p_{k+1}^b
\]

are obtained for \( n \leq 5 \).

In this paper, we prove the following

Theorem. Equation (1) has no solutions for \( n \geq 6 \).

One can check that equation (1) has no solutions for \( 5 < n \leq 11 \). From now on, we work with a potential solution of (1) with \( n \geq 12 \).

2. An elementary lemma

The following elementary result turns out to be helpful when searching for the values of \( n \).

Lemma. In equation (1), one has \( ab \neq 0 \).

Proof of the Lemma. Assume that this is not so and write

\[
n! + 1 = p^a
\]

for some \( p \in \{ p_k, p_{k+1} \} \).

Let \( a = 2^a a_1 \) where \( a_1 \geq 1 \) is odd. Then,

\[
\text{ord}_2(n!) = \text{ord}_2(p^a - 1) \leq \max(\text{ord}_2(p \pm 1)) + i \leq \log_2(p_{k+1} + 1) + \log_2(a).
\]

From equation (2), we know that

\[
n^a < p^a = n! + 1 < n^n,
\]

therefore \( a < n \). Since the interval \([ n + 1, 2n ]\) contains at least two primes for \( n \geq 12 \), we get \( p_{k+1} + 1 \leq 2n \). Hence, inequality (3) implies

\[
\text{ord}_2(n!) < \log_2(2n) + \log_2(n) = 2 \log_2(n) + 1.
\]
From Lemma 1 in [1], we know that
\[
\text{ord}_2(n!) \geq n - \log_2(n + 1).
\]
From inequalities (5) and (6), we get
\[
n - \log_2(n + 1) < 2 \log_2(n) + 1,
\]
which implies \( n \leq 11 \). This contradicts the assumption on \( n \geq 12 \).

3. A linear form in logarithms and a bound on \( n \)

Write
\[
n! = p_k^a p_{k+1}^b - 1 = p_{k+1}^b \left( p_k^a - \frac{1}{p_{k+1}^a} \right).
\]
We find an upper bound for \( \text{ord}_2(n!) \). We apply Théorème 4 in [1] with the choices
\[
p = 2, \quad D = 1, \quad g = 1,
\]
\[
\alpha_1 = p_k, \quad \alpha_2 = \frac{1}{p_{k+1}}, \quad b_1 = a, \quad b_2 = b,
\]
\[
A_1 = p_k, \quad A_2 = p_{k+1}
\]
and
\[
\mu = 15, \quad \nu = 10, \quad c(\mu, \nu) = 18.
\]
From the result in [1], it follows that
\[
\text{ord}_2(n!) \leq \frac{36}{(\log 2)^4} \left( \max \{ \log b' + \log \log 2 + 0.4, 15 \log 2 \} \right)^2 \log p_k \log p_{k+1},
\]
where
\[
b' = \frac{a}{\log p_{k+1}} + \frac{b}{\log p_k}.
\]
We now find a bound on \( b' \) in terms on \( n \). Since
\[
p_k p_{k+1} = n! + 1 < n^n,
\]
it follows that
\[
a \log p_k + b \log p_{k+1} = \log p_k^a p_{k+1}^b < \log n^n = n \log n.
\]
Hence,
\[
b' = \frac{a}{\log p_{k+1}} + \frac{b}{\log p_k} = \frac{a \log p_k + b \log p_{k+1}}{\log p_k \log p_{k+1}} < \frac{n \log n}{\log p_k \log p_{k+1}} < \frac{n}{\log n},
\]
Since the interval \([n + 1, 2n] \) contains at least two primes, it follows that \( p_k < p_{k+1} < 2n \). Inequality (9) now implies
\[
\text{ord}_2(n!) \leq \frac{36}{(\log 2)^4} \left( \max \left\{ \log \left( \frac{n}{\log n} \right) + \log \log 2 + 0.4, 15 \log 2 \right\} \right)^2 \log^2(2n).
\]
When
\[
\log \left( \frac{n}{\log n} \right) + \log \log 2 + 0.4 \leq 15 \log 2,
\]
we get \( n < 409 \) 506. When
\[
\log\left(\frac{n}{\log n}\right) + \log \log 2 + 0.4 > 15 \log 2,
\]
we get, by inequalities (6) and (13), that
\[
(14) \quad n - \log_2(n + 1) < \frac{36}{(\log 2)^2}\left(\log\left(\frac{n}{\log n}\right) + \log \log 2 + 0.4\right)^2 \log^2(2n),
\]
which implies \( n < 7 \) 242 116. The conclusion is that \( n < p_k < p_{k+1} < 7.5 \cdot 10^6 \).

4. THE REMAINING COMPUTATIONS

For the remaining computations, we used the following result due to Erdős and Obláth (see [2]).

**Theorem EO.** The equation
\[
(15) \quad x^p \pm y^p = n!
\]
has no solutions such that \( p > 2 \) is prime and \( \gcd(x, y) = 1 \).

**Case 1.** \( n > 193 \).

The idea here was to check, computationally, that if \( n \) leads to a solution of (1), then \( a \equiv b \equiv 0 \pmod{3} \). Once we prove this, the impossibility of (1) follows from Theorem EO for \( p = 3 \).

Assume, for example, that (1) has a solution such that either \( 3 \nmid a \) or \( 3 \nmid b \). Write
\[
(16) \quad n! + 1 = Ax^3 \quad \text{where} \quad A = p^{\delta_1}_{k_1} p^{\delta_2}_{k_2+1} \quad \text{for some} \delta_1, \delta_2 \in \{0, 1, 2\} \quad \text{with} \quad (\delta_1, \delta_2) \neq (0, 0).
\]
Let \( q \leq 193 \) be a prime congruent to 1 modulo 3. Equation (1) implies that \( Ax^3 \equiv 1 \pmod{q} \) for every such \( q \). It now follows that \( A \) is a cubic residue modulo \( q \) for every \( q \leq 193 \) which is congruent to 1 modulo 3. Since a number \( y \) is a cubic residue modulo \( q \) if and only if \( y^2 \) is a cubic residue modulo \( q \), it follows that we need to identify only those numbers \( A \) of the form
\[
(17) \quad A = p_k \quad \text{or} \quad A = p_k p_{k+1} \quad \text{or} \quad A = p_k^2 p_{k+1}
\]
in the range \( 193 < p_k < p_{k+1} < 7.5 \cdot 10^6 \) which are cubic residues with respect to every prime \( q \leq 193 \) which is congruent to 1 modulo 3. Achim Flammenkamp wrote a computer program which checked in a few minutes that there are no such \( A \)'s. Hence, \( n \leq 193 \).

**Case 2.** \( n \leq 193 \).

By the Lemma, we know that if \( n \) leads to a solution of (1), then \( ab > 0 \). Achim Flammenkamp wrote another computer program which checked in less than a second that in this range \( n! + 1 \not\equiv 0 \pmod{p_k p_{k+1}} \).

The Theorem is therefore proved.

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REFERENCES


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