MATHEMATICS OF COMPUTATION
Volume 70, Number 234, Pages 893–896
S 0025-5718(00)01178-9
Article electronically published on March 8, 2000

ON A CONJECTURE OF ERDŐS AND STEWART

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ABSTRACT. For any $k \geq 1$, let p_k be the kth prime number. In this paper, we confirm a conjecture of Erdős and Stewart concerning all the solutions of the diophantine equation $n! + 1 = p_k^a p_{k+1}^b$, when $p_{k-1} \leq n < p_k$.

1. Introduction

For any $k \geq 1$ let p_k be the kth prime number. From [3], we found out that Erdős and Stewart conjectured that the only solutions of the equation

$$n! + 1 = p_k^a p_{k+1}^b \quad \text{for some } a \geq 0, b \geq 0 \text{ and } p_{k-1} \leq n < p_k \tag{1}$$

are obtained for $n \leq 5$.

In this paper, we prove the following

Theorem. *Equation* (1) *has no solutions for* $n \geq 6$.

One can check that equation (1) has no solutions for $5 < n \leq 11$. From now on, we work with a potential solution of (1) with $n \geq 12$.

2. An elementary lemma

The following elementary result turns out to be helpful when searching for the values of n.

Lemma. *In equation* (1), *one has* $ab \neq 0$.

Proof of the Lemma. Assume that this is not so and write

$$n! + 1 = p^a \quad \text{for some } p \in \{p_k, p_{k+1}\}. \tag{2}$$

Let $a = 2^i a_1$ where $a_1 \geq 1$ is odd. Then,

$$\text{ord}_2(n!) = \text{ord}_2(p^a - 1) \leq \max(\text{ord}_2(p \pm 1)) + i \leq \log_2(p_{k+1} + 1) + \log_2(a). \tag{3}$$

From equation (2), we know that

$$n^a < p^a = n! + 1 < n^n, \tag{4}$$

therefore $a < n$. Since the interval $[n + 1, 2n]$ contains at least two primes for $n \geq 12$, we get $p_{k+1} + 1 \leq 2n$. Hence, inequality (3) implies

$$\text{ord}_2(n!) < \log_2(2n) + \log_2(n) = 2\log_2(n) + 1. \tag{5}$$

Received by the editor January 4, 1999.
2000 *Mathematics Subject Classification.* Primary 11D61.
Key words and phrases. p-adic linear forms in two logarithms.

From Lemma 1 in [1], we know that

$$\operatorname{ord}_2(n!) \geq n - \log_2(n+1). \tag{6}$$

From inequalities (5) and (6), we get

$$n - \log_2(n+1) < 2\log_2(n) + 1, \tag{7}$$

which implies $n \leq 11$. This contradicts the assumption on $n \geq 12$.

3. A linear form in logarithms and a bound on n

Write

$$n! = p_k^a p_{k+1}^b - 1 = p_{k+1}^b \left(p_k^a - \left(\frac{1}{p_{k+1}} \right)^b \right). \tag{8}$$

We find an upper bound for $\operatorname{ord}_2(n!)$. We apply Théoréme 4 in [1] with the choices

$$p = 2, \quad D = 1, \quad g = 1,$$
$$\alpha_1 = p_k,\ \alpha_2 = \frac{1}{p_{k+1}},\ b_1 = a,\ b_2 = b,$$
$$A_1 = p_k,\ A_2 = p_{k+1}$$

and

$$\mu = 15, \quad \nu = 10, \quad c(\mu,\nu) = 18.$$

From the result in [1], it follows that

$$\operatorname{ord}_2(n!) \leq \frac{36}{(\log 2)^4} (\max\{\log b' + \log\log 2 + 0.4, 15\log 2\})^2 \log p_k \log p_{k+1}, \tag{9}$$

where

$$b' = \frac{a}{\log p_{k+1}} + \frac{b}{\log p_k}. \tag{10}$$

We now find a bound on b' in terms on n. Since

$$p_k^a p_{k+1}^b = n! + 1 < n^n,$$

it follows that

$$a \log p_k + b \log p_{k+1} = \log p_k^a p_{k+1}^b < \log n^n = n \log n. \tag{11}$$

Hence,

$$b' = \frac{a}{\log p_{k+1}} + \frac{b}{\log p_k} = \frac{a\log p_k + b \log p_{k+1}}{\log p_k \log p_{k+1}} < \frac{n \log n}{\log p_k \log p_{k+1}} < \frac{n}{\log n}. \tag{12}$$

Since the interval $[n+1, 2n]$ contains at least two primes, it follows that $p_k < p_{k+1} < 2n$. Inequality (9) now implies

(13)
$$\operatorname{ord}_2(n!) < \frac{36}{(\log 2)^4} \left(\max\left\{ \log\left(\frac{n}{\log n} \right) + \log\log 2 + 0.4, 15\log 2 \right\} \right)^2 \log^2(2n).$$

When

$$\log\left(\frac{n}{\log n}\right) + \log\log 2 + 0.4 \leq 15 \log 2,$$

we get $n < 409\,506$. When

$$\log\left(\frac{n}{\log n}\right) + \log\log 2 + 0.4 > 15\log 2,$$

we get, by inequalities (6) and (13), that

$$n - \log_2(n+1) < \frac{36}{(\log 2)^4}\left(\log\left(\frac{n}{\log n}\right) + \log\log 2 + 0.4\right)^2 \log^2(2n), \tag{14}$$

which implies $n < 7\,242\,116$. The conclusion is that $n < p_k < p_{k+1} < 7.5 \cdot 10^6$.

4. The remaining computations

For the remaining computations, we used the following result due to Erdős and Obláth (see [2]).

Theorem EO. *The equation*

$$x^p \pm y^p = n! \tag{15}$$

has no solutions such that $p > 2$ *is prime and* $\gcd(x, y) = 1$.

Case 1. $n > 193$.

The idea here was to check, computationally, that if n leads to a solution of (1), then $a \equiv b \equiv 0 \pmod 3$. Once we prove this, the impossibility of (1) follows from Theorem EO for $p = 3$.

Assume, for example, that (1) has a solution such that either $3 \nmid a$ or $3 \nmid b$. Write

$$n! + 1 = Ax^3 \quad \text{where } A = p_k^{\delta_1} p_{k+1}^{\delta_2} \text{ for some } \delta_1,\ \delta_2 \in \{0, 1, 2\} \text{ with } (\delta_1, \delta_2) \neq (0, 0). \tag{16}$$

Let $q \leq 193$ be a prime congruent to 1 modulo 3. Equation (1) implies that $Ax^3 \equiv 1 \pmod q$ for every such q. It now follows that A is a cubic residue modulo q for every $q \leq 193$ which is congruent to 1 modulo 3. Since a number y is a cubic residue modulo q if and only if y^2 is a cubic residue modulo q, it follows that we need to identify only those numbers A of the form

$$A = p_k \quad \text{or} \quad A = p_k p_{k+1} \quad \text{or} \quad A = p_k^2 p_{k+1} \tag{17}$$

in the range $193 < p_k < p_{k+1} < 7.5 \cdot 10^6$ which are cubic residues with respect to every prime $q \leq 193$ which is congruent to 1 modulo 3. Achim Flammenkamp wrote a computer program which checked in a few minutes that there are no such A's. Hence, $n \leq 193$.

Case 2. $n \leq 193$.

By the Lemma, we know that if n leads to a solution of (1), then $ab > 0$. Achim Flammenkamp wrote another computer program which checked in less than a second that in this range $n! + 1 \not\equiv 0 \pmod{p_k p_{k+1}}$.

The Theorem is therefore proved.

Acknowledgments

We thank Achim Flammenkamp who carried out the computations described in the last section. We also thank Professor Andreas Dress and his research group in Bielefeld for their hospitality during the period when this paper was written, and the Alexander von Humboldt Foundation for support.

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