

POWER SERIES EXPANSIONS FOR MATHIEU FUNCTIONS WITH SMALL ARGUMENTS

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ABSTRACT. Power series expansions for the even and odd angular Mathieu functions $\text{Se}_m(h, \cos \theta)$ and $\text{So}_m(h, \cos \theta)$, with small argument h , are derived for general integer values of m . The expansion coefficients that we evaluate are also useful for the calculation of the corresponding radial functions of any kind.

1. INTRODUCTION

Mathieu functions are used in many applications in engineering and physics [1]–[12] originating, principally, from the solution of the wave equation in elliptic cylindrical coordinates. Studies on these functions can be found in [13]–[17], among other places. Numerical computation of Mathieu functions with integer orders is examined in [18]–[25].

In this paper we produce power series expansions for the even and odd angular Mathieu functions $\text{Se}_m(h, \cos \theta)$ and $\text{So}_m(h, \cos \theta)$, with small argument h , for general integer values of m . In [13]–[16], [26], [27] the lower order expansion coefficients were given only for the special values $m = 0, \dots, 4$.

The expansion coefficients that we calculate here can also be used to evaluate the corresponding radial Mathieu functions of any kind.

An analogous method was used in [28] for the evaluation of the spheroidal wave functions.

The calculation of the expansion coefficients for the even and odd functions is developed in Sections 2 and 3, respectively, while in Section 4 we give some relations connecting the various expansion coefficients of the even and odd functions. Finally, in Section 5 we compare our results, for special values of the parameters, with previous ones produced by other methods. These comparisons confirm the correctness of our formulas, at least for those values of the parameters, giving a very good check on the validity of our procedure.

2. CALCULATION OF THE EXPANSION COEFFICIENTS FOR THE EVEN FUNCTIONS

The angular Mathieu functions are solutions of Mathieu's differential equation

$$(1) \quad \frac{d^2 y}{d\theta^2} + \left(b - \frac{h^2}{2} \cos 2\theta \right) y = 0,$$

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where b and h^2 are constants. These functions are periodic, even (Se_m) or odd (So_m), only for certain discrete characteristic values (eigenvalues) of b , with symbols be_m or bo_m , respectively, and m an integer. In this section we examine the even functions, which are given by the series [13]–[15],

$$(2) \quad Se_m(h, \cos \theta) = \sum_{n=0}^{\infty} B_n^e(h, m) \cos n\theta, \quad m \geq 0.$$

Here m and n are both even or both odd. Substituting from (2) into (1), we obtain the following second order recurrence relation for the expansion coefficients $B_n^e(h, m)$ [13]–[15],[17]:

$$(3) \quad (-n^2 + be_m)B_n^e - \frac{h^2}{4} \left[B_{n+2}^e + \frac{2}{\varepsilon_{n-2}} B_{n-2}^e + (2 - \varepsilon_{n-1})B_n^e \right] = 0, \quad m, n \geq 0,$$

where $\varepsilon_0 = 1$ and $\varepsilon_i = 2$ for $i \neq 0$ is the Neumann factor, and $B_{-n}^e = 0$ for $n > 0$.

When $h \rightarrow 0$, $Se_m(h, \cos \theta) \rightarrow \cos m\theta$ and $be_m \rightarrow m^2$. In this case, the only nonzero coefficient is $B_m^e(h, m) \rightarrow 1$.

Approximations for Se_m , valid for small values of h , follow by expanding $B_n^e(h, m)$ in power series in h . Keeping in mind the above remarks, we see that this expansion must be of the form

$$B_{m \pm 2q}^e(h, m) = [\alpha_{2q,0}^{\pm} h^{2q} + \alpha_{2q,2}^{\pm} h^{2q+2} + \alpha_{2q,4}^{\pm} h^{2q+4} + \dots] B_m^e(h, m),$$

$$(4) \quad m \geq 0 (+), \quad m \geq 2q(-), \quad q = 0, 1, 2, \dots,$$

where $\alpha_{2q,2k}^- = 0$ ($k \geq 0$) if $0 \leq m < 2q$, while $\alpha_{0,0}^{\pm} = 1$ and $\alpha_{0,2k}^{\pm} = 0$ if $k \geq 1$.

The expansion for $be_m(h)$ must be

$$(5) \quad be_m(h) = m^2 + l_2 h^2 + l_4 h^4 + l_6 h^6 + \dots$$

In (4), (5), and in what follows we omit the indices m and e from the various expansion coefficients α and l , as well as from any others that may be used, for reasons of simplicity. Certainly, these coefficients are independent of h .

By substituting (4) and (5) into (3) and equating the coefficients of h^2, h^4, \dots to zero separately, we obtain with $n = m \geq 0$, first from the coefficient of h^2

$$(6) \quad l_2 = (2 - \varepsilon_{m-1})/4,$$

and next from that of h^{2k+4}

$$(7) \quad \frac{1}{4} \left(\alpha_{2,2k}^+ + \frac{2}{\varepsilon_{m-2}} \alpha_{2,2k}^- \right) = l_{2k+4}, \quad k = 0, 1, 2, \dots$$

Setting now $n = m \pm 2q$, where $q \geq 1$ ($m \geq 2q$ for the lower sign), in (3) and using (4) and (5), we obtain by equating to zero, first the coefficient of h^{2q}

$$(8) \quad \alpha_{2q,0}^{\pm} = f^{\pm}(2q - 2)\alpha_{2q-2,0}^{\pm}, \quad q \geq 1,$$

next that of h^{2q+2}

$$(9) \quad \alpha_{2q,2}^{\pm} = f^{\pm}(2q - 2)\alpha_{2q-2,2}^{\pm} + g^{\pm}(2q)\alpha_{2q,0}^{\pm}, \quad q \geq 1,$$

and finally that of h^{2q+2k}

$$(10) \quad \alpha_{2q,2k}^\pm = f^\pm(2q-2)\alpha_{2q-2,2k}^\pm + g^\pm(2q)\alpha_{2q,2k-2}^\pm + t^\pm(2q) \sum_{j=2}^k l_{2j}\alpha_{2q,2k-2j}^\pm + p^\pm(2q+2)\alpha_{2q+2,2k-4}^\pm, \quad q \geq 1, k \geq 2,$$

where we have made the substitutions ($q \geq 1$)

$$(11) \quad t^\pm(2q) = \pm \frac{1}{4q(m \pm q)},$$

$$(12) \quad f^+(2q-2) = -\frac{1}{2\varepsilon_{m+2q-2}}t^+(2q), \quad f^-(2q-2) = -\frac{1}{4}t^-(2q),$$

$$(13) \quad g^+(2q) = \frac{2-\varepsilon_{m-1}}{4}t^+(2q), \quad g^-(2q) = -\frac{2-\varepsilon_{m-2q-1}}{4}t^-(2q),$$

$$(14) \quad p^+(2q+2) = -\frac{1}{4}t^+(2q), \quad p^-(2q+2) = -\frac{1}{2\varepsilon_{m-2q-2}}t^-(2q).$$

From (9) with $q = 1$ we obtain

$$(15) \quad \alpha_{2,2}^\pm = g^\pm(2)\alpha_{2,0}^\pm = v_2^\pm(2)\alpha_{2,0}^\pm,$$

where

$$(16) \quad v_2^\pm(2) = g^\pm(2).$$

Setting, for $q \geq 2$,

$$(17) \quad \alpha_{2q,2}^\pm = v_2^\pm(2q)\alpha_{2q,0}^\pm,$$

we get from (8), (9), (17) that

$$(18) \quad \alpha_{2q,2}^\pm = [v_2^\pm(2q-2) + g^\pm(2q)]\alpha_{2q,0}^\pm.$$

From (17) and (18) we get the recurrence relation

$$(19) \quad v_2^\pm(2q) = v_2^\pm(2q-2) + g^\pm(2q), \quad q \geq 1,$$

where $v_2^\pm(0) = 0$.

Using (13), (16), (19), we obtain

$$(20a) \quad v_2^+(2q) = \sum_{i=1}^q g^+(2i) = \begin{cases} \frac{1}{16} \sum_{i=1}^q \left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{q}{16(q+1)}, & m = 1, \\ 0, & m \neq 1, \end{cases}$$

for $q \geq 0$, and, for $q \geq 1$,

$$(20b) \quad v_2^-(2q) = \sum_{i=1}^q g^-(2i) = \begin{cases} \frac{1}{16q(q+1)}, & m = 2q + 1, \\ 0, & m \neq 2q + 1. \end{cases}$$

In (20b) $g^-(2i) \neq 0$ only for $m = 2i + 1$, and as $m \geq 2q$ in this case, the only value of i giving a nonzero result is $i = q$.

Setting now

$$(21) \quad \alpha_{2q,2k}^\pm = v_{2k}^\pm(2q)\alpha_{2q,0}^\pm$$

for $q \geq 1$ and $k \geq 2$, we find from (8), (10) and (21) that

$$(22) \quad \alpha_{2q,2k}^\pm = \left[v_{2k}^\pm(2q-2) + g^\pm(2q)v_{2k-2}^\pm(2q) + t^\pm(2q) \sum_{j=2}^k l_{2j}v_{2k-2j}^\pm(2q) + p^\pm(2q+2)v_{2k-4}^\pm(2q+2)f^\pm(2q) \right] \alpha_{2q,0}^\pm \quad (q \geq 1, k \geq 2).$$

It is evident from (21) that $v_0^\pm(2q) = 1$. From (21), (22) we obtain the following recurrence relation for the calculation of $v_{2k}^\pm(2q)$:

$$(23) \quad v_{2k}^\pm(2q) = v_{2k}^\pm(2q-2) + g^\pm(2q)v_{2k-2}^\pm(2q) + t^\pm(2q) \sum_{j=2}^k l_{2j}v_{2k-2j}^\pm(2q) + p^\pm(2q+2)v_{2k-4}^\pm(2q+2)f^\pm(2q) \quad (q \geq 1, k \geq 2).$$

From (23) with $q = 1$ we get that

$$(24) \quad v_{2k}^\pm(2) = g^\pm(2)v_{2k-2}^\pm(2) + t^\pm(2) \sum_{j=2}^k l_{2j}v_{2k-2j}^\pm(2) + p^\pm(4)v_{2k-4}^\pm(4)f^\pm(2),$$

where we have used the result $v_{2k}^\pm(0) = 0$ for $k \geq 1$.

From (23), (24) we find that

$$(25) \quad v_{2k}^\pm(2q) = \sum_{i=1}^q \left[g^\pm(2i)v_{2k-2}^\pm(2i) + t^\pm(2i) \sum_{j=2}^k l_{2j}v_{2k-2j}^\pm(2i) + p^\pm(2i+2)v_{2k-4}^\pm(2i+2)f^\pm(2i) \right] \\ = \beta_{2k}^\pm(2q) + \sum_{j=2}^k l_{2j}\gamma_{2k-2j}^\pm(2q) + \delta_{2k}^\pm(2q), \quad q \geq 1, k \geq 2,$$

where

$$(26) \quad \beta_{2k}^\pm(2q) = \sum_{i=1}^q g^\pm(2i)v_{2k-2}^\pm(2i), \quad k \geq 1,$$

$$(27) \quad \gamma_{2k-2j}^\pm(2q) = \sum_{i=1}^q t^\pm(2i)v_{2k-2j}^\pm(2i), \quad k \geq j \geq 2,$$

$$(28) \quad \delta_{2k}^\pm(2q) = \sum_{i=1}^q p^\pm(2i+2)v_{2k-4}^\pm(2i+2)f^\pm(2i), \quad k \geq 2.$$

Formula (28) for $\delta_{2k}^\pm(2q)$ is correct for $m \geq 2q + 2$. For $m = 2q + 1$ or $m = 2q$ the term with $i = q$ should be omitted from the sum because it corresponds to the expansion coefficient $\alpha_{2q+2,2k-4}^-$ (see (10), (22), (23), (25), (28)), which is equal to zero for $m < 2q + 2$. The sum in the last two aforementioned cases, extended from $i = 1$ to $i = q - 1$, is considered as zero if its upper limit is less than 1, i.e., for $q = 1$.

The recurrence relation (25) can be used for the evaluation of $v_{2k}^\pm(2q)$, and consequently of $\alpha_{2q,2k}^\pm$ from (21), by using the expressions of $v_{2k-2s}^\pm(2i)$ ($i \geq 1, 1 \leq s \leq k$). The expansion coefficients l_{2j} ($2 \leq j \leq k$) are calculated from (7) by using $\alpha_{2,2j-4}^\pm$ calculated from (21) with $q = 1$. Their expressions for $j \leq 8$ can be found also in [13], [15], [17].

For great values of k , $v_{2k}^\pm(2q)$ can easily be obtained numerically from (25). The process must be repeated for each different value of m and q . However, for small values of k , analytic closed-form expressions are obtained, valid for each m and q . So, for $k = 2$, we find from (25) that

$$(29) \quad v_4^\pm(2q) = \beta_4^\pm(2q) + l_4\gamma_0^\pm(2q) + \delta_4^\pm(2q).$$

Substituting (13) and (20) into (26), we obtain

$$(30a) \quad \beta_4^+(2q) = \begin{cases} \frac{1}{256} \sum_{i=1}^q \frac{1}{(i+1)^2}, & m = 1, \\ 0, & m \neq 1, \end{cases}$$

$$(30b) \quad \beta_4^-(2q) = \begin{cases} \frac{1}{256q^2(q+1)^2}, & m = 2q + 1, \\ 0, & m \neq 2q + 1. \end{cases}$$

In (30b) the remarks made after (20b) have been used.

Next, substituting (11) into (27) with $k = j = 2$, we obtain

$$(31) \quad \gamma_0^\pm(2q) = \pm \sum_{i=1}^q \frac{1}{4i(m \pm i)}.$$

Finally, substituting (12) and (14) into (28), with $k = 2$, we obtain

$$(32a) \quad \delta_4^+(2q) = \frac{1}{256} \sum_{i=1}^q \frac{1}{i(i+1)(i+m)(i+m+1)},$$

$$(32b) \quad \delta_4^-(2q) = \begin{cases} \frac{1}{256} \sum_{i=1}^q \frac{1}{i(i+1)(i-m)(i-m+1)}, & m \geq 2q + 3, \\ \frac{1}{256} \sum_{i=1}^{q-1} \frac{1}{i(i+1)(i-m)(i-m+1)} + \frac{1}{128q(q+1)(q-m)(q-m+1)}, & m = 2q + 2, \\ \frac{1}{256} \sum_{i=1}^{q-1} \frac{1}{i(i+1)(i-m)(i-m+1)}, & m = 2q + 1, \\ \frac{1}{256} \sum_{i=1}^{q-2} \frac{1}{i(i+1)(i-m)(i-m+1)} + \frac{1}{128(q-1)q(q-1-m)(q-m)}, & m = 2q, \quad q \neq 1, \\ 0, & m = 2 \quad (q = 1). \end{cases}$$

In the sums of (32b) we have used the remarks after (28). Also, for $m = 2q + 2$ or $m = 2q$ their last term (with $i = q$ or $i = q - 1$, respectively) is written separately, because in these cases the Neumann factor appearing in (14) is equal to 1, while in the rest it is equal to 2. Sums whose upper limit is less than 1 should be taken as zero.

We expand the fractions after the summation signs in (31) and (32) into sums of partial fractions, and we next substitute (30)–(32) in (29) by using the expressions

for l_4 calculated from (7) and (8) with $k = 0$ and $q = 1$, respectively:

$$(33) \quad l_4 = \begin{cases} -1/128, & m = 1, \\ 5/192, & m = 2, \\ 1/[32(m^2 - 1)], & m = 0 \text{ or } m \geq 3. \end{cases}$$

So, we finally obtain, after straightforward but lengthy calculations, the results

$$(34a) \quad v_4^+(2q) = \begin{cases} \frac{q}{1024(q+2)}, & m = 1, \\ \frac{q(47q^2+222q+247)}{9216(q+1)(q+2)(q+3)}, & m = 2, \\ \frac{q(m^2+3m+mq+3q+4)}{256(m^2-1)(m+1)(q+1)(m+q+1)}, & m = 0 \text{ or } m \geq 3, \end{cases}$$

for $q \geq 0$, and, for $q \geq 1$,

$$(34b) \quad v_4^-(2q) = \begin{cases} -\frac{5}{768}, & m = 2, \quad q = 1, \\ \frac{q(m^2-3m-mq+3q+4)}{256(m^2-1)(m-1)(q+1)(m-1-q)}, & \begin{cases} m = 2q \ (q \neq 1), \\ \text{or } m = 2q + 1, \\ \text{or } m \geq 2q + 3, \end{cases} \\ \frac{2q^5+7q^4+16q^3+24q^2+14q+3}{256q(q+1)^2(q+2)(2q+1)^2(2q+3)}, & m = 2q + 2. \end{cases}$$

From (8), (11), (12) we get $(\alpha_{0,0}^\pm = 1)$

$$(35a) \quad \alpha_{2q,0}^+ = -\frac{1}{8\varepsilon_{m+2q-2q}(m+q)}\alpha_{2q-2,0}^+, \quad m \geq 0, \quad q \geq 1,$$

$$(35b) \quad \alpha_{2q,0}^- = \frac{1}{16q(m-q)}\alpha_{2q-2,0}^-, \quad m \geq 2q, \quad q \geq 1.$$

From (35) we easily find that

$$(36a) \quad \alpha_{2q,0}^+ = (-1)^q \frac{2m!}{\varepsilon_m 16^q q!(m+q)!}, \quad m \geq 0, \quad q \geq 1,$$

$$(36b) \quad \alpha_{2q,0}^- = \frac{(m-q-1)!}{16^q q!(m-1)!}, \quad m \geq 2q, \quad q \geq 0.$$

From (9), (11)–(13), (17), (20a) and (36a), we obtain

$$(37) \quad \alpha_{2q,2}^+ = \begin{cases} -\frac{1}{1024}, & m = 1, \quad q = 1, \\ -\frac{q}{16(q-1)(q+1)^2}\alpha_{2q-2,2}^+, & m = 1, \quad q \geq 2, \\ 0, & m \neq 1, \quad q \geq 1. \end{cases}$$

From (37), as well as from (17), (20a) and (36a), we obtain, for $q \geq 0$,

$$(38a) \quad \alpha_{2q,2}^+ = \begin{cases} (-1)^q \frac{q}{16^{q+1}[(q+1)!]^2}, & m = 1, \\ 0, & m \neq 1. \end{cases}$$

Using (17), (20b) and (36b), we get, for $q \geq 1$,

$$(38b) \quad \alpha_{2q,2}^- = \begin{cases} \frac{1}{16^{q+1}q(q+1)(2q)!}, & m = 2q + 1, \\ 0, & m \neq 2q + 1. \end{cases}$$

Using (21) for $k = 2$ with (34) and (36), we easily find $\alpha_{2q,4}^+$ and $\alpha_{2q,4}^-$, respectively:

$$(39a) \quad \alpha_{2q,4}^+ = \begin{cases} (-1)^q \frac{4q}{16^{q+3} q! (q+2)!}, & m = 1, \\ (-1)^q \frac{q(47q^2 + 222q + 247)}{16^{q+2} 18(q+2)! (q+3)!}, & m = 2, \\ (-1)^q \frac{2q(m^2 + 3m + mq + 3q + 4)m!}{\varepsilon_m 16^{q+2} (m^2 - 1)(m+1)(q+1)!(m+q+1)!}, & m = 0 \text{ or } m \geq 3 \end{cases}$$

for $q \geq 0$, and, for $q \geq 1$,

$$(39b) \quad \alpha_{2q,4}^- = \begin{cases} -\frac{5}{12288}, & m = 2, q = 1, \\ \frac{q(m^2 - 3m - mq + 3q + 4)(m - q - 2)!}{16^{q+2} (m^2 - 1)(m - 1)(q + 1)!(m - 1)!}, & \begin{cases} m = 2q (q \neq 1), \\ \text{or } m = 2q + 1, \\ \text{or } m \geq 2q + 3, \end{cases} \\ \frac{2q^5 + 7q^4 + 16q^3 + 24q^2 + 14q + 3}{16^{q+2} q(q+1)(q+2)(2q+1)^2(2q+3)(2q+1)!}, & m = 2q + 2. \end{cases}$$

The analytic evaluation of $v_{2k}^\pm(2q)$ becomes more and more lengthy and tedious, as k increases (see (25)). After some manipulation we can obtain from (25) the analytic expressions for $v_6^\pm(2q)$ ($k = 3$) which are given in the Appendix.

The explicit values of the B 's depend on the normalization used. Here it will be assumed [14] that $\text{Se}_m(h, 1) = 1$, or equivalently,

$$(40) \quad \sum_{n=0}^{\infty} B_n^e(h, m) = 1,$$

with n and m both even or both odd. Substituting (4) into (40), we obtain

$$(41) \quad B_m^e(h, m) \left[1 + \sum_{q=1}^{\infty} \sum_{k=0}^{\infty} (\alpha_{2q,2k}^+ + \alpha_{2q,2k}^-) h^{2q+2k} \right] = 1.$$

The expansion of $B_m^e(h, m)$ for small values of h has the expression

$$(42) \quad B_m^e(h, m) = 1 + g_2 h^2 + g_4 h^4 + g_6 h^6 + \dots$$

Substituting (42) into (41), we carry out the various multiplications and equate the coefficients of like powers of h , calculating successively the coefficients g_{2s} , which are independent of h and given by the formula

$$(43) \quad g_{2s} = - \sum_{q=1}^s \sum_{k=0}^{s-q} g_{2k} [\alpha_{2q,2(s-q-k)}^+ + \alpha_{2q,2(s-q-k)}^-], \quad s = 1, 2, \dots,$$

while $g_0 = 1$.

Using, finally, (4) and (42) in (2), we obtain, after some manipulation, the power series expansion for $\text{Se}_m(h, \cos \theta)$:

$$(44) \quad \text{Se}_m(h, \cos \theta) = \sum_{l=0}^{\infty} g_{2l} h^{2l} \left\{ \cos m\theta + \sum_{q=1}^{\infty} \sum_{k=0}^{\infty} \{ \alpha_{2q,2k}^+ \cos[(m + 2q)\theta] + \alpha_{2q,2k}^- \cos[(m - 2q)\theta] \} h^{2(q+k)} \right\}.$$

We recall that in each case $\alpha_{2q,2k}^- = 0$, if $0 \leq m < 2q$.

The above-calculated expansion coefficients are also useful for the evaluation of the even radial Mathieu functions of any kind. These functions are solutions of the equation

$$(45) \quad \frac{d^2y}{d\mu^2} - \left(b - \frac{h^2}{2} \cosh 2\mu \right) y = 0,$$

which is merely (1) with $\theta = i\mu$ (in this case $b = be_m$), and are expressed as [14]

$$(46) \quad Ze_m(h, \cosh \mu) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} (-1)^{(n-m)/2} B_n^e(h, m) Z_n(h \cosh \mu).$$

Here Z_n is the cylindrical Bessel function of any kind and Ze_m the corresponding even radial Mathieu function of the same kind. By using (4), (42) and (43) in (46) we can find its expansion for small values of h .

3. CALCULATION OF THE EXPANSION COEFFICIENTS FOR THE ODD FUNCTIONS

The odd angular Mathieu functions of integral order are given by the series [13]–[15]

$$(47) \quad So_m(h, \cos \theta) = \sum_{n=1}^{\infty} B_n^o(h, m) \sin n\theta, \quad m \geq 1,$$

where n and m are both even or both odd. Substituting (47) into (1), with $b = bo_m$, we obtain the following second order recurrence relation for the expansion coefficients $B_n^o(h, m)$ [13]–[15], [17]:

$$(48) \quad (-n^2 + bo_m)B_n^o - \frac{h^2}{4}[B_{n+2}^o + B_{n-2}^o - (2 - \varepsilon_{n-1})B_n^o] = 0, \quad m, n \geq 1,$$

where $B_{-n}^o = 0$ for $n \geq 0$.

The expansions (4) and (5) are also valid in this case, with e replaced by o . Now, in (4), $m \geq 1$ for the upper (+) sign and $m \geq 2q + 1$ for the lower (−) sign.

Following steps identical to those for the even functions, we obtain, with $n = m \geq 1$, in place of (6) and (7), the relations

$$(49) \quad l_2 = -(2 - \varepsilon_{m-1})/4$$

and

$$(50) \quad \frac{1}{4}(\alpha_{2,2k}^+ + \alpha_{2,2k}^-) = l_{2k+4}, \quad k = 0, 1, 2, \dots$$

Setting next $n = m \pm 2q$, where $q \geq 1$ ($m \geq 2q + 1$ for the lower sign), in (48) and using (4) and (5) with e replaced by o , we again obtain (8)–(14), with the only differences that ε_{m+2q-2} in f^+ (equation (12)) and ε_{m-2q-2} in p^- (equation (14)) should be replaced by 2, and that g^\pm (equation (13)) should change their sign. For this reason, (15)–(19) and (21)–(28) remain the same (the remarks after (28) for $\delta_{2k}^-(2q)$ are also valid in this case, with $2q + 2$, $2q + 1$ and $2q$ replaced by $2q + 3$, $2q + 2$ and $2q + 1$, respectively) while $v_2^\pm(2q)$ in (20) simply change their sign. So, in this case,

$$(51a) \quad v_2^+(2q) = \begin{cases} -\frac{q}{16(q+1)}, & m = 1, \\ 0, & m \neq 1, \end{cases}$$

for $q \geq 0$, and, for $q \geq 1$,

$$(51b) \quad v_2^-(2q) = \begin{cases} -\frac{1}{16q(q+1)}, & m = 2q + 1, \\ 0, & m \neq 2q + 1, \end{cases}$$

while the expansion coefficients l_{2j} ($2 \leq j \leq k$) should be calculated from (50) (with the use of $\alpha_{2,2j-4}^\pm$ calculated from (21) with $q = 1$). Their expressions for $j \leq 8$ can also be found in [13], [15], [17].

Setting $k = 2$ in (25), we again obtain (29)–(32a), with the only differences that now $m \geq 1$ for the superscript $+$ and $m \geq 2q + 1$ of the superscript $-$. Equations (32b) are replaced by

$$(52) \quad \delta_4^-(2q) = \begin{cases} \frac{1}{256} \sum_{i=1}^q \frac{1}{i(i+1)(i-m)(i-m+1)}, & m \geq 2q + 3, \\ \frac{1}{256} \sum_{i=1}^{q-1} \frac{1}{i(i+1)(i-m)(i-m+1)}, & m = 2q + 1 \text{ or } 2q + 2, \quad q \neq 1, \\ 0, & q = 1, m = 3 \text{ or } 4. \end{cases}$$

Substituting the aforementioned equations in (29) and using the expressions for l_4 found from (50) and (8) with $k = 0$ and $q = 1$, respectively,

$$(53) \quad l_4 = \begin{cases} -1/128, & m = 1, \\ -1/192, & m = 2, \\ 1/[32(m^2 - 1)], & m \geq 3, \end{cases}$$

we obtain, after some manipulation, the results

$$(54a) \quad v_4^+(2q) = \begin{cases} \frac{q}{1024(q+2)}, & m = 1, \\ -\frac{q(7q+23)}{9216(q+2)(q+3)}, & m = 2, \\ \frac{q(m^2+3m+mq+3q+4)}{256(m^2-1)(m+1)(q+1)(m+q+1)}, & m \geq 3, \end{cases}$$

for $q \geq 0$, and, for $q \geq 1$,

$$(54b) \quad v_4^-(2q) = \begin{cases} \frac{q(m^2-3m-mq+3q+4)}{256(m^2-1)(m-1)(q+1)(m-1-q)}, & m \neq 2q + 2, \\ \frac{2q^5+7q^4-16q^2-14q-3}{256q(q+1)^2(q+2)(2q+1)^2(2q+3)}, & m = 2q + 2. \end{cases}$$

From (8), (11), (12) (keeping in mind the modification for f^+) we have in this case ($\alpha_{0,0}^\pm = 1$)

$$(55) \quad \alpha_{2q,0}^\pm = \mp \frac{1}{16q(m \pm q)} \alpha_{2q-2,0}^\pm, \quad \left\{ \begin{array}{l} m \geq 1 \text{ (upper sign)} \\ m \geq 2q + 1 \text{ (lower sign)} \end{array} \right\}, \quad q \geq 1.$$

From this we easily obtain

$$(56a) \quad \alpha_{2q,0}^+ = (-1)^q \frac{m!}{16^q q! (m+q)!}, \quad m \geq 1, \quad q \geq 0,$$

$$(56b) \quad \alpha_{2q,0}^- = \frac{(m-q-1)!}{16^q q! (m-1)!}, \quad m \geq 2q + 1, \quad q \geq 0.$$

From (9), (11)–(13), (17), (51a) and (56a), keeping in mind the aforementioned modifications, we find that

$$(57) \quad \alpha_{2q,2}^+ = \begin{cases} \frac{1}{1024}, & m = 1, q = 1, \\ -\frac{q}{16(q-1)(q+1)^2} \alpha_{2q-2,2}^+, & m = 1, q \geq 2, \\ 0, & m \neq 1, q \geq 1. \end{cases}$$

From this and from (17), (51a) and (56a), we obtain

$$(58a) \quad \alpha_{2q,2}^+ = \begin{cases} (-1)^{q+1} \frac{q}{16^{q+1}[(q+1)!]^2}, & m = 1, \\ 0, & m \neq 1, \end{cases} \quad (q \geq 0).$$

Using (17), (51b) and (56b), we get

$$(58b) \quad \alpha_{2q,2}^- = \begin{cases} -\frac{1}{16^{q+1}q(q+1)(2q)!}, & m = 2q + 1, \\ 0, & m \neq 2q + 1, \end{cases} \quad (q \geq 1).$$

From (21) with $k = 2$ and (54), (56) we easily obtain $\alpha_{2q,4}^+$ and $\alpha_{2q,4}^-$:

$$(59a) \quad \alpha_{2q,4}^+ = \begin{cases} (-1)^q \frac{4q}{16^{q+3}q!(q+2)!}, & m = 1, \\ (-1)^{q+1} \frac{q(q+1)(7q+23)}{16^{q+2} \cdot 18(q+2)!(q+3)!}, & m = 2, \\ (-1)^q \frac{q(m^2+3m+mq+3q+4)m!}{16^{q+2}(m^2-1)(m+1)(q+1)(m+q+1)!}, & m \geq 3, \end{cases}$$

for $q \geq 0$, and, for $q \geq 1$,

$$(59b) \quad \alpha_{2q,4}^- = \begin{cases} \frac{q(m^2-3m-mq+3q+4)(m-q-2)!}{16^{q+2}(m^2-1)(m-1)(q+1)(m-1)!}, & m \neq 2q + 2, \\ \frac{2q^5+7q^4-16q^2-14q-3}{16^{q+2}q(q+1)(q+2)(2q+1)^2(2q+3)(2q+1)!}, & m = 2q + 2. \end{cases}$$

As in Section 2, the analytic evaluation of $v_{2k}^{\pm}(2q)$ becomes lengthier and more tedious as k increases. By using (25), we have found also the analytic expressions for $v_6^{\pm}(2q)$ ($k = 3$) in the case of the odd functions. These expressions are given in the Appendix.

The normalization used for the calculation of the explicit values of B 's is [14] $d \text{So}_m(h, 1)/d\theta = 1$, or, equivalently,

$$(60) \quad \sum_{n=1}^{\infty} n B_n^o(h, m) = 1,$$

with n and m both even or both odd. Substituting (4) (with e replaced by o) into (60), we obtain

$$(61) \quad B_m^o(h, m) \left\{ m + \sum_{q=1}^{\infty} \sum_{k=0}^{\infty} [(m+2q)\alpha_{2q,2k}^+ + (m-2q)\alpha_{2q,2k}^-] h^{2q+2k} \right\} = 1.$$

The expansion of $B_m^o(h, m)$ for small values of h has the expression

$$(62) \quad B_m^o(h, m) = \frac{1}{m} + g_2 h^2 + g_4 h^4 + g_6 h^6 + \dots$$

Substituting (62) into (61), we follow the same procedure as for the even functions, and finally get

$$(63) \quad g_{2s} = -\frac{1}{m} \sum_{q=1}^s \sum_{k=0}^{s-q} g_{2k} [(m+2q)\alpha_{2q,2(s-q-k)}^+ + (m-2q)\alpha_{2q,2(s-q-k)}^-], \quad s = 1, 2, \dots,$$

with $g_0 = 1/m$.

$So_m(h, \cos \theta)$ is given by (44), with \cos replaced by \sin . We recall that in each case $\alpha_{2q,2k}^- = 0$ ($k \geq 0$) if $1 \leq m \leq 2q + 1$.

The evaluation of the odd radial Mathieu functions $Zo_m(h, \cosh \mu)$ of any kind, which are solutions of (45) with $b = bo_m$, is made, in analogy to (46), by the formula [14]

$$(64) \quad Zo_m(h, \cosh \mu) = \sqrt{\frac{\pi}{2}} \tanh \mu \sum_{n=1}^{\infty} (-1)^{(n-m)/2} n B_n^o(h, m) Z_n(h \cosh \mu).$$

4. RELATIONS AMONG THE EXPANSION COEFFICIENTS FOR EVEN AND ODD FUNCTIONS

In this section we shall use the superscripts e and o to distinguish the various expansion coefficients used for the even and odd functions.

By comparing the expansion coefficients $l_{2j}^{e(o)}$, given in [13], [15], [17] for $j \leq 8$, we can easily find the following relations connecting them (their validity for general j will be proved at the end of this section):

$$(65) \quad \begin{aligned} l_{2j}^e &= l_{2j}^o = 0, & j \text{ odd} & & \left\{ \begin{array}{l} m \geq 0 \text{ (} m \text{ even)} \\ m > j \text{ (} m \text{ odd)} \end{array} \right\}, & (a) \\ l_{2j}^e &= (-1)^j l_{2j}^o, & m \text{ odd}, & & (b) \\ l_{2j}^e &= l_{2j}^o, & j \text{ even} < m \text{ even}, & & (c) \end{aligned}$$

From (20a), (51a), (34a), (54a), as well as (A1) and its analog for odd functions given in the Appendix, we observe that for $k = 1, 2, 3$

$$(66) \quad v_{2k}^{o+}(2q) = (-1)^k v_{2k}^{e+}(2q), \quad \left\{ \begin{array}{l} m = 1, 3, \dots, k \text{ (} k \text{ odd)}, \\ m \neq 2, 4, \dots, k \text{ (} k \text{ even)}, \end{array} \right.$$

while for k odd and m even or m odd $> k$, (66) is also valid, but with both v 's equal to zero, as will be proved later in this section. We suppose now that (66) is valid for $k = 1, 2, \dots, r$, and we shall prove it for $k = r + 1$. We use (25), with $k = r + 1$, for odd functions:

$$(67) \quad v_{2r+2}^{o+}(2q) = \sum_{i=1}^q \left[g^{o+}(2i) v_{2r}^{o+}(2i) + t^{o+}(2i) \sum_{j=2}^{r+1} l_{2j}^o v_{2r+2-2j}^{o+}(2i) + p^{o+}(2i+2) v_{2r-2}^{o+}(2i+2) f^{o+}(2i) \right].$$

Using the relations $g^{o+} = -g^{e+}$, $t^{o+} = t^{e+}$, $p^{o+} = p^{e+}$, $f^{o+} = f^{e+}$ ($m \geq 1$), as well as (65) and (66) in (67), we very easily prove (66) for $k = r + 1$ and m odd (r even or odd). If m is even, $g^{o+} = g^{e+} = 0$; in this case for j odd, $l_{2j}^o = l_{2j}^e = 0$, while for j even $l_{2j}^o = l_{2j}^e$, if $j \leq r + 1 < m$ (see (65)). Using (66), as

well as the aforementioned relations in (67), we also prove (66) for $k = r + 1$ and $m = \text{even} > r + 1$ (r even or odd).

From (20b), (51b), (34b), (54b), as well as (A2) and its analog for odd functions given in the Appendix, we observe that for $k = 1, 2, 3$

$$(68) \quad v_{2k}^{o-}(2q) = (-1)^k v_{2k}^{e-}(2q), \quad \begin{cases} m = 2q + 1, 2q + 3, \dots, 2q + k \text{ (} k \text{ odd),} \\ m \neq 2q + 2, 2q + 4, \dots, 2q + k \text{ (} k \text{ even),} \end{cases}$$

while for k odd and m even ($> 2q + 1$), or m odd ($> 2q + k$), (68) is also valid, with both v 's equal to zero, as will be proved in what follows.

Following the same procedure as in the previous case (with the superscript $+$ replaced by $-$), we prove (68) very easily for $k = r + 1$ when m is odd, or when m is even $> 2q + r + 1$ (r even or odd in both cases).

Next we examine the relation

$$(69) \quad v_{2k}^{e\pm}(2q) = 0 \quad k \text{ odd}, \quad \begin{cases} m \text{ even or } m \text{ odd} > k(+), \\ m \text{ even or } m \text{ odd} > 2q + k(-), \end{cases}$$

which is valid for $k = 1, 3$ (equations (20) and (A1), (A2)). By supposing that it is also valid for $k = 1, 3, \dots, r$, with r odd, we shall prove it for $k = r + 2$.

Substituting in (25), we obtain

$$(70) \quad v_{2r+4}^{e\pm}(2q) = \sum_{i=1}^q \left[g^{e\pm}(2i) v_{2r+2}^{e\pm}(2i) + t^{e\pm}(2i) \sum_{j=2}^{r+2} l_{2j}^e v_{2r+4-2j}^{e\pm}(2i) + p^{e\pm}(2i + 2) v_{2r}^{e\pm}(2i + 2) f^{e\pm}(2i) \right].$$

Using (13), (65a) and (69) in (70), we very easily prove (69) for $k = r + 2$ and m even, as well as for m odd, with $m > r + 2(+)$ or $m > 2q + r + 2(-)$.

Finally, for $v_{2k}^{o\pm}(2q)$, equations (69) and (70), as well as the aforementioned proof, remain the same, with the only difference that e is replaced by o .

Relations (66), (68), as well as (69) and its corresponding one for odd functions, remain valid if we replace $v_{2k}(2q)$ (with the proper superscripts in each case) by the corresponding $\alpha_{2q,2k}$ (see (21)), because $\alpha_{2q,0}^{e+} = \alpha_{2q,0}^{o+}$ for $m \geq 1$ (by (36a), (56a)) and $\alpha_{2q,0}^{e-} = \alpha_{2q,0}^{o-}$ for $m \geq 2q + 1$ (by (36b)), (56b)).

Now we turn to the proof of the relations (65) for general j . We suppose that they are valid for $j = 1, 2, \dots, r$, and we shall prove them for $j = r + 1$. Setting $k + 2 = r + 1$ ($k = r - 1$) in (7) and (50), we obtain, respectively,

$$(71) \quad l_{2r+2}^e = \frac{1}{4} \left(\alpha_{2,2r-2}^{e+} + \frac{2}{\varepsilon_{m-2}} \alpha_{2,2r-2}^{e-} \right),$$

$$(72) \quad l_{2r+2}^o = \frac{1}{4} (\alpha_{2,2r-2}^{o+} + \alpha_{2,2r-2}^{o-}).$$

If we use the remarks in the paragraph before (71), we get

$$(73) \quad l_{2r+2}^e = \frac{(-1)^{r-1}}{4} \left(\alpha_{2,2r-2}^{o+} + \frac{2}{\varepsilon_{m-2}} \alpha_{2,2r-2}^{o-} \right),$$

with the limitations given for (66) and (68). By comparing (72) and (73), with the use of these limitations, we prove (65) for $j = r + 1$ (r even or odd).

5. COMPARISONS WITH EXISTING DATA

In this section we discuss various checks and comparisons made to confirm the correctness of our results.

Formulas (36) and (56) are the same as the corresponding ones in [13], [15], [17] for each q and m . In the special cases with $m = 0 - 2$, (38), (58), as well as (39), (59) give the results found in [13], [15], [16]. The orders of the first terms omitted there can also be confirmed by the use of (A1), (A2) and the analogous formulas for odd functions.

Using (21), (36), (38), (39) and (A1), (A2), all with $q = 1$, in (7) we obtain, for each value of m , the expansion coefficients l_4, l_6, l_8 and l_{10} (with $k = 0, 1, 2$ and 3 , respectively) of the characteristic values be_m , which are also given in [13], [15], [17]. The same is valid also for bo_m , by using (21), (56), (58), (59) and the corresponding versions of (A1), (A2) for odd functions, in (50). For $m = 0, \dots, 4$ our coefficients are the same as the ones given in [26], [27].

Finally, by using (43) and (44) we verify the expansions for Se_m in the special cases with $m = 0, \dots, 4$ examined in [14], [26], [27]. The same is valid also for So_m ($m = 1, \dots, 4$) [14], [26], [27] by virtue of (63) and the formula corresponding to (44) for odd functions.

It is obvious that our results are much more general than the ones used above for the confirmation of their validity. The latter constitute special cases of our formulas, for $m = 0, \dots, 4$ (with the exception of those corresponding to l_{2k+4} , $k = 0, \dots, 3$, which are valid for each m).

It should be noticed here that (5) can also be used for the evaluation of the initial estimates for be_m and bo_m , in the iterative algorithms described in [25] for their computation.

APPENDIX

The analytic expressions for $v_6^\pm(2q)$ ($k = 3$) obtained from (25) after some manipulation are the following.

1) **Even functions.** For $q \geq 0$ we have

$$(A1) \quad v_6^+(2q) = \begin{cases} -\frac{q(3q+8)}{32768(q+2)^2}, & m = 1, \\ \frac{q(11q^2+48q+49)}{294912(q+1)(q+2)(q+3)}, & m = 3, \\ 0, & m \neq 1, 3. \end{cases}$$

Moreover,

$$(A2) \quad v_6^-(2q) = \begin{cases} -\frac{1}{32768}, & q = 1, m = 3, \\ \frac{q^2-q+2}{16384(q^2-1)q^3(q+2)}, & m = 2q + 1, q \geq 2, \\ \frac{1}{4096q(q+1)^2(q+2)^2(q+3)}, & m = 2q + 3, q \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For the evaluation of (A1) and (A2) we have used l_4 from (33), while

$$(A3) \quad l_6 = \begin{cases} -1/4096, & m = 1, \\ 1/4096, & m = 3, \\ 0, & m \neq 1, 3, \end{cases}$$

calculated from (7) and (38) with $q = 1$.

2) **Odd functions.** For the odd functions, $v_6^+(2q)$, $v_6^-(2q)$ and l_6 are opposite to the corresponding ones for the even functions given in (A1)–(A3).

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