

ON ITERATES OF MÖBIUS TRANSFORMATIONS ON FIELDS

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ABSTRACT. Let p be a quadratic polynomial over a splitting field K , and S be the set of zeros of p . We define an associative and commutative binary relation on $G \equiv K \cup \{\infty\} - S$ so that every Möbius transformation with fixed point set S is of the form x “plus” c for some c . This permits an easy proof of Aitken acceleration as well as generalizations of known results concerning Newton’s method, the secant method, Halley’s method, and higher order methods. If K is equipped with a norm, then we give necessary and sufficient conditions for the iterates of a Möbius transformation m to converge (necessarily to one of its fixed points) in the norm topology. Finally, we show that if the fixed points of m are distinct and the iterates of m converge, then Newton’s method converges with order 2, and higher order generalizations converge accordingly.

Consider the Fibonacci sequence $F_1 = F_2 = 1$ and, for $n \geq 1$, $F_{n+2} = F_{n+1} + F_n$. It is known that the ratios $r_n \equiv F_{n+1}/F_n$ converge (and in fact are continued fraction convergents) to the “golden ratio” $\frac{1+\sqrt{5}}{2}$ (see, for example, [1], [2]). If $m(x) = 1 + 1/x$, then the sequence (r_n) satisfies the recursion $r_{n+1} = m(r_n)$ and so, letting n approach ∞ , the iterates of m converge to a fixed point of m . We associate to m its *characteristic polynomial* $\theta(x) \equiv x^2 - x - 1$ (the monic polynomial whose zeros are the fixed points of m). Hence the iterates of m (starting with 1) converge to a zero of θ . Iteration by Newton’s method (applied to θ and starting with 1) also converges to this zero and, in fact, gives the 1st, 2nd, 4th, 8th, 16th, \dots iterates of m (see [2]). Iteration by the secant method (applied to θ and starting with 1,2) gives the 1st, 2nd, 3rd, 5th, 8th, 13th, \dots iterates of m (see [2]).

This paper grew out of an attempt to understand these and other (known [1], [2], [3], [4], [5]) phenomena. We shall first generalize some of the results of [2], [3], [4], [5] to the case where m is a Möbius transformation (i.e., function of the form $x \mapsto \frac{ax+b}{cx+d}$) where a, b, c, d, x are elements of an arbitrarily chosen field K . We shall generalize results of [2], [4] to our case as well as introduce generalizations of Newton’s method. We shall also derive a generalization of *Aitken acceleration* (a main result of [3], [5]). Our proofs are different (and perhaps simpler) than the extant proofs.

Next, we shall assume that K is equipped with a norm or absolute value and that K contains the fixed points of m . We give necessary and sufficient conditions for the iterates of m (with a given starting point) to converge to a given fixed

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point of m . Given convergence of the iterates of m , we show that Newton's method converges quadratically and that its generalizations converge with correspondingly high order.

Let $\theta(x) \equiv x^2 - ax - b$. We shall assume that K contains the two zeros (ξ_1 and ξ_2) of θ ; we allow for the possibility that $\xi_1 = \xi_2$. For any c , it is easily seen that θ is the characteristic polynomial of the Möbius transformations

$$(1) \quad m(x) = \frac{cx + b}{x - a + c} \quad \text{and} \quad m^{-1}(x) = \frac{(a - c)x + b}{x - c}.$$

As in the real case, we introduce two conditions regarding ∞ : $m(\infty) = c$ and $m(a - c) = \infty$. Let $G = K \cup \{\infty\} - \{\xi_1, \xi_2\}$. Given any r_0 , we can form a sequence (r_n) in *both* directions:

$$(2) \quad r_{n+1} = m(r_n), \quad r_{n-1} = m^{-1}(r_n).$$

Interestingly, the numbers r_n are ratios of "generalized Fibonacci numbers". Given initial values $G_0 = 1$ and $G_1 = r_0$, define

$$G_{n+2} = \frac{cG_{n+1} + bG_n}{G_{n+1} + (c - a)G_n}.$$

Note that when $c = a$, $G_{n+2} = aG_{n+1} + bG_n$. In any case, $r_0 = G_1/G_0$ and, if $r_n = G_{n+1}/G_n$, then $r_{n+1} = m(r_n) = m(G_{n+1}/G_n) = G_{n+2}/G_{n+1}$ so that $r_n = G_{n+1}/G_n$ for all n .

Given $x, y \in G$, let

$$x \oplus y = \frac{xy + b}{x + y - a}.$$

Here the conventions regarding ∞ are $x \oplus (a - x) = \infty$ and $x \oplus \infty = x$. Although it is clear that the binary relation \oplus is commutative, it is perhaps less clear that it is associative. In the real case, it is a challenging problem to show *geometrically* that this is so. (The connection to geometry in this case is that the line through $(x, \theta(x))$ and $(y, \theta(y))$ has the x -intercept $x \oplus y$.)

Theorem 1. *The relation \oplus is associative.*

Proof. Given any two-by-two matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define an associated Möbius transformation $\Phi_A(x) = \frac{ax+b}{cx+d}$. It is well known (and easy to verify) that the composition of such functions corresponds to matrix multiplication; that is, $\Phi_A \circ \Phi_B = \Phi_{AB}$.

Let $M = \begin{pmatrix} 0 & b \\ 1 & -a \end{pmatrix}$. Note that $\Phi_{M+xI}(y) = x \oplus y$. Hence

$$(x \oplus y) \oplus z = \Phi_{M+zI}(x \oplus y) = \Phi_{M+zI}(\Phi_{M+xI}(y)) = \Phi_{(M+zI)(M+xI)}(y).$$

Since $M + zI$ and $M + xI$ commute, we have $(x \oplus y) \oplus z = (z \oplus y) \oplus x$. \square

We remark that by this theorem (G, \oplus) is an abelian group with identity ∞ such that $a - x$ is the inverse of x . By the proof above, $(M + xI)(M + yI)$ is a scalar multiple of $M + (x \oplus y)I$, and so the map $x \mapsto M + xI$ is a projective representation of G into $GL_2(K)$.

We let $x^{\oplus n}$ denote the n -fold "sum" of x (i.e., $x^{\oplus 1} = x$ and $x^{\oplus(n+1)} = x \oplus x^{\oplus n}$). This definition extends to any integer n by $x^{\oplus 0} = \infty$ and $x^{\oplus(-n)} = (a - x)^{\oplus n}$. Note that, by (1),

$$m(x) = x \oplus c \quad \text{and} \quad m^{-1}(x) = x \oplus (a - c).$$

Therefore, with r_k defined above,

$$(3) \quad r_{n+k} = r_k \oplus c^{\oplus n}$$

for any n and k . By associativity and commutativity,

$$\frac{r_{l-i}r_i + b}{r_{l-i} + r_i - a} = r_{l-i} \oplus r_i = r_k \oplus r_k \oplus c^{\oplus(l-2k)} = r_k \oplus r_{l-k}.$$

Hence, for any i and j

$$\frac{r_{l-i}r_i + b}{r_{l-i} + r_i - a} = \frac{r_{l-j}r_j + b}{r_{l-j} + r_j - a}.$$

Since they are equal, they are also equal to the ratio of differences (i.e., if $A/B = C/D$, then $A/B = (A - C)/(B - D)$) and we have

Theorem 2. *For all i, j and k such that the denominator of the fraction below is nonzero,*

$$\frac{r_{l-i}r_i - r_{l-j}r_j}{r_{l-i} + r_i - r_{l-j} - r_j} = r_k \oplus r_{l-k}.$$

Note that if $r_1 = c$ (equivalently, $r_0 = \infty$), then $r_n = c^{\oplus n}$. We shall use this in the next four results. The reader is invited to extend those results to the case when $r_1 \neq c$. The following is a generalization of the Aitken acceleration formula (see [3], [5])

Corollary 3. *If $r_1 = c$, then for all n and l ,*

$$\frac{r_{n+l}r_{n-l} - r_n^2}{r_{n+l} - 2r_n + r_{n-l}} = r_{2n}.$$

Proof. Replace i, j, k , and l in Theorem 2 by $n - l, n, l$, and $2n$, respectively. □

When $K = \mathbb{R}$, Newton’s method to approximate the zeros of θ is, given a starting point t_0 , to define a sequence inductively

$$t_{n+1} = t_n - \frac{\theta(t_n)}{\theta'(t_n)},$$

which converges (in many cases) to a zero of θ . In our case, this boils down to

$$t_{n+1} = \frac{t_n^2 + b}{2t_n - a} = t_n \oplus t_n.$$

We take this to be the definition of Newton’s method in the general case.

If we take $t_0 = c$, then a simple induction argument shows that $t_n = c^{\oplus 2^n}$. If (r_n) is defined as in (2) above, then $r_n = r_0 \oplus c^{\oplus n}$ and we have

Theorem 4. *If $t_0 = c$, then $t_n = (a - r_0) \oplus r_{2^n}$.*

One may generalize further. Let $g^{(n)}(x) = x^{\oplus n}$. For example,

$$g^{(3)}(x) = \frac{x^3 + 3bx - ab}{3x^2 - 3ax + b + a^2}, \quad g^{(4)}(x) = \frac{x^4 + 6bx^2 - 4abx + b(a^2 + b)}{4x^3 - 6ax^2 + 4(a^2 + b)x + a^3 - 2ab}.$$

Iteration of $g^{(3)}$ is Halley’s method applied to θ . The rational functions $g^{(n)}$ appear, with different notation, in [4]. In that paper, a larger family of iterative procedures is introduced; our family is that of [4] when the parameter d introduced there is 1. This will be clear from a closed form for $g^{(n)}$ in terms of the numbers u_n defined

by $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = au_{n+1} + bu_n$. Note that these numbers are a special case of the “generalized Fibonacci numbers” G_n introduced earlier.

We define polynomials P_n and Q_n to be the unique polynomials satisfying $x^{\oplus n} = P_n(x)/Q_n(x)$, where P_n is monic and of minimal degree.

Hence $P_0(x) = 1$ and $Q_0(x) = 0$. Letting M be the matrix in the proof of Theorem 1, note that

$$\begin{pmatrix} P_{n+1}(x) \\ Q_{n+1}(x) \end{pmatrix} = (M + xI) \begin{pmatrix} P_n(x) \\ Q_n(x) \end{pmatrix}.$$

Let $\begin{pmatrix} v_n \\ w_n \end{pmatrix} = (-1)^n M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then

$$\begin{pmatrix} P_n(x) \\ Q_n(x) \end{pmatrix} = (M + xI)^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sum_{k=0}^n \binom{n}{k} (-1)^k x^{n-k} \begin{pmatrix} v_k \\ w_k \end{pmatrix}.$$

Since M satisfies its characteristic equation, $M^2 + aM - bI = 0$ and thus (v_n) and (w_n) satisfy the difference equation $x_{n+2} = ax_{n+1} + bx_n$. Since $v_0 = 1$, $w_0 = 0 = v_1$, and $w_1 = -1$, we may write $v_n = bu_{n-1}$ and $w_n = -u_n$. Hence,

Proposition 5. $x^{\oplus n} = P_n(x)/Q_n(x)$, where $P_n(x) = b \sum_{k=0}^n \binom{n}{k} x^{n-k} (-1)^k u_{k-1}$ and $Q_n(x) = - \sum_{k=0}^n \binom{n}{k} x^{n-k} (-1)^k u_k$.

Iterates of $g^{(k)}$ give an exponential subsequence of iterates of m . Let $t_{n+1}^{(k)} = g^{(k)}(t_n^{(k)})$. As for Theorem 4,

Theorem 6. If $t_0 = c$, then $t_n^{(k)} = (a - r_0) \oplus r_{k^n}$.

The secant method is, given two starting points s_0 and s_1 , to construct a sequence defined by,

$$s_{n+1} = s_n - \frac{\theta(s_n)(s_n - s_{n-1})}{\theta(s_n) - \theta(s_{n-1})},$$

which, in our case, boils down to

$$s_{n+1} = \frac{s_n s_{n-1} + b}{s_n + s_{n-1} - a} = s_n \oplus s_{n-1}.$$

As above, we take this to be the definition of the secant method in the general case.

The Fibonacci sequence (F_n) defined at the beginning shows up in a perhaps surprising way (see [2]).

Theorem 7. $s_n = s_0^{\oplus F_{n-1}} \oplus s_1^{\oplus F_n}$.

Proof. Taking $F_{-1} = 1$ and $F_0 = 0$, the theorem clearly holds for $n = 0, 1$. Supposing it holds for all $k \leq n$, $s_{n+1} = s_n \oplus s_{n-1} = s_0^{\oplus F_{n-1}} \oplus s_1^{\oplus F_n} \oplus s_0^{\oplus F_{n-2}} \oplus s_1^{\oplus F_{n-1}} = s_0^{\oplus F_n} \oplus s_1^{\oplus F_{n+1}}$. By induction, the theorem is proven. \square

To discuss convergence, we assume that K has a topology defined by a norm (or absolute value) $|\cdot|$. That is, for all $x, y \in K$,

- a) $|x| = 0$ if and only if $x = 0$,
- b) $|x + y| \leq |x| + |y|$, and
- c) $|xy| = |x||y|$.

If $r_n \rightarrow \xi$, then, by (2) and the definition of absolute value, $\xi = m(\xi)$ and so ξ must be a zero of θ . We still assume then that the zeros $(\xi_1$ and $\xi_2)$ of θ are in K .

We now define a function on G . Let $f(\infty) = 1$ and, for $x \in K - \{\xi_1, \xi_2\}$,

$$f(x) = \left| \frac{x - \xi_1}{x - \xi_2} \right|.$$

Lemma 8. For all $x, y \in K - \{\xi_1, \xi_2\}$,

$$f(x \oplus y) = f(x)f(y).$$

Proof. Since $x^2 - ax - b = (x - \xi_1)(x - \xi_2)$, we have $\xi_1 + \xi_2 = a$ and $\xi_1\xi_2 = -b$. Hence, $z = x \oplus y = \frac{xy - \xi_1\xi_2}{x + y - \xi_1 - \xi_2}$, which implies

$$\frac{z - \xi_1}{z - \xi_2} = \frac{xy - (x + y)\xi_1 + \xi_1^2}{xy - (x + y)\xi_2 + \xi_2^2}.$$

Taking the absolute value of both sides,

$$f(z) = \left| \frac{(x - \xi_1)(y - \xi_1)}{(x - \xi_2)(y - \xi_2)} \right| = f(x)f(y).$$

□

We remark that f is a group homomorphism from (G, \oplus) into the group of positive real numbers under multiplication. If $K = \mathbb{R}$, then G is an example of a disconnected Lie group and f is two-to-one.

We are now able to say some things about the convergence of (r_n) .

Theorem 9. Let $m(z) = z \oplus c$ and m_n be the n -th iterate of m .

a) If $|c - \xi_1| > |c - \xi_2|$, then, for $z \neq \xi_1$, $m_n(z)$ converges to ξ_2 in the norm topology.

b) If $|c - \xi_1| = |c - \xi_2|$ but $\xi_1 \neq \xi_2$, then, for all $z \notin \{\xi_1, \xi_2\}$, $m_n(z)$ does not converge.

c) If ξ is the only zero of θ , then, for all $z \neq \xi$, $m_n(z)$ converges to ξ if and only if K is Archimedean (i.e., $\lim_{n \rightarrow \infty} |\tilde{n}| = \infty$ where \tilde{n} denotes the n -fold sum of the unit in K).

Proof. a) If $|c - \xi_1| > |c - \xi_2|$, then $f(c) > 1$ and, by Lemma 8 and induction, $f(m_n(z)) = f(z)f(c)^n$. Unless $f(z) = 0$ (equivalently, $z = \xi_1$), $f(m_n(z)) \rightarrow \infty$. Since f is bounded outside any neighborhood of ξ_2 (triangle inequality), the result follows.

b) If $|c - \xi_1| = |c - \xi_2|$, then $f(c) = 1$ and so, for any $z \notin \{\xi_1, \xi_2\}$, $f(m_n(z))$ is nonzero and independent of n . Since $m_n(z)$ can converge only to ξ_1 or ξ_2 (in which case $f(m_n(z))$ would converge to 0 or ∞), the result follows.

c) If ξ is the only zero of θ , then $x \oplus y = \frac{xy - \xi^2}{x + y - 2\xi}$ and a simple calculation gives

$$\frac{1}{x \oplus y - \xi} = \frac{1}{x - \xi} + \frac{1}{y - \xi}.$$

Hence, by induction,

$$\left| \frac{1}{m_n(x) - \xi} - \frac{1}{x - \xi} \right| = \frac{|\tilde{n}|}{|c - \xi|},$$

and the result follows.

□

We say x_n converges to x with order k if $\frac{|x_{n+1}-x|}{|x_n-x|^k}$ converges to a nonzero constant. For example, we shall see that Newton's method converges with order two and Halley's method converges with order three.

Let $t_n^{(k)}$ be defined as above.

Theorem 10. *If θ has distinct zeros and $r_n \rightarrow \xi$, then $t_n^{(k)} \rightarrow \xi$ with order k .*

Proof. Suppose $r_n \rightarrow \xi$. We write $f(x) \asymp g(x)$ if $\lim_{x \rightarrow \xi} \frac{f(x)}{g(x)}$ exists and is nonzero.

Suppose y depends on x and $y \rightarrow \xi$ as $x \rightarrow \xi$. Using the fact that $\xi^2 = a\xi + b$,

$$(x - \xi)(y - \xi) = xy + b - (x + y - a)\xi = (x + y - a)(x \oplus y - \xi).$$

Since the zeros of θ are assumed distinct, $2\xi \neq \xi_1 + \xi_2 = a$ and so $|x + y - a| \asymp 1$. Hence $|x - \xi||y - \xi| \asymp |x \oplus y - \xi|$. If $|y - \xi| \asymp |x - \xi|^k$, then $|x \oplus y - \xi| \asymp |x - \xi|^{k+1}$ and so, by induction,

$$|x^{\oplus k} - \xi| \asymp |x - \xi|^k.$$

The result follows. □

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