

## BREEDING AMICABLE NUMBERS IN ABUNDANCE. II

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ABSTRACT. In a first article of this title, new procedures were described to compute many amicable numbers by “breeding” them in several generations. An extensive computer search was later performed (in 1988), and demonstrated the remarkable effectiveness of this breeding method: the number of known amicable pairs was easily quadrupled by this search. As we learnt recently (1999) from the internet, Pederson and te Riele have again multiplied that number roughly by ten. While they give no information on their method of search, we publish here our method and summarize the computations. Our results provide some evidence for the conjecture that the number of amicable pairs is infinite.

### 1

Two numbers  $n$  and  $m$  are amicable, iff the sum of the proper divisors of  $n$  is  $m$  and conversely. The first example, 220 and 284, is attributed to Pythagoras.

The method of bilinear diophantine equations (BDE-method) was used in the past by E.J. Lee [5] and many others, in special cases already by L. Euler around 1750, to produce many amicable pairs of the form  $a_1r_1r_2$ ,  $a_2r_3$  from suitable inputs  $a_1, a_2$ . Here  $r_1 \neq r_2, r_3$  are three primes prime to  $a_1, a_2$ .

The great art of the great number hunters like Euler, Poulet, Escott, Lee was to find suitable inputs  $a_1, a_2$ .

### 2

The BDE-method (explained below) is modified and extended here to the “breeding method” [2]. Here a *breeder* is a pair of positive integers  $a_1, a_2$  such that

$$\sigma(a_1)(r_1 + 1) = a_1r_1 + a_2r_2 = \sigma(a_2)(r_2 + 1)$$

has a solution in positive integers  $r_1, r_2$ . Here  $\sigma$  denotes the sum of divisors. Note that if  $r_1, r_2$  are primes prime to  $a_1, a_2$ , then these equations mean that  $a_1r_1, a_2r_2$  are amicable.

Obviously for the experts, breeders are very good *inputs* for the BDE-method. But another point of the breeder method is that the same BDE-method also yields new breeders as *output* as a by-product. So the procedure can be iterated to produce various generations of breeders, and simultaneously, amicable number pairs are produced as the usual output of the BDE-method. The overwhelming success of numerical examples (see below) demonstrates how effective this breeder method is producing breeders and simultaneously amicable pairs in abundance.

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In more detail following [2, 2.1], for an input pair  $a_1, a_2$ , e.g., a breeder of generation  $n$ , the following procedure is carried out: If the determinant  $D := a_1 a_2 - (\sigma(a_1) - a_1)(\sigma(a_2) - a_2)$  is positive, the number

$$C := \sigma(a_2) (D(a_2 - a_1) + a_1^2 \sigma(a_2))$$

(which coincides with [2, 2.1, (21)], up to change of notation) is computed and completely factorized. Common divisors like the greatest common divisors  $(a_1, \sigma(a_1))$  and  $((\sigma(a_1), \sigma(a_2)))$  are cancelled in the sequel. The cancellation is indicated by a bar “ $\overline{\phantom{x}}$ ”. If  $(a_1, \sigma(a_1))$  does not divide  $a_2$ , or if  $(a_2, \sigma(a_2))$  does not divide  $a_1$ , then the start-value  $(a_1, a_2)$  is “sterile” in the sense that it gives no amicable pairs. So in this case, the procedure is stopped at this point. For each factorization  $C = d_1 d_2$  the numbers

$$r_i := (a_1 \sigma(a_2) + d_i) / D - 1$$

are computed for  $i = 1, 2$  and, if integer, are tested for primality. Whenever  $r_1$  is prime and prime to  $a_1$ , then a *breeder*  $(a_1 r_1, a_2)$  of the next generation  $n + 1$  is found. These are the first outputs. Whenever  $r_1, r_2$  and

$$r_3 := \sigma(a_1)(r_1 + 1)(r_2 + 1) / \sigma(a_2) - 1$$

are simultaneously prime, different, and prime to  $a_i$ , then an *amicable pair*  $(a_1 r_1 r_2, a_2 r_3)$  of generation  $n$  is found. This gives the second outputs. After finishing this procedure, the same is carried out for the input  $a_2, a_1$  (in reversed order). Then the computation of generation  $n + 1$  is completed. Next the generation  $n + 2$  is computed in the same way from all breeders of generation  $n + 1$  as new inputs, as far as possible (i.e., as far as no overflows occur). This procedure is iterated as far as possible.

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For example, take start-values  $(a_1, a_2) = (4, 4)$ . Then, as the first generation of breeders, the four pairs  $(4, 4p)$  with  $p = 5, 7, 11, 19$  result. These breed 18 breeders in second generation, viz.

- $(4, 4 \cdot 5 \cdot 11)$     $(4, 4 \cdot 5 \cdot 13)$     $(4, 4 \cdot 5 \cdot 101)$     $(4 \cdot 5, 4 \cdot 13)$     $(4 \cdot 5, 4 \cdot 17)$
- $(4 \cdot 5, 4 \cdot 19)$     $(4 \cdot 5, 4 \cdot 23)$     $(4 \cdot 5, 4 \cdot 43)$     $(4 \cdot 5, 4 \cdot 59)$     $(4 \cdot 5, 4 \cdot 107)$
- $(4 \cdot 7, 4 \cdot 11)$     $(4 \cdot 11, 4 \cdot 11)$     $(4 \cdot 11, 4 \cdot 13)$     $(4 \cdot 11, 4 \cdot 17)$     $(4 \cdot 11, 4 \cdot 29)$
- $(4 \cdot 11, 4 \cdot 37)$     $(4 \cdot 11, 4 \cdot 53)$     $(4 \cdot 11, 4 \cdot 101)$

Again, these 18 breeders lead to 56 breeders in third generation. We further computed completely the fourth and fifth generation and, not quite completely (because of overflows), the sixth generation. The results follow.

generation:	0	1	2	3	4	5	6
number of breeders:	4	18	56	208	874	$\geq 4686$	
number of amicable pairs found:	1	2	1	3	8	$\geq 2$	$\geq 0$

Note that the list of 17 amicable pairs found here begins with Pythagoras’ pair  $(220, 284)$  which occurs in generation 0. Note also that the number of breeders seems to grow exponentially, while this is not yet clear for the number of amicable pairs found.

## 5

The breeder method can be started with any number pair  $a_1, a_2$ , but only a few are successful. In a first extensive search, start-values  $a_1 = a_2 = a$  were chosen. In the range  $a \leq 50\,000\,000$  only those  $a$  were taken, for which the “cancelled determinant”  $\overline{D} = (2a - \sigma(a)) / (a, \sigma(a))$  had a small value  $< 20$ . There are exactly 376 such numbers  $a$  which produce at least one breeder in the second generation. For each of these 376 start-values, the above breeder algorithm was applied as far as no overflows (see below) occurred, usually up to the fourth or third generation. For each of these 376 start-values, the breeders and amicable pairs were (computed and) counted in each generation. These numbers were tabulated in the first author’s Diplom–Arbeit [1] (unpublished). The table is too extensive to be reproduced here, but we do give these data in Table 1 for those 20 start-values  $a$  which were most productive, i.e., which bred the largest numbers of amicable pairs. The figures in the table indicate that for these start-values the number of resulting breeders, resp. amicable pairs, seem often to grow explosively (exponentially or even more) from generation to generation. We expect that this tendency would continue, if one could compute some even higher generations.

Note that the higher generations are only partially computed here because of overflows. The general limit for numbers processed in our search was  $10^{29}$  (but the resulting amicable pairs were often larger). Whenever this limit was exceeded, e.g., by  $\overline{C}$  or  $r_1, r_2, r_3$ , the algorithm was stopped and the search was continued with the next items. Factorizations of  $\overline{C}$  were fully carried out only up to prime factors  $\leq 10^8$ , so sometimes some factorizations may have been missed by overflow of this limit.

## 6

From Table 1 we see that the most successful start-value was  $a = 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 79$ , which produced 5428 amicable pairs by our algorithm, that is more than 10% of the total number of amicable pairs known then (in 1988). Of these 5428 pairs, 5287 were produced by only five breeders.

**Example.** The champion breeder was

$$(a_1, a_2) = (3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 79, \quad 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 79 \cdot 1013 \cdot 6180283 \cdot 2091919367).$$

In the BDE-method for this input, one had to factorize

$$\overline{C} = 2^{19} \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11^3 \cdot 13^4 \cdot 19 \cdot 23^4 \cdot 31^2 \cdot 79 \cdot 89^2 \cdot 197^2$$

into two factors. Of the 4 320 000 factorizations, 4 083 964 could be fully processed and led to 2910 amicable pairs!

## 7

The whole search for the above-mentioned 376 start-values  $a$  produced a total of almost a million breeders (most of which are too large for the amicable pair search within our limits) and a total of 26 684 amicable pairs. Note that more than half of these (13 996) were produced by the 20 best start-values as given in Table 1. Together with other searches, which we do not describe here in detail, a total of 42 900 amicable pairs were bred. These pairs were listed and sorted, and compared with te Riele’s list of all ca. 13 760 amicable pairs known in 1987 (most of them in [3]). As a result, a total of 37 803 of those amicable pairs were new in 1988 (see Battiato’s Diplom–Arbeit [1]).

TABLE 1. The 20 best inputs  $a \leq 5 \cdot 10^7$ , with  $\overline{D} < 20$ , producing the most amicable pairs by our iterated breeding process as described in the text. For each generation (0 to 4) the upper entry given is the number of breeders and the lower entry is the number of amicable pairs produced. A “ $\geq$ ” sign indicates that this generation was only partially processed because of overflows. A “-” sign means the generation was not computed.

No.	input $a$	$\overline{D}$	number of breeders amicable pairs per generation				total number of amicable pairs produced
			0	1	2	3	
1.	$3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 79$	6	4	164	$\geq 5113$	---	$\geq 5428$
			0	2	$\geq 24$	$\geq 5402$	---
2.	$3 \cdot 5^2 \cdot 7 \cdot 19 \cdot 181$	2	13	943	$\geq 8565$	---	$\geq 1453$
			1	4	$\geq 27$	$\geq 1421$	---
3.	$3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$	1	15	637	$\geq 14421$	---	$\geq 689$
			1	6	$\geq 83$	$\geq 599$	---
4.	$3^2 \cdot 5^2 \cdot 11 \cdot 43$	2	8	137	$\geq 6953$	---	$\geq 623$
			0	1	$\geq 34$	$\geq 588$	---
5.	$3^2 \cdot 5 \cdot 7 \cdot 137 \cdot 547$	18	2	117	$\geq 1712$	---	$\geq 519$
			0	3	$\geq 10$	$\geq 506$	---
6.	$2^{10}$	1	9	315	$\geq 3218$	---	$\geq 488$
			0	6	$\geq 30$	$\geq 452$	---
7.	$2^4$	1	7	113	736	$\geq 9036$	$\geq 476$
			1	11	23	$\geq 166$	$\geq 275$
8.	$3^4 \cdot 5 \cdot 11 \cdot 59 \cdot 179$	2	1	20	$\geq 654$	---	$\geq 431$
			0	2	$\geq 4$	$\geq 425$	---
9.	$3^4 \cdot 5^3 \cdot 19 \cdot 53$	14	2	146	$\geq 2852$	---	$\geq 407$
			0	0	$\geq 7$	$\geq 400$	---
10.	$2^9 \cdot 1031$	1	6	336	$\geq 1516$	---	$\geq 374$
			0	0	$\geq 3$	$\geq 371$	---
11.	$3^6 \cdot 5 \cdot 11 \cdot 53$	2	15	671	$\geq 10156$	---	$\geq 360$
			2	0	$\geq 27$	$\geq 331$	---
12.	$3^4 \cdot 5 \cdot 11^3 \cdot 71$	2	6	150	$\geq 1564$	---	$\geq 360$
			0	2	$\geq 16$	$\geq 342$	---
13.	$2^7 \cdot 257$	1	7	194	$\geq 1500$	---	$\geq 347$
			0	0	$\geq 8$	$\geq 339$	---
14.	$2^8$	1	7	295	3954	---	$\geq 335$
			1	6	43	$\geq 285$	---
15.	$3^2 \cdot 5 \cdot 7$	2	8	133	1579	$\geq 17894$	$\geq 302$
			1	2	50	$\geq 185$	$\geq 64$
16.	$2^4 \cdot 41 \cdot 131$	1	1	30	$\geq 99$	---	$\geq 300$
			0	1	$\geq 3$	$\geq 296$	---
17.	$3^2 \cdot 5 \cdot 7 \cdot 113 \cdot 1319$	2	3	174	$\geq 18106$	---	$\geq 284$
			0	0	$\geq 284$	$\geq 0$	---
18.	$3^4 \cdot 5 \cdot 11 \cdot 79 \cdot 101$	2	4	328	$\geq 5274$	---	$\geq 276$
			0	3	$\geq 11$	$\geq 262$	---
19.	$2^6$	1	8	122	1101	$\geq 17404$	$\geq 275$
			0	1	28	$\geq 149$	$\geq 97$
20.	$2^3 \cdot 23 \cdot 47$	2	2	49	274	$\geq 9705$	$\geq 269$
			0	0	3	$\geq 265$	$\geq 1$
sum							13996

The complete list of new amicable pairs was then sent to Herman te Riele in Amsterdam, who used to keep track of all new discoveries all over the world for years. Te Riele confirmed the list. Meanwhile, the total of ca. 51 560 amicable pairs thus known in 1988 was again roughly multiplied by ten by Pederson and te Riele and others, as we learnt from the internet (August 1999), address: <http://www.vejlehs.dk/staff/jmp/aliquot/kwnap.htm>.

Let us mention that our breeder method was also of use in finding the first odd amicable pairs not divisible by three [4], although in this case the first difficulty was to find the common factor  $a$ . Meanwhile, even amicable numbers prime to 30 have been found (see the internet).

## 8

Table 1 provides some modest evidence for the following conjecture.

**Conjecture 1.** For some start-values  $a$  the number of breeders resp. amicable pairs produced by our algorithm in generation  $n$  increases at least exponentially with  $n$ .

This is a very specific version of saying:

**Conjecture 2.** The number of amicable pairs is infinite.

## REFERENCES

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