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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the American Mathematical Society classification scheme. The 2000 Mathematics Subject Classification can be found in print starting with the 1999 annual index of *Mathematical Reviews*. The classifications are also accessible from `www.ams.org/msc/`.

2[65N30, 75V05, 76M10]—*p- and hp-finite element methods. Theory and applications in solid and fluid mechanics*, by Ch. Schwab, Oxford Science Publications, Clarendon Press, 1998, xii+374 pp., hardcover, $85.00

Over the past half century, the finite element method has emerged as the method of choice for the numerical approximation of elliptic boundary value problems, particularly those arising in structural mechanics and particularly amongst the engineering community. Although mathematicians like to cite the work of Courant as representing the beginnings of the method, the major part of the credit must be attributed to figures from the engineering community, such as Clough, Irons, Taig, etc., who were responsible for producing highly innovative numerical procedures that paved the way for the development of computer codes that could tackle problems of industrial magnitude with comparatively modest computational power.

The finite element method produces an approximation based on piecewise polynomial approximation on an underlying mesh. As with many classical numerical methods, the basic premise behind the finite element method is that improved accuracy is sought through reduction of the mesh-size h through mesh refinement (adaptive or otherwise) based on low order piecewise polynomial approximation.

In the mid-seventies, an alternative strategy was investigated by Szabó and his group in St. Louis, based on fixing the underlying mesh and seeking convergence by increasing the degree p of the polynomials used to construct the approximation. Numerical experiments on linear elasticity problems showed that this p-version approach was often superior to the h-version of the finite element method. In particular, for problems with smooth solutions, the rate of convergence appeared to be *exponential* while for problems with corner singularities, the rate of convergence appeared to be *double* that of the h-version on a sequence of quasi-uniform meshes. In the first theoretical analysis of the p-version, Babuška, Szabó and Katz (1981) proved these conclusions to be true. The analysis of the singular case exploits the fact that the solutions of linear elasticity problems have a specific structure that lends itself to approximation by polynomials, resulting in the doubling of the rate of convergence.

Encouraged by these results, investigations were made into how the advantages of h and p refinement might be combined to best effect in the so-called hp-version. The first theoretical analysis of the hp-version was given by Babuška and Dorr (1981). The exponential convergence of the hp-version for linear elliptic problems with piecewise analytic data was established in a series of landmark papers by Guo and Babuška. The key to the analysis again lies in the fact that the solutions of such problems have a specific structure that lends itself to efficient approximation by appropriately constructed hp-finite element spaces. In particular, this body of work included the introduction of so-called *countably normed spaces* and the demonstration that the solutions of linear elliptic problems with piecewise analytic data belong to such spaces. Approximation theoretic results and guidelines for the construction of the *hp* spaces were developed for approximation of these countably normed spaces culminating in the proof of the exponential convergence.

The success of *hp*-methods for the approximation of singularities that arise in elliptic problems has led to investigations, particularly by Schwab and coworkers, into what advantages the methodology might offer for the resolution of other local features such as boundary layers. Schwab and Suri (1996) analysed the performance of the h, p, and hp versions for a singularly perturbed two-point boundary value problem and showed the possibility of obtaining *robust exponential convergence*, where the constants and rate of convergence are independent of the perturbation parameter. This was later extended to include model problems in two and three dimensions by Schwab's group at Zürich, including convection dominated problems.

Another important area where the *hp*-methodology offers significant advantages includes numerical approximation of dimensionally reduced models for elastic plates, where the so-called *locking effects* can render low order approximations practically useless. Here, higher-order methods result in the alleviation of locking without the need to resort to numerical remedies such as reduced integration. These and other advantages of the *hp*-methodology have led to a number of commercial finite element codes such as MSC Nastran, proPHLEX, PolyFEM, Pro/MECHANICA, STRESSCHECK and STRIPE, now providing facilities for p and hp refinement.

The present book by Ch. Schwab is the first to discuss the theoretical aspects of *hp* finite element methods in depth. Chapter 1 deals with the variational formulation of boundary value problems and starts out innocently enough with a second order self-adjoint problem in one space dimension. Nevertheless, by the end of the 42-page chapter, the reader will have seen the generalised Lax–Milgram lemma and its application to a number of variational formulations of the bar problem, and met the trace spaces $H^{1/2}$ and $H^{1/2}_{00}$.

A brief introduction to the finite element method in Chapter 2, is followed by Chapter 3 (just under 100 pages) consisting of a detailed treatment of the h, p, and *hp*-version applied to problems in one space dimension. The coverage includes the approximation theory for singular and boundary layer solutions, mesh grading, and a posteriori error estimation. The chapter begins with an elementary discussion of the computation of element matrices, but one soon reaches a level found in a journal article. Indeed, the sections on boundary layers and convection diffusion problems are reproductions of the author's joint publications in these areas.

The reader would be well advised to thoroughly master the content of Chapter 3 before proceeding to the two dimensional case in Chapter 4. This is another substantial chapter of 67 pages that is chiefly concerned with establishing the exponential convergence of the *hp*-version for scalar elliptic problems, following in the footsteps of Babuška and Guo.

The application of *hp*-methods to elliptic systems forms the remainder of the book. Chapter 5 deals with mixed finite element methods for Stokes equations and has a full discussion of the standard background theory on the numerical treatment of saddle point problems and the Babuška–Brezzi condition. The chapter culminates with a proof of stability of various families of mixed p-finite elements. The final chapter deals with *hp*-methods for linear elasticity, with an emphasis on the modelling of thin domains and approximation of the plate and shell models that arise. An extended discussion of the locking phenomenon is given. The book concludes with two short appendixes on Sobolev spaces, Hilbert space interpolation, and orthogonal polynomials.

The book complements other texts in the area [1, 2] that are at a more elementary level and focus more on the practical implementation aspects. The manuscript is based on graduate lectures presented by the author to an audience of engineers and mathematicians at the ETH Zürich. In principle, the inclusion of background material means that the book should be accessible to a graduate student with quite a modest background in numerical analysis of elliptic partial differential equations. However, the demanding pace of the text would leave many UK graduate students in mathematics trailing in its wake. Exercises are included in the text, ranging from trivial computations to deeper applications of the theory.

My only real criticism of the book lies in the number of minor typographical errors and inconsistencies that should have easily been detected by a copy editor. At a cost of $85.00 for 374 pages, I would expect the publisher to produce a far more polished product. Nevertheless, this is a detailed and authoritative account of the theory of *hp*-version finite element methods at the end of the 1990s, and provides a much needed reference source for theoreticians in this area.

References

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3[41A10, 42A10, 65M70, 65T10]—*Spectral methods in Matlab*, by Lloyd N. Trefethen, SIAM, Philadelphia, PA, 2000, xvi+165 pp., 23 1/2 cm, softcover, $36.00

This book is published within the series "Software, Environments, Tools"; in other words it is meant to be a "cookbook" for someone who is curious about learning spectral methods but does not want to go through a more comprehensive spectral method book or course, at least not at the beginning. It builds on the powerful Matlab platform and brings the essentials of spectral collacation methods with just forty short Matlab "M-files". These Matlab codes will also generate intriguing graphics to vividly illustrate the numerical results.

Spectral methods have been under rapid development in the last twenty-five years. There are many books written in this period, most notably the pioneering book by Gottlieb and Orszag in 1977 and the comprehensive book by Canuto, Hussaini, Quarteroni and Zang in 1988. The book under review is different from these comprehensive books. Although it does explain the essential background of spectral methods, in order to give the readers the basic ideas before letting them play with the Matlab codes, the emphasis here is clearly not on a comprehensive coverage of spectral method but on a practical and rapid introduction to the readers of how the spectral method works through the Matlab codes. The advantage of this approach is that many more students and researchers are expected to learn the basics of the spectral methods through this book and the Matlab codes in it. Perhaps some of them will get so interested in the method that they will ask deeper questions which this book cannot answer, but then they will already be adequately prepared to move on to read a more comprehensive spectral method book.

There are fourteen chapters in the book. The first six chapters cover the basic topics in spectral methods, such as the differentiation matrices and fast Fourier transforms. Chapters 7 through 14 give more applications. A reader who cares less about the underlying ideas of the spectral method but more about the applications could probably skip the first six chapters. However, it would be much more effective if one went through all the chapters and played with the Matlab codes as soon as they appear in the book.

This book is a very nice addition to the collection of books on spectral methods, from a totally different angle. It should attract more students and researchers to the powerful spectral methods.

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4[65-02, 65N06, 65N30]—*Generalized difference methods for differential equations. Numerical analysis of finite volume methods*, by Ronghua Li, Zhongying Chen, and Wei Wu, Marcel Dekker, New York, NY, 2000, xv+442 pp., 23 1/2 cm, hardcover, $175.00

This book provides a framework for construction and analysis of finite volume approximations of partial differential equations. The approach falls into the general class of Petrov–Galerkin methods that presents the boundary value problem in a weak form with the corresponding bilinear form defined over two different spaces: the solution space and the test space. In the book, the approximate solution is taken in the finite element space of piecewise polynomial functions over a partition of the domain into simplices or quadrilaterals (in most of the cases these are conforming spaces), while the test space consists of piecewise constant functions over a different (a dual) partition of the domain. Integrating a convection-diffusion-reaction equation over a particular finite volume produces a balance equation, which is the sum of surface (line) integrals of the diffusive and convective flux through the volume boundary and volume integrals of the reaction and the source terms. Replacing the derivatives in the balance equation by finite differences has been successfully used in the last 50 years. Alternatively, one can replace the exact solution by its finite element interpolant. This approach, consistently used in the book, is often called finite volume element method, a term that describes quite accurately its essence. The finite volume method and its applications to problems in science and engineering has been a major direction in computational mathematics in the last fifteen years (see, e.g., the Proceedings of the First and Second International Conferences on Finite Volumes for Complex Applications [5, 6]).

In the book the finite volume element approach is applied to second and fourth order elliptic equations, to parabolic and hyperbolic equations as well as convection-dominated diffusion problems, elasticity and Maxwell's equations. A merit of the book is that it gives a general framework for presenting this approach in a unified and consistent way. In this respect the book is quite timely and useful. The presentation is clear and compact (except some lengthy computations that could have been left as exercises).

Chapter 1 contains the usual material of Sobolev spaces and the corresponding abstract Hilbert theory of elliptic problems. Chapter 2 is the most elaborate part of the book and describes the main steps in the construction and analysis of the schemes on one-dimensional problems. In my opinion, this chapter is the least interesting since most of the schemes and theory are the same as in the standard finite element method, including the results on convergence and super-convergence error estimates. At the same time this part also shows the limitations of the method. For example, this method is missing important schemes with harmonic averaging of the coefficients of the differential equation and other high-order 3-point finite difference schemes of the sixties (see, e.g., [3]).

Chapter 3 is the most important and useful part of the book. The Petrov–Galerkin approach is developed fully and demonstrated on self-adjoint elliptic equations of second order in a polygonal domain. The construction of the test spaces based on finite volumes are given for linear, quadratic, and cubic elements, and many constructions are carried out to an explicit form. The coercivity of the bilinear form is studied in detail which is not so trivial since the bilinear form is defined on the product of the solution and test spaces and, therefore, one has to verify the corresponding inf-sup condition. Further, error estimates in a discrete energy, L^2- and maximum-norms are derived. After reading this chapter, one realizes that the construction of the schemes and their analysis depend heavily on the finite element method and its theoretical tools.

Subsequently, the same approach is applied to fourth order elliptic equations (Chapter 4), and to parabolic (Chapter 5) and hyperbolic equations (Chapter 6). Chapter 7 contains the construction and analysis of finite volume methods for convection-dominated diffusion problems. In my opinion, this is an interesting part of the book. It demonstrates the flexibility of the method and its capability to easily generate various schemes that combine characteristic and upwind approximation. It is shown that for diffusion and convection-diffusion problems, the presented Petrov–Galerkin method produces discretizations that have the property of local volume-by-volume mass conservation, an important and desired property for many applications. The analysis also proves stability in maximum norm by the maximum principle argument, a technique that has been widely used in the theory of finite differences (see, e.g., [3]).

I found numerous misprints and minor inaccuracies. For example, the term interpolation projection that is used in the book sounds very strange. In fact, almost everywhere it has the meaning of the finite element interpolant (see, e.g., [1]). On page 118 alone I found two inaccuracies: as stated, the estimate (3.2.8) does not make sense for $m = 2$, while in (3.2.12) the interpolation projection is not well defined for $u \in H^1$. There are also claims (in the introduction) that the estimates in the finite difference schemes theory are "usually not optimal". In the Russian literature there are a number of papers on this subject, and the optimality of the error estimates has been resolved by using the fundamental Bramble–Hilbert lemma. This research has been summarized in [4] for rectangular and in [2] for simplicial and quadrilateral meshes. The book under review does an excellent job by giving a fairly complete bibliography on the finite volume research published in Chinese. The references to the English literature are far from complete, but they represent the mainstream research in the area, while the references to the Russian literature are scarce and misleading.

As noted above, the book has various merits and is a useful and timely text in the area of finite volume methods for partial differential equations. It could be used as a textbook for an advanced course on finite volume methods or as a supplement to a course on discretization methods for differential equations.

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5[65-01, 65L05, 65L06, 65L10, 65L12, 65L99]—*Computer methods for ordinary differential equations and differential-algebraic equations*, by Uri M. Ascher and Linda R. Petzold, SIAM, Philadelphia, PA, 1998, xvii+314 pp., 25 cm, softcover, $36.50

This book is intended to be a textbook introducing students of mathematics as well as of other fields like computer science, mechanical, electrical, and chemical engineering, physics, into the field of the numerical solution of ordinary differential equations (ODEs).

In addition to initial value problems (IVPs) and boundary value problems (BVPs) in regular ODEs, also specific, singular ODEs, namely, differential-algebraic equations (DAEs) are considered. While regular IVPs are discussed on about 140 pages, about 70 pages each are devoted to regular BVPs and DAEs.

Each attempt to write a concise textbook on such a broad and complex subject constitutes a hazardous business. It always means a tightrope walk trying to present facts as simple as possible and as complex as necessary. The selection of the topics and the necessary restriction of the material to be presented will be subjective in any case.

Many nice, partly quite extensive monographs are available on the numerical solution of regular IVPs, even on single methods, like the good, old Adams book [1] by Lawrence F. Shampine and Marilyn Gordon. Flicking through this book is of great benefit still today. More recent books already contain sections on DAEs. Corresponding extensive monographs are available on regular BVPs, too. Concerning the integration of those three aspects, IVPs, BVPs, and DAEs, in the necessary brevity but in adequate detail for an independent course, this new textbook by Uri M. Ascher and Linda R. Petzold is a welcome and useful novelty.

A predecessor worth mentioning is the now classical textbook of Josef Stoer and Roland Bulirsch [2] where the above three aspects are discussed within the framework of a general introduction to numerical analysis (IVPs and BVPs on 70 pages each, DAE on 5 pages).

In the preface the two author outline what they aim at with this book: "It is designed for people who want to gain a practical knowledge of the techniques used today. The course aims to achieve a thorough understanding of the issues and methods involved and of the reasons for the successes and failures of existing software. On one hand, we avoid an extensive, thorough, theorem-proof-type exposition: we try to get to current methods, issues, and software as quickly as possible. On the other hand, this is not a quick recipe book, as we feel that a deeper understanding than can usually be gained by a recipe course is required to enable students or researchers to use their knowledge to design their own solution approaches for any nonstandard problems they may encounter in future work." Although I do not at all dislike "theorem-proof-type expositions", I also like and use a textbook that is, so to say, written in prose, and that conveys knowledge in a relatively relaxed way. This book reminds me of another classical textbook by Gene H. Golub and James M. Ortega [3].

Uri M. Ascher and Linda R. Petzold, both of them teachers with experience and highly acknowledged experts, have structured this book didactically and formally in an excellent way. The great variety of applications considered in expositions as well as in examples motivate the reader for further investigation. Well thought-out and well understandable examples lead to individual questions and insights. This, too, reminds me of [3].

The whole layout turned out very well. It is quite convenient that the necessary fundamentals (Newtown's method, Taylor's theorem, matrix eigenvalues, etc.) are arranged as a short framed review within the text and not in a separate appendix.

In particular, students with diverse intentions will like the helpful guide-boxes, which also indicate whether certain paragraphs can possibly be skipped.

The book is divided into four parts and ten chapters. Part I is an introduction to all three problem areas. Part II (Chapters 2–5), Part III (Chapters 6–8), and Part IV (Chapters 9–10) refer to the mentioned areas, namely IVPs, BVPs, and DAEs. For experts, the headlines of the chapters clearly indicate what can be expected there. In Part II on IVPs we have On Problem Stability; Basic Concepts; One-step Methods; Multistep Methods. Part III on BVPs is subdivided into More Boundary Value Problems; Theory and Application; Shooting; Finite Difference Methods for BVPs. Part IV on DAEs contains More on DAEs; Numerical Methods for DAEs.

Each chapter concludes with informative sections on "Software, Notes and References" as well as a section presenting a whole host of exercises of different level, which are all very instructive and interesting.

As most books for beginners (e.g., [1]), the parts devoted to regular ODEs sensibly start from the most simple relations in the ODEs to be solved: a global Lipschitz condition is assumed for the vector field. More complex problems and the resulting difficulties are only sketched additionally. This sovereignty to restrict oneself to the essentials for beginners characterizes Parts II and III in a very pleasant way. In particular, this holds true for the stability discussion, although I am wondering here whether the authors have gone a bit too far in simplifying. I would prefer to make a primal distinction in notation, e.g., of the condition numbers of linearly bounded bijective mappings (as for BVPs for example) and of the qualitative, asymptotic properties of flows (like stability in the sense of Lyapunov for IVPs).

In my opinion the representation of the practical applicability of (multiple) shooting methods for regular ordinary BVPs is somewhat misleading. The much older representation in [2] seems to me to be more mature.

Writing a good textbook always requires, in addition to the authors' expert knowledge, that the development in the field concerned has been finished to a certain extent. A certain distance is necessary to be able to restrict oneself to the most essential things. In the case of the DAEs in Part IV, the stage of development essential for a really good textbook has not yet been reached, in my opinion, and the two authors are themselves too strongly involved in this development to keep the necessary distance. Hence, the nice character of a textbook gets lost in Part IV. This is rather a part of a monograph with a great amount of subproblems and approaches strung together. For example, in spite of the mentioned sound restriction to globally Lipschitz-continuous vector fields in the beginning, the authors do not introduce a global notion of index for DAEs then. Just for these already complicated equations, they start with a local notion of index, which is confusing not only for beginners.

As intended by the authors, this new textbook is a strongly advisable aid for introductory courses to the numerics of regular ODEs. In particular, I consider the IVP part to be so exceptionally successful that I will advise students to use it as first literature for studying on one's own. For the BVP part it requires a few additional comments to achieve a more balanced education.

The abundant source of instructive examples and exercises in all parts of this book will be extremely valuable for all teachers.

Altogether, a gain!

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6[65L05, 34A50, 58F99]—*Geometric integration: numerical solution of differential equations*, C. J. Budd and A. Iserles (Editors), Philosophical Transactions of the Royal Society, Mathematical, Physical and Engineering Sciences, The Royal Society, UK, April 1999, vol. 157, no. 1754, pp. 943–1133

The present issue of volume 357 of the *Philosophical Transactions of the Royal Society of London, Series A*, is entirely devoted to *geometric integration*. Under this heading, several recent developments in the numerical treatment of differential equations are collected. They have as a common theme the idea to preserve as far as possible structures (symmetries) of the exact flow in numerical discretization.

This book starts with a paper by C. J. Budd and A. Iserles with the title of the book, "Geometric integration: numerical solution of differential equations on manifolds". This article provides an interesting introduction to the topic of geometric integration: it briefly mentions different numerical approaches that have been developed in this context, and it addresses their importance for a large number of real applications. The remaining contributions of this issue treat different aspects of geometric integration; they are research articles, and they are independent of each other.

Much recent research is devoted to the solution of differential equations on manifolds having a Lie group action. Their discretization by a numerical algorithm usually involves computations in the corresponding Lie algebra. For reasons of efficiency, the number of appearing commutators has to be kept as small as possible. The article by H. Munthe-Kaas and B. Owren "Computations in a free Lie algebra" extends Witt's formula on the number of commutators of a fixed length (of the Hall basis) to graded Lie algebras. This allows one to get upper bounds on the number of necessary commutators in a numerical method. A substantial improvement of Runge–Kutta methods for Lie-type equations has been obtained by suitably regrouping the arguments of the commutators.

Differential equations of the form $y' = a(t)y$, where the solution $y(t)$ evolves in a Lie group and where $a(t)$ is a smooth function in its Lie algebra, are the subject of the paper "On the solution of linear differential equations in Lie groups" by A. Iserles and S. P. Nørsett. It is based on an explicit formula of the solution given by the Magnus series. This contribution presents a new one-to-one correspondence between the individual terms of the Magnus series and binary trees, which allows one to derive explicit recurrence relations as well as a convergence proof of the series. By suitably truncating the series and by ingeniously evaluating the appearing multiple integrals and commutators, new efficient methods are proposed for this class of methods.

In the contribution "Geometric integration using discrete gradients" by R. I. McLachlan, G. R. W. Quispel, and N. Robidoux, it is pointed out that ordinary differential equations with first integrals and/or Lyapunov functions can be written as "linear-gradient systems" $\dot{x} = L(x)\nabla V(x)$, where $L(x)$ is a matrix-valued function. Using discrete gradients, numerical methods are derived that preserve exactly first integrals and Lyapunov functions. This method is successfully applied to Hamiltonian, Poisson, and gradient systems, and also to many dissipative systems.

Geometric aspects of partial differential equations are the subject of the article "Self-similar numerical solutions of the porous-medium equation using moving mesh methods" by C. J. Budd, G. J. Collins, W. Z. Huang, and R. D. Russel. The role of conservation laws and similarity solutions is discussed, and adaptive numerical discretizations are studied which admit discrete forms of conservation laws and which have discrete self-similar solutions. It is shown that such methods capture correctly the long-time dynamics of the underlying partial differential equation.

The composition of simple methods with favorable geometric properties (symplecticity, volume preservation, ...) automatically inherits these properties, and it also allows us to increase the order of accuracy. One possibility to get the corresponding order conditions is by using the Baker-Campbell-Hausdorff formula. The article "Order conditions for numerical integrators obtained by composing simpler integrators" by A. Murua and J. M. Sanz-Serna presents a different approach based on a new type of rooted tree. As a result, the authors derive a simple presentation of the order conditions, which is obtained by a simple transcription of the structure of the corresponding graph.

The stable computation of trajectories of N-body problems is important for astronomical applications as well as for studies of atomic systems. The article "Reversible adaptive regularization: perturbed Kepler motion and classical atomic trajectories" by B. Leimkuhler discusses the impact of symplectic and time-reversible integrators on the energy error, it studies time transformations and the use of variable step sizes, and it presents a series of interesting numerical experiments with a new code that is especially written for the simulation of perturbed Kepler motions and classical atomic trajectories.

To sum up, this issue shows several important aspects of the wide field of geometric integration, all of which are written by experts in this topic.

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7[37Mxx, 65Pxx]—*Dynamical systems and numerical analysis*, by A. M. Stuart and A. R. Humphries, Cambridge University Press, New York, NY, 1998, xxi+685 pp., 23 cm, hardcover, $64.95, softcover $39.95

To come to the point first: This is an extraordinarily well-made monograph on dynamical systems which are generated by application of linear multistep methods and Runge–Kutta methods to explicit ordinary differential equations (ODEs). In particular, emphasis is put on the qualitatively correct reflection of properties, resp. the structure of the dynamical system generated by the ODE itself.

The principle aim of this book is—according to the two authors—to address two questions:

I. Assume that the differential equation has a particular invariant set. Does the numerical method have a corresponding invariant set which converges to the true invariant set as $\Delta t \to 0$? If so, what is the rate of convergence?
II. Assume that the vector field defining the differential equation has a particular structural property which confers certain properties on the dynamical behaviour of the equation. Find special numerical methods which inherit these structural properties under mild or no restrictions on the time-step or, for general numerical methods, find conditions on Δt under which these structural properties are inherited.

During the last 15 years, considerable progress has been made in answering these questions. Numerous articles have been written, by numerical mathematicians as well as analysts, among them, not least the two authors of the present monograph. In this 700-page monograph they collect these results to form a comprehensive and cogent account which "is intended to be accessible to anyone familiar with either dynamical systems or numerical analysis theory and, with a little work, to someone familiar with neither."

No doubt, this book is a considerable gain for all those who want to get familiar with this field of mathematics as well as for someone requiring a reference text for the major results in the field. However, someone looking for a guide for the immediately practicable numerical solution of initial value problems or for the appropriate modelling of genuine initial value problems should better consult other sources.

The ODEs allowed here are always explicit and autonomous where the concerned vector field is globally defined on $\mathbb{R}^p$ and Lipschitz-continuous, resp. locally Lipschitz-continuous with additional global structural assumptions. This is the price that has to be paid for the really nice theory, which would not have been possible otherwise.

As far as numerical integration methods are concerned, exclusively constant step-sizes are assumed, if necessary sufficiently small ones with a relatively detailed specification of this condition. In case of implicit integration methods it is assumed that the resulting nonlinear equations can be solved exactly. Problems occurring in practice, like rounding errors, defects and error estimations, are not an issue here.

I read the book with mixed feelings. On the one hand, I realized the very large distance from the starting point and from the intention of the investigations the authors explain in the preface: "Dynamical systems are pervasive in the modelling of naturally occurring phenomena. Problems as diverse as the simulation of planetary interactions, fluid flow, chemical reactions, biological pattern formation and economic markets can all be modelled as dynamical systems. The theory of dynamical systems is concerned primarily with making qualitative predictions about the behaviour of systems which evolve in time, as parameters which control the system, and the initial state of the system itself, are varied. Most of the models arising in practice cannot be completely solved by analytic techniques and thus numerical simulations are of fundamental importance in gleaning understanding of dynamical systems. Hence it is crucial to understand the behaviour of numerical simulations of dynamical systems in order that we may interpret the data obtained from such simulations and in order to facilitate the design of algorithms which provide correct qualitative information without being unduly expensive. These issues are at the heart of this book." Problems of in how far or where the mathematically useful structural properties are relevant for the real world are not touched. Neither is it mentioned which out of these lots of nice results are applicable and useful in practice. On the other hand, I take pleasure in the beauty and high degree of perfection the mathematical theory of dynamical systems generated by numerical integration formulas as documented in this monograph has obviously now reached.

The monograph is properly divided into the following chapters:

1. Finite dimensional maps
2. Ordinary differential equations
3. Numerical methods for initial value problems
4. Numerical methods as dynamical systems
5. Global stability
6. Convergence of invariant sets
7. Global properties and attractors under discretization
8. Hamiltonian and conservative systems

All chapters are completed by a detailed and competent discussion of references. Various simple and well-obvious examples serve for illustration and motivation.

Based on their experience, the authors provide hints for lectures as well as numerous exercises, which is valuable for teachers as well as students.

The book is admirably carefully written. The unreasonable references (e.g., "Proof of Theorem XY: See Exercise MN/Exercise MN: Proof of Theorem XY"), however, I found somewhat bothersome. Otherwise, the representation is clear and as didactically well structured as can already be assumed from the subdivision into the eight chapters mentioned. A pleasant layout contributes to the pleasure one surely has reading the book.

Roswitha März

8[65-02, 65Fxx]—*Computer solution of large linear systems*, by Gerard Meurant, North Holland, Amsterdam, 1999, xxii+753 pp., 23 cm, hardcover, $149.50

Two characteristics of this book are immediately evident: the table of contents is very detailed and the list of references is large, which makes the book a useful reference for someone working in the area. The chapters are well organized and it is easy to find something quickly through the table of contents if you know the name of a technique, or through browsing in the appropriate chapter if not. The index in the back of the book is rather sparse and not very helpful.

This book contains many results on estimates of eigenvalues and error bounds for symmetric positive definite systems. Some proofs are presented in each chapter to give an idea of the typical arguments used. For many theorems the reader is referred to the original work for the proof. This allows the author to present a lot of material without making the book overly large. This is not a defect as most researchers are interested in the original citation anyway. Also, since the book carries a fairly complete and up-to-date (for the mid- to late-1990s) set of algorithms with descriptions of implementations, it is easy to search in the appropriate chapter and answer the question, "Who has done work in this area before?"

This book is not intended as a text as there are no exercises, and if a proof is not presented, it tells you where to find it. Some of the material is developed in detail and is easy to follow for someone new to the field. Some of the results are not as well motivated.

Chapter 1 covers introductory material, beginning with topics in linear algebra and graph theory and progressing to issues involved in solving linear systems on computers, where exact arithmetic is not available. Sparse storage schemes, a short history on BLAS routines, and parallel machines are also presented.

The second chapter is devoted to Gaussian elimination on dense matrices. Six different algorithms are presented, which may be a bit of overkill. A lot of attention is paid to tridiagonal and block tridiagonal systems, which are the types of large systems one would solve this way. Otherwise, the chapter is rather brief. Some parallel implementation details are presented at the end.

Chapter 3 covers Gaussian elimination for sparse matrices. Graph theory is used to motivate reordering strategies. Most of this chapter is devoted to symmetric positive definite matrices, with just a little bit on nonsymmetric matrices. Admittedly, this is where the theory is best developed. There is some discussion of parallel implementations also.

Chapter 4 is a rather short chapter on fast solvers for separable PDEs. It covers mostly FFT and cyclic reduction methods.

Chapter 5 contains the classical iterative methods, such as Jacobi, Gauss–Seidel, SOR, and variations. Convergence criteria are derived, as well as convergence rates for each method. There is very little new material in this chapter, but it belongs in the book for completeness.

Chapter 6 has conjugate gradient type methods. There is a particularly detailed set of proofs on the optimality of PCG and convergence rates. There are also a lot of details on a posteriori error bounds. All of these relatively short algorithms are presented in detail.

Chapter 7 is on Krylov subspace methods for nonsymmetric systems. This chapter contains a lot of details on the algorithms, but only a few short proofs of convergence theorems are presented, and the reader is directed elsewhere for longer ones. The Arnoldi and Hessenberg algorithms are presented, along with descriptions of, and reasoning for, using modified Gram Schmidt orthogonalization and Givens rotations. FOM and GMRES and relations between the two are discussed, as is BiGCG. Generalizations of methods are discussed, including restarted and flexible versions and stabilization techniques. There are a number of plots depicting the norm of the error as a function of iteration count. These could be useful to someone for selecting a method, or for comparing results of a new method. The chapter ends with a quick presentation of some algorithms for complex linear systems and a short discussion of why parallel implementations pose problems.

Chapter 8 on preconditioning is the largest chapter in the book. Most of the chapter is devoted to methods for symmetric matrices, and mostly symmetric positive definite. It begins with a nice description of the properties sought in a preconditioner and the trade-offs involved between then. Then it launches into simple diagonal preconditioning, followed by point and block SSOR. This section contains perhaps more details than are necessary on finding an optimum relaxation parameter, including a derivation on estimates of the eigenvalues using Fourier analysis of the Poisson equation with periodic boundary conditions. Similar detail is included in analysis of incomplete Cholesky preconditioning.

There is a large section on incomplete Cholesky and its usefulness as a preconditioner for various types of matrices. Many results are presented from Axelsson's book, and the reader is referred there for some proofs. There is a good description of some reordering techniques complete with examples on rectangular domains detailing the new unknown orderings and iteration counts, computational costs, and preconditioner sizes.

The section on sparse approximate inverse preconditioners gives a good description of the guiding principles behind the most popular methods. However, there is little mention of the costs associated with constructing them, and no mention at all of their advantages over other preconditioners on parallel machines.

This chapter concludes with a discussion, including a few implementation details, of the differences between parallel and vector computers. As vector computers have been going in and out of fashion, it is nice to see at least some mention of them.

Chapter 9 covers multigrid methods. This chapter is a little disappointing. The order in which the material is presented is not as smooth as in other chapters. There is very little mention of algebraic multigrid methods.

The last chapter covers domain decomposition methods. This chapter starts, as domain decomposition did historically, with a description of an alternating Schwarz method and some convergence results for the Poisson problem. From there it heads into additive and multiplicative Schwarz methods and Schur complement methods. The most useful part of the chapter describes how domain decomposition methods can be used to construct preconditioners. The examples are all related to PDEs solved on rectangular, or at least conceptually rectangular domains. One would expect to see an example of how a domain can be broken up for solving a PDE using finite elements on an unstructured mesh. The chapter ends with a short discussion of multilevel ILU preconditioners. This again is restricted to SPD matrices.

All in all, *Computer solution of large linear systems* would make a fine reference book for engineers, computer scientists, and mathematicians working with large systems. It is most useful for those working with symmetric positive definite systems. The book presents a large number of useful algorithms, along with the important theory governing their behavior. It also does a wonderful job of citing other sources where one can fill in the details.

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9[11L05, 11L40, 65C10, 94A60, 94B05]—*Character sums with exponential functions and their applications*, by Sergei Konyagin and Igor Shparlinski, Cambridge University Press, New York, NY, 1999, viii+163 pp., 23 1/2 cm, hardcover, \$49.95

The main theme of this monograph is the distribution of the powers of an integer g with $1 < g < p$ modulo a prime p. Character sums with exponential functions in the argument form the most important tool in the analysis. Many of the problems considered here are motivated by applications, and a good part of the book is devoted to applications. The book collects known results, most of them of fairly recent origin, but also presents new theorems not published before. One stated aim of the book is to stimulate further research, and this goal has certainly been reached, for instance through the many open problems that the authors pose along the way.

The first two chapters set the stage via introductory remarks and auxiliary results. Chapters 3 to 6 are devoted to the core of the theory, namely bounds for character sums with exponential functions in the argument and related bounds for Gaussian sums. Chapters 7 to 10 deal with number-theoretic applications; for instance, to multiplicative translates of sets modulo p and to class numbers of cyclotomic fields. Chapter 11 considers the important problem of the occurrence of given strings in digit expansions of rational numbers, which is connected with the Blum-Blum-Shub pseudorandom bit generator in cryptography. Applications to linear congruential pseudorandom numbers, in particular the question of the existence of good multipliers, are treated in Chapter 12. Chapters 13 and 14 are more of number-theoretic interest, whereas the distribution theorems in Chapter 15 allow very interesting applications to a "pseudo-randomized" version of the QuickSort algorithm. In the last three chapters, the treatment of upper bounds for the dimension of BCH codes is of particular relevance from the viewpoint of applications.

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10[11-02, 11Dxx]—*The algorithmic resolution of Diophantine equations*, by Nigel P. Smart, Cambridge University Press, New York, NY, 1999, xvi+243 pp., 23 cm, hardcover, \$69.95, softcover \$26.95

1. Introduction

A hundred years has passed since Hilbert posed his tenth problem: whether there exists an algorithm to determine if a given Diophantine equation has finitely or infinitely many solutions. By any reasonable measure, Hilbert would be pleased with the progress made in the last century. Although it has since been shown by Matijasevič that the general problem is undecidable, there have been significant advances on these problems for many classes of Diophantine equations. The most notable along these lines include the pioneering work of Thue and later work by Siegel and Roth on Diophantine approximation of algebraic numbers and its application to bivariate homogeneous form equations, the finite basis theorem for elliptic curves of Mordell and Weil, Baker's results on linear forms in the logarithms of algebraic numbers. Tijdeman's application of Baker's method to Catalan's equation, the results of Győry and later Evertse on S-unit equations, Schmidt's subspace theorem, Falting's proof of the Mordell conjecture, and without question the most notable being Wiles' proof that all semi-stable elliptic curves over the rational numbers are modular, yielding a proof of Fermat's Last Theorem.

Given the apparent renaissance of this subject area in modern times, together with the development of sophisticated computing technology, it is natural to consider many of these problems from an algorithmic perspective. In particular, if a given Diophantine equation is known to have only finitely many solutions, then one may attempt to construct an algorithm which will find all of the solutions. This is precisely what Smart's book is all about.

The author describes his text as a recipe book for solving Diophantine equations. This is perhaps inspired by the fundamental thesis of de Weger, and also perhaps by the lack of textbooks on this subject, especially given the outstanding developments in recent years. This book is long overdue, and the author has succeeded in producing a valuable asset for the shelf of anyone interested in learning about solving Diophantine equations. The only prerequisite for this book is some basic algebraic number theory, and so it would be suitable as a graduate textbook. This reviewer would certainly recommend it as such, given that the author has included many exercises and worked examples. In fact, the examples go a long way in making the book a success.

The content is based on fundamental theorems arising from Baker's work on estimates for linear forms in the logarithms of algebraic numbers, the p-adic case by Yu, and the elliptic case by David. Armed with this machinery, there are large classes of Diophantine equations for which there exist computable upper bounds for the size of solutions. The primary examples of these include bivariate homogeneous form equations, bivariate equations whose curve $F(x, y) = 0$ defines a curve of genus one, and superelliptic Diophantine equations $y^n = P(x)$. Unfortunately, the upper bounds deduced from Baker's theory are, in most cases, much too large for a computer to search for small solutions. Thus, a technique first devised by Baker and Davenport, and later refined by de Weger using methods from the geometry of numbers via lattice basis reduction, can be employed to reduce the upper bounds by an order of magnitude. In many cases this procedure can be iterated sufficiently many times to deduce an upper bound for the original problem which is amenable to a computer search for all solutions.

The text begins with a historical perspective on the subject of Diophantine equations, with reference to some of the well-known topics that have shaped the subject as we now know it. Fermat receives enormous coverage, not only in relation to the Last Theorem, but also for his observation concerning the problem of integer factorization and his method of infinite descent. This immediately indicates the author's desire to incorporate algorithms into his presentation. A discussion on this topic is the content of the next section, and some words concerning computational complexity attempt to bring the reader into the mindset of a computational number theorist. The introductory chapter finishes off with an abstract description of Diophantine equation, and then a concrete example of one: the elliptic curve $Y^2 = X^3 - 4$. A nonalgorithmic proof that $(X, Y) = (5, \pm 11)$ and $(2, \pm 2)$ constitute all of the integer solutions to this equation is provided.

Part I of the book discusses a smattering of topics, some being in preparation for the main topics of the book and others of independent interest. Chapter Two discusses local theory: p-adic numbers and their extensions, p-adic numerical analysis, Hensel's lemma, the Newton–Raphson method in this context, and p-adic power series. These topics are used in the subsequent chapter where one is shown how to use local methods to solve some types of Diophantine equations. The method of Skolem is used to solve the equation $X^4 - 2Y^4 = \pm 1$. The author briefly covers the Hasse (Local-Global) principle, and Selmer's famous example $3X^3 + 4X^3 + 5X^3 = 0$. The chapter ends on an algorithmic note, as the author discusses the concept of sieving as it applies to finding (or rather ruling out) small solutions to Diophantine equations of the shape $y^2 = P(x)$. This will prove to be a useful tool, required for some of the algorithms in the later chapters. Chapter Four is devoted to solving ternary quadratic forms, with reference to local and global conditions for solvability, along with a sieving algorithm to find small solutions. In Chapter Five we finally arrive at the machinery which provides the basis for all of the recipes in this book. Methods from Diophantine approximation and the geometry of numbers are presented clearly and in considerable detail. In particular, the theory of continued fractions, approximation lattices, the lattice basis reduction method of Lenstra, Lenstra and Lovász, and its integer variant by de Weger are the topics of this chapter. Some important applications of these topics are presented in the next and final chapter of Part I. In particular, the theoretical basis for de Weger's reduction procedure via approximation of linear forms is described in the global and local cases, with details given for both the homogeneous and inhomogeneous cases. This is a crucial part of the book, and the author does well by providing a very explicit example to illuminate the procedure.

Of the three parts of the book, Part II is, in the reviewer's opinion, the most essential. Through the work of de Weger, Stroeker, Tzanakis and others, there have been significant developments in recent years of techniques which use linear forms in logarithms to deduce upper bounds for Diophantine equations, and then lattice basis reduction methods to lower these upper bounds. Chapter Seven provides a detailed description of how these methods apply to Thue equations (bivariate homogeneous Diophantine equations of the form $F(x, y) = 0$). The presentation is clear, and the examples exhibit the method of approximation lattices and lattice basis reduction very well. The author even describes a recent method due to Bilu and Hanrot, which lends itself to the complete solution of higher degree Thue equations. In Chapter Eight, these methods are applied to Thue–Mahler equations. A general description of the methodology is presented along with the method as it applies to the specific example $X^3 - X^2Y + XY^2 + Y^3 = \pm 11^s$. In Chapter Nine the methods are applied to the more general problem of S-unit equations. Also, a sieving procedure for the determination of small solutions to S-unit equations is described. The problem of solving S-unit equations has developed into an important area of study within Diophantine approximation, with the pioneering work of Győry and Evertse. The myriad of applications of this pursuit are described at the end of Chapter Nine, and on into Chapter 10 and 11: triangularly connected decomposable form equations, discriminant form equations, and index form equations. Thue equations are a very special case of all of these. Complete algorithms for solving all of these types of Diophantine equations are described in considerable detail, along with many examples and exercises.

The last part of the book covers the topic of elliptic curves, with the primary goal of determining all integer points on an elliptic curve. The author provides a fairly detailed expository of the Mordell–Weil theorem, along with the computation of 2-Selmer groups, and the conjecture of Birch and Swinnerton-Dyer. The main topic in Part III is Chapter 13, in which all integer points on an elliptic curve are determined using David's estimates for linear forms in elliptic logarithms, and reduction techniques similar in nature as those described in earlier chapters. The author once again succeeds in presenting the method very clearly with illuminating examples. The final chapter superficially covers an assortment of topics that the reader may wish to go off and see more of via the literature provided. These include Faltings' famous theorem on the finiteness of the number of rational points on curves of genus two, an effective method going back to Chabauty for finding rational points on such curves via their Jacobians (the reader should be aware that the subject of Diophantine analysis fails from the shortcoming of an almost complete lack of effective methods for finding all integer points on a curve of genus $g > 1$). Unfortunately, some recent work of Bruin on this particular topic did not manage to make it into the book. The chapter is completed by some remarks on Fermat curves, and Catalan's equation. Two appendices are provided for the reader, with the first of these providing the theoretical basis for the book. In particular, precise estimates for linear forms in complex logarithms of algebraic numbers due to Baker and Wustholz are given, along with the p-adic version due to Yu, and David's result for linear forms in elliptic logarithms.

The basic philosophy in this book is to follow a mathematical recipe, combining theory and computation, to solve a single Diophantine equation. There are many examples provided which convey the techniques very nicely. One should keep in mind that it can be the case that a given Diophantine problem, or classes thereof, can be completely solved without the brute force of a computer. For example, results of Ljunggren and Cohn have completely solved the family of elliptic curves given by quartic models of the form $X^4 - dY^2 = 1$ without perhaps a single keystroke on a computer. Nevertheless, the main point of Smart's book is to exhibit how the reduction techniques of de Weger can be used to minimize the required computation, which is clearly worthwhile. There is also a surprising absence of reference to recent work by Coppersmith, in which lattice basis reduction methods are used to find small solutions to certain classes of Diophantine equations with applications to integer factorization.

The Algorithmic resolution of Diophantine equations is full of interesting and fundamental mathematics. It has the advantage of holding both the theoretical and the computational mathematician's attention. The chapters do seem to be presented at varying depths, and there is some discrepancy as to precisely who the author is writing to, but overall the presentation is certainly motivating.

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