MATHEMATICS OF COMPUTATION
Volume 70, Number 236, Pages 1713–1718
S 0025-5718(01)01348-5
Article electronically published on March 22, 2001

DENSE ADMISSIBLE SEQUENCES

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ABSTRACT. A sequence of integers in an interval of length x is called admissible if for each prime there is a residue class modulo the prime which contains no elements of the sequence. The maximum number of elements in an admissible sequence in an interval of length x is denoted by $\varrho^*(x)$. Hensley and Richards showed that $\varrho^*(x) > \pi(x)$ for large enough x. We increase the known bounds on the set of x satisfying $\varrho^*(x) \leq \pi(x)$ and find smaller values of x such that $\varrho^*(x) > \pi(x)$. We also find values of x satisfying $\varrho^*(x) > 2\pi(x/2)$. This shows the incompatibility of the conjecture $\pi(x+y)-\pi(y) \leq 2\pi(x/2)$ with the prime k-tuples conjecture.

1. Introduction

A sequence of integers $b_1 < b_2 < \cdots < b_k$ is called admissible if for each prime p, there is some residue class modulo p which contains none of the b_i. Hardy and Littlewood [2] conjectured two relations between admissible sequences and sequences of prime numbers in intervals.

Conjecture A (Prime k-tuples Conjecture)**.** *Let $b_1 < b_2 < \cdots < b_k$ be an admissible sequence. Then there exist infinitely many integers n for which $n + b_1$, $n + b_2$, ... , $n + b_k$ are prime.*

Conjecture B. $\pi(x + y) - \pi(y) \leq \pi(x)$.

One way of interpreting Conjecture B is that no interval of length x contains more primes than the initial interval $[1, x]$. We will consider the two functions, $\varrho(x) = \limsup_{x\to\infty} (\pi(x + y) - \pi(x))$ and $\varrho^*(x)$ which is defined to be the maximum number k of elements in an admissible sequence $y < b_1 < b_2 < \cdots < b_k \leq y+x$ of length x. The prime k-tuples conjecture implies that $\varrho^*(x) = \varrho(x)$. Hensley and Richards [3] showed that for large enough x, $\varrho^*(x) > \pi(x)$. That is, they showed that Conjecture B is incompatible with the prime k-tuples conjecture. In this paper we increase the known bounds on the set of x satisfying $\varrho^*(x) \leq \pi(x)$ and find smaller values of x such that $\varrho^*(x) > \pi(x)$.

Since there is stronger evidence for the prime k-tuples conjecture, the general opinion is that Conjecture B is false. Erdös [1] stated a weaker conjecture to replace Conjecture B.

Conjecture C. $\pi(x + y) - \pi(y) \leq 2\pi(x/2)$.

Received by the editor August 5, 1996 and, in revised form, April 18, 1997.
2000 *Mathematics Subject Classification.* Primary 11B83, 11N13.

After this paper was submitted, the authors learned that Dan Gordon and Gene Rodemich have extended the calculation of $\rho^*(n)$ to $n = 1600$.

This conjecture implies that the centered interval $(-x/2, x/2)$ contains more primes than any other interval of length x. In this paper we find values of x satisfying $\varrho^*(x) > 2\pi(x/2)$. This shows the incompatibility of Conjecture C with the prime k-tuples conjecture.

2. Conjecture B: Extending the range of validity

Schinzel [5] showed that $\varrho^*(x) \le \pi(x)$ for $x \le 146$ using tables published by Smith [6]. This result implies that Conjecture B holds for $x \le 146$. Selfridge has shown that Conjecture B holds for $x \le 500$ (see [4]).

Denote the odd integers in an interval of length x by

$$n_0, n_1, \ldots, n_m,$$

where $m = [([x]-1)/2]$. To produce admissible sequences in this interval, we use the erasing sieve. That is, we prescribe a residue class for each prime less than $x/4$ and sift all elements of the interval in this residue class. Residue classes may be chosen for the remaining primes such that the unsifted elements are not in these residue classes. By the pigeon-hole principle it is only necessary to sift for primes less than k, where k is the number of elements in the admissible sequence. However, in order to introduce an ordering on admissible sequences, we sift a residue class for all primes less than $x/4$.

We assume that n_0 is in the admissible sequence. We denote the ith prime by p_i. For each prime $3 \le p_i < x/4$ we choose an integer a_i, $1 \le a_i \le p-1$, and sift n_{a_i} together with each pth successive element of the sequence. We denote the resulting admissible sequence by $(x, \{a_2, a_3, \ldots, a_r\})$, where p_r is the largest prime less than $x/4$.

We can order admissible sequences in the following way:

$$\begin{aligned}
&(x,\{1,1,\ldots,1,1\}),(x,\{1,1,\ldots,1,2\}),\ldots,(x,\{1,1,\ldots,1,p_r-1\}),\\
&(x,\{1,1,\ldots,2,1\}),(x,\{1,1,\ldots,2,2\}),\ldots,(x,\{1,1,\ldots,2,p_r-1\}),\\
&\cdots\\
&(x,\{2,4,\ldots,p_{m-1}-1,1\}),(x,\{2,4,\ldots,p_{m-1}-1,2\}),\\
&\qquad\ldots,(x,\{2,4,\ldots,p_{r-1}-1,p_r-1\}).
\end{aligned}$$

Note that one admissible sequence may occur several times in the ordering.

To compute $\varrho^*(x)$, we use the following algorithm.

Algorithm for computing $\varrho^*(x)$.

Step 0. Input interval length x, upper bound U for $\rho^*(x)$ and lower bound L for $\varrho^*(x)$. Compute $p_0, p_1, \ldots, p_k$, the primes less than or equal to U. Let r_i be the residue class being sifted modulo p_i. Let u_i be the number of unsifted elements at level i.

Step 1. Initialize $i = 0$, i is the level at which sifting is occuring modulo p_i.

Step 2. Increment i. Set $r_i = 1$. If $i \le k$, check for a residue class modulo p_i for which all elements in the interval have previously been sifted.

Step 3. If a sifted residue class is not found at this level, sift the residue class r_i modulo p_i. If $u_i < L$, go to Step 4. Otherwise, return to Step 2.

Step 4. If $i = k$ and $u_k > L$, set $L = u_k$.

Table 1.

x	$\varrho^*(x)$	x	$\varrho^*(x)$	x	$\varrho^*(x)$	x	$\varrho^*(x)$
6	2	8	3	12	4	16	5
20	6	26	7	30	8	32	9
36	10	42	11	48	12	50	13
56	14	60	15	66	16	70	17
76	18	80	19	84	20	90	21
94	22	100	23	110	24	114	25
120	26	126	27	130	28	136	29
140	30	146	31	152	32	156	33
158	34	162	35	168	36	176	37
182	38	186	39	188	40	196	41
200	42	210	43	212	44	216	45
226	46	236	47	240	48	246	49
252	50	254	51	264	52	270	53
272	54	278	55	282	56	288	57
300	58	304	59	310	60	320	61
324	62	330	63	336	64	342	65
350	66	356	67	366	68	370	69
378	70	384	71	390	72	392	73
398	74	410	75	420	76	422	77
426	78	432	79	438	80	446	81
450	82	452	83	462	84	470	85
476	86	482	87	486	88	494	89
504	90	506	91	512	92	516	93
518	94	530	95	536	96	546	97
552	98	558	99	572	100	576	101
578	102	590	103	600	104	602	105
606	106	612	107	616	108	628	109
634	110	640	111	646	112	654	113
656	114	662	115	672	116	680	117
686	118	692	119	702	120	708	121
714	122	722	123	732	124	740	125
746	126	750	127	760	128	768	129
774	130	780	131	784	132	794	133
804	134	808	135	812	136	816	137
818	138	828	139	840	140	842	141
848	142	856	143	864	144	872	145
878	146	882	147	892	148	902	149
908	150	912	151	926	152	930	153
934	154	946	155	952	156	960	157
970	158	974	159	986	160	990	161
998	162	1002	163	1012	164	1022	165
1026	166	1032	167	1036	168	1044	169
1050	170						

TABLE 2.

$$
\begin{aligned}
\varrho^*(1066) &\le \varrho^*(1050) + \varrho^*(16) = 170 + 5 = 175 = \pi(1039)\\
\varrho^*(1070) &\le \varrho^*(1050) + \varrho^*(20) = 170 + 6 = 176 = \pi(1049)\\
\varrho^*(1076) &\le \varrho^*(1050) + \varrho^*(26) = 170 + 7 = 177 = \pi(1051)\\
\varrho^*(1080) &\le \varrho^*(1050) + \varrho^*(30) = 170 + 8 = 178 = \pi(1061)\\
\varrho^*(1082) &\le \varrho^*(1050) + \varrho^*(32) = 170 + 9 = 179 = \pi(1063)\\
\varrho^*(1086) &\le \varrho^*(1050) + \varrho^*(36) = 170 + 10 = 180 = \pi(1069)\\
\varrho^*(1092) &\le \varrho^*(1050) + \varrho^*(42) = 170 + 11 = 181 = \pi(1087)\\
\varrho^*(1098) &\le \varrho^*(1050) + \varrho^*(48) = 170 + 12 = 182 = \pi(1091)\\
\varrho^*(1100) &\le \varrho^*(1050) + \varrho^*(50) = 170 + 13 = 183 = \pi(1093)\\
\varrho^*(1106) &\le \varrho^*(1050) + \varrho^*(56) = 170 + 14 = 184 = \pi(1097)\\
\varrho^*(1110) &\le \varrho^*(1050) + \varrho^*(60) = 170 + 15 = 185 = \pi(1103)\\
\varrho^*(1116) &\le \varrho^*(1050) + \varrho^*(66) = 170 + 16 = 186 = \pi(1109)\\
\varrho^*(1120) &\le \varrho^*(1050) + \varrho^*(70) = 170 + 17 = 187 = \pi(1117)
\end{aligned}
$$

Step 5. Decrement i.

Step 6. If there is a sifted residue class at level i, go to Step 5. Otherwise, increment r_i. If $r_i \le p_i - 1$, sift the residue class r_i modulo p_i in the interval. If $u_i < L$, go to Step 4. If $r_i = p$, go to Step 4.

Step 7. Return to Step 2 until all cases have been checked.

Step 8. Output L as $\varrho^*(x)$.

The values for $\varrho^*(x)$ in Table 1 were calculated using this algorithm. The x values are the lengths of the largest interval with an admissible sequence of $\varrho^*(x)$ elements.

We have made our implementation of this algorithm in C available for public ftp access at `math.byu.edu` in the directory `/pub/clark/dense`. Other programs described below may also be found in this directory.

Since we are interested mainly in the set of x satisfying $\varrho^*(x) \le \pi(x)$, we can use the method that Schinzel [5] applied to Smith's [6] results. Schinzel used the inequality $\varrho^*(x+y) \le \varrho^*(x) + \varrho^*(y)$ to find a bound on $\varrho^*(x+y)$. The bounds for $\varrho^*(x)$ in Table 2 are derived from Table 1 using this inequality.

These inequalities show that for $2 \le x \le 1120$ that $\varrho^*(x) < \pi(x)$; that is, Conjecture B holds for $x \le 1120$.

If we set $L = \pi(x)$ in algorithm 1, we can further extend this range. This change allows us to ignore admissible sequences containing less than $\pi(x)$ points. Using this modification we calculated that for $1121 \le x \le 1426$, $\varrho^*(x) \le \pi(x)$. One interesting result of this computation is that $\varrho^*(1422) = \pi(1422) = 223$.

TABLE 3. Admissible sequence of length 1422

(1422,{2, 4, 4, 8, 11, 3, 11, 2, 9, 4, 32, 33, 32, 16, 24, 16, 34, 56, 17,
23, 18, 66, 16, 65, 54, 14, 24, 30, 33, 62, 9, 35, 23, 52, 14, 25, 2,
26, 3, 11, 2, 2, 2, 2, 2, 16, 18, 5, 3, 8, 2, 3, 3, 4, 3, 5, 3, 4, 2, 4, 3,
3, 4, 3, 16, 2, 2, 3, 11, 3, 2, 2, 2, 3, 3, . . . , 2})

The running time for this example was approximately eleven days. The programs described in this paper were run on an SGI Challenge L with 200 MHz R4400 processors. All running times refer to this computer.

3. Conjecture B: Dense admissible sequences

In the previous section, we were interested in extending the range of validity of Conjecture B. In this section, we want to find values of x such that $\varrho^*(x) > \pi(x)$. Hensley and Richards [3] showed that $\varrho^*(x) > \pi(x)$ for large enough x. In collaboration with Stenberg, Hensley and Richards found that $\varrho^*(20000) > \pi(20000)$ (see the note at the end of [3]). Vekha and Richards [7] found an admissible sequence of 1412 points in an interval of length 11763. Since $\pi(11763) = 1409$, this gives $\varrho^*(x) > \pi(x)$ for smaller x.

For intervals of length greater than 2000, it is not computationally feasible to make a thorough search in order to determine $\varrho^*(x)$. In our search for smaller x satisfying $\varrho^*(x) > \pi(x)$, we considered only those x for which $\pi(x) - \mathrm{Li}(x)$ is small. That is, we chose intervals with fewer primes than would be expected on average. For the first n primes, we considered all possible combinations of sifted residue classes. For the remaining primes, we chose the residue class which removes the fewest number of unsifted elements from the sequence. Choosing $n = 9$, we found an admissible sequence of 715 points with interval length 5380 and $\pi(5380) = 708$. The running time for this computation was approximately nine days.

Given a dense admissible sequence, one can search for dense subsequences with shorter interval length. For this example, we found an admissible subsequence of 657 points in an interval of length 4916 and $\pi(4916) = 656$.

Table 4. Admissible sequence of length 4916

(4916, {1, 3, 3, 5, 4, 10, 14, 21, 1, 25, 23, 22, 40, 29, 24, 7, 19, 55, 8, 18, 48,
68,9, 51, 57, 93, 18, 9, 91, 121, 51, 34, 104, 137, 104, 128, 146, 57, 118, 178,
142, 172, 19, 1, 17, 22, 125, 109, 21, 104, 97, 60, 69, 19, 130, 53, 46, 5, 4, 9,
29, 77, 53, 7, 14, 10, 28, 1, 23, 4, 1, 51, 24, 14, 3, 3, 4, 9, 8, 18, 4, 4, 1, 1, 5, 1,
1, 21, 5, 30, 7, 8, 4, 5, 1, 3, 1, 1, 4, 1, 1, 10, 8, 7, 1, 4, 3, 1, 4, 5, 7, 4, 1, . . . , 1})

4. Incompatibility of Conjecture C and prime k-tuples conjecture

Schinzel (see [5]) showed that

$$\varrho^*(x) - \pi(x) \geq (2\log 2 - \epsilon)x/\log^2 x$$

assuming a special sifting hypothesis. Since $2\pi(x/2) - \pi(x)$ is asymptotically equal to $\log 2 \times x/\log^2 x$, we suspected that Conjecture C and the prime k-tuples conjecture would be incompatible. That is, for large values of x, $\varrho^*(x) > 2\pi(x/2)$. We investigated this problem using the method employed by Schinzel in his proof. We sifted the residue class n_i with $i \equiv -1 \pmod p$ for all primes p up to a certain value which we denote by s in Table 5. For the remaining primes we chose the residue class so that the fewest remaining elements of the interval are sifted as in our previous computations. Let $S(x)$ be the number of elements in the admissible sequence produced by this computation.

Table 5.

x	s	$2\pi(x/2)$	$S(x)$	$2\pi(x/2) - S(x)$
1355252	5273	109752	107451	2301
3065252	11119	232888	229247	3641
8040390	23687	568944	563250	5694
13584312	39419	926712	919689	7023
29250264	71209	1896054	1887469	8585
44904720	111733	2831608	2823187	8421
73834196	176041	4513606	4507094	6512
109865792	228881	6556668	6554038	2630
130808636	277169	7725840	7725926	-86
160471116	343327	9364504	9369426	-4922
367702770	654697	20464876	20509567	-44691

The last three lines in the table indicate the incompatibility of Conjecture C and the prime k-tuples conjecture. The running time for $x = 130808636$ was approximately eleven days. We have placed an output of the residue classes sifted for the primes between s and $S(x)$ in the public ftp directory referred to above.

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