MERGING THE BRAMBLE-PASCIAK-STEINBACH
AND THE CROUZEIX-THOMÉE CRITERION
FOR $H^1$-STABILITY OF THE $L^2$-PROJECTION
ONTO FINITE ELEMENT SPACES

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ABSTRACT. Suppose $S \subset H^1(\Omega)$ is a finite-dimensional linear space based on
a triangulation $T$ of a domain $\Omega$, and let $\Pi : L^2(\Omega) \to L^2(\Omega)$ denote the
$L^2$-projection onto $S$. Provided the mass matrix of each element $T \in T$ and
the surrounding mesh-sizes obey the inequalities due to Bramble, Pasciak, and
Steinbach or that neighboring element-sizes obey the global growth-condition
due to Crouzeix and Thomée, $\Pi$ is $H^1$-stable: For all $u \in H^1(\Omega)$ we have
$\| \Pi u \|_{H^1(\Omega)} \leq C \| u \|_{H^1(\Omega)}$ with a constant $C$ that is independent of, e.g.,
the dimension of $S$.

This paper provides a more flexible version of the Bramble-Pasciak-
Steinbach criterion for $H^1$-stability on an abstract level. In its general ver-
sion, (i) the criterion is applicable to all kind of finite element spaces and
yields, in particular, $H^1$-stability for nonconforming schemes on arbitrary
(shape-regular) meshes; (ii) it is weaker than (i.e., implied by) either
the Bramble-Pasciak-Steinbach or the Crouzeix-Thomée criterion for regular tri-
angulations into triangles; (iii) it guarantees $H^1$-stability of $\Pi$ a priori for a
class of adaptively-refined triangulations into right isosceles triangles.

1. THE $L^2$-PROJECTION IN A FINITE ELEMENT SPACE

Suppose the bounded Lipschitz domain $\Omega$ in $\mathbb{R}^d$ is partitioned into a triangulation
$T$, i.e., $\Omega = \bigcup T$ for a finite set $T$ of elements $T$ which are closed and whose
interiors are Lipschitz domains. The intersection of two distinct elements has zero
dimensional Lebesgue measure. To describe nonconforming finite elements, let $H$
be a closed subset of $H^1(T)$,

$$H_0^0(\Omega) \subseteq H \subseteq H^1(T) := \{ u \in L^2(\Omega) : \forall T \in T, u|_T \in H^1(T) \},$$

closed with respect to the semi-norm $\| \nabla_T \cdot \|$, where $\| \cdot \|$ denotes the $L^2(\Omega)$-norm
and $\nabla_T$ is the $T$-piecewise action of the gradient $\nabla$ (different from the distributional
gradient for discontinuous arguments). For instance, in the conforming setting, the
choice of $H = H_0^0(\Omega)$ or $H = H^1(\Omega)$ is a typical example.

Suppose that $S \subset H$ is an $n$-dimensional subspace with a (not necessarily nodal)
basis $(\varphi_1, \varphi_2, \ldots, \varphi_n)$, and let $\Pi$ denote the $L^2(\Omega)$-projection defined, for all $u \in H$,
by

\[ \Pi u \in \mathcal{S} \quad \text{and} \quad \int_{\Omega} (u - \Pi u) \varphi_j \, dx = 0 \quad \text{for all} \ j = 1, \ldots, n. \]

In this context, the \( L^2 \)-projection \( \Pi \) is called \( H^1 \)-stable if there exists a constant \( c_1 > 0 \) with

\[ \| \nabla_T \Pi u \| \leq c_1 \| \nabla_T u \| \quad \text{for all} \ u \in H. \]

Two sets of parameters, the \( n \) positive parameters \( (d_1, d_2, \ldots, d_n) \) and the \( T \)-piecewise constant weight \( h_T \), defined on \( T \in T \) by \( h_T > 0 \), will provide the link between the triangulation \( T \) and the discrete space \( \mathcal{S} \). Their choice is arbitrary up to the severe restriction of inequality (7) below.

To verify \( H^1 \)-stability of the \( L^2 \)-projection (3) we suppose that there exist a (possibly nonlinear) mapping \( P : H \to \mathcal{S} \) and a constant \( c_2 > 0 \) that satisfy, for all \( u \in H \),

\[ \| \nabla_T P(u) \| + \| h_T^{-1}(u - P(u)) \| \leq c_2 \| \nabla_T u \|. \]

Remark 1. In Sections 4, 5, and 6, \( h_T \) will be the element-size and \( d_\ell \) a measure for the size of \( \text{supp} \varphi_\ell \).

Remark 2. Approximation operators which satisfy (4) for \( h_T = \text{diam}(T) \) can be found in [Ca, CP, Cl].

2. Mass matrices and two inequalities

To define the mass matrix for a given \( T \in T \), let \( \ell(T, 1), \ell(T, 2), \ldots, \ell(T, m(T)) \) denote exactly those indices of basis functions whose restrictions \( \psi_{T,j} := \varphi_{\ell(T,j)} \big|_T \in H^1(T), 1 \leq j \leq m(T) \), on \( T \) are nonzero. Then the shape functions \( \psi_{T,j} : j = 1, \ldots, m(T) \) on \( T \) satisfy an inverse inequality (by equivalence of norms),

\[ \| \sum_{j=1}^{m(T)} \xi_j \nabla \psi_{T,j} \|_{L^2(T)} \leq c_3 h_T^{-1} \| \sum_{j=1}^{m(T)} \xi_j \psi_{T,j} \|_{L^2(T)} \]

for all \( (\xi_1, \ldots, \xi_{m(T)}) \in \mathbb{R}^{m(T)} \).

The (local) \( m(T) \times m(T) \)-dimensional mass matrix \( M(T) \) and the diagonal matrix \( \Lambda(T) \),

\[ \Lambda(T)_{jk} = \frac{h_T}{d_{\ell(T,j)}} \delta_{jk} \quad \text{and} \quad M(T)_{jk} = \int_T \psi_{T,j} \psi_{T,k} \, dx \quad \text{for all} \ j, k = 1, \ldots, m(T), \]

(\( \delta_{jk} \in \{0, 1\} \) denotes Kronecker’s symbol) are supposed to satisfy, for constants \( c_4, c_5 > 0 \),

\[ c_4 x \cdot \Lambda(T)^2 M(T) \Lambda(T)^2 x \leq x \cdot M(T) x \leq c_5 x \cdot \Lambda(T)^2 M(T) x \quad \text{for all} \ x \in \mathbb{R}^{m(T)}. \]

Remark 3. Inverse estimates [BS, Cl] provide (5) for a size-independent constant \( c_5 \) if \( h_T = \text{diam}(T) \).

Remark 4. The first inequality of (7) merely reflects a proper scaling of \( d_{\ell(T,j)} \) and \( h_T \).
Remark 5. The second inequality of (7) implies that $\Lambda(T)^2 M(T)$ has positive definite symmetric part. This is the crucial condition and relates the mass-matrix $M(T)$ to neighboring mesh-sizes.

Remark 6. We stress that (7) can always be satisfied even with $c_4 = c_5 = 1$ if we let $h_T = d_{\ell(T,j)}$ be equal to a global discretization parameter. For quasi-uniform meshes this implies (7).

Remark 7. In the original version [BPS, 8], $d_j$ is fixed as the arithmetic mean of all $h_T$ with $T \subset \text{supp} \varphi_j$, where $h_T^d$ is the $d$-dimensional volume of an element $T \in \mathcal{T}$. Then, the Bramble-Pasciak-Steinbach criterion [BPS] (4.2) implies the crucial second inequality in (7) (and is, in particular situations, equivalent).

3. A modified Bramble-Pasciak-Steinbach criterion for $H^1$-stability

Under the present assumptions (1)-(2) and (4)-(7) we have $H^1$-stability of $\Pi$.

Theorem 1. We have (3) with $c_1 = c_2 \max\{1, c_3 c_5 / c_4\}$.

Proof. Given $u \in H$, define $q_h := P(u) - \Pi u = \sum_{\ell=1}^n q_\ell \varphi_\ell \in \mathcal{S}$ and $p_h := \sum_{\ell=1}^n q_\ell d_{\ell,}\varphi_\ell \in S$ so that

$$q_h|_T = \sum_{\ell=1}^n q_\ell \varphi_\ell|_T = \sum_{j=1}^{m(T)} \xi_{T,j} \psi_{T,j} \quad \text{on } T \in \mathcal{T}$$

for certain coefficient vectors $x_T = (\xi_{T,1}, \ldots, \xi_{T,m(T)}) = (q_{\ell(T,1)}, \ldots, q_{\ell(T,m(T))})$. The triangle inequality for $\Pi u = P(u) - q_h$ and (1)-(4) show that

$$\|\nabla_T \Pi u\| \leq \|\nabla_T P(u)\| + \|\nabla_T q_h\| \leq c_2 \|\nabla_T u\| + c_3 \|h_T^{-1} q_h\|.$$

According to direct calculations with coefficients from (8), the second inequality in (7) yields

$$c_5^{-1} \|h_T^{-1} q_h\|^2 = c_5^{-1} \sum_{T \in \mathcal{T}} h_T^{-2} x_T \cdot M(T) x_T \leq \sum_{T \in \mathcal{T}} h_T^{-2} x_T \cdot \Lambda(T)^2 M(T) x_T$$

$$= \sum_{T \in \mathcal{T}} \sum_{j=1}^{m(T)} \frac{q_{\ell(T,j)}}{d_{\ell(T,j)}} \int_T \varphi_{\ell(T,j)} q_h \, dx = \int_{\Omega} p_h q_h \, dx$$

$$= \int_{\Omega} p_h (P(u) - u) \, dx \leq c_2 \|h_T p_h\| \|\nabla_T u\|$$

because of (2), Cauchy’s inequality, and (3). Similar arguments and (7) lead to

$$c_4^2 \|h_T p_h\|^2 = c_4^2 \sum_{T \in \mathcal{T}} h_T^{-2} x_T \cdot \Lambda(T)^2 M(T) \Lambda(T)^2 x_T$$

$$\leq \sum_{T \in \mathcal{T}} h_T^{-2} x_T \cdot M(T) x_T = \|h_T^{-1} q_h\|^2.$$ 

Utilizing this in (10), we obtain a bound of $\|h_T^{-1} q_h\|$, which we need in (3) to see (4).
4. Examples for Courant triangles

Suppose $T$ is a regular triangulation (in the sense of Ciarlet [BS, Ci]) of the bounded Lipschitz domain $\Omega$ in the plane into triangles. Homogeneous Dirichlet conditions may apply on a (relatively closed and possibly empty) boundary part $\Gamma_D$ (matched exactly by edges). Each node $z \in \mathcal{N}$ with nodal function $\varphi_z$ involves a positive real number $d_z$ such that $h_T/d_z + d_z/h_T \leq c_0$ for all triangles $T \in \mathcal{T}$ of diameter $h_T$ with vertex $z$. Let $\mathcal{S} := \text{span} \{ \varphi_z : z \in \mathcal{K} \}$, where $\mathcal{K} := \mathcal{N} \setminus \Gamma_D$ denotes the set of free nodes, and for the preceding notation identify $(\varphi_z : z \in \mathcal{K})$ and the parameters $(d_z : z \in \mathcal{K})$ with $(\varphi_1, \varphi_2, \ldots, \varphi_n)$ and $(d_1, d_2, \ldots, d_n)$, respectively.

**Theorem 2.** Suppose that $d_z/d_\zeta \leq \kappa < \sqrt{2} + \sqrt{3} \approx 3.1462$ for all vertices $z$ and $\zeta$ of some triangle $T \in \mathcal{T}$. Then we have $[4]$.

**Proof.** The mass-matrix of a fixed $T \in \mathcal{T}$ is a multiple of the $3 \times 3$ matrix $M$ with $M_{jk} = 1 + \delta_{jk}$ and $\Delta(T)$ has diagonal entries $\lambda_1, \lambda_2, \lambda_3 > 0$ with $\lambda_j/\lambda_k \leq \kappa$. The eigenvalues of $\Delta(T)^{-1} \Delta(T)^{-1}$ for $A := (\Delta(T)^2 M + M \Delta(T)^2)/2$ can be calculated [BPS, S], and their smallest value is $(5 - \mu)$ for $\mu^2 := \sum_{j,k=1}^3 \lambda_j^2/\lambda_k^2$. A straightforward analysis reveals that $\mu^2 \leq 3 + 2(1 + \kappa^2 + 1/\kappa^2) < 25$, which shows that $A$ is positive definite. Therefore, $(x \cdot Ax)^{1/2}$ defines a norm which is equivalent to $|x|$ in $\mathbb{R}^3$. This and $h_T/d_z \leq c_0$ yield $[7]$. $\square$

**Remark 8.** The proof shows that $\sum_{j,k=1}^3 \lambda_j^2/\lambda_k^2 \leq \nu < 22$ for some constant $\nu$ suffices for $[4]$. Given $d_z$ as in Remark 4 this is the a posteriori criterion of [BPS, S] for two dimensions.

The technical assumption on the artificial, extended triangulation in the following theorem merely reduces the consideration to interior triangles for brevity.

**Theorem 3.** Suppose $T \subset \hat{T}$ for some regular triangulation $\hat{T}$ of a Lipschitz domain $\bar{\Omega} \supset \Omega$ such that $\hat{T}$ consists of right isosceles triangles only, there are no hanging nodes, and each free node on the boundary is an interior node of $\bar{\Omega}$. Then we have $[4]$.

**Proof.** Theorem 2 yields the assertion if we take

$$d_z = \min \{|z - \zeta| : \zeta \in \mathcal{N}, \delta(z, \zeta) = 1\},$$

where $\delta(z, \zeta) = 1$ characterizes neighboring vertices $z$ and $\zeta$, i.e., $z, \zeta \in T \cap \mathcal{N}$ for at least one $T \in \mathcal{T}$. Since (up to scaling, transition, and rotation) there are only

![Figure 1: Part of a mesh as a smallest neighborhood of the reference triangle.](image-url)
a finite number of possible configurations, it can be checked by a finite number of figures that $d_z/d_\zeta \leq \sqrt{8}$. Figure 1 illustrates a deduction: Suppose $T$ has the vertices $(0,0), (1,0),$ and $(0,1)$. Then, the patch $\text{supp} \varphi_z$ of $z = (0,0)$ must include the polygonal domain with vertices $(0,1), (-.5,.5), (-.5,0), (0,-.5), (.5,-.5), (1,0)$. This shows that $1/\sqrt{8} \leq d_{(1,0)} \leq 1$. Similarly, the patch $\text{supp} \varphi_{(1,0)}$ must include the polygon $(0,0), (.5,-.5), (1,-.5), (1.5,0), (1.5,3), (1,1), (0,1)$, whence $1/\sqrt{8} \leq d_{(1,0)}, d_{(0,1)} \leq 1$. Consequently, $d_z/d_\zeta \leq \sqrt{8}$ for any choice of two vertices $z$ and $\zeta$ of $T$.

\textbf{Example 1.} Let $T$ be the mesh of Figure 2 that consists of 8 triangles in a regular pattern that match the square $\Omega := (0,H)^2$ for positive $H = 1 + \lambda$, where non-diagonals' lengths are either $\lambda < 1$ or 1. For the nodes 1, 2, and 3 of Figure 2 the choice of $(d_1, d_2, d_3)$ from [BPS], mentioned in Remark 7, is

\begin{align*}
((\lambda + 2\lambda^{1/2})/3, (\lambda^2 + \lambda^{1/2} + 1)/3, \lambda)/\sqrt{2}.
\end{align*}

The conditions of the Bramble-Pasciak-Steinbach criterion (cf. Remark 8) and those of Theorem 2 are violated for $\lambda < .1349$, which corresponds to an aspect ratio larger than 7.4122 However, Theorem 1 with the parameters from Remark 6 guarantees 13 for any positive $\lambda$ (with a $\lambda$-dependent constant $c_1 = c_1(\lambda)$).

\textbf{Example 2.} Take a scaled copy of $\Omega$ and the mesh from Example 1 and extend it by reflection about the $x_1$-axis, the $x_2$-axis, and about the anti-diagonal through the origin to $h(-1,1)^2$; and then extend it $2h$-periodically to the entire plane. The calculations of Example 1 remain valid and we conclude that, for a fixed $\lambda < .1349$, the Bramble-Pasciak-Steinbach criterion is not applicable, but Remark 6 (or the Crouzeix-Thomee criterion) guarantees 14 with an $h$-independent constant $c_1 = c_1(\lambda)$.

The nonconforming Crouzeix-Raviart finite element (cf., e.g., BS, C4) concludes our first series of applications.

\textbf{Theorem 4.}\hspace{1em}Suppose $T$ is an arbitrary shape-regular triangulation into triangles and $S$ denotes the $T$-piecewise affine functions which are continuous at midpoints of edges. Then we have 15.

\textbf{Proof.} The mass-matrices are diagonal, so 15 is a consequence of shape-regularity. The operator $P$ can be chosen exactly as in the conforming case. \hfill $\square$
5. WEAKENING OF THE CROUZEIX-THOMÉE CRITERION FOR $H^1$-STABILITY

Part of the Crouzeix-Thomée criterion \([CT]\) is the existence of $c_7$ and $1 \leq \kappa := \sqrt{\alpha} < \sqrt{2} + \sqrt{3}$ such that
\begin{equation}
|T_1|/|T_2| \leq c_7 \alpha^{l(T_1,T_2)} \tag{11}
\end{equation}
for all $T_1, T_2 \in \mathcal{T}$.

Here, $|T_j|$ is the area of $T_j \in \mathcal{T}$ and the neighbor-index $l(T_1, T_2)$ might be defined via a metric $\delta$ on the nodes $\mathcal{N}$: For two distinct nodes $z$ and $\zeta$, $\delta(z, \zeta)$ is the smallest integer $j$ such that there exists a polygon $(z_1, z_2, \ldots, z_j)$ of nodes $z_1, \ldots, z_j \in \mathcal{N}$ which connects $z = z_1$ with $\zeta = z_j$ along edges, i.e., $\{z_i, z_{i+1}\} \subset \partial T_i$ for some $T_i \in \mathcal{T}$ and all $i = 1, \ldots, j - 1$; $\delta(z, z) := 0$. For any $T, K \in \mathcal{T}$ and $z \in \mathcal{N}$, let $\delta(z, T) := \min_{\zeta \in T \cap \mathcal{N}} \delta(z, \zeta)$ and $\delta(K, T) = \min_{z \in K \cap \mathcal{N}} \delta(z, T)$. Then, $l(T_1, T_2) = \delta(T_1, T_2) + 1$ if $T_1 \neq T_2$, while $l(T_1, T_2) = 0$ if and only if $T_1 = T_2$.

At first glance, the local Bramble-Pasciak-Steinbach and the global Crouzeix-Thomée criteria appear incomparable: a large constant $c_7$ prohibits a direct application of (11) in the spirit of Theorems 2 and 3 (as $d_z/d_\zeta \leq c_8 |(T_1|/|T_2|)^{1/2} \leq c_7^{1/2} c_{8 \kappa} < \sqrt{2} + \sqrt{3}$ for $\delta(z, \zeta) = 1$). However, all necessities are provided by
\begin{equation}
d_z := \min_{T \in \mathcal{T}} h_T \kappa^{\delta(z, T)} = \frac{h_K}{h_T \kappa^{\delta(z, T)}} = \frac{|K|^{1/2}}{|T|^{1/2}} \kappa^{-\delta(z, T)} \leq \sqrt{c_7} \alpha^{l(T, K) - \delta(z, T)} \leq \sqrt{c_7} \alpha^{l(T, K)}.
\end{equation}

To bound $d_z/d_\zeta$ for $z, \zeta \in \mathcal{N}$ with $\delta(z, \zeta) = 1$, let $K \in \mathcal{T}$ satisfy $d_\zeta = h_K \kappa^{\delta(z, K)}$. The definition (12) and $\delta(z, K) - \delta(\zeta, K) \leq 1$ show that
\begin{equation}
d_z/d_\zeta = \frac{d_z}{h_K \kappa^{\delta(z, K)}} \leq \frac{h_K \kappa^{\delta(z, K)}}{h_K \kappa^{\delta(\zeta, K)}} = \kappa^{\delta(z, K) - \delta(\zeta, K)} \leq \kappa. \quad \Box
\end{equation}

Example 3. There exists an adaptively-refined mesh \([CV]\) Figure 1] of right isosceles triangles where the modified Bramble-Pasciak-Steinbach criterion guarantees $H^1$-stability (cf. \([S]\) or Theorem 3) while the Crouzeix-Thomée criterion is not applicable.

REFERENCES


\[ H^1\text{-}\text{stability of the } L^2\text{-projection in FEM} \]


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