

SZEGÖ QUADRATURE FORMULAS FOR CERTAIN JACOBI-TYPE WEIGHT FUNCTIONS

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ABSTRACT. In this paper we are concerned with the estimation of integrals on the unit circle of the form $\int_0^{2\pi} f(e^{i\theta})\omega(\theta)d\theta$ by means of the so-called Szegő quadrature formulas, i.e., formulas of the type $\sum_{j=1}^n \lambda_j f(x_j)$ with distinct nodes on the unit circle, exactly integrating Laurent polynomials in subspaces of dimension as high as possible. When considering certain weight functions $\omega(\theta)$ related to the Jacobi functions for the interval $[-1, 1]$, nodes $\{x_j\}_{j=1}^n$ and weights $\{\lambda_j\}_{j=1}^n$ in Szegő quadrature formulas are explicitly deduced. Illustrative numerical examples are also given.

1. INTRODUCTION

When considering the approximate calculation of integrals of the form

$$I_\psi(f) = \int_a^b f(x)d\psi(x), \quad (-\infty \leq a < b \leq +\infty)$$

by means of a quadrature formula like

$$I_n(f) = \sum_{j=1}^n A_j f(x_j)$$

it is well known that distinct nodes $\{x_j\}_{j=1}^n$ in (a, b) and positive weights $\{A_j\}_{j=1}^n$ can be uniquely determined so that $I_\psi(p) = I_n(p)$ for any p in Π_{2n-1} (the space of polynomials of degree $2n - 1$ at most). In this case, $I_n(f)$ represents the n th Gauss-Christoffel quadrature formula so that its nodes $\{x_j\}_{j=1}^n$ are the zeros of the n th orthogonal polynomial with respect to $d\psi$.

In this paper, we shall be concerned with the estimation of integrals on the unit circle i.e.,

$$I_\psi(f) = \int_0^{2\pi} f(e^{i\theta})d\psi(\theta),$$

Received by the editor February 11, 2000 and, in revised form, July 10, 2000.

2000 *Mathematics Subject Classification.* Primary 41A55, 33C45.

Key words and phrases. Weight functions, quadrature formulas, orthogonal polynomials, Szegő polynomials, error bounds.

The work of the first author was performed as part of a grant of the Gobierno de Canarias.

The work of the second author was supported by the Scientific Research Project of the Spanish D.G.E.S. under contract PB96-1029.

where ψ is a positive Borel measure on $[0, 2\pi]$. For our purpose, instead of a measure ψ , we could deal with a distribution function on $[0, 2\pi]$ i.e., a real valued non-decreasing function with infinitely many points of increase on $[0, 2\pi]$, (see [7]). In order to make the paper consistent throughout, measures will be used.

We shall make use of the so-called Szegő quadrature formulas which are to some extent analogous on the unit circle to Gauss quadrature formulas on the real line. The Szegő quadrature formulas were introduced by Jones, Njåstad and Thron in [6] in connection with the trigonometric moment problem, where orthogonal polynomials on the unit circle with respect to ψ (Szegő polynomials), become crucial. Except for very special measures ψ , it is difficult to obtain explicit expressions for these polynomials. For instance, it is known that for the Lebesgue measure, the sequence of monic Szegő polynomials is given by $\rho_n(z) = z^n$ for all n . If we consider the measure

$$(1.1) \quad d\psi(\theta) = \frac{d\theta}{2\pi|h(e^{i\theta})|^2},$$

where $h(z) = \prod_{i=1}^k (z - \alpha_i)$ with $|\alpha_i| < 1$, then, (1.1) represents a rational modification of the Lebesgue measure and the system of Szegő polynomials is given explicitly in [11], pp. 289-290. For further details see [5].

Our main interest is centered on calculating explicit expressions for Szegő polynomials for certain Jacobi-type weight functions, which allows us to obtain an explicit formula for Szegő quadrature formulas.

We shall use the notation $\mathbb{T} = \{z : |z| = 1\}$ and $\mathbb{D} = \{z : |z| < 1\}$ for the unit circle and the open unit disc, respectively. Also, for p and q nonnegative integers, $p \leq q$, $\Lambda_{p,q}$ will denote the space of Laurent polynomials of the form $L(z) = \sum_{j=p}^q \alpha_j z^j$, $\alpha_j \in \mathbb{C}$. Λ will denote the space of all Laurent polynomials and Π the space of algebraic polynomials.

The paper is organized as follows. In Section 2, preliminary results concerning Szegő polynomials, associated polynomials, Szegő quadrature formulas and second kind measures are given. In Sections 3 we obtain explicit representations for Szegő quadrature formulas with respect to the measures $d\psi(\theta) = \frac{\sin^2(\theta)}{2\pi}d\theta$, $d\psi(\theta) = \frac{1+\cos\theta}{2\pi}d\theta$ and $d\psi(\theta) = \frac{1-\cos\theta}{2\pi}d\theta$, respectively, along with integral error expressions and computable error bounds when considering analytic integrands. Finally, in Section 4, illustrative numerical examples are given. For other results concerning Szegő quadrature formulas see [3].

2. PRELIMINARY RESULTS

Szegő polynomials. Let ψ be a positive Borel measure on $[0, 2\pi]$. Let us consider the following inner product in the linear space Π of polynomials with complex coefficients:

$$(f, g)_\psi = \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\psi(\theta).$$

By applying the Gram-Schmidt orthogonalization process to $\{1, z, \dots, z^n\}$, an orthogonal basis $\{\rho_k(z)\}_{k=0}^n$ can be deduced such that $\deg(\rho_k) = k$ and

$$(\rho_j, \rho_k)_\psi = K_j \delta_{j,k}, \quad K_j > 0, \quad 0 \leq j, k \leq n.$$

Thus, when taking $\rho_n(z)$ monic for each n , the so-called monic orthogonal polynomials on the unit circle \mathbb{T} or Szegő polynomials are obtained. These polynomials

satisfy recurrence relations

$$(2.1) \quad \begin{aligned} \rho_0(z) &= \rho_0^*(z) = 1, \\ \rho_n(z) &= z\rho_{n-1}(z) + \delta_n\rho_{n-1}^*(z), \quad n = 1, 2, 3, \dots, \\ \rho_n^*(z) &= \bar{\delta}_nz\rho_{n-1}(z) + \rho_{n-1}^*(z), \quad n = 1, 2, 3, \dots, \end{aligned}$$

where $\delta_n := \rho_n(0), n = 1, 2, 3, \dots$, are called the reflection coefficients and $\rho_n^*(z) = z^n \overline{\rho_n(1/\bar{z})}$. Conversely, if a sequence $\{\rho_n\}$ such that, for each n the polynomial ρ_n has exact degree n and satisfies (2.1), is given, then there exists a measure ψ so that $\{\rho_n\}$ is the corresponding sequence of monic Szegő polynomials.

In general, it is difficult to obtain an explicit expression for these polynomials. If we want to calculate them, we can make use of the so-called Levinson algorithm (see e.g., [2]).

Associated Szegő polynomials. Let $\mu_k = \int_0^{2\pi} e^{-ki\theta} d\psi(\theta), k \in \mathbb{Z}$, then $\{\mu_k\}_k$ is called the moment sequence with respect to ψ . Note that $\mu_{-k} = \overline{\mu_k}$. Let $\{\pi_n\}_n$ be a sequence of polynomials defined in terms of the sequence $\{\rho_n\}_n$ and the moments $\{\mu_k\}$, by means of

$$(2.2) \quad \pi_n(z) := \begin{cases} \int_0^{2\pi} \frac{z+e^{i\theta}}{z-e^{i\theta}} (\rho_n(e^{i\theta}) - \rho_n(z)) d\psi(\theta) & \text{if } n = 1, 2, 3, \dots, \\ -\mu_0 & \text{if } n = 0. \end{cases}$$

These polynomials are called the associated Szegő polynomials and they satisfy the relations

$$(2.3) \quad \begin{aligned} \pi_0(z) &:= -\mu_0, \\ z\pi_{n-1}(z) - \delta_n\pi_{n-1}^*(z) &= \pi_n(z), \quad n = 1, 2, \dots, \\ \delta_nz\pi_{n-1}(z) - \pi_{n-1}^*(z) &= -\pi_n^*(z), \quad n = 1, 2, \dots, \end{aligned}$$

where δ_n are the reflection coefficients and $\pi_n^*(z) = z^n \overline{\pi_n(1/\bar{z})}$.

Szegő quadrature formulas. We are concerned with the estimation of integrals on the unit circle $\mathbb{T} = \{z : |z| = 1\}$ of the form

$$I_\psi(f) = \int_0^{2\pi} f(e^{i\theta})d\psi(\theta)$$

by a quadrature formula

$$(2.4) \quad I_n(f) = \sum_{j=1}^n \lambda_j f(x_j),$$

where the parameters λ_j and $x_j, 1 \leq j \leq n$, are determined so that $I_\psi(f) = I_n(f)$ for all $f \in \Lambda_{-(n-1),n-1}$. Furthermore, the nodes $\{x_j\}_{j=1}^n$ should lie on the unit circle. We know that the zeros of ρ_n lie in \mathbb{D} (see [1], p. 184). In order to construct a polynomial with zeros on \mathbb{T} , we define

$$(2.5) \quad B_n(z, \tau_n) = \rho_n(z) + \tau_n\rho_n^*(z), \quad |\tau_n| = 1, \quad \forall n.$$

This polynomial has n simple zeros on \mathbb{T} (see [6]) and $\{B_n(z, \tau_n)\}_n$ is called a sequence of para-orthogonal polynomials with respect to ψ . If $x_j, 1 \leq j \leq n$, are the zeros of $B_n(z, \tau_n)$, then (2.4) is called the n -point Szegő quadrature formula.

Let us define

$$(2.6) \quad A_n(z, \tau_n) = \pi_n(z) - \tau_n\pi_n^*(z), \quad |\tau_n| = 1, \quad \forall n,$$

where π_n is the polynomial associated with ρ_n with respect to ψ , then the weights λ_j , $1 \leq j \leq n$, of the n -point Szegő quadrature formula, can be expressed as ([5])

$$(2.7) \quad \lambda_j = \frac{-1}{2x_j} \frac{A_n(x_j, \tau_n)}{B'_n(x_j, \tau_n)}, \quad 1 \leq j \leq n,$$

where x_j , $1 \leq j \leq n$, are the zeros of $B_n(z, \tau_n)$.

To the measure ψ , we can associate Carathéodory functions F_ψ that is, functions analytic in $\mathbb{D} = \{z : |z| < 1\}$ with $\Re(F_\psi) \geq 0$ for $z \in \mathbb{D}$. We can express F_ψ as

$$(2.8) \quad F_\psi(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\psi(\theta) + ic, \quad c \in \mathbb{R},$$

which is known as the Herglotz-Riesz transform for the measure ψ . Clearly $c = \Im F_\psi(0)$.

Conversely, given a Carathéodory function F , there exists a positive Borel measure ψ on $[0, 2\pi]$ such that (2.8) holds.

Let us define the rational functions

$$(2.9) \quad R_n(z, \tau_n) = \frac{A_n(z, \tau_n)}{B_n(z, \tau_n)}, \quad |\tau_n| = 1, \quad \forall n,$$

where B_n and A_n are as in formulas (2.5) and (2.6), respectively. The role played by $F_\psi(z)$ and $R_n(z, \tau_n)$ with respect to the Szegő formula is displayed in the following

Lemma 2.1 (See [5]). *Let G be a neighbourhood of \mathbb{T} with boundary Γ consisting of a finite number of rectifiable Jordan curves, and let $\frac{f(z)}{z}$ be analytic in G . Then, for each $n = 1, 2, \dots$ it holds that*

$$E_n(f) = I_\psi(f) - I_n(f) = \frac{1}{2\pi i} \int_\Gamma (F_\psi(z) - R_n(z, \tau_n)) g(z) dz,$$

where $g(z) = -\frac{f(z)}{2z}$ and the rational function $R_n(z, \tau_n)$ is given as in formula (2.9).

Remark 2.2. The Lemma in particular covers the situation where G contains the whole of $\hat{\mathbb{C}} - \mathbb{D}$, and $\frac{f(z)}{z}$ is analytic at $z = \infty$. Furthermore, assuming that $\frac{f(z)}{z}$ is analytic in G is not a restriction, since when $f(0) \neq 0$, we can deal with the function $f(z) - f(0)$.

So, it is easy to see that

$$(2.10) \quad |E_n(f)| \leq \frac{1}{4\pi} \max_{\xi \in \Gamma} \left\{ \left| \frac{f(\xi)}{\xi} \right| \right\} \int_\Gamma |F_\psi(z) - R_n(z, \tau_n)| dz.$$

Second kind measure. From the recurrence relations (2.1), and the Favard Theorem ([6]) one sees that the sequence $\{\pi_n\}$ of associated polynomials is also orthogonal with respect to a measure $\tilde{\psi}$ which is said to be of the second kind associated with ψ . In order to calculate this measure, the following result is required,

Theorem 2.3 ([9]). *Let F be analytic in \mathbb{T} and suppose that F has simple poles at $\{z_k\}_{k=1}^n$ with $|z_k| = 1$ in such a way that the limits*

$$(2.11) \quad \lim_{z \rightarrow z_k} (z - z_k)F(z) = \gamma_k$$

exist and $\bar{z}_k \gamma_k \in \mathbb{R}$. Assume that the nontangential boundary values

$$\lim_{z \rightarrow e^{i\theta}} \Re \left\{ F(z) - \sum_{k=1}^n \frac{\gamma_k}{z - z_k} \right\}$$

exist a.e. on $[0, 2\pi]$ and are L_p -integrable on $[0, 2\pi]$ with $p \in (1, +\infty)$. Then,

$$F(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\psi(\theta) + i\Im(F(0))$$

with

$$(2.12) \quad d\psi(\theta) = (\Re(F(z))) d\theta - \sum_{k=1}^n \frac{2\pi\gamma_k}{z_k} d\delta(e^{i\theta} - z_k),$$

where $\Re(F(e^{i\theta})) = \lim_{z \rightarrow e^{i\theta}} \Re(f(z))$ means the nontangential limit and where

$$d\delta(e^{i\theta} - z_k) = \begin{cases} 1 & \text{if } e^{i\theta} = z_k, \\ 0 & \text{if } e^{i\theta} \neq z_k. \end{cases}$$

It can also be shown (see [10]) that

Theorem 2.4. *If F is the Carathéodory function corresponding to the measure ψ and $\tilde{\psi}$ is the measure of the second kind associated with ψ , then, $G = \frac{1}{F}$ is the Carathéodory function corresponding to $\tilde{\psi}$.*

3. CHEBYSHEV WEIGHT FUNCTIONS

Throughout the rest of the paper we will restrict ourselves to an absolutely continuous measure ψ , i.e., $d\psi(\theta) = \omega(\theta)d\theta$, $\omega(\theta) > 0$ a.e. on $[0, 2\pi]$. Thus, instead of a measure ψ we will deal with a weight function $\omega(\theta)$. A special case is the so-called Jacobi-type weight functions.

By a Jacobi-type weight function we mean a function of the form $\omega(\theta) = h(\cos \theta) |\sin \theta|$, $\theta \in [0, 2\pi]$ where $h(x) = (1 - x)^\alpha (1 + x)^\beta$, $x \in [-1, 1]$, $\alpha, \beta > -1$.

Thus,

$$\begin{aligned} \omega(\theta) &= h(\cos \theta) |\sin \theta| \\ &= (1 - \cos \theta)^\alpha (1 + \cos \theta)^\beta \sqrt{1 - \cos^2 \theta} \\ &= (1 - \cos \theta)^\alpha (1 + \cos \theta)^\beta \sqrt{(1 - \cos \theta)(1 + \cos \theta)} \\ &= (1 - \cos \theta)^{\alpha+1/2} (1 + \cos \theta)^{\beta+1/2}. \end{aligned}$$

Observe that we can write

$$(3.1) \quad \omega(\theta) = 2^{\gamma_1+\gamma_2} |e^{i\theta} - 1|^{2\gamma_1} |e^{i\theta} + 1|^{2\gamma_2} = 2^{\gamma_1+\gamma_2} \left| \sin \frac{\theta}{2} \right|^{2\gamma_1} \left| \cos \frac{\theta}{2} \right|^{2\gamma_2}$$

with $\gamma_1 = \alpha + \frac{1}{2}$; and $\gamma_2 = \beta + \frac{1}{2}$. Since $\alpha, \beta > -1$, then $\gamma_1, \gamma_2 > -\frac{1}{2}$. In this paper we shall restrict ourselves to the following cases

$$(3.2) \quad \begin{aligned} \alpha = \frac{1}{2}, \quad \beta = \frac{1}{2}, \quad \omega(\theta) &= \sin^2 \theta, \\ \alpha = -\frac{1}{2}, \quad \beta = \frac{1}{2}, \quad \omega(\theta) &= 1 + \cos \theta, \\ \alpha = \frac{1}{2}, \quad \beta = -\frac{1}{2}, \quad \omega(\theta) &= 1 - \cos \theta. \end{aligned}$$

These are three of the so-called Chebyshev weight functions, i.e., $\alpha, \beta \in \{\pm\frac{1}{2}\}$. The remaining case $\alpha = \beta = -\frac{1}{2}$ gives rise to $\omega(\theta) = 1$ (Lebesgue measure) where Szegö quadrature formulas are well known (see e.g., [5]).

On the other hand, taking advantage of the connection between orthogonal polynomials on the unit circle and the interval $[-1, 1]$, in [8] it is proved that for the weight functions (3.1), the sequence $\rho_n(z)$ of monic polynomials satisfies

$$(3.3) \quad \rho_n(0) = \frac{\alpha + \frac{1}{2} + (-1)^n(\beta + \frac{1}{2})}{n + \alpha + \beta + 1}.$$

We will consider first the weight function: $\omega(\theta) = \frac{\sin^2(\theta)}{2\pi}$. Its moment sequence is given by $\mu_0 = 1/2$, $\mu_1 = 0$, $\mu_2 = -1/4$ and $\mu_k = 0 \forall k \geq 3$. From (3.3) the reflection coefficients are $\delta_n := \rho_n(0) = \frac{1+(-1)^n}{n+2}$. The Szegő polynomials for some values of n in this case are

$$\begin{aligned}\rho_2(z) &= \frac{1}{2} + z^2, & \rho_3(z) &= z\left(\frac{1}{2} + z^2\right), \\ \rho_4(z) &= \frac{1}{3} + \frac{2}{3}z^2 + z^4, & \rho_5(z) &= z\left(\frac{1}{3} + \frac{2}{3}z^2 + z^4\right).\end{aligned}$$

As a general rule, we have deduced the following

Proposition 3.1. *The sequence $\{\rho_n\}$, where*

$$(3.4) \quad \rho_n(z) = \begin{cases} \frac{2}{n+2} \sum_{i=0}^{n/2} (i+1)z^{2i} & \text{if } n \text{ is even,} \\ z \frac{2}{n+1} \sum_{i=0}^{(n-1)/2} (i+1)z^{2i} & \text{if } n \text{ is odd,} \end{cases}$$

is the sequence of monic Szegő polynomials with respect to the weight function $\omega(\theta) = \frac{\sin^2(\theta)}{2\pi}$, for all n .

Proof. Suppose that n is even, then

$$\begin{aligned}\langle \rho_n(z), z^k \rangle_\omega &= \int_0^{2\pi} \rho_n(e^{i\theta}) e^{-ik\theta} \frac{\sin^2(\theta)}{2\pi} d\theta \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\left(\frac{-1}{2n+4} \sum_{i=0}^{n/2} (i+1)z^{2i}\right) (z^2-1)^2}{z^{k+3}} dz \\ &= \text{Res}(h, 0),\end{aligned}$$

where $h(z) = \frac{\left(\frac{-1}{2n+4} \sum_{i=0}^{n/2} (i+1)z^{2i}\right) (z^2-1)^2}{z^{k+3}}$. Therefore,

$$h(z) = \frac{-1}{2n+4} \left(\sum_{i=0}^{n/2} \frac{i+1}{z^{k-2i-1}} + \sum_{i=0}^{n/2} \frac{-2(i+1)}{z^{k-2i+1}} + \sum_{i=0}^{n/2} \frac{i+1}{z^{k-2i+3}} \right).$$

Note that if k is odd, then $\text{Res}(h, 0) = 0$. So, if $k = 0$, then

$$\text{Res}(h, 0) = \frac{-1}{2n+4} (-2+2) = 0.$$

If $2 \leq k \leq n-2$, then

$$\text{Res}(h, 0) = \frac{-1}{2n+4} \left(\frac{k-2}{2} + 1 - 2 \left(\frac{k}{2} + 1 \right) + \frac{k+2}{2} + 1 \right) = 0.$$

If $k = n$, then

$$\text{Res}(h, 0) = \frac{-1}{2n+4} \left(\frac{n}{2} - n - 2 \right) = \frac{n+4}{4(n+2)} \neq 0.$$

If n is odd, we have

$$\begin{aligned}\langle \rho_n(z), z^k \rangle_\omega &= \langle z\rho_{n-1}(z), z^k \rangle_\omega \\ &= \langle \rho_{n-1}(z), z^{k-1} \rangle_\omega \\ &= \begin{cases} 0 & \text{if } 1 \leq k \leq n-1 \\ \frac{n+4}{4(n+2)} & \text{if } k = n. \end{cases}\end{aligned}$$

If $k = 0$,

$$\begin{aligned}\langle \rho_n(z), 1 \rangle_\omega &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{z\rho_{n-1}(z)(z^2-1)^2}{-4z^3} dz \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\rho_{n-1}(z)(z^2-1)^2}{-4z^2} dz \\ &= \text{Res}(h, 0),\end{aligned}$$

where

$$h(z) = \frac{-1}{2(n+1)} \left(\sum_{i=0}^{(n-1)/2} (i+1)(z^{2i+2}) + \sum_{i=0}^{(n-1)/2} (-2(i+1))(z^{2i}) + \sum_{i=0}^{(n-1)/2} (i+1)(z^{2i-2}) \right).$$

Thus, $\text{Res}(h, 0) = 0$. Therefore, for all n , we have that

$$\langle \rho_n(z), z^k \rangle_\omega = \begin{cases} 0 & \text{if } 0 \leq k \leq n-1, \\ \frac{n+4}{4(n+2)} & \text{if } k = n. \end{cases}$$

□

The associated polynomials for some values of n are

$$\begin{aligned} \pi_0(z) &= -\frac{1}{2}, & \pi_1(z) &= -\frac{1}{2}z, \\ \pi_2(z) &= \frac{1}{4} - \frac{1}{2}z^2, & \pi_3(z) &= z \left(\frac{1}{4} - \frac{1}{2}z^2 \right), \\ \pi_4(z) &= \frac{1}{6} + \frac{1}{6}z^2 - \frac{1}{2}z^4, & \pi_5(z) &= z \left(\frac{1}{6} + \frac{1}{6}z^2 - \frac{1}{2}z^4 \right). \end{aligned}$$

Proposition 3.2. *The sequence $\{\pi_n\}$, where*

$$(3.5) \quad \begin{aligned} \pi_n(z) &= \frac{1}{n+2} \frac{1-z^n}{1-z^2} - \frac{1}{2}z^n && \text{for } n \text{ even,} \\ \pi_n(z) &= z \left(\frac{1}{n+1} \frac{1-z^{n-1}}{1-z^2} - \frac{1}{2}z^{n-1} \right) = z\pi_{n-1}(z) && \text{for } n \text{ odd,} \end{aligned}$$

is the sequence of associated polynomials with respect to the weight function $\omega(\theta) = \frac{\sin^2(\theta)}{2\pi}$, for all n .

Proof. For n even,

$$\begin{aligned} z\pi_{n-1}(z) - \delta_n \pi_{n-1}^*(z) &= z^2\pi_{n-2}(z) - \delta_n \pi_{n-2}^*(z) \\ &= z^2 \left(\frac{1}{n} \frac{1-z^{n-2}}{1-z^2} - \frac{1}{2}z^{n-2} \right) - \frac{2}{n+2} \left(\frac{1}{n} \frac{z^2(1-z^{n-2})}{1-z^2} - \frac{1}{2} \right) \\ &= \frac{1}{n} \frac{z^2(1-z^{n-2})}{1-z^2} - \frac{1}{n(n+2)} \frac{z^2(1-z^{n-2})}{1-z^2} + \frac{1}{n+2} - \frac{1}{2}z^{n-2} \\ &= \frac{1}{n(n+2)(1-z^2)} \left((n+2)z^2(1-z^{n-2}) - 2z^2(1-z^{n-2}) + (1-z^2)n \right) - \frac{1}{2}z^n \\ &= \frac{1}{n(n+2)(1-z^2)} \left((n+2)z^2 - (n+2)z^n - 2z^2 + 2z^n + n - nz^2 \right) - \frac{1}{2}z^n \\ &= \frac{1}{n(n+2)(1-z^2)} \left(-nz^n + n \right) - \frac{1}{2}z^n \\ &= \frac{1}{n+2} \frac{1-z^n}{1-z^2} - \frac{1}{2}z^n \\ &= \pi_n(z). \end{aligned}$$

On the other hand,

$$\begin{aligned}
\delta_n z \pi_{n-1}(z) - \pi_{n-1}^*(z) &= \delta_n z^2 \pi_{n-2}(z) - \pi_{n-2}^*(z) \\
&= \frac{2}{n+2} z^2 \left(\frac{1}{n} \frac{1-z^{n-2}}{1-z^2} - \frac{1}{2} z^{n-2} \right) - \left(\frac{1}{n} \frac{z^2(1-z^{n-2})}{1-z^2} - \frac{1}{2} \right) \\
&= \frac{2}{n(n+2)} \frac{z^2(1-z^{n-2})}{1-z^2} - \frac{1}{n} \frac{z^2(1-z^{n-2})}{1-z^2} - \frac{1}{n+2} z^n + \frac{1}{2} \\
&= \frac{1}{n(n+2)(1-z^2)} \left(2z^2(1-z^{n-2}) - (n+2)z^2(1-z^{n-2}) - (1-z^2)nz^n \right) + \frac{1}{2} \\
&= \frac{1}{n(n+2)(1-z^2)} \left(-nz^2 + nz^n - nz^n + nz^{n+2} \right) + \frac{1}{2} \\
&= \frac{1}{n(n+2)(1-z^2)} \left(-nz^2(1-z^n) \right) + \frac{1}{2} \\
&= -\frac{1}{n+2} \frac{z^2(1-z^n)}{1-z^2} + \frac{1}{2} \\
&= -\pi_n^*(z).
\end{aligned}$$

For n odd we have, since $\delta_n = 0$,

$$z\pi_{n-1}(z) - \delta_n \pi_{n-1}^*(z) = z\pi_{n-1}(z) = \pi_n(z),$$

$$\delta_n z \pi_{n-1}(z) - \pi_{n-1}^*(z) = -\pi_{n-1}^*(z) = -\pi_n^*(z).$$

We have proved that the polynomial π_n , defined above, satisfies the recurrence relations

$$z\pi_{n-1}(z) - \delta_n \pi_{n-1}^*(z) = \pi_n^*(z),$$

$$\delta_n z \pi_{n-1}(z) - \pi_{n-1}^*(z) = -\pi_n^*(z).$$

Thus, since $\pi_0(z) = -\frac{1}{2}$, this proves the proposition. \square

If $\{\rho_n\}$ is the sequence of monic Szegő polynomials given in formula (3.4), we can construct the para-orthogonal polynomials $B_n(z, \tau_n) = \rho_n(z) + \tau_n \rho_n^*(z)$, where $|\tau_n| = 1$. We have calculated them for $\tau_n = 1$ and $\tau_n = -1$, $\forall n$ as given in the following

Proposition 3.3.

$$(3.6) \quad B_{2n}(z, 1) = \frac{(n+2)}{n+1} \frac{1-z^{2n+2}}{1-z^2}, \quad B_{2n}(z, -1) = \frac{1}{n+1} \sum_{i=0}^n (2i-n)z^{2i}$$

and

$$\begin{aligned}
(3.7) \quad B_{2n+1}(z, 1) &= \frac{1}{n+1} \left(\sum_{i=0}^n (i+1)z^{2i+1} + \sum_{i=0}^n (n-i+1)z^{2i} \right), \\
B_{2n+1}(z, -1) &= \frac{1}{n+1} \left(\sum_{i=0}^n (i+1)z^{2i+1} - \sum_{i=0}^n (n-i+1)z^{2i} \right).
\end{aligned}$$

Proof.

$$\begin{aligned}
B_{2n}(z, 1) &= \frac{1}{n+1} \sum_{i=0}^n (i+1)z^{2i} + \frac{1}{n+1} \sum_{i=0}^n (n-i+1)z^{2i} \\
&= \frac{1}{n+1} \sum_{i=0}^n ((i+1) + (n-i+1)) z^{2i} \\
&= \frac{n+2}{n+1} \sum_{i=0}^n z^{2i} \\
&= \frac{(n+2)}{n+1} \frac{1-z^{2n+2}}{1-z^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned} B_{2n}(z, -1) &= \frac{1}{n+1} \sum_{i=0}^n (i+1)z^{2i} - \frac{1}{n+1} \sum_{i=0}^n (n-i+1)z^{2i} \\ &= \frac{1}{n+1} \sum_{i=0}^n ((i+1) - (n-i+1)) z^{2i} \\ &= \frac{1}{n+1} \sum_{i=0}^n (2i-n)z^{2i} \end{aligned}$$

and

$$\begin{aligned} B_{2n+1}(z, 1) &= \rho_{2n+1}(z) + \rho_{2n+1}^*(z) = z\rho_{2n}(z) + \rho_{2n}^*(z) \\ &= \frac{1}{n+1} \sum_{i=0}^n (i+1)z^{2i+1} + \frac{1}{n+1} \sum_{i=0}^n (n-i+1)z^{2i} \\ &= \frac{1}{n+1} \left(\sum_{i=0}^n (i+1)z^{2i+1} + \sum_{i=0}^n (n-i+1)z^{2i} \right), \end{aligned}$$

$$\begin{aligned} B_{2n+1}(z, -1) &= \rho_{2n+1}(z) - \rho_{2n+1}^*(z) = z\rho_{2n}(z) - \rho_{2n}^*(z) \\ &= \frac{1}{n+1} \sum_{i=0}^n (i+1)z^{2i+1} - \frac{1}{n+1} \sum_{i=0}^n (n-i+1)z^{2i} \\ &= \frac{1}{n+1} \left(\sum_{i=0}^n (i+1)z^{2i+1} - \sum_{i=0}^n (n-i+1)z^{2i} \right). \end{aligned}$$

□

Similarly, we can construct the polynomial $A_n(z, \tau_n) = \pi_n(z) - \tau_n \pi_n^*(z)$, where again, $|\tau_n| = 1, \forall n$. If the sequence $\{\pi_n\}$ is given by formula (3.5), then we have

Proposition 3.4.

(3.8)

$$A_{2n}(z, 1) = \frac{n+2}{2(n+1)}(1 - z^{2n}), \quad A_{2n}(z, -1) = \frac{n(z^{2n+2}-1)}{2(n+1)(1-z^2)} + \frac{(n+2)z^2(1-z^{2n-2})}{2(n+1)(1-z^2)}$$

and

$$\begin{aligned} (3.9) \quad A_{2n+1}(z, 1) &= \frac{z(1 - z^{2n})}{2(n+1)(1+z)} + \frac{1}{2}(1 - z^{2n+1}), \\ A_{2n+1}(z, -1) &= \frac{z(1 - z^{2n})}{2(n+1)(1-z)} - \frac{1}{2}(1 + z^{2n+1}). \end{aligned}$$

Proof.

$$\begin{aligned} A_{2n}(z, 1) &= \left(\frac{1}{2n+2} \frac{1-z^{2n}}{1-z^2} - \frac{1}{2} z^{2n} \right) - \left(\frac{1}{2n+2} \frac{z^2(1-z^{2n})}{1-z^2} - \frac{1}{2} \right) \\ &= \frac{1}{2(2n+2)(1-z^2)} \left(2(1 - z^{2n}) - 2z^2(1 - z^{2n}) + (1 - z^{2n})(2n + 2)(1 - z^2) \right) \\ &= \frac{1}{2(2n+2)(1-z^2)} \left((2n + 4) - (2n + 4)z^2 - (2n + 4)z^{2n} + (2n + 4)z^{2n+2} \right) \\ &= \frac{n+4}{2(2n+2)(1-z^2)} \left(1 - z^2 - z^{2n} + z^{2n+2} \right) \\ &= \frac{2n+4}{2(2n+2)(1-z^2)} \left((1 - z^2)(1 - z^{2n}) \right) \\ &= \frac{n+2}{2(n+1)}(1 - z^{2n}), \end{aligned}$$

$$\begin{aligned} A_{2n}(z, -1) &= \left(\frac{1}{2n+2} \frac{1-z^{2n}}{1-z^2} - \frac{1}{2} z^{2n} \right) + \left(\frac{1}{2n+2} \frac{z^2(1-z^{2n})}{1-z^2} - \frac{1}{2} \right) \\ &= \frac{1}{2(2n+2)(1-z^2)} \left(2(1 - z^{2n}) + 2z^2(1 - z^{2n}) - (1 + z^{2n})(2n + 2)(1 - z^2) \right) \\ &= \frac{1}{2(2n+2)(1-z^2)} \left(-2n - (2n + 4)z^2 - (2n + 4)z^{2n} + 2nz^{2n+2} \right) \\ &= \frac{n(z^{2n+2}-1)}{2(n+1)(1-z^2)} + \frac{(n+2)z^2(1-z^{2n-2})}{2(n+1)(1-z^2)}. \end{aligned}$$

Similarly,

$$\begin{aligned}
 A_{2n+1}(z, 1) &= \pi_{2n+1}(z) - \pi_{2n+1}^*(z) = z\pi_{2n}(z) - \pi_{2n}^*(z) \\
 &= z\left(\frac{1}{2(n+1)}\frac{1-z^{2n}}{1-z^2} - \frac{1}{2}z^{2n}\right) - \left(\frac{1}{2(n+1)}\frac{z^2(1-z^{2n})}{1-z^2} - \frac{1}{2}\right) \\
 &= \frac{1}{2(n+1)(1-z^2)}(z(1-z^{2n}) - z^2(1-z^{2n}) + (1-z^{2n+1})(n+1)(1-z^2)) \\
 &= \frac{1}{2(n+1)(1-z^2)}(z(1-z^{2n})(1-z) + (1-z^{2n+1})(n+1)(1-z^2)) \\
 &= \frac{z(1-z^{2n})}{2(n+1)(1+z)} + \frac{1}{2}(1-z^{2n+1}), \\
 A_{2n+1}(z, -1) &= \pi_{2n+1}(z) + \pi_{2n+1}^*(z) = z\pi_{2n}(z) + \pi_{2n}^*(z) \\
 &= z\left(\frac{1}{2(n+1)}\frac{1-z^{2n}}{1-z^2} - \frac{1}{2}z^{2n}\right) + \left(\frac{1}{2(n+1)}\frac{z^2(1-z^{2n})}{1-z^2} - \frac{1}{2}\right) \\
 &= \frac{1}{2(n+1)(1-z^2)}(z(1-z^{2n}) + z^2(1-z^{2n}) - (1+z^{2n+1})(n+1)(1-z^2)) \\
 &= \frac{1}{2(n+1)(1-z^2)}(z(1-z^{2n})(1+z) - (1+z^{2n+1})(n+1)(1-z^2)) \\
 &= \frac{z(1-z^{2n})}{2(n+1)(1-z)} - \frac{1}{2}(1+z^{2n+1}).
 \end{aligned}$$

□

We know that the coefficients of the Szegő quadrature formula can be written as in (2.7). If A_n and B_n are as in formulas (3.8) and (3.6), respectively, for n even, we have

$$\begin{aligned}
 B'_n(z, 1) &= \frac{n+4}{n+2} \frac{-(n+2)z^{n+1}(1-z^2) - (1-z^{n+2})(-2z)}{(1-z^2)^2} \\
 &= \frac{n+4}{n+2} \frac{-(n+2)z^{n+1} + (n+2)z^{n+3} + 2z - 2z^{n+3}}{(1+z)^2} \\
 &= \frac{n+4}{n+2} z \left(\frac{2-(n+2)z^n + nz^{n+2}}{(1-z^2)^2} \right).
 \end{aligned}$$

In this case, the nodes are $x_k = e^{\frac{2k\pi i}{n+2}}$, $k = 1, \dots, n+1$, $k \neq \frac{n}{2} + 1$. Thus, for all $k = 1, \dots, n+1$, $k \neq \frac{n}{2} + 1$ and by formula (2.7),

$$\begin{aligned}
 \lambda_k &= \frac{-1}{2e^{\frac{2k\pi i}{n+2}}} \frac{1 - \left(-e^{\frac{2k\pi i}{n+2}}\right)^n}{\frac{2k\pi i}{2e^{\frac{2k\pi i}{n+2}}} \frac{2k\pi i}{-(n+2)} \left(\frac{2k\pi i}{e^{\frac{2k\pi i}{n+2}}}\right)^n + n+2} \\
 &= -\frac{1}{4e^{\frac{2k\pi i}{n+2}}} \frac{1 - e^{\frac{2kn\pi i}{n+2}}}{(n+2)(1 - e^{\frac{2kn\pi i}{n+2}})} \left(1 - e^{\frac{4kn\pi i}{n+2}}\right)^2 \\
 &= -\frac{1}{4(n+2)} \frac{1 - 2e^{\frac{4k\pi i}{n+2}} + e^{\frac{8k\pi i}{n+2}}}{e^{\frac{4k\pi i}{n+2}}} \\
 &= -\frac{1}{4(n+2)} \left(-2 + 2\Re\left(e^{\frac{4k\pi i}{n+2}}\right)\right) \\
 &= \frac{1 - \cos\left(\frac{4k\pi}{n+2}\right)}{2(n+2)}.
 \end{aligned}$$

In short, the following theorem has been shown

Theorem 3.5. *The coefficients of the n -point Szegő quadrature formula for n even corresponding to $\tau_n = 1$, $n = 1, 2, \dots$ for the weight function $\omega(\theta) = \frac{\sin^2 \theta}{2\pi}$ are*

$$\lambda_k = \frac{1 - \cos\left(\frac{4k\pi}{n+2}\right)}{2(n+2)}, \quad k = 1, \dots, n+1, \quad k \neq \frac{n}{2} + 1,$$

and the nodes are given by $x_k = e^{\frac{2k\pi i}{n+2}}$, $k = 1, \dots, n+1$, $k \neq \frac{n}{2} + 1$.

When $\tau_n \neq 1$ explicit formulas for the nodes and the weights are not available. For the purpose of illustration we have calculated the weights and nodes for the n -point Szegő quadrature formula for some values of even n and $\tau_n = -1$. If we choose $n = 4$, then

$$B_4(z, -1) = \frac{2}{3}(z^4 - 1) \quad \text{and} \quad A_4(z, -1) = -\frac{1}{3}(z^4 - z^2 + 1).$$

Thus, the nodes $\{x_k\}_{k=1}^4$ and the weights $\{\lambda_k\}_{k=1}^4$ will be given by

$$x_1 = 1, \quad x_2 = -1, \quad x_3 = i, \quad x_4 = -i,$$

$$\lambda_1 = \lambda_2 = 0.0625, \quad \lambda_3 = \lambda_4 = 0.1875.$$

If we take $n = 6$, then in this case we have

$$B_6(z, -1) = \frac{1}{4}(3z^6 + z^4 - z^2 - 3) \quad \text{and} \quad A_6(z, -1) = -\frac{1}{8}(3z^6 - 2z^4 - 2z^2 + 3),$$

and the nodes $\{x_k\}_{k=1}^6$ and the weights $\{\lambda_k\}_{k=1}^6$ will be given by

$$\begin{aligned} x_1 = 1, & & x_2 = -1, \\ x_3 = 0.408248 + 0.912871i, & & x_4 = 0.408248 - 0.912871i, \\ x_5 = -0.408248 - 0.912871i, & & x_6 = -0.408248 + 0.912871i, \end{aligned}$$

$$\begin{aligned} \lambda_k = 0.025 & & \text{if } k = 1, 2, \\ \lambda_k = 0.1125000066 & & \text{if } 3 \leq k \leq 6. \end{aligned}$$

In order to obtain the upper bound given in formula (2.10) and the second kind measure associated with the weight function $\omega(\theta) = \frac{\sin^2 \theta}{2\pi}$, we have calculated the Herglotz-Riesz transform

$$\begin{aligned} F_\omega(z) &= \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \frac{\sin^2(\theta)}{2\pi} d\theta \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{-(w+z)(w^2-1)^2}{4w^3(w-z)} dw \\ &= \begin{cases} \text{Res}(h, 0) + \text{Res}(h, z) & \text{if } |z| < 1, \\ \text{Res}(h, 0) & \text{if } |z| > 1, \end{cases} \end{aligned}$$

where $h(w) = \frac{-(w+z)(w^2-1)^2}{4w^3(w-z)}$. Since

$$\frac{1}{w-z} = \sum_{i=0}^{\infty} \left(\frac{-1}{z^{i+1}}\right) w^i$$

and

$$\frac{-(w+z)(w^2-1)^2}{4w^3(w-z)} = \frac{-w^2}{4} + \frac{1}{2} + \frac{-1}{4w^2} + \frac{-zw}{4} + \frac{z}{2w} + \frac{-z}{4w^3},$$

then

$$\text{Res}(h, 0) = \frac{1}{4z^2} + \frac{-z}{2z} + \frac{z}{4z^3} = \frac{1-z^2}{2z^2} \quad \text{and} \quad \text{Res}(h, z) = \frac{-2z(z^2-1)^2}{4z^3} = \frac{-(z^2-1)^2}{2z^2}.$$

Therefore

$$(3.10) \quad F_\omega(z) = \begin{cases} \frac{1-z^2}{2} & \text{if } |z| < 1, \\ \frac{1-z^2}{2z^2} & \text{if } |z| > 1. \end{cases}$$

In short, we have the following

Proposition 3.6. *The Herglotz-Riesz transform associated with the weight function $\omega(\theta) = \frac{\sin^2 \theta}{2\pi}$ is given by*

$$(3.11) \quad F_\omega(z) = \begin{cases} \frac{1-z^2}{2} & \text{if } |z| < 1, \\ \frac{1-z^2}{2z^2} & \text{if } |z| > 1. \end{cases}$$

Then we have

Corollary 3.7. $d\tilde{\psi} = d\theta + \pi (d\delta(e^{i\theta} - z_1) + d\delta(e^{i\theta} - z_2))$ is the second kind measure associated with $d\psi(\theta) = \frac{\sin^2 \theta}{2\pi} d\theta$, where $z_1 = 1$ and $z_2 = -1$.

Proof. By (3.10), if $|z| < 1$, then $F(z) = \frac{1-z^2}{2}$ is the Carathéodory function corresponding to the measure $d\psi(\theta) = \frac{\sin^2 \theta}{2\pi} d\theta$ and then, by Theorem 1.3, $G(z) = \frac{2}{1-z^2}$ is the Carathéodory function corresponding to the measure $\tilde{\psi}$ given in Theorem 1.2. By formula (2.11),

$$\begin{aligned} z_1 &= 1, \quad \gamma_1 = -1, \\ z_2 &= -1, \quad \gamma_2 = 1 \end{aligned}$$

and

$$\begin{aligned} G(e^{i\theta}) &= 2 \frac{1}{1 - \cos 2\theta - i \sin 2\theta} = 2 \frac{1 - \cos 2\theta + i \sin 2\theta}{2 - 2 \cos 2\theta} \\ &= 1 + \frac{\sin 2\theta}{1 - \cos 2\theta} i. \end{aligned}$$

Thus, $\Re(G(e^{i\theta})) = 1$ and, by formula (2.12),

$$d\tilde{\psi} = d\theta + \pi (d\delta(e^{i\theta} - z_1) + d\delta(e^{i\theta} - z_2))$$

and the sequence $\{\pi_n\}$, given as in formula (3.5), is orthogonal with respect to $d\tilde{\psi}$. □

If A_n and B_n are as in formulas (3.8) and (3.6), respectively, for n even, we can compute the modified approximants: $R_n(z, 1) = \frac{A_n(z, 1)}{B_n(z, 1)}$ and we obtain

$$R_n(z, 1) = \frac{(1 - z^n)(1 - z^2)}{2(1 - z^{n+2})}$$

and the error for the Herglotz-Riesz transform is given by

$$(3.12) \quad F_\omega(z) - R_n(z, 1) = \begin{cases} \frac{z^n(1-z^2)^2}{2(1-z^{n+2})} & \text{if } |z| < 1, \\ \frac{(1-z^2)^2}{2z^2(1-z^{n+2})} & \text{if } |z| > 1. \end{cases}$$

According to Lemma 2.1, if $G = \{z : r < |z| < R, r < 1, R > 1\}$, then $\Gamma = \Gamma_1 \cup \Gamma_2$, where

$$(3.13) \quad \Gamma_1 = \{z : |z| = r, 0 < r < 1\} \quad \text{and} \quad \Gamma_2 = \{z : |z| = R, R > 1\}.$$

In this case, by formula (3.12) and (2.10)

$$\begin{aligned} |E_n(f)| &\leq \frac{1}{4\pi} \max_{\xi \in \Gamma} \left\{ \left| \frac{f(\xi)}{\xi} \right| \right\} \left(\int_{\Gamma_1} \frac{|z|^n |1-z^2|^2}{2|1-z|^{n+2}} + \int_{\Gamma_2} \frac{|1-z^2|^2}{2|z|^2(1-|z|^{n+2})} \right) \\ &= \frac{1}{4\pi} \max_{\xi \in \Gamma} \left\{ \left| \frac{f(\xi)}{\xi} \right| \right\} \left(\int_0^{2\pi} \frac{r^n(1-2r^2 \cos 2\theta + r^4)}{2(1-r^{n+2})} r d\theta + \int_0^{2\pi} \frac{1-2R^2 \cos 2\theta + R^4}{2R^2(R^{n+2}-1)} R d\theta \right) \\ &= \frac{1}{4} \max_{\xi \in \Gamma} \left\{ \left| \frac{f(\xi)}{\xi} \right| \right\} \left(\frac{r^{n+1}(1+r^4)}{1-r^{n+2}} + \frac{(1+R^4)}{R(R^{n+2}-1)} \right). \end{aligned}$$

Thus, we have proved the following

Theorem 3.8. *Let f be analytic in $G = \{z : r < |z| < R, r < 1, R > 1\}$. Then, for each n even*

$$(3.14) \quad |E_n(f)| \leq \frac{1}{4} \max_{\xi \in \Gamma} \left\{ \left| \frac{f(\xi)}{\xi} \right| \right\} \left(\frac{r^{n+1}(1+r^4)}{1-r^{n+2}} + \frac{(1+R^4)}{R(R^{n+2}-1)} \right),$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$. Γ_1 and Γ_2 as in formula (3.13).

Remark 3.9. Note that if $f(z)$ is analytic in $\{z : |z| < R; R > 1\}$, then we can make $r \rightarrow 0$ and

$$|E_n(f)| \leq \frac{1}{4} \max_{\xi \in \Gamma_2} \left\{ \left| \frac{f(\xi)}{\xi} \right| \right\} \left(\frac{(1+R^4)}{R(R^{n+2}-1)} \right).$$

On the other hand, if $f(z)$ is analytic in $\{z : r < |z| \leq \infty; 0 < r < 1\}$, now we can make $R \rightarrow \infty$ and one has

$$|E_n(f)| \leq \frac{1}{4} \max_{\xi \in \Gamma_1} \left\{ \left| \frac{f(\xi)}{\xi} \right| \right\} \left(\frac{r^{n+1}(1+r^4)}{1-r^{n+2}} \right).$$

Since for the remaining Chebyshev weight functions, i.e., $\omega(\theta) = \frac{1+\cos\theta}{2\pi}$ and $\omega(\theta) = \frac{1-\cos\theta}{2\pi}$, the calculations are quite similar to the previous case, we will omit them and give only the results.

First, we consider the Chebyshev weight function: $\omega(\theta) = \frac{1+\cos\theta}{2\pi}$. In this case the moment sequence is given by $\mu_0 = 1$, $\mu_1 = 1/2$ and $\mu_k = 0, \forall k \geq 2$. From (3.3), the reflection coefficients are $\delta_n := \frac{(-1)^n}{n+1}, \forall n$ and we have the following expression for the Szegő polynomials

Proposition 3.10. *The sequence $\{\rho_n\}$, given by*

$$(3.15) \quad \rho_n(z) = \frac{(-1)^n}{n+1} \sum_{i=0}^n (-1)^i (i+1) z^i,$$

is the sequence of monic Szegő polynomials with respect to the weight function $\omega(\theta) = \frac{1+\cos\theta}{2\pi}$, for all n .

For the associated Szegő polynomials one has

Proposition 3.11. *The sequence $\{\pi_n\}$, where*

$$(3.16) \quad \pi_n(z) = \frac{(-1)^n}{n+1} \frac{1 + (-1)^{n-1} z^n}{1+z} - z^n$$

is the sequence of associated polynomials with respect to the weight function $\omega(\theta) = \frac{1+\cos\theta}{2\pi}$, for all n .

Let $\{\rho_n\}$ be the sequence of monic Szegő polynomials given in formula (3.15). We have calculated the para-orthogonal polynomial $B_n(z, \tau_n)$ for $\tau_n = (-1)^n, \forall n$ and $\tau_n = (-1)^{n+1}, \forall n$:

Proposition 3.12.

$$(3.17) \quad B_n(z, (-1)^n) = \frac{(-1)^n (n+2)}{n+1} \frac{1 + (-1)^n z^{n+1}}{1+z}$$

and

$$(3.18) \quad B_n(z, (-1)^{n+1}) = \frac{(-1)^n}{n+1} \sum_{i=0}^n (-1)^i (2i-n) z^i.$$

Similarly, if the sequence $\{\pi_n\}$ is given by formula (3.16), for $A_n(z, (-1)^n)$ and $A_n(z, (-1)^{n+1})$ we have the following

Proposition 3.13.

$$(3.19) \quad A_n(z, (-1)^n) = \frac{(-1)^n(n+2)}{n+1}(1 - (-1)^n z^n)$$

and

$$(3.20) \quad A_n(z, (-1)^{n+1}) = \frac{(-1)^n n(1 - (-1)^{n+1} z^{n+1})}{(n+1)(1+z)} + \frac{(-1)^n(n+2)z(1 + (-1)^n z^{n-1})}{(n+1)(1+z)}.$$

If we choose B_n as in formula (3.17) and A_n as in formula (3.19), we have,

Theorem 3.14. *The coefficients of the n -point Szegő quadrature formula corresponding to $\tau_n = (-1)^n$ for the weight function $\omega(\theta) = \frac{1+\cos\theta}{2\pi}$ are*

$$\lambda_k = \begin{cases} \frac{1+\cos(\frac{2k\pi}{n+1})}{n+1} & \text{if } n \text{ is odd, } 0 \leq k \leq n, k \neq \frac{n+1}{2}, \\ \frac{1+\cos(\frac{\pi+2k\pi}{n+1})}{n+1} & \text{if } n \text{ is even, } 0 \leq k \leq n, k \neq \frac{n}{2}, \end{cases}$$

and the nodes are given by

$$(3.21) \quad x_k = \begin{cases} e^{\frac{2k\pi i}{n+1}} & \text{if } n \text{ is odd, } 0 \leq k \leq n, k \neq \frac{n+1}{2}, \\ e^{\frac{(\pi+2k\pi)i}{n+1}} & \text{if } n \text{ is even, } 0 \leq k \leq n, k \neq \frac{n}{2}. \end{cases}$$

In order to obtain the upper bound given in formula (2.10) and the second kind measure associated with the weight function $\omega(\theta) = \frac{1+\cos\theta}{2\pi}$, we have calculated the Herglotz-Riesz transform:

Proposition 3.15. *The Herglotz-Riesz transform associated with the weight function $\omega(\theta) = \frac{1+\cos\theta}{2\pi}$ is given by*

$$(3.22) \quad F_\omega(z) = \begin{cases} 1+z & \text{if } |z| < 1 \\ -\frac{1+z}{z} & \text{if } |z| > 1. \end{cases}$$

As a corollary of this proposition we have

Corollary 3.16. *$d\tilde{\psi} = \frac{1}{2}d\theta + \pi d\delta(e^{i\theta} - z_1)$ is the second kind measure associated with $d\psi(\theta) = \frac{1+\cos\theta}{2\pi}d\theta$, where $z_1 = -1$.*

If A_n and B_n are as in formulas (3.19) and (3.17), respectively, we can compute the modified approximants: $R_n(z, (-1)^n) = \frac{A_n(z, (-1)^n)}{B_n(z, (-1)^n)}$ and we obtain

$$R_n(z, (-1)^n) = \frac{(1 - (-1)^n z^n)(1+z)}{1 + (-1)^n z^{n+1}}$$

and the error for the Herglotz-Riesz transform is given by

$$(3.23) \quad F_\omega(z) - R_n(z, (-1)^n) = \begin{cases} \frac{(-1)^n z^n (1+z)^2}{1+(-1)^n z^{n+1}} & \text{if } |z| < 1, \\ \frac{-(1+z)^2}{z(1+(-1)^n z^{n+1})} & \text{if } |z| > 1. \end{cases}$$

In this case, by formula (3.23) and (2.10)

Theorem 3.17. *Let f be analytic in $G = \{z : r < |z| < R, r < 1, R > 1\}$. Then, for each n*

$$(3.24) \quad |E_n(f)| \leq \frac{1}{2} \max_{\xi \in \Gamma} \left\{ \left| \frac{f(\xi)}{\xi} \right| \right\} \left(\frac{r^{n+1}(1+r^2)}{1-r^{n+1}} + \frac{(1+R^2)}{R^{n+1}-1} \right),$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$. Γ_1 and Γ_2 as in formula (3.13).

For this weight function a result similar to that given in Remark 3.9 also holds.

Finally, for the Chebyshev weight function $\omega(\theta) = \frac{1-\cos\theta}{2\pi}$, we have the following: its moment sequence is given by $\mu_0 = 1$, $\mu_1 = -1/2$ and $\mu_k = 0 \forall k \geq 2$. Note that in this case, from (3.3), the reflection coefficients are $\delta_n := \rho_n(0) = \frac{1}{n+1}, \forall n$.

Proposition 3.18. *The sequence $\{\rho_n\}$, where*

$$(3.25) \quad \rho_n(z) = \frac{1}{n+1} \sum_{i=0}^n (i+1)z^i,$$

is the sequence of monic Szegő polynomials with respect to the weight function $\omega(\theta) = \frac{1-\cos\theta}{2\pi}$, for all n .

Remark 3.19. Proceeding as in proof of Proposition 3.1, it can be checked that

$$\langle \rho_n(z), z^k \rangle_\omega = 0, \quad 0 \leq k \leq n-1.$$

However, the weight function under consideration is a particular case of $|\sin \frac{\theta}{2}|^{2\alpha}$, $\alpha > -\frac{1}{2}$ where the sequence $\{\rho_n(z)\}$ of monic orthogonal polynomials is explicitly known. Indeed, it holds [8] that

$$(3.26) \quad \rho_n(z) = \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(\alpha+n-k)\Gamma(\alpha+k+1)}{\Gamma(\alpha+n+1)\Gamma(\alpha)} z^k.$$

Thus, when taking $\alpha = 1$ in (3.26), formula (3.25) follows.

The associated polynomials are now given in the following

Proposition 3.20. *The sequence $\{\pi_n\}$, where*

$$(3.27) \quad \pi_n(z) = \frac{1}{n+1} \frac{1-z^n}{1-z} - z^n,$$

is the sequence of associated polynomials with respect to the weight function $\omega(\theta) = \frac{1-\cos\theta}{2\pi}$, for all n .

If $\{\rho_n\}$ is the sequence of monic Szegő polynomials given in formula (3.25), the para-orthogonal polynomials $B_n(z, \tau_n)$ for $\tau_n = 1, \forall n$ and $\tau_n = -1, \forall n$, are given by

Proposition 3.21.

$$(3.28) \quad B_n(z, 1) = \frac{(n+2)}{n+1} \frac{1-z^{n+1}}{1-z}$$

and

$$(3.29) \quad B_n(z, -1) = \frac{1}{n+1} \sum_{i=0}^n (2i-n)z^i.$$

If the sequence $\{\pi_n\}$ is given by formula (3.27), then, for $A_n(z, 1)$ and $A_n(z, -1)$ one can write

Proposition 3.22.

$$(3.30) \quad A_n(z, 1) = \frac{n+2}{n+1}(1-z^n)$$

and

$$(3.31) \quad A_n(z, -1) = \frac{n(z^{n+1}-1)}{(n+1)(1-z)} + \frac{(n+2)z(1-z^{n-1})}{(n+1)(1-z)}.$$

If we choose B_n as in formula (3.28) and A_n as in formula (3.30), we have,

Theorem 3.23. *The coefficients of the n -point Szegő quadrature formula corresponding to $\tau_n = 1$ for the weight function $\omega(\theta) = \frac{1-\cos\theta}{2\pi}$ are*

$$\lambda_k = \frac{1 - \cos\left(\frac{2k\pi}{n+1}\right)}{n+1}$$

and the nodes are $x_k = e^{\frac{2k\pi i}{n+1}}$, $k = 1, \dots, n$.

For the weight function $\omega(\theta) = \frac{1-\cos\theta}{2\pi}$, the Herglotz-Riesz transform is given by

Proposition 3.24. *The Herglotz-Riesz transform associated with the weight function $\omega(\theta) = \frac{1-\cos\theta}{2\pi}$ is given by*

$$(3.32) \quad F_\omega(z) = \begin{cases} 1-z & \text{if } |z| < 1, \\ \frac{1-z}{z} & \text{if } |z| > 1. \end{cases}$$

Again we have

Corollary 3.25. $d\tilde{\psi} = \frac{1}{2}d\theta + \pi d\delta(e^{i\theta} - z_1)$ is the second kind weight function associated with $d\psi(\theta) = \frac{1-\cos\theta}{2\pi}d\theta$, where $z_1 = 1$.

If B_n and A_n are as in formulas (3.28) and (3.30), respectively, we can compute the modified approximants: $R_n(z, 1) = \frac{A_n(z, 1)}{B_n(z, 1)}$ and we obtain

$$R_n(z, 1) = \frac{(1-z^n)(1-z)}{1-z^{n+1}}$$

and the error for the Herglotz-Riesz transform is

$$(3.33) \quad F_\omega(z) - R_n(z, 1) = \begin{cases} \frac{z^n(1-z)^2}{1-z^{n+1}} & \text{if } |z| < 1, \\ \frac{(1-z)^2}{z(1-z^{n+1})} & \text{if } |z| > 1. \end{cases}$$

Therefore, the error bound in this case is given by the following

Theorem 3.26. *Let f be analytic in $G = \{z : r < |z| < R, r < 1, R > 1\}$. Then for each n ,*

$$(3.34) \quad |E_n(f)| \leq \frac{1}{2} \max_{\xi \in \Gamma} \left\{ \left| \frac{f(\xi)}{\xi} \right| \right\} \left(\frac{r^{n+1}(1+r^2)}{1-r^{n+1}} + \frac{(1+R^2)}{R^{n+1}-1} \right),$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$. Γ_1 and Γ_2 as in formula (3.13).

Finally, it should be indicated that a similar result to Remark 3.9 can be given for this case.

4. NUMERICAL RESULTS

Let

$$f_1(z) = \frac{\sin(z)}{4-z}, \quad f_2(z) = \frac{\sin(\frac{1}{z})}{z-\frac{1}{4}}, \quad \text{and} \quad f_3(z) = \frac{\sin(z)}{(4-z)(z-\frac{1}{4})}.$$

Note that $\frac{f_1(z)}{z}$ is analytic for $0 \leq |z| \leq R$ when $R < 4$, $\frac{f_2(z)}{z}$ is analytic for $r \leq |z| \leq \infty$ when $r > \frac{1}{4}$, and $\frac{f_3(z)}{z}$ is analytic for $r \leq |z| \leq R$ when $\frac{1}{4} < r < R < 4$.

For these functions, we have calculated the exact error of the Szegő quadrature formula with respect to the weight functions $\omega(\theta) = \frac{\sin^2 \theta}{2\pi}$, $\omega(\theta) = \frac{1+\cos \theta}{2\pi}$ and $\omega(\theta) = \frac{1-\cos \theta}{2\pi}$, respectively, and compared them with the error for the well-known Gauss-Legendre quadrature formulas. Such quadrature formulas should be understood in the following sense (see [4],[5]). Set $f(e^{i\theta}) = f_1(\theta) + if_2(\theta)$. Thus,

$$I_\omega(f) = \int_0^{2\pi} F_1(\theta)d\theta + i \int_0^{2\pi} F_2(\theta)d\theta,$$

where $F_i(\theta) = f_i(\theta)\omega(\theta)$, $i = 1, 2$. If we estimate these latter integrals by means of the n -point Gauss-Legendre formula for $[0, 2\pi]$, $\sum_{j=1}^n A_j F_i(\theta_j)$, $i = 1, 2$ we can write

$$I_\omega(f) \approx \sum_{j=1}^n A_j (F_1(\theta_j) + iF_2(\theta_j)) = \sum_{j=1}^n A_j f(e^{i\theta_j})\omega(\theta_j).$$

We also have calculated the error bounds given by formulas (3.14), (3.24) and (3.34). For the computation of Szegő quadrature formulas we have used Theorem 3.5, Theorem 3.14 and Theorem 3.23 for the weights functions $\omega(\theta) = \frac{\sin^2 \theta}{2\pi}$, $\omega(\theta) = \frac{1+\cos \theta}{2\pi}$ and $\omega(\theta) = \frac{1-\cos \theta}{2\pi}$, respectively.

For the weight function $\omega(\theta) = \frac{\sin^2 \theta}{2\pi}$, we have the results appearing in Tables 1–3. For the weight function $\omega(\theta) = \frac{1+\cos \theta}{2\pi}$, we have the numerical results displayed in Tables 4–6. Finally, for the weight function $\omega(\theta) = \frac{1-\cos \theta}{2\pi}$, we have the corresponding numerical results given in Tables 7–9.

From the numerical results we see that Szegő formulas compete very favorably with Gauss-Legendre formulas. On the other hand, sharpness of the error bounds is also established in all of the examples.

TABLE 1. $f_1(z)$

Nodes	Exact error (Szegő)	Error bound ($R=3.8$)	Exact error (Gauss-Legendre)
$n=8$	1.24397935E-06	2.431076E-05	1.71024787E-03
$n=16$	9.68002240E-12	5.591511E-10	1.48264889E-07
$n=24$	1.54390389E-16	1.286057E-14	1.18979947E-11

TABLE 2. $f_2(z)$

Nodes	Exact error (Szegő)	Error bound ($r=0.27$)	Exact error (Gauss-Legendre)
$n=8$	1.990366E-05	4.071163E-04	3.156266E-02
$n=16$	1.548800E-10	1.149814E-08	4.977121E-06
$n=24$	2.178812E-15	3.247415E-13	7.953741E-08

TABLE 3. $f_3(z)$

Nodes	Exact error (Szegö)	Error bound ($R=3.8, r=0.27$)	Exact error (Gauss-Legendre)
$n=8$	7.400192E-07	3.161199E-04	3.116131E-04
$n=16$	1.091954E-11	7.403863E-09	2.943992E-10
$n=24$	2.775557E-17	1.740482E-13	4.424738E-09

TABLE 4. $f_1(z)$

Nodes	Exact error (Szegö)	Error bound ($R=3.8$)	Exact error (Gauss-Legendre)
$n=8$	3.34996165E-06	5.174076E-05	3.50150187E-04
$n=16$	3.44166917E-11	1.190039E-09	1.43798826E-08
$n=24$	5.27355936E-16	2.737112E-14	1.26720897E-10

TABLE 5. $f_2(z)$

Nodes	Exact error (Szegö)	Error bound ($r=0.27$)	Exact error (Gauss-Legendre)
$n=8$	6.462231E-05	8.689770E-04	1.718254E-02
$n=16$	5.506783E-10	2.454229E-08	9.102588E-07
$n=24$	8.409939E-15	6.931467E-13	8.015859E-07

TABLE 6. $f_3(z)$

Nodes	Exact error (Szegö)	Error bound ($R=3.8, r=0.27$)	Exact error (Gauss-Legendre)
$n=8$	2.614782E-06	6.729565E-04	1.895428E-04
$n=16$	3.882527E-11	1.576203E-08	1.192112E-07
$n=24$	4.996003E-16	3.705510E-13	9.867432E-08

TABLE 7. $f_1(z)$

Nodes	Exact error (Szegö)	Error bound ($R=3.8$)	Exact error (Gauss-Legendre)
$n=8$	2.08782660E-06	5.174076E-05	9.61621164E-04
$n=16$	1.23908591E-11	1.190039E-09	3.25547073E-08
$n=24$	1.90819582E-16	2.737112E-14	2.32242974E-11

TABLE 8. $f_2(z)$

Nodes	Exact error (Szegö)	Error bound ($r=0.27$)	Exact error (Gauss-Legendre)
$n=8$	2.238229E-05	8.689770E-04	6.482502E-03
$n=16$	1.982434E-10	2.454229E-08	8.908710E-07
$n=24$	2.955968E-15	6.931467E-13	5.565306E-08

TABLE 9. $f_3(z)$

Nodes	Exact error (Szegö)	Error bound ($R=3.8, r=0.27$)	Exact error (Gauss-Legendre)
$n=8$	9.413287E-07	6.729565E-04	1.330823E-04
$n=16$	1.397712E-11	1.576203E-08	2.980441E-08
$n=24$	2.220446E-16	3.705510E-13	7.450711E-08

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