

APPROXIMATION OF THE HILBERT TRANSFORM ON THE REAL LINE USING HERMITE ZEROS

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ABSTRACT. The authors study the Hilbert Transform on the real line. They introduce some polynomial approximations and some algorithms for its numerical evaluation. Error estimates in uniform norm are given.

1. INTRODUCTION

Let us consider the integral

$$(1.1) \quad H(G, t) := \int_{\mathbb{R}} \frac{G(x)}{x-t} dx = \lim_{\epsilon \rightarrow 0^+} \int_{|x-t| \geq \epsilon} \frac{G(x)}{x-t} dx,$$

where $t \in \mathbb{R}$. If we assume that the limit on the right-hand side exists, then H is called Hilbert Transform. It appears in several mathematical problems and, essentially, it is the main part of the singular integral equations on \mathbb{R} [28]. Therefore approximations and numerical evaluations of $H(G, t)$ are of great interest.

It is well known that H is a bounded map in the L^p spaces, $1 < p < \infty$, and that it is usually considered in the L^2 spaces where $\pi^{-1}H$ is an isometric isomorphism [1]. In the space of the continuous functions equipped with the uniform metric, the Hilbert Transform is an unbounded operator.

However, if we assume that the Dini type condition

$$\int_0^1 \frac{\omega(G, u)}{u} du < \infty$$

holds, where ω is the ordinary modulus of smoothness on \mathbb{R} , then $H(G, t)$ is a continuous function on \mathbb{R} [14, Theorem 2.24, p. 218].

In the last decade several papers have dealt with the numerical approximation of the Hilbert Transform in the case of bounded intervals and the reader can refer to [4], [5], [6], [7], [13] [18], [20], [25] and [26]. The algorithms proposed in these papers are mainly two: Gauss-type quadrature rules and product quadrature rules. The first ones subtract the singularity and apply an ordinary Gauss quadrature rule using an additional algorithm to control the term of the quadrature sum containing

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the knot closest to the singularity [4], [13], [27]. The second ones consist in substituting the integrand function by its interpolating polynomial having Lebesgue constants of order $\log m$.

On the other hand the literature concerning the numerical integration on unbounded intervals is by far poorer than the one on bounded intervals. For instance, in the case of integral transforms with continuous or weakly singular kernels, the convergence of some product quadrature rules has been proved in [19], [29], [33] and [34]. Estimates of the quadrature error have been recently proved in [21], [22], [23] and [27]. The case of the Hilbert Transform has been considered very little and the reader can consult [3], [9], [15], [16], [35], [36], [37], [38] and the references given there.

In particular in [16] the authors assume that the function G is analytic in the strip $\{\mathbb{C} : |\Im z| < d\}$, in which case they show that the series

$$2 \sum_{\substack{\nu=-\infty \\ \nu \neq \text{even}}}^{\infty} \frac{G(t + \nu h)}{\nu},$$

converges to $H(G, t)$ at the rate $\mathcal{O}(e^{-\frac{\pi d}{h}})$ as $h \rightarrow 0$. Successively they approximate $H(G, t)$ by a partial sum of the above series. Obviously the error depends on the decay to $\pm\infty$ of the function G .

In [3] the author replaces the above series with the following one

$$\sum_{k=-\infty}^{\infty} \frac{G(t + kh + \frac{h}{2})}{k + \frac{1}{2}},$$

for a suitable choice of the step $h \rightarrow 0$. Proceeding as in [16] and making suitable transformations (Sinc Method), this procedure can be used for piece-wise analytic functions (see [35], [36], [37], [38]) which frequently appear in the applications.

In this paper we propose to approximate $H(G, t)$ by using the zeros of Hermite polynomials. More precisely, we write $G(x) = [G(x)e^{(px)^2}]e^{-(px)^2} := f(x)e^{-(px)^2}$, $p \neq 0$, and we assume the function f belonging to suitable Sobolev spaces and such that, for $r \geq 1$ and $|x| > x_0 > 0$, $|f^{(r)}(x)e^{-(px)^2}|$ decays to $\pm\infty$ algebraically. Since the zeros of the polynomial $H_m(x)$ are in the interval $[-p\sqrt{2m}, p\sqrt{2m}]$, taking into account the decay of $|f^{(r)}(x)e^{-(px)^2}|$, the computation of $H(G, t)$ is reduced to the computation of an analogous integral on a finite interval (depending on m). This circumstance allows us to use the procedures proposed in [4], [5], [6], [7], [13], [18], [20], [25] and [26].

In Section 3 we propose a Gauss-type quadrature rule and a product quadrature rule.

The Gauss-type quadrature rule is useful because it has the computational cost of an ordinary Gaussian formula and it controls the term of the formula containing the zero closest to the singularity. This circumstance requires additional information on the distance of the zeros of two consecutive Hermite polynomials (see Lemma 2.1).

The product quadrature rules have not been studied till now, because interpolation processes having Lebesgue constants (in W_0^∞) of order $\log m$ were not available. Using an idea by J. Szabados [39], we construct a simple interpolating Lagrange polynomial and we prove that it is optimal in W_0^∞ (see Theorem 2.2). Subsequently we approximate $H(G, t)$ by replacing f with the above-mentioned polynomial. We

obtain a formula having greater computational cost than the one of the Gauss-type formula, but it proves to be more useful in collocation methods for solving singular integral equations on \mathbb{R} . In Theorem 3.2, we establish convergence conditions for this procedure. In the same section we propose a simple algorithm when the parameter t is “large”.

Extensive numerical testing and comparisons with other procedures have been performed and all the results confirm our theoretical estimates. Finally, in Section 5 we show some significant examples.

2. PRELIMINARY RESULTS

Functional spaces. In the following \mathcal{C} denotes a positive constant which may assume different values in different formulas. In the sequel, $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ means that \mathcal{C} is independent of a, b, \dots . Moreover we write $A \sim B$, for $A, B > 0$, iff there exist two positive constants M_1, M_2 , independent of A and B , such that

$$M_1 \leq \left(\frac{A}{B}\right)^{\pm 1} \leq M_2.$$

As we have already announced in the introduction, the main aim of this paper is the numerical approximation of the integral

$$\int_{\mathbb{R}} \frac{G(x)}{x-t} dx, \quad t \in \mathbb{R},$$

by using Hermite zeros. In order to do this, we write the function G as $G(x) = [G(x)e^{(px)^2}]e^{-(px)^2} := f(x)e^{-(px)^2}$, $p \neq 0$, and we successively construct the weighted polynomial approximation of the function f . To this end the following preliminary results are useful.

With respect to $w_p(x) = e^{-(px)^2}$, let us consider the following set of locally continuous functions

$$(2.1) \quad W_0^\infty := W_0^\infty(w_p) = \{f \in C_{\text{LOC}}^0(\mathbb{R}) : \lim_{|x| \rightarrow \infty} w_p(x)f(x) = 0\}$$

which, equipped with the norm

$$\|f\|_{W_0^\infty} := \|fw_p\|_\infty = \max_{\mathbb{R}} |f(x)w_p(x)|,$$

is a Banach space. For smoother functions we consider the usual Sobolev space $W_r^\infty, r \geq 1$, given by

$$(2.2) \quad W_r^\infty := \{f \in W_0^\infty : \|f^{(r)}w_p\|_\infty < \infty\},$$

with the norm

$$\|f\|_{W_r^\infty} := \|fw_p\|_\infty + \|f^{(r)}w_p\|_\infty.$$

For all $f \in W_0^\infty$ we define the following weighted modulus of smoothness

$$(2.3) \quad \Omega^k(f, t)_{w_p, \infty} := \sup_{0 < h \leq t} \max_{|x| \leq \frac{1}{h}} |\Delta_h^k f(x)|_{w_p(x)},$$

where $k \geq 1$ and

$$\Delta_h^k f(x) := \sum_{i=0}^k (-1)^i \binom{k}{i} f\left(x + \frac{h}{2}[k-2i]\right).$$

Moreover, we consider the error of best weighted approximation of the function f given by

$$E_m(f)_{w_p, \infty} := \inf_{P \in \mathbb{P}_m} \|(f - P)w_p\|_\infty,$$

where \mathbb{P}_m denotes the set of all algebraic polynomials of degree at most m . Then [11, p. 185]

$$(2.4) \quad E_m(f)_{w_p, \infty} \leq C \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega^k(f, t)_{w_p, \infty}}{t} dt$$

and

$$(2.5) \quad \Omega^k(f, t)_{w_p, \infty} \leq C t^k \sum_{1 \leq i \leq \frac{1}{t^2}} i^{\frac{k}{2}-1} E_i(f)_{w_p, \infty}$$

hold. The first inequality is a weaker version of the Jackson theorem; the second one is a Stechkin-type inequality. In particular, if $f \in W_r^\infty$, then we have

$$(2.6) \quad E_m(f)_{w_p, \infty} \leq C \frac{\|f^{(r)}\|_{w_p, \infty}}{(\sqrt{m})^r}.$$

In the sequel we will denote by \tilde{W}_r^∞ the space of the functions f belonging to W_r^∞ , $r \geq 0$, such that, for some $x_0 > 0$, we have

$$(2.7) \quad |f^{(r)}(x)w_p(x)| \leq C \frac{\|f^{(r)}\|_{w_p, \infty}}{|x|^{1+\lambda}}, \quad \lambda > 1, \quad |x| > x_0.$$

Such functions frequently appear in the applications and sometimes $|x|^{1+\lambda}$ is replaced by $e^{|x|}$ (see for example [3], [16]).

Orthonormal polynomials. Since we have to recall some properties of the Hermite polynomials, we assume $w_p(x) = w(x) := e^{-x^2}$ (i.e., $p = 1$), but we remark that the same properties can be easily extended by dilation in the more general case $w_p(x) = e^{-(px)^2}$, with $p \neq 0$. Let $\{p_m(w)\}_{m \in \mathbb{N}}$ be the sequence of the Hermite orthonormal polynomials with positive leading coefficient, i.e.,

$$p_m(w) = \gamma_m(w)x^m + \dots, \quad \gamma_m(w) > 0,$$

$$\int_{\mathbb{R}} p_m(w, x)p_n(w, x)w(x)dx = \delta_{m,n}.$$

The zeros $x_k := x_{m,k}(w)$, $k = 1, \dots, m$, of $p_m(w)$ satisfy

$$-\sqrt{2m} = x_0 < x_1 < \dots < x_m < x_{m+1} = \sqrt{2m}.$$

Furthermore, setting $\phi_m(x) := \left(\sqrt{2m - x^2 + (2m)^{\frac{1}{3}}}\right)^{-1}$, then

$$(2.8) \quad \Delta x_k = x_{k+1} - x_k \sim \phi_m(x_k) \sim \frac{1}{\sqrt{2m - x_k^2}}, \quad k = 0, \dots, m,$$

and

$$(2.9) \quad \phi_m(x_k) \sim \phi_m(\xi) \sim \phi_m(x_{k+1}), \quad \xi \in [x_k, x_{k+1}], \quad k = 0, \dots, m,$$

hold uniformly with respect to m and k . The previous relations can be easily deduced from [17] and [39]. The following lemma is new and will be useful in the sequel.

Lemma 2.1. *Let $x_{m+1,k}, k = 1, \dots, m + 1$, be the zeros of $p_{m+1}(w)$. Then*

$$\frac{1}{x_{m+1,k+1} - x_{m,k}} \leq C m^{\frac{3}{2}} \phi_m^2(x_{m+1,k+1}), \quad k = 1, \dots, m,$$

holds uniformly with respect to m and k .

Consequently, if $x_{m+1,k+1}, x_{m,k} \in (-\theta\sqrt{2m}, \theta\sqrt{2m})$, with $0 < \theta < 1$ fixed, then

$$(2.10) \quad \frac{1}{x_{m+1,k+1} - x_{m,k}} \leq C\sqrt{m}.$$

The Christoffel functions $\lambda_m(w, x)$ are defined by

$$\lambda_m(w, x) = \left[\sum_{i=0}^{m-1} p_i^2(w, x) \right]^{-1}$$

and $\lambda_k(w) := \lambda_{m,k}(w) = \lambda_m(w, x_k), k = 1, \dots, m$, are the Christoffel numbers. In the sequel we will use the well-known estimates [17]:

$$(2.11) \quad \lambda_m(w, x) \sim e^{-x^2} \phi_m(x)$$

and

$$(2.12) \quad \lambda_k(w) \sim \frac{e^{-x_k^2}}{\sqrt{2m - x_k^2 + (2m)^{\frac{1}{3}}}} \sim \Delta x_k e^{-x_k^2},$$

where the constants in “ \sim ” are independent of m, x and k .

Lagrange interpolation. Let us denote by $L_{m+2}(w, f), f \in W_0^\infty$, the Lagrange polynomial based on the $m + 2$ knots

$$-\sqrt{2m} = x_0 < x_1 < \dots < x_m < x_{m+1} = \sqrt{2m},$$

where $x_k := x_{m,k}(w), k = 1, \dots, m$, are the zeros of $p_m(w)$. For this interpolation process we can state the following

Theorem 2.2. *For all $f \in W_0^\infty$, the estimate*

$$(2.13) \quad \|[f - L_{m+2}(w, f)]\sqrt{w}\|_\infty \leq CE_{m+1}(f)_{\sqrt{w}, \infty} \log m$$

holds, where C is a positive constant independent of m and f .

Theorem 2.2 is a simple but useful modification of a previous result by J. Szabados [39]. In Section 4 we will give the proof of this theorem not only for completeness, but also because we can deduce a slightly more general estimate. In fact, for $w_p(x) := e^{-(px)^2}, p \neq 0$, we have $\{p_m(w_p, x)\}_m = \{\sqrt{p} p_m(w, px)\}_m$ and

$$p_m(w_p, z_k) = 0 \Leftrightarrow z_k = \frac{x_k}{p}, \quad k = 1, \dots, m.$$

If we denote by $L_{m+2}(w_p, f)$ the Lagrange polynomial based on the knots $\{x_k/p\}_{k=0,1,\dots,m+1}$, then, from Theorem 2.2, we can easily deduce the estimate

$$(2.14) \quad \|[f - L_{m+2}(w_p, f)]\sqrt{w_p}\|_\infty \leq CE_{m+1}(f)_{\sqrt{w_p}, \infty} \log m,$$

where C is a positive constant independent of m and f .

Finally, we want to state a simple proposition that will be useful in the sequel. Let $f \in W_r^\infty, r > 0$, and define f_r as follows

$$(2.15) \quad f_r(x) = \begin{cases} T_{r-1}(-\frac{\sqrt{2m}}{p}), & x \leq -\frac{\sqrt{2m}}{p}, \\ f(x), & |x| < \frac{\sqrt{2m}}{p}, \\ \tilde{T}_{r-1}(\frac{\sqrt{2m}}{p}), & x \geq \frac{\sqrt{2m}}{p}, \end{cases}$$

where T_{r-1} and \tilde{T}_{r-1} are the Taylor polynomials of f of degree $r - 1$ with starting points $-\frac{\sqrt{2m}}{p}$ and $\frac{\sqrt{2m}}{p}$, respectively.

Obviously $f_r \in W_r^\infty$ and the next proposition holds.

Proposition 2.3. *For all $f \in \tilde{W}_r^\infty$ and $\sqrt{2m} > x_0$, we have*

$$(2.16) \quad \sup_{|t| \leq \frac{\sqrt{2m}}{p}} \left| \int_{\mathbb{R}} \frac{f(x) - f_r(x)}{x - t} w_p(x) dx \right| \leq C \frac{\|f^{(r)} w_p\|_\infty}{(\sqrt{m})^{r+\lambda-1}},$$

where $\lambda > 1$ and C is a positive constant independent of m, f and t .

If the decay condition (2.7) in the definition of \tilde{W}_r^∞ is replaced by the following

$$|f^{(r)}(x) w_p(x)| \leq C \frac{\|f^{(r)} w_p\|_\infty}{e^{|x|}}, \quad |x| > x_0,$$

then the quantity $(\sqrt{m})^r$ on the right-hand side of (2.16) is replaced by $o(e^{-\sqrt{m}})$.

Now we are able to establish the main results of this paper.

3. MAIN RESULTS

Pointwise approximation of $H(fw, t)$. In this subsection we introduce some simple procedures useful for the numerical computation of $H(fw, t), w(x) = e^{-x^2}$. Following an idea in [13], let us consider the following identity

$$(3.1) \quad H(fw, t) = f(t) \int_{\mathbb{R}} \frac{e^{-x^2}}{x - t} dx + \int_{\mathbb{R}} \frac{f(x) - f(t)}{x - t} e^{-x^2} dx.$$

To compute the first integral on the right-hand side, the reader can consult [31]. For the second one we apply the Gauss quadrature rule

$$(3.2) \quad \int_{\mathbb{R}} f(x) e^{-x^2} dx = \sum_{k=1}^m \lambda_k(w) f(x_k), \quad \forall f \in \mathbb{P}_{2m-1},$$

where $x_k, k = 1, \dots, m$, are the Hermite zeros and $\lambda_k(w), k = 1, \dots, m$, are the Christoffel numbers. After some simple computations we obtain the following formula

$$(3.3) \quad \begin{aligned} H(fw, t) &= f(t) \left[\int_{\mathbb{R}} \frac{e^{-x^2}}{x - t} dx - \sum_{k=1}^m \frac{\lambda_k(w)}{x_k - t} \right] + \sum_{k=1}^m \lambda_k(w) \frac{f(x_k)}{x_k - t} + r_m(f, t) \\ &:= \Phi_m(fw, t) + r_m(f, t), \end{aligned}$$

where $t \neq x_k, k = 1, \dots, m$, and $r_m(f)$ is the remainder term. $\Phi_m(fw)$ is a Gauss-type quadrature rule and it has degree of exactness $2m$ (i.e., $r_m(f, t) = 0, \forall f \in \mathbb{P}_{2m}$). Unfortunately the relation $t \neq x_k, k = 1, \dots, m$, is not always verified. But even when $t \neq x_k, k = 1, \dots, m$, the point t could be too close to one of the

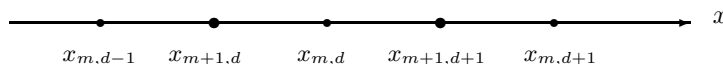
Hermite zeros and this produces numerical instability. On the other hand the only term in (3.3) causing numerical instability is

$$\frac{f(x_d) - f(t)}{x_d - t} \lambda_d(w),$$

where x_d is the zero closest to t . Following an argument in [27], we now introduce an algorithm in order to control this term.

For every fixed t , choose $m_0 = m_0(t) \in \mathbb{N}$ such that, for $m \geq m_0$, we have $x_{m,d} \leq t \leq x_{m,d+1}$ for some $d \in \{1, 2, \dots, m - 1\}$.

Moreover, because of the interlacing properties of the zeros $x_{m+1,k}, k = 1, \dots, m + 1$, of $p_{m+1}(w)$, we have



Thus, two cases are possible:

- (a) $x_{m+1,d+1} \leq t \leq x_{m,d+1}$ or
- (b) $x_{m,d} \leq t \leq x_{m+1,d+1}$.

In case (a),

if $t < \frac{x_{m+1,d+1} + x_{m,d+1}}{2}$, then we use the quadrature rule $\Phi_m(fw)$;

if $t \geq \frac{x_{m+1,d+1} + x_{m,d+1}}{2}$, then we use the quadrature rule $\Phi_{m+1}(fw)$.

Similarly in case (b). Thus, for every fixed t , we have defined the numerical sequence $\{\Phi_{m^*}(fw, t)\}$, $m^* \in \{m, m + 1\}$.

Moreover the algorithm for the choice of m^* is based on Lemma 2.1 and it assures us that the knot of $\Phi_{m^*}(fw)$ closest to t is sufficiently far from t . In fact, for m sufficiently large, $|t| \leq \theta\sqrt{2m}$, with $0 < \theta < 1$ fixed, we have

$$|x_{m^*,d} - t| \geq \frac{C}{\sqrt{m}}.$$

The next theorem deals with the convergence of the numerical sequence $\{\Phi_{m^*}(fw, t)\}$. We assume $f \in \tilde{W}_r^\infty$, $r > 0$, but, with minor effort, we can estimate the error for different classes of functions.

Theorem 3.1. *Let t be fixed on \mathbb{R} and let m be such that $|t| \leq \theta\sqrt{2m}$, with $0 < \theta < 1$. Then, for all $f \in \tilde{W}_r^\infty, r > 0$, we have*

$$(3.4) \quad |r_{m^*}(f, t)| := |H(fw, t) - \Phi_{m^*}(fw, t)| \leq C \frac{\|f^{(r)}w\|_\infty}{(\sqrt{m})^r} \log m,$$

where C is a positive constant independent of m and f .

Theorem 3.1 shows also that, for any fixed t on \mathbb{R} , the numerical sequence $\{\Phi_{m^*}(fw, t)\}$ converges to $H(fw, t)$. In particular, if $t = 0$, then it is easy to see that $\Phi_{m^*}(fw, 0) = \Phi_{2m}(fw, 0)$. More generally, if for any m we choose

$$t = t_k = \frac{x_k + x_{k+1}}{2}, \quad k = 1, \dots, m - 1,$$

then we can evaluate $H(fw, t_k)$ with the required accuracy and we can reconstruct the function $H(fw, t)$, in the interval $(-\theta\sqrt{2m}, \theta\sqrt{2m})$, by means of suitable interpolating splines.

Numerical considerations. Theorem 3.1 holds under the condition $|t| < \theta\sqrt{2m}$, i.e., since t is fixed, for $m > \frac{1}{2} \left(\frac{t}{\theta}\right)^2$. Thus, when t is “large” (but not too large), then the computation of the Christoffel numbers and of the function f on the zeros of $p_m(w)$ is too expensive and sometimes impossible (for example, if $t = 100$, then we need $m > 5000$). When this happens, i.e., t is “large”, we propose to approximate $H(fw, t)$ by the formula

$$(3.5) \quad H(fw, t) = \sum_{k=1}^m \frac{f(x_k)}{x_k - t} \lambda_k(w) + \rho_m(f, t),$$

where $1 \leq x_m \leq t - 1$ and $\rho_m(f, t)$ is the remainder term.

Moreover, in [10] it has been proved that, for all $f \in W_r^\infty$, we have

$$|\rho_m(f, t)| \leq \frac{C}{(\sqrt{m})^r} \max_{0 \leq k \leq r} \|f^{(k)}w\|_\infty.$$

So, using (3.3) together with (3.5), we obtain an efficient procedure for the computation of $H(fw, t)$, for different domains of t . Obviously (3.3) and (3.5) can also be used after making suitable transformations in order to regularise the density function f .

Uniform approximation of $H(fw, t)$. In order to construct a uniform approximation of $H(fw)$ by means of algebraic polynomials, we recall that, at the present time, the only polynomial processes convergent in W_0^∞ are the de la Vallée Poussin means [30]. Obviously, the computation of the Fourier coefficients does not allow us to use such polynomials. Then, taking into account the estimate (2.14), with $p = \sqrt{2}$, it seems natural to replace f by $L_{m+2}(w^2, f)$ in $H(fw, t)$ and to define the sequence $\{H_m(fw, t)\}_{m \in \mathbb{N}}$, where

$$H_m(fw, t) = H(L_{m+2}(w^2, f)w, t).$$

We can also write $H_m(fw, t)$ as

$$\begin{aligned} H_m(fw, t) &= \left[\frac{f(\sqrt{m})}{2\sqrt{m}p_m(w^2, \sqrt{m})} - \frac{f(-\sqrt{m})}{2\sqrt{m}p_m(w^2, -\sqrt{m})} \right] A_m \\ &+ \left[\frac{f(\sqrt{m})(\sqrt{m} + t)}{2\sqrt{m}p_m(w^2, \sqrt{m})} + \frac{f(-\sqrt{m})(\sqrt{m} - t)}{2\sqrt{m}p_m(w^2, -\sqrt{m})} \right] q_m(w, t) \\ &+ \sum_{k=1}^m \frac{f(x_k)\lambda_k(w^2)}{(x_k^2 - m)} \sum_{i=0}^{m-1} p_i(w^2, x_k)[tA_i + B_i] \\ &+ \sum_{k=1}^m \frac{f(x_k)\lambda_k(w^2)(t^2 - m)}{(x_k^2 - m)} \sum_{i=0}^{m-1} p_i(w^2, x_k)q_i(w, t), \end{aligned}$$

where $A_i = \int_{\mathbb{R}} p_i(w^2, x)w(x)dx$, $q_i(w, t) = H(p_i(w^2)w, t)$, $i = 0, \dots, m$, and $B_i = \int_{\mathbb{R}} xp_i(w^2, x)w(x)dx$, $i = 0, \dots, m - 1$. The coefficients $A_i, i = 0, \dots, m$, and $B_i, i = 0, \dots, m - 1$, can be computed by using (3.2). To compute $q_i(w, t), i = 0, \dots, m$, we can use the following recurrence relation:

$$\begin{cases} q_0(w, t) = \frac{2^{\frac{1}{4}}}{h_0} \int_{\mathbb{R}} \frac{e^{-x^2}}{x - t} dx, & q_1(w, t) = \frac{2^{\frac{7}{4}}}{h_1} \sqrt{\pi} + 2\sqrt{2}t \frac{h_0}{h_1} q_0(w, t), \\ q_i(w, t) = 2\sqrt{2}t \frac{h_{i-1}}{h_i} q_{i-1}(w, t) - 2(i - 1) \frac{h_{i-2}}{h_i} q_{i-2}(w, t), & i = 2, \dots, m \end{cases}$$

where $h_i = \left(\sqrt{\sqrt{\pi}2^i i!}\right)^{-1}$, $i = 0, 1, \dots, m$. To compute $q_0(w, t)$ some good routines are available (see for instance [31]).

The convergence and the estimate of the approximation error $e_m(f, t) := H(fw, t) - H_m(fw, t)$ are shown by the following

Theorem 3.2. *For all $f \in \tilde{W}_r^\infty, r > 0$, we have*

$$(3.6) \quad \sup_{|t| \leq \sqrt{2m}} |e_m(f, t)| \leq C \frac{\|f^{(r)}w\|_\infty}{(\sqrt{m})^r} \log^2 m,$$

where $\sqrt{2m} > x_0$ and C is a positive constant independent of m and f .

Theorem 3.2 shows also that $H_m(fw, t)$ is a stable approximation of $H(fw, t)$ (except for a $\log^2 m$ factor) and, even if it has a greater computational cost than the previous procedure, it can be used for the computation of $H(fw, t)$.

On the other hand $H_m(fw, t)$ can prove to be useful for another kind of problem. In fact, as already mentioned in the Introduction, sometimes $H(fw, t)$ appears as the main part of the singular integral equations and, if we use a collocation method, then $H(fw, t)$ can be replaced by $H_m(fw, t)$. In this case we have to compute $f(x_i)$ ($|x_i| \leq \sqrt{m}$) by a linear system. Thus, Theorem 3.2 is essential for the stability and the well-conditioning of the matrix of the resulting linear system.

4. PROOFS

Proof of Lemma 2.1. Let $Q_{2m+1}(x) = p_{m+1}(w, x)p_m(w, x)$. Since the zeros of $p_m(w)$ interlace with the zeros of $p_{m+1}(w)$, then $Q'_{2m+1}(x_{m+1,k}) > 0$, $Q'_{2m+1}(x_{m,k}) < 0$ and

$$0 < Q'_{2m+1}(x_{m+1,k+1}) - Q'_{2m+1}(x_{m,k}) = (x_{m+1,k+1} - x_{m,k})Q''_{2m+1}(\xi_k),$$

where $x_{m,k} < \xi_k < x_{m+1,k+1}$. Consequently

$$\frac{1}{x_{m+1,k+1} - x_{m,k}} < \frac{|Q''_{2m+1}(\xi_k)|}{Q'_{2m+1}(x_{m+1,k+1})}.$$

It is easily seen that

$$Q''_{2m+1}(\xi_k) \leq C\sqrt{m(m+1)}p_m^2(w, \xi_k).$$

On the other hand, by (2.12) and $\frac{\gamma_{m+1}}{\gamma_m} = \frac{1}{\sqrt{2(m+1)}} \sim \frac{1}{\sqrt{m}}$, we get

$$\begin{aligned} Q'_{2m+1}(x_{m+1,k+1}) &= \frac{\gamma_{m+1}}{\gamma_m} \frac{1}{\lambda_{m+1,k+1}(w)} \\ &\geq \frac{1}{\sqrt{2(m+1)}} \frac{\sqrt{2(m+1) - x_{m+1,k+1}^2}}{e^{-x_{m+1,k+1}^2}}. \end{aligned}$$

Thus

$$\frac{1}{x_{m+1,k+1} - x_{m,k}} < Cm\sqrt{m+1} \frac{e^{-x_{m+1,k+1}^2}}{\sqrt{2(m+1) - x_{m+1,k+1}^2}} p_m^2(w, \xi_k).$$

Moreover, taking into account (4.4), we obtain

$$(4.1) \quad \frac{1}{x_{m+1,k+1} - x_{m,k}} \leq C \frac{m^{3/2}}{2m - x_{m+1,k+1}^2},$$

which proves Lemma 2.1. □

We recall some equivalences that are useful in the sequel:

$$(4.2) \quad |\sqrt{w(x_k)}p'_m(w, x_k)| \sim \sqrt[4]{2m - x_k^2 + m^{\frac{1}{3}}}, \quad k = 1, \dots, m,$$

$$(4.3) \quad \max_{x \in \mathbb{R}} |\sqrt{w(x)}p_m(w, x)\sqrt[4]{2m - x^2 + m^{\frac{1}{3}}}| \sim 1$$

and

$$(4.4) \quad w(x)p_m^2(w, x)\sqrt{2m - x^2 + m^{\frac{1}{3}}} \sim \left(\frac{x - x_d}{x_d - x_{d\pm 1}}\right)^2,$$

$|x| \leq \sqrt{2m}$. The first two equivalences follow by [8, Lemma 4.2 (4.6) and (4.11), p. 266]; the third is equivalent to

$$\frac{w(x)}{w(x_d)}l_{m,d}^2(x) \sim 1,$$

where x_d is the zero of $p_m(w)$ closest to t , i.e.,

$$|x_d - t| = \min_{k=1, \dots, m} |t - x_k|$$

and

$$l_{m,d}(x) = \frac{p_m(w, x)}{p'_m(w, x_d)(x - x_d)}.$$

Proof of Theorem 2.2. Now, for $|x| \leq \sqrt{2m}$, we can write

$$(4.5) \quad \begin{aligned} L_{m+2}(w, f; x)\sqrt{w(x)} &= \frac{(\sqrt{2m} - x)\sqrt{w(x)}p_m(w, x)f(-\sqrt{2m})}{2\sqrt{2m}p_m(w, -\sqrt{2m})} \\ &\quad + \sqrt{w(x)}\sum_{k=1}^m \frac{(2m - x^2)l_{m,k}(x)f(x_k)}{(2m - x_k^2)} \\ &\quad + \frac{(\sqrt{2m} + x)\sqrt{w(x)}p_m(w, x)f(\sqrt{2m})}{2\sqrt{2m}p_m(w, \sqrt{2m})} \\ &:= A + B + C. \end{aligned}$$

We recall that, from the Rahmanov-Mhaskar-Saff identity, we have

$$\max_{x \in \mathbb{R}} |L_{m+2}(w, f; x)\sqrt{w(x)}| = \max_{x \in [-\sqrt{2m}, \sqrt{2m}]} |L_{m+2}(w, f; x)\sqrt{w(x)}|$$

and from (4.3) we get

$$|p_m(w, x)\sqrt{w(x)}| \leq Cm^{-\frac{1}{12}}.$$

Moreover, from (4.4) with $x = \pm\sqrt{2m}$ (i.e., $x_d = x_m$ or x_1), we can also deduce

$$|p_m(w, \pm\sqrt{2m})\sqrt{w(\pm\sqrt{2m})}| \geq Cm^{-\frac{1}{12}}.$$

Therefore

$$\begin{aligned}
 |A| + |C| &\leq \|f\sqrt{w}\|_\infty \left\{ \frac{\|\sqrt{w}p_m(w)\|_\infty}{|\sqrt{2m}\sqrt{w(-\sqrt{2m})}p_m(w, -\sqrt{2m})|} \right. \\
 (4.6) \qquad &\qquad \qquad \left. + \frac{\|\sqrt{w}p_m(w)\|_\infty}{|\sqrt{2m}\sqrt{w(\sqrt{2m})}p_m(w, \sqrt{2m})|} \right\} \\
 &\leq C\|f\sqrt{w}\|_\infty.
 \end{aligned}$$

Now we estimate B . By (4.3), we have

$$|\sqrt[4]{2m-x^2}\sqrt{w(x)}p_m(w, x)| \leq C$$

and by (4.2) and (2.8) we get

$$|\sqrt[4]{2m-x_k^2}\sqrt{w(x_k)}p'_m(w, x_k)| \leq C|x_k - x_{k-1}|^{-1} := C\Delta^{-1}x_k.$$

Recalling that $|l_{m,d}(x)| \sim 1$ and $\frac{2m-x^2}{2m-x_d^2} \sim 1$, we deduce

$$(4.7) \qquad |B| \leq C\|f\sqrt{w}\|_\infty \left[1 + \sum_{\substack{k=1 \\ k \neq d}}^m \left(\frac{2m-x^2}{2m-x_k^2} \right)^{\frac{3}{4}} \frac{\Delta x_k}{|x-x_k|} \right].$$

We have only to prove that the sum on the right-hand side of (4.7) is dominated by $\log m$. But it has been proved in [39].

Finally, substituting (4.6) and (4.7) into (4.5), for $|x| \leq \sqrt{2m}$, we obtain

$$(4.8) \qquad |L_{m+2}(w, f; x)\sqrt{w(x)}| \leq C\|f\sqrt{w}\|_\infty \log m$$

and, from the Rahmanov-Mhaskar-Saff identity, it follows that

$$\|L_{m+2}(w, f)\sqrt{w}\|_\infty \leq C\|f\sqrt{w}\|_\infty \log m, \quad C \neq C(m, f).$$

Then the inequality (2.13) easily follows. □

Proof of Proposition 2.3. By the definition of f_r , we have

$$\begin{aligned}
 \left| \int_{\mathbb{R}} \frac{f(x) - f_r(x)}{x-t} w_p(x) dx \right| &\leq \left\{ \int_{-\infty}^{-\frac{\sqrt{2m}}{p}} \frac{|f(x) - f_r(x)|}{t-x} w_p(x) dx \right. \\
 &\qquad \qquad \qquad \left. + \int_{\frac{\sqrt{2m}}{p}}^{\infty} \frac{|f(x) - f_r(x)|}{x-t} w_p(x) dx \right\} \\
 (4.9) \qquad &= I_1 + I_2.
 \end{aligned}$$

Taking into account that

$$|f(x) - f_r(x)| \leq \int_{\frac{\sqrt{2m}}{p}}^x \frac{(x-u)^{r-1}}{(r-1)!} |f^{(r)}(u)| du,$$

we get

$$\begin{aligned}
 I_2 &\leq C \int_{\frac{\sqrt{2m}}{p}}^{\infty} |f^{(r)}(u)| \int_u^{\infty} w_p(x) \frac{(x-u)^{r-1}}{x-t} dx du \\
 &\leq C \int_{\frac{\sqrt{2m}}{p}}^{\infty} |f^{(r)}(u)| \int_u^{\infty} w_p(x) (x-u)^{r-2} dx du \\
 &\leq C \int_{\frac{\sqrt{2m}}{p}}^{\infty} \frac{|f^{(r)}(u)|}{u^{r-1}} w_p(u) du \\
 (4.10) \quad &\leq \frac{C}{(\sqrt{2m})^{r+\lambda-1}} \|f^{(r)} w_p\|_{\infty}.
 \end{aligned}$$

Analogously, we can prove that

$$(4.11) \quad I_1 \leq \frac{C}{(\sqrt{2m})^{r+\lambda-1}} \|f^{(r)} w_p\|_{\infty}.$$

Substituting (4.10) and (4.11) into (4.9), we get (2.16). \square

In order to prove Theorems 3.1 and 3.2 we need the following lemmas.

Lemma 4.1. *If $-a < -a' < t < a' < a$, $a > 0$, we have*

$$(4.12) \quad \int_{-a}^a \frac{f(x)}{x-t} w(x) dx \leq C \left\{ \|fw\|_{[-a,a]} \log \frac{a}{a-a'} + \int_0^1 \frac{\Omega(f, u)_{w, \infty}}{u} du \right\},$$

where C is a positive constant independent of f, a and t . Obviously we assume that the integral on the right-hand side exists.

Proof. Letting $\varepsilon := \frac{(a-a')}{2}$, we use the following decomposition

$$\begin{aligned}
 \int_{-a}^a \frac{f(x)}{x-t} w(x) dx &= \left\{ \int_{-a}^{t-\varepsilon} + \int_{t-\varepsilon}^{t+\varepsilon} + \int_{t+\varepsilon}^a \right\} \frac{f(x)}{x-t} w(x) dx \\
 (4.13) \quad &:= I_1 + I_2 + I_3.
 \end{aligned}$$

We have

$$(4.14) \quad |I_1| \leq \int_{-a}^{t-\varepsilon} \frac{|f(x)w(x)|}{t-x} dx \leq \|fw\|_{[-a, t-\varepsilon]} \log \frac{t+a}{\varepsilon} \leq \|fw\|_{[-a, a]} \log \frac{2(a+a')}{a-a'}.$$

Analogously

$$(4.15) \quad |I_3| \leq \|fw\|_{[-a, a]} \log \frac{2(a+a')}{a-a'}.$$

About I_2 , letting $G(x) := f(x)e^{-x^2}$, we can write

$$|I_2| = \left| \int_0^{2\varepsilon} \frac{G(t + \frac{u}{2}) - G(t - \frac{u}{2})}{u} du \right| \leq \left\{ \int_0^1 + \int_1^{2\varepsilon} \right\} |\Delta_{\frac{u}{2}} G(t)| \frac{du}{u}.$$

If $2\varepsilon < 1$, then we let the second integral on the right-hand side equal zero. Applying the definition (2.3), we obtain

$$\begin{aligned}
 |I_2| &\leq \int_0^1 \frac{\Omega(f, t)_{w, \infty}}{u} du + \|fw\|_{[0,1]} \int_0^1 \left| \frac{\Delta_{\frac{u}{2}} w(t)}{uw(t - \frac{u}{2})} \right| du \\
 &\quad + \mathcal{C} \|fw\|_{[-a,a]} \\
 (4.16) \quad &\leq \int_0^1 \frac{\Omega(f, t)_{w, \infty}}{u} du + \mathcal{C} \|fw\|_{[-a,a]}.
 \end{aligned}$$

Finally, combining (4.14), (4.16) and (4.15) with (4.13), we deduce (4.12). \square

Lemma 4.2. For any $f \in W_r^\infty$ and for $P \in \mathbb{P}_m$ the polynomial of best approximation of the function f , we have

$$(4.17) \quad \int_0^1 \frac{\Omega(f - P, u)_{w, \infty}}{u} du \leq \frac{\mathcal{C}}{(\sqrt{m})^r} \|f^{(r)}w\|_\infty \log m,$$

where \mathcal{C} is a positive constant independent of m, f and P .

Proof. We can write

$$\begin{aligned}
 \int_0^1 \frac{\Omega(f - P, u)_{w, \infty}}{u} du &\leq \left\{ \int_0^{\frac{1}{\sqrt{m}}} + \int_{\frac{1}{\sqrt{m}}}^1 \right\} \frac{\Omega(f - P, u)_{w, \infty}}{u} du \\
 (4.18) \quad &\leq \mathcal{C} \|[f - P]w\|_\infty \log m + \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega(f - P, u)_{w, \infty}}{u} du.
 \end{aligned}$$

By using (2.5) and proceeding as in [24, Proof of Proposition 4.2, pp. 280-281], we get

$$\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega(f - P, u)_{w, \infty}}{u} du \leq \mathcal{C} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega^r(f, u)_{w, \infty}}{u} du.$$

On the other hand by (2.4) we deduce

$$\|(f - P)w\|_\infty \leq \mathcal{C} \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega^r(f, u)_{w, \infty}}{u} du.$$

Finally, taking into account

$$\Omega^r(f, u)_{w, \infty} \leq \mathcal{C} \|f^{(r)}w\|_\infty u^r,$$

we get (4.17). \square

We recall that if a function g is such that $g^{(i)}(x) \geq 0, i = 0, 1, \dots, 2m - 1, m > 1$, for $x \in (-\infty, x_d], d = 2, \dots, m$, then

$$(4.19) \quad \sum_{i=1}^{d-1} \lambda_i(w)g(x_i) \leq \int_{-\infty}^{x_d} g(x)w(x)dx \leq \sum_{i=1}^d \lambda_i(w)g(x_i).$$

If $(-1)^i g^{(i)}(x) \geq 0, i = 0, 1, \dots, 2m - 1, m > 1$, for $x \in [x_d, \infty), d = 1, \dots, m - 1$, then

$$(4.20) \quad \sum_{i=d+1}^m \lambda_i(w)g(x_i) \leq \int_{x_d}^{\infty} g(x)w(x)dx \leq \sum_{i=d}^m \lambda_i(w)g(x_i)$$

(see [8, Proof of Lemma 5.1 (b), pp. 271-272]).

Letting

$$A_m(t) = \int_{\mathbb{R}} \frac{e^{-x^2}}{x-t} dx - \sum_{\substack{k=1 \\ k \neq d}}^m \frac{\lambda_k(w)}{x_k - t},$$

we can prove the following

Lemma 4.3. *Let $|t| \leq \theta\sqrt{2m}$, with $0 < \theta < 1$, then*

$$(4.21) \quad A_m(t) \leq \mathcal{C}e^{-t^2}$$

and

$$(4.22) \quad \frac{\lambda_{m^*,d}(w)}{|x_{m^*,d} - t|} \leq \mathcal{C}e^{-x_d^2} \leq \mathcal{C}e^{-t^2}$$

hold, where \mathcal{C} is a positive constant independent of m and f and $m^* \in \{m, m + 1\}$.

Proof. Consider the case $x_{d-1} < x_d \leq t < x_{d+1}$, $d \in \{2, \dots, m - 1\}$. By using (4.19) and (4.20), we have

$$(4.23) \quad \begin{aligned} A_m(t) &\leq \frac{\lambda_{d-1}(w)}{t - x_{d-1}} + \int_{x_{d-1}}^{x_{d+1}} \frac{e^{-x^2}}{x-t} dx \\ &:= A + B. \end{aligned}$$

Since $t - x_{d-1} \geq (x_d - x_{d-1}) = \Delta x_{d-1}$, by (2.8) and (2.12), we deduce

$$|A| \leq \mathcal{C}e^{-x_{d-1}^2}.$$

Taking into account that $t^2 - x_{d-1}^2 = (t - x_{d-1})(t + x_{d-1}) \leq \frac{2\theta\sqrt{2m}}{\sqrt{2m}\sqrt{1-\theta}} \leq \mathcal{C}$, we get

$$(4.24) \quad |A| \leq \mathcal{C}e^{-t^2} e^{t^2 - x_{d-1}^2} \leq \mathcal{C}e^{-t^2}.$$

To estimate B , we note that

$$(4.25) \quad \begin{aligned} B &= \int_{x_{d-1}}^{x_{d+1}} \frac{e^{-x^2} - e^{-t^2}}{x-t} dx + e^{-t^2} \int_{x_{d-1}}^{x_{d+1}} \frac{dx}{x-t} \\ &\leq 2 \int_{x_{d-1}}^{x_{d+1}} \xi_x e^{-\xi_x^2} dx + e^{-t^2} \log \frac{x_{d+1} - t}{t - x_{d-1}} \\ &\leq 2x_{d+1} e^{-x_{d-1}^2} (x_{d+1} - x_{d-1}) + e^{-t^2} \log \frac{x_{d+1} - t}{t - x_{d-1}} \\ &\leq \mathcal{C}\sqrt{2m} e^{-x_{d-1}^2} (t - x_{d-1}) + \mathcal{C}e^{-t^2} \\ &\leq \mathcal{C}e^{-t^2}. \end{aligned}$$

Thus, replacing (4.24) and (4.25) into (4.23), we prove (4.21).

To prove (4.22), we note that

$$|x_{m^*,d} - t| > \mathcal{C}|x_{m^*+1,d+1} - x_{m^*,d}|.$$

Thus, applying Lemma 2.1 and (2.12), we obtain

$$\frac{\lambda_d(w)}{|x_{m^*,d} - t|} \leq \mathcal{C}e^{-x_d^2} \left(\frac{m}{2m - x_d^2 + (2m)^{\frac{1}{3}}} \right)^{\frac{2}{3}} \leq \mathcal{C}e^{-x_d^2} \leq \mathcal{C}e^{-t^2} e^{t^2 - x_d^2} \leq \mathcal{C}e^{-t^2},$$

that is the thesis. □

Proof of Theorem 3.1. Let $p_m \in \mathbb{P}_m$ be the polynomial of best approximation of the function f_r . Recalling definition (2.15), we note that

$$(4.26) \quad r_m(f, t) = \int_{\mathbb{R}} \frac{f(x) - f_r(x)}{x - t} w(x) dx + r_m(f_r, t),$$

where

$$(4.27) \quad \begin{aligned} r_m(f_r, t) &= \int_{\mathbb{R}} \frac{f_r(x) - p_m(x)}{x - t} w(x) dx \\ &\quad - [f_r(t) - p_m(t)] A_m(t) + [f_r(t) - p_m(t)] \frac{\lambda_d(w)}{x_d - t} \\ &\quad - [f_r(x_d) - p_m(x_d)] \frac{\lambda_d(w)}{x_d - t} - \sum_{\substack{k=1 \\ k \neq d}}^m \frac{f_r(x_k) - p_m(x_k)}{x_k - t} \lambda_k(w) \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

We can write

$$(4.28) \quad \begin{aligned} I_1 &= \left\{ \int_{-\infty}^{-\sqrt{2m}} + \int_{\sqrt{2m}}^{\infty} + \int_{-\sqrt{2m}}^{\sqrt{2m}} \right\} \frac{f_r(x) - p_m(x)}{x - t} w(x) dx \\ &= A_1 + A_2 + A_3. \end{aligned}$$

Since $|t| \leq \theta\sqrt{2m}$, then $x - t > (1 - \theta)\sqrt{2m}$. Therefore, we have

$$\begin{aligned} |A_2| &\leq \frac{1}{(1 - \theta)\sqrt{2m}} \int_{\sqrt{2m}}^{\infty} |f_r(x) - p_m(x)| w(x) dx \\ &\leq \frac{1}{(1 - \theta)\sqrt{2m}} \int_{-\infty}^{\infty} |f_r(x) - p_m(x)| w(x) dx. \end{aligned}$$

Finally, by (2.6) we get

$$(4.29) \quad |A_2| \leq \frac{1}{(1 - \theta)(\sqrt{2m})^{r+1}} \int_{-\sqrt{2m}}^{\sqrt{2m}} |f^{(r)}(x) w(x)| dx \leq \frac{\mathcal{C}}{(\sqrt{m})^r} \|f^{(r)} w\|_{\infty}.$$

Analogously we can prove that

$$(4.30) \quad |A_1| \leq \frac{\mathcal{C}}{(\sqrt{m})^r} \|f^{(r)} w\|_{\infty}.$$

About A_3 , applying Lemma 4.1, with $a = \sqrt{2m}$ and $a' = \theta\sqrt{2m}$, we get

$$|A_3| \leq \mathcal{C} \left\{ \|(f_r - p_m)w\|_{\infty} + \int_0^1 \frac{\Omega(f_r - p_m, u)_{w, \infty}}{u} du \right\}.$$

Moreover, taking into account Lemma 4.2 and (2.6), we get

$$(4.31) \quad |A_3| \leq \frac{\mathcal{C}}{(\sqrt{m})^r} \|f^{(r)} w\|_{\infty} \log m.$$

Substituting (4.30), (4.29), (4.31) into (4.28), we obtain

$$(4.32) \quad |I_1| \leq \frac{\mathcal{C}}{(\sqrt{m})^r} \|f^{(r)} w\|_{\infty} \log m.$$

Applying Lemma 4.3 and (2.6), we obtain

$$(4.33) \quad |I_2| + |I_3| + |I_4| \leq \mathcal{C} \|(f_r - p_m)w\|_{\infty} \leq \frac{\mathcal{C}}{(\sqrt{m})^r} \|f^{(r)} w\|_{\infty}.$$

Moreover, taking into account (2.12) we get

$$|I_5| \leq C \|(f_r - p_m)w\|_\infty \sum_{\substack{k=1 \\ k \neq d}}^m \frac{\Delta x_k}{|x_k - t|}.$$

Since it is easy to see that

$$\sum_{\substack{k=1 \\ k \neq d}}^m \frac{\Delta x_k}{|x_k - t|} \leq C \log m,$$

applying (2.6), we have

$$(4.34) \quad |I_5| \leq \frac{C}{(\sqrt{m})^r} \|f^{(r)}w\|_\infty \log m.$$

Combining (4.32), (4.33) and (4.34) with (4.27), we get

$$(4.35) \quad r_m(f_r, t) \leq \frac{C}{(\sqrt{m})^r} \|f^{(r)}w\|_\infty \log m.$$

Finally, substituting (4.35) into (4.26) and applying Proposition 2.3, with $p = 1$, we deduce (3.4). □

Proof of Theorem 3.2. We can write

$$(4.36) \quad e_m(f, t) = \int_{\mathbb{R}} \frac{f(x) - f_r(x)}{x - t} w(x) dx + e_m(f_r, t),$$

where $e_m(f_r, t)$

$$\begin{aligned} &= \int_{\mathbb{R}} \frac{f_r(x) - L_{m+2}(w^2, f_r, x)}{x - t} w(x) dx \\ &= \left\{ \int_{-\infty}^{-\sqrt{2m-1}} + \int_{\sqrt{2m+1}}^{\infty} + \int_{-\sqrt{2m-1}}^{\sqrt{2m+1}} \right\} \frac{f_r(x) - L_{m+2}(w^2, f_r, x)}{x - t} w(x) dx \\ &= A_1 + A_2 + A_3. \end{aligned}$$

Now, we can proceed analogously to the proof of (4.28), replacing p_m by $L_{m+2}(w^2, f_r)$. The only difference consists in the evaluation of A_3 , for which we apply Lemma 4.1 with $a = \sqrt{2m} + 1$ and $a' = \sqrt{2m}$. Thus, we obtain

$$(4.37) \quad |e_m(f_r, t)| \leq \frac{C}{(\sqrt{m})^r} \|f^{(r)}w\|_\infty \log m.$$

Substituting (4.37) into (4.36) and applying Proposition 2.3, with $p = 1$, (3.6) follows. □

5. NUMERICAL EVALUATIONS

In this section we show some approximate values for the integral $H(fw, t), t \in \mathbb{R}$, obtained by using the algorithm described in Section 3.

The density functions we choose are representatives of the functional spaces (e.g., Sobolev spaces) on which we want to test our method, i.e., we do not exclude the integrals that could be better calculated otherwise.

Since in the following examples the exact values of the integrals are not known, the results on the last line of our tables are thought to be exact to the number of

TABLE 1.

m	Φ_m		
	$t = 0.1$	$t = 5$	$t = 10$
8	-0.261315425408	-0.470154	-0.2293312798
16	-0.261315425408597	-0.47015461500803	-0.2293312798756

TABLE 2.

m	H_m		
	$t = 0.1$	$t = 5$	$t = 10$
8	-0.26131	-0.47015461	-0.2293312798
16	-0.261315425408	-0.4701546150080	-0.2293312798
32	-0.261315425408597	-0.47015461500803	-0.229331279875

TABLE 3.

t	m	$Q_{m,m}^{(1)}$	m	g_m
0.1	32	-0.26131542540859	63	-0.26131542540859
5	32	-0.47015461500803	113	-0.47015461500803
10	64	-0.22933127987563	163	-0.2293312798756

figures shown. Moreover, in all the tables we have reported only the digits which are correct according to these exact values.

Example 1. We want to evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{\cosh x}{x - t} e^{-x^2} dx.$$

Since the function $f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$ is an analytic function with an exponential growth, we obtain very accurate results. In Table 1 we can see that, for different values of the parameter t , we need only 16 points to obtain machine precision.

In Table 2 we show the corresponding results obtained by using the quadrature formula H_m .

We can note that the two formulas Φ_m and H_m give results almost comparable, but H_m has a more expensive computational cost than Φ_m .

In Table 3 we compare our results with the ones obtained by using the quadrature rules proposed in [16] and [3]. We denote by

$$Q_{m,m}^{(1)} = \sum_{k=-m}^m \frac{G(t + kh + \frac{h}{2})}{k + \frac{h}{2}},$$

where $h = \sqrt{\frac{2\pi}{m}}$, the quadrature rule proposed in [3], and by

$$g_m = 2 \sum_{\substack{\nu=-m \\ \nu \neq \text{even}}}^m \frac{G(t + \nu h)}{\nu},$$

where $h = 0.1$, the quadrature rule proposed in [16].

TABLE 4.

$t = 0.25$		$t = 0.3$		$t = 15$	
m	Φ_m	m	Φ_m	m	Φ_m
9	-480.	9	-470.1	8	-229.
19	-480.4897	19	-470.1335	17	-229.2247
39	-480.489721	38	-470.133527	35	-229.224743
78	-480.48972131	76	-470.13352728	71	-229.22474324
157	-480.4897213129	153	-470.1335272874	143	-229.2247432478
314	-480.48972131296	307	-470.133527287461	286	-229.224743247854

TABLE 5. $\alpha = 2$

$t = -1.5$		$t = 5$		$t = 15$	
m	Φ_m	m	Φ_m	m	Φ_m
32	1.1711	32	-0.3	32	-0.105
65	1.17112	65	-0.32	64	-0.1051
131	1.171126	131	-0.3253	128	-0.1051
262	1.1711263	262	-0.32531	257	-0.105179

TABLE 6. $\alpha = 3$

$t = 0.5$		$t = 8$		$t = 18$	
m	Φ_m	m	Φ_m	m	Φ_m
37	-1.77814	32	-0.14	32	-6.551
74	-1.77814144	65	-0.14	64	-6.5517
149	-1.77814144	131	-0.148	128	-6.55177
298	-1.7781414419	262	-0.1480	257	-6.55177

Example 2. Now we consider the following integral

$$\int_{-\infty}^{\infty} \frac{|x - \frac{1}{2}|^{\frac{7}{2}} |x - \frac{1}{3}|^{\frac{9}{2}} |x - \frac{1}{4}|^{\frac{9}{2}}}{x - t} e^{-x^2} dx.$$

Since $f(x) = |x - \frac{1}{2}|^{\frac{7}{2}} |x - \frac{1}{3}|^{\frac{9}{2}} |x - \frac{1}{4}|^{\frac{9}{2}} \in \tilde{W}_3^\infty$, the theoretical error is $m^{-\frac{3}{2}} \log m$. The results shown in Table 4 confirm that we have to increase m to reach significant digits.

Example 3. Finally we evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{e^{x^2}}{(1+x^2)^\alpha} \frac{e^{-x^2} dx}{x - t}, \quad \alpha = 2, 3.$$

If $\alpha = 2$, then the function $f(x) = \frac{e^{x^2}}{(1+x^2)^2} \in \tilde{W}_3^\infty$ and the theoretical error is $m^{-\frac{3}{2}} \log m$; if $\alpha = 3$, then the function $f(x) = \frac{e^{x^2}}{(1+x^2)^3} \in \tilde{W}_4^\infty$ and the theoretical error is $m^{-2} \log m$. In Tables 5 and 6 we can see that the numerical results agree with the theoretical ones.

All the computations were done in Double Precision Arithmetic on the Digital Ultimate Workstation 533au².

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