ESTIMATES OF $\theta(x; k, l)$ FOR LARGE VALUES OF $x$

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Abstract. We extend a result of Ramaré and Rumely, 1996, about the Chebyshev function $\theta$ in arithmetic progressions. We find a map $\epsilon(x)$ such that $|\theta(x; k, l) - x/\varphi(k)| < x\epsilon(x)$ and $\epsilon(x) = O\left(\frac{1}{\ln x}\right)$ (for $a > 0$), whereas $\epsilon(x)$ is a constant. Now we are able to show that, for $x \geq 1531$,

$$|\theta(x; 3, l) - x/2| < \frac{0.262 x}{\ln x}$$

and, for $x \geq 151$,

$$\pi(x; 3, l) > \frac{x}{2\ln x}.$$

1. Introduction

Let $R = 9.645908801$ and $X = \sqrt{\frac{\ln x}{R}}$. Rosser [6] and Schoenfeld [7, Th. 11 p. 342] showed that, for $x \geq 101$,

$$|\theta(x) - x|, |\psi(x) - x| < x\epsilon(x),$$

where

$$\epsilon(x) = \sqrt{\frac{8}{17\pi}}X^{1/2}\exp(-X).$$

We adapt their work to the case of arithmetic progressions. Let us recall the usual notations for nonnegative real $x$:

$$\theta(x; k, l) = \sum_{\substack{p \equiv l \mod k \leq x}} \ln p,$$

where $p$ is a prime number,

$$\psi(x; k, l) = \sum_{\substack{n \equiv l \mod k \leq x}} \Lambda(n),$$

where $\Lambda$ is Von Mangold’s function,

and $\varphi$ is Euler’s function. We show, for $x \geq x_0(k)$ where $x_0(k)$ can be easily computed, that

$$|\theta(x; k, l) - x/\varphi(k)|, |\psi(x; k, l) - x/\varphi(k)| < x\epsilon(x),$$

where

$$\epsilon(x) = 3\sqrt{\frac{k}{\varphi(k)C_1(k)}}X^{1/2}\exp(-X).$$

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for an explicit constant $C_1(k)$. We apply the above results for $k = 3$. For small values, we use Ramaré and Rumely’s results \cite{4}. We show that for $x \geq 1531$,

\begin{equation}
| \theta(x; 3, l) - x/2 | < 0.262 \frac{x}{\ln x}.
\end{equation}

If we assume that the Generalized Riemann Hypothesis is true, then we can show that, for $x > 1$ and $k \leq 432$,

\begin{equation}
| \psi(x; k, l) - x/\varphi(k) | < \frac{1}{4\pi} \sqrt{x} \ln^2 x.
\end{equation}

Let us define, as usual, $\pi(x)$ the number of primes not greater than $x$. In 1962, Rosser and Schoenfeld \cite{5, p. 69} found a lower bound for $\pi(x)$:

\begin{equation}
\pi(x) > \frac{x}{\ln x} \quad \text{for } x \geq 17.
\end{equation}

Letting

$$
\pi(x; k, l) = \sum_{p \leq x, \, p \equiv l \mod k} 1,
$$

we show an analogous result in the case of arithmetic progression with $k = 3$ and $l = 1$ or 2,

$$
\pi(x; 3, l) > \frac{x}{2\ln x} \quad \text{for } x \geq 151.
$$

This result, inferred from (1), implies (2) and cannot be proved with Ramaré and Rumely’s results.

The method used for $k = 3$ can also be applied for other fixed integers $k$.

2. Preliminary lemmas

Notations. We will always denote by $\rho$ a nontrivial zero of Dirichlet’s function $L$, that is to say a zero such that $0 < \Re \rho < 1$. We write $\rho = \beta + i\gamma$. Let $\varphi(\chi)$ be the set of the zeros $\rho$ of the function $L(s, \chi)$, with $0 < \beta < 1$.

For a positive real $H$, following Ramaré and Rumely, we say that GRH($k, H$) holds\footnote{Note that our GRH is an acronym for the usual Generalized Riemann Hypothesis.} if, for all $\chi$ modulo $k$, all the nontrivial zeros of $L(s, \chi)$ with $|\gamma| \leq H$ are such that $\beta = 1/2$.

As in Rosser and Schoenfeld (in \cite{5, 7} where the case $k = 1$ is studied), we must know the distribution of $L(s, \chi)$’s zeros; namely, find a real $H$ such that GRH($k, H$) is satisfied and is a zero-free region.

2.1. Zero-free region.

Theorem 1 (Ramaré and Rumely \cite{3}). If $\chi$ is a character with conductor $k$, $H \geq 1000$, and $\rho = \beta + i\gamma$ is a zero of $L(s, \chi)$ with $|\gamma| \geq H$, then there exists a computable constant $C_1(\chi, H)$ such that

$$
1 - \beta \geq \frac{1}{R \ln(k|\gamma|/C_1(\chi, H))}.
$$
Examples. Some examples, extracted from [3, p. 409], appear in the following table.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$H_k$</th>
<th>$C_1(\chi, H_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5450000000</td>
<td>38.31</td>
</tr>
<tr>
<td>3</td>
<td>10000</td>
<td>20.92</td>
</tr>
<tr>
<td>420</td>
<td>2500</td>
<td>56.59</td>
</tr>
</tbody>
</table>

Proof. See Theorem 3.6.3 of Ramaré and Rumely [3, p. 409].

Remark. For $k \geq 1$ and $H_k \geq 1000$, $C_1(\chi, H) \geq C_1(\chi_0, 1000) \geq 9.14$.

As $C_1(\chi, H)$ could be large, we limit $C_1(\chi, H_k)$ up to $32\pi$ to make some computations. So we have in our hypothesis

$$9.14 \leq C_1(\chi, H) \leq 32\pi.$$ 

From now on,

$$C_1(k) = \min(\min_{\chi \mod k} C_1(\chi, H_\chi), 32\pi).$$

2.2. GRH($k, H$) and $N(T, \chi)$.

Lemma 1 (McCurley [1]). Let $C_2 = 0.9185$ and $C_3 = 5.512$. Write $F(y, \chi) = \frac{y}{\pi} \ln \left( \frac{k y}{2\pi e} \right)$ and $R(y, \chi) = C_2 \ln(ky) + C_3$. If $\chi$ is a character of Dirichlet with conductor $k$, if $T \geq 1$ is a real number, and if $N(T, \chi)$ denotes the number of zeros $\beta + i\gamma$ of $L(s, \chi)$ in the rectangle $0 < \beta < 1, |\gamma| \leq T$, then

$$|N(T, \chi) - F(T, \chi)| \leq R(T, \chi).$$

Lemma 2 (deduced from [3] Theorem 2.1.1, p. 399) and [3].

- GRH($1, H$) is true for $H = 5.45 \times 10^8$.
- GRH($k, H$) is true for $H = 10000$ and $k \leq 13$.
- GRH($k, 2500$) is true for sets
  - $E_1 = \{k \leq 72\}$,
  - $E_2 = \{k \leq 112, k \text{ not prime}\}$,
  - $E_3 = \{116, 117, 120, 121, 124, 125, 128, 132, 140, 143, 144, 156, 163, 169, 180, 216, 243, 256, 360, 420, 432\}$.

2.3. Estimates of $|\psi(x; k, l) - x/\varphi(k)|$ using properties of zeros of $L(s, \chi)$.

As in Ramaré and Rumely, we remove the zeros with $\beta = 0$ and we consider only primitive $L$-series by adding small terms. Here we take the version stated in [3] Theorem 4.3.1 which is deduced from [1].

Theorem 2 (McCurley [1]). Let $x > 2$ be a real number, $m$ and $k$ two positive integers, $\delta$ a real number such that $0 < \delta < \frac{x - 2}{m^2}$, and $T$ a positive real. Let

$$A(m, \delta) = \frac{1}{\delta^m} \sum_{j=0}^{m} \binom{m}{j} (1 + j\delta)^{m+1}.$$
Assume GRH\((k, 1)\). Then
\[
\frac{\varphi(k)}{x} \max_{1 \leq y \leq x} |\psi(y; k, l) - \frac{y}{\varphi(k)}| < A(m, \delta) \sum_{\chi} \sum_{\rho \in \mu(k) \text{ with } |\gamma| < 2} |\rho(\rho + 1) \cdots (\rho + m)|^{x^{\beta - 1}} + \left(1 + \frac{m\delta}{2}\right) \sum_{\chi} \sum_{\rho \in \mu(k) \text{ with } |\gamma| < 2} |\rho|^{x^{\beta - 1}} + \frac{m\delta}{2} + \tilde{R}/x,
\]
where \(\sum_{\chi}\) denotes the summation over all characters modulo \(k\), \(\tilde{R} = \varphi(k)\left[ (f(k) + 0.5) \ln x + 4 \ln k + 13.4 \right]\) and \(f(k) = \sum_{\rho \mid k} \frac{1}{\rho - 1}\).

2.4. **One more explicit form of estimates.** The next lemma can be found in [3] with the difference that the authors assumed GRH\((k, H)\) but in fact they used only GRH\((k, 1)\). Since we must apply it with \(T > H\), we repeat the proof.

**Lemma 3.** Let \(\chi\) be a character modulo \(k\). Assume GRH\((k, 1)\). Then, for any \(T \geq 1\), we have
\[
\sum_{|\gamma| \leq T \atop \rho \in \mu(k)} \frac{1}{|\gamma|} \leq \tilde{E}(T)
\]
with \(\tilde{E}(T) = \frac{1}{2\pi} \ln^2(T) + \frac{\ln(n(k)}){\pi} \ln(T) + C_2 + 2 \left( \frac{1}{2} \ln \left( \frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right)\).

**Proof.** For \(|\gamma| \leq 1\), we have GRH\((k, 1)\) and so
\[
\sum_{|\gamma| \leq 1 \atop \rho \in \mu(k)} \frac{1}{|\gamma|} \leq \sum_{|\gamma| \leq 1 \atop \rho \in \mu(k)} \frac{1}{|1/2 + i\gamma|} \leq 2N(1, \chi).
\]
For \(|\gamma| > 1\),
\[
\sum_{1 < |\gamma| \leq T \atop \rho \in \mu(k)} \frac{1}{|\gamma|} \leq \int_1^T dN(t, \chi) = \int_1^T \frac{N(t, \chi)}{t^2} dt + \frac{N(T, \chi)}{T} - \frac{N(1, \chi)}{1}.
\]
Thus,
\[
\sum_{|\gamma| \leq T \atop \rho \in \mu(k)} \frac{1}{|\gamma|} \leq \int_1^T \frac{N(t, \chi)}{t^2} dt + \frac{N(T, \chi)}{T} + N(1, \chi).
\]
We conclude by Lemma [1] that
\[
\int_1^T \frac{N(t, \chi)}{t^2} dt \leq \int_1^T F(t, \chi) + R(t, \chi) dt = \frac{1}{\pi} \int_1^T \frac{\ln(kt/(2\pi e))}{t} dt + C_2 \int_1^T \frac{\ln(kt)}{t^2} dt + C_3 \int_1^T \frac{1}{t^2} dt
\]
\[
= \frac{1}{\pi} \left[ \frac{1}{2} \ln^2 \left( \frac{kt}{2\pi e} \right) \right]_1^T + C_2 \left[ -\frac{\ln(kt)}{t} \right]_1^T + \int_1^T \frac{1}{t^2} dt + C_3 \left[ -1/t \right]_1^T
\]
\[
= \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) \ln T + C_2 \left( -\frac{\ln(kt)}{T} + \ln k - \frac{1}{T} + 1 \right)
\]
\[
+ C_3 (1 - 1/T).
\]
In the same way, we have an upper bound of
\[ \frac{N(T, \chi)}{T} \] with \[ \frac{F(T, \chi) + R(T, \chi)}{T} \] and
\[ N(1, \chi) \] with \[ F(1, \chi) + R(1, \chi) \].

Finally, we obtain
\[ \sum_{|\gamma| \leq T \atop \rho \in \rho(\chi)} \frac{1}{|\rho|} \leq \frac{1}{2\pi} \ln^2(T) + \frac{\ln \left( \frac{k}{2\pi} \right)}{\pi} \ln(T) \]
\[ + C_2 + 2 \left( \frac{1}{\pi} \ln \left( \frac{k}{2\pi} \right) + C_2 \ln k + C_3 \right) - \frac{C_2}{T} \]

Using the facts that
\begin{itemize}
  \item if \( \rho \) is a zero of \( L(s, \chi) \) then \( \overline{\rho} \) is zero of \( L(s, \overline{\chi}) \),
  \item these zeros are symmetrical with to the line \( \Re(z) = 1/2 \),
\end{itemize}
we obtain Lemma 4 by examining the proof of [3, Lemma 4.1.3].

**Lemma 4** ([3]). Let
\[ \phi_m(t) = \frac{1}{|t|^{m+1}} \exp \left( -\frac{\ln x}{R \ln(k|t|/C_1(k))} \right) \]
with \( R = 9.645908801 \). Let \( T \geq H \). We have
\[ \sum_{|\gamma| \leq T \atop \rho \in \rho(\chi)} |\gamma|^m + \sum_{|\gamma| \leq T \atop \rho \in \rho(\chi)} |\gamma|^m x^\beta \leq x \sum_{|\gamma| \leq T \atop \rho \in \rho(\chi)} \phi_m(\gamma) + \sqrt{x} \sum_{|\gamma| \leq T \atop \rho \in \rho(\chi)} \frac{1}{|\gamma|^{m+1}}. \]

Let us rewrite Lemma 7 of [6] to adapt it to the new functions \( F(y, \chi) \) and \( R(y, \chi) \) which we use.

**Lemma 5.** Write \( N(y) = N(y, \chi) \), \( F(y) = F(y, \chi) \), and \( R(y) = R(y, \chi) \). Let \( 1 < U \leq V \) and \( \phi(y) \) be a positive and differentiable function for \( U \leq y \leq V \). Let \( (W - y)\phi'(y) \geq 0 \) for \( U < y < V \), where \( W \) does not necessarily belong to \( [U, V] \). Let \( Y \) be that one of the numbers \( U, V, W \) which is not numerically the least or greatest (or is the repeated one, if two among \( U, V, W \) are equal). Take \( j = 0 \) or \( 1 \), accordingly as \( W < V \) or \( W \geq V \). Then
\[ \sum_{|\gamma| \leq V \atop U < |\gamma| \leq V} \phi(|\gamma|) \leq \frac{1}{\pi} \int_U^V \phi(y) \ln \left( \frac{ky}{2\pi} \right) dy + (-1)^j C_2 \int_U^V \frac{\phi(y)}{y} dy + B_j(Y, U, V), \]
where
\[ B_0(Y, U, V) = 2R(Y)\phi(Y) + \{N(V) - F(V) - R(V)\}\phi(V) - \{N(U) - F(U) + R(U)\}\phi(U), \]
\[ B_1(Y, U, V) = \{N(V) - F(V) + R(V)\}\phi(V) - \{N(U) - F(U) + R(U)\}\phi(U). \]
Proof. We have

\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) = \int_U^V \phi(y) dN(y) = -\int_U^V N(y) \phi'(y) dy + N(V) \phi(V) - N(U) \phi(U).
\]

- \( j = 1 \). We have \( W > V \) and so \( Y = \min(V, W) = V \). According to Theorem 1, \( N(y) \geq F(y) - R(y) \).

\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) \leq [(N(y) - F(y) + R(y)) \phi(y)]_U^V + \frac{1}{\pi} \int_U^V \ln \left( \frac{ky}{2\pi} \right) \phi(y) dy - \int_U^V R'(y) \phi(y) dy
\]

because \( F'(y) = \frac{1}{\pi} \left( \ln \left( \frac{ky}{2\pi} \right) + 1 \right) = \frac{1}{\pi} \ln \left( \frac{ky}{2\pi} \right) \). Moreover,

\[
- \int_U^V R'(y) \phi(y) dy = -C_2 \int_U^V \frac{\phi(y)}{y} dy.
\]

- \( j = 0 \). We have \( V > W \). Take \( Y = \max(U, W) \). Split the integral at \( Y \). Then \(-\phi'(y) \leq 0 \) for \( y \in [U, Y] \) and \(-\phi'(y) \geq 0 \) for \( y \in [Y, V] \). Replacing \( N(y) \) by \( F(y) - R(y) \) in the first part and by \( F(y) + R(y) \) in the second part, we obtain

\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) \leq \frac{1}{\pi} \int_U^V \ln \left( \frac{ky}{2\pi} \right) \phi(y) dy + \int_Y^V R'(y) \phi(y) dy - \int_U^V R'(y) \phi(y) dy
\]

\[
+ B_0(Y, U, V).
\]

Moreover,

\[
\int_Y^V R'(y) \phi(y) dy \leq (-1)^j C_2 \int_U^V \frac{\phi(y)}{y} dy
\]

and

\[
- \int_U^Y R'(y) \phi(y) dy \leq 0.
\]

We want to apply Lemma 5 with \( \phi = \phi_m \) defined by (5) and with \( W = W_m \) being the root of \( \phi_m' \). Let

\[
X = \sqrt{\frac{\ln x}{R}}
\]

and, for \( m \geq 0 \),

\[
W_m = \frac{C_1(k)}{k} \exp(X/\sqrt{m + 1}).
\]

**Corollary 1** (Corollary from Lemma 5). Under the hypothesis of Lemma 5 if moreover \( \frac{X}{k} \leq U \), then

\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) \leq \left\{ \frac{1}{\pi} + (-1)^j q(Y) \right\} \int_U^V \phi(y) \ln(ky/2\pi) dy + B_j(Y, U, V),
\]

where \( q(y) = \frac{C_2}{y \ln(\frac{ky}{2\pi})} \).
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Proof. The map $y \mapsto 1/(y \ln(ky/2\pi))$ is decreasing if $y \geq 2\pi/(ke)$.

- Case $(j = 0)$, then $Y = \max(U, W)$.

$$
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) < B_0(Y, U, V) + \frac{1}{\pi} \int_U^V \phi(y) \ln \left( \frac{ky}{2\pi} \right) dy + \int_Y^V R'(y) \phi(y) dy.
$$

$$
\int_Y^V R'(y) \phi(y) dy = C_2 \int_Y^V \frac{\phi(y)}{y} dy = C_2 \int_Y^V \frac{\phi(y) \ln(ky/2\pi)}{y \ln(ky/2\pi)} dy 
\leq \frac{C_2}{Y \ln(kY/2\pi)} \int_Y^V \phi(y) \ln(ky/2\pi) dy.
$$

- Case $(j = 1)$, then $Y = V$.

$$
- \int_U^V R'(y) \phi(y) dy \leq -\frac{C_2}{V \ln(kV/2\pi)} \int_U^V \phi(y) \ln(ky/2\pi) dy.
$$

\[\square\]

Theorem 3. Let $k \geq 1$ an integer, $H \geq 1000$ a real number. Assume GRH($k, H$).

Let $x_0 > 2$ be a real number, $m$ a positive integer, and $\delta$ a real number such that $0 < \delta < (x_0 - 2)/(mx_0)$ and let $Y$ be defined as in Lemma 5. We write

$$
\tilde{A}_H = \frac{1}{\pi} \int_H^\infty \phi_m(y) \ln \left( \frac{ky}{2\pi} \right) dy + C_2 \int_H^\infty \frac{\phi_m(y)}{y} dy,
$$

$$
\tilde{B}_H = B_0(Y, H, \infty),
$$

$$
\tilde{C}_H = \frac{1}{m\pi H^m} \left( \ln \left( \frac{kH}{2\pi} \right) + 1/m \right),
$$

$$
\tilde{D}_H = \left( 2C_2 \ln(kH) + 2C_3 + \frac{C_2}{m + 1} \right) / H^{m+1}.
$$

Then for all $x \geq x_0$, we have

$$
\frac{\varphi(k)}{x} \max_{1 \leq y \leq x} |\psi(y; k, l) - \frac{y}{\varphi(k)}| \leq A(m, \delta) \frac{\varphi(k)}{2} \left( \tilde{A}_H + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H) / \sqrt{x} \right) 
+ \left( 1 + \frac{m\delta}{2} \right) \varphi(k) E(H) / \sqrt{x} + \frac{m\delta}{2} + \tilde{R}/x.
$$

Remark. We find a version of Theorem 4.3.2 of [3] where $x_0$ is replaced by $x$ in $\tilde{A}$ and $\tilde{B}$.

Proof. According to Theorem 1,

$$
\frac{\varphi(k)}{x} \max_{1 \leq y \leq x} |\psi(y; k, l) - \frac{y}{\varphi(k)}| \leq A(m, \delta) \sum_{\chi \mid \rho \in \rho(k)} \sum_{|\gamma| > H} \frac{x^{\rho-1}}{|\rho+1| \cdots (\rho+m)|} 
+ \left( 1 + \frac{m\delta}{2} \right) \sum_{\chi \mid \rho \in \rho(k)} \sum_{|\gamma| \leq H} \frac{x^{\rho-1}}{|\rho|} + \frac{m\delta}{2} + \tilde{R}/x.
$$
We separately examine the different parts:

- We have
  \[ \sum_{\chi \rho \in \mu_{\chi}} \sum_{|\gamma| > H} |\rho(\rho + 1) \cdots (\rho + m)| \lesssim \sum_{\chi \rho \in \mu_{\chi}} \sum_{|\gamma| > H} x^{\beta - 1} |\gamma|^{m+1}. \]

By Lemma 4

\[ \sum_{\chi} \sum_{\rho \in \mu_{\chi}} |\gamma|^{m+1} = \sum_{\chi} \frac{1}{2} \left( \sum_{\rho \in \mu_{\chi}} |\gamma|^{m+1} + \sum_{\rho \in \mu_{\chi}} |\gamma|^{m+1} \right) \leq \frac{1}{2} \sum_{\chi} \left( \sum_{|\gamma| > H} \phi_m(\gamma) + \frac{1}{x} \sum_{|\gamma| > H} \frac{1}{|\gamma|^{m+1}} \right). \]

Using Lemma 5 with \( U = H, V = 1, \phi = \phi_m, \) and \( W = W_m, \)

\[ \sum_{|\gamma| > H} \phi_m(\gamma) \leq \tilde{A}_H + \tilde{B}_H. \]

Integration by parts gives

\[ \sum_{|\gamma| > H} \frac{1}{|\gamma|^{m+1}} \leq \tilde{C}_H + \tilde{D}_H. \]

- By GRH \((k, H)\) we have \( \beta = 1/2 \) for all \( |\gamma| \leq H, \) and by Lemma 3

\[ \sum_{\rho \in \mu_{\chi}, |\gamma| < H} x^{\beta - 1} |\rho| \leq \tilde{E}(H)/\sqrt{x}. \]

2.5. The leading term \((\tilde{A}_H)\). To obtain an upper bound for the leading term, we proceed like Rosser and Schoenfeld with upper bounds on the integrals. The next three lemmas are issued directly from \( \mathbb{R} \) p. 251-255.

**Lemma 6** (Functions of incomplete Bessel type). Let

\[ K_\nu(z, u) = \frac{1}{2} \int_u^\infty t^{\nu-1} H^z(t) dt, \]

where \( z > 0, u \geq 0, \) and

\[ H^z(t) = \{H(t)\}^z = \exp\{-\frac{z}{2}(t + 1/t)\}. \]

Further, write \( K_\nu(z, 0) = K_\nu(z). \) Then

\[ K_1(z) \leq \sqrt{\frac{\pi}{2z}} \exp(-z) \left( 1 + \frac{3}{8z} \right), \]

\[ K_2(z) \leq \sqrt{\frac{\pi}{2z}} \exp(-z) \left( 1 + \frac{15}{8z} + \frac{105}{128z^2} \right). \]

**Lemma 7.**

\[ K_\nu(z, x) + K_{-\nu}(z, x) = K_\nu(z). \]

Hence, \( K_\nu(z, x) \leq K_\nu(z) \) \((\nu \geq 0)\).
Lemma 8. Let
\[ Q_\nu(z, x) = \frac{x^{\nu+1}}{z(x^2 - 1)} \exp\{-z(x + 1)/2\}. \]
If \( z > 0 \) and \( x > 1 \), then
\[ K_1(z, x) < Q_1(z, x) \]
and
\[ K_2(z, x) < (x + 2/z)Q_1(z, x). \]

The term \( \tilde{A}_H \) can be expressed using incomplete Bessel functions.

Lemma 9. Let \( X \) be defined by (6). Let \( z_m = 2X \sqrt{m} = 2\sqrt{\frac{m \ln x}{R}} \) and \( U_m = \frac{2m}{z_m} \ln \left( \frac{kH}{C_1(k)} \right) = \sqrt{\frac{Rm}{\ln x}} \ln \left( \frac{kH}{C_1(k)} \right) \).
\[ \tilde{A}_H = \frac{2}{\pi} \ln \frac{x}{Rm} \left( \frac{k}{C_1(k)} \right)^m K_2(z_m, U_m) \]
\[ + \frac{2}{\pi} \ln \left( \frac{C_1(k)}{2\pi} \right) \sqrt{\frac{\ln x}{Rm}} \left( \frac{k}{C_1(k)} \right)^m K_1(z_m, U_m) \]
\[ + 2C_2 \sqrt{\frac{\ln x}{R(m + 1)}} \left( \frac{k}{C_1(k)} \right)^{m+1} K_1(z_{m+1}, U_{m+1}). \]

Proof. This is by straightforward algebraic manipulation; for example, we write
\[ I = \int_{\tilde{H}}^{\infty} \frac{C_2}{y^{m+1}} \exp \left( -\frac{\ln x}{R \ln(ky/C_1(k))} \right) dy. \]
Changing variables:
\[ t = \sqrt{\frac{R(m + 1)}{\ln x}} \ln \left( \frac{ky}{C_1(k)} \right), \]
\[ dt = \frac{\sqrt{R(m + 1)}}{\ln x} \frac{dy}{y}. \]
Now
\[ \exp \left( -\frac{\ln x}{R \ln(ky/C_1(k))} \right) = \exp \left( -\frac{\ln x}{Rt/\sqrt{\frac{R(m + 1)}{\ln x}}} \right) \]
\[ = \exp \left( -\frac{(m + 1) \ln x}{Rt} \right) = \exp \left( -\frac{z_{m+1}}{2t} \right) \]
and
\[ \frac{1}{y^{m+1}} \left( \frac{k}{C_1(k)} \right)^{m+1} \exp \left( -\frac{(m + 1)t}{\sqrt{R(m + 1)}} \right) = \left( \frac{k}{C_1(k)} \right)^{m+1} \exp \left( -\frac{t}{2} \right). \]
Consequently,
\[ I = \int_{U_{m+1}}^{\infty} C_2 \sqrt{\frac{\ln x}{R(m + 1)}} \left( \frac{k}{C_1(k)} \right)^{m+1} \exp \left( -\frac{z_{m+1}}{2} (t + 1/t) \right). \]
2.6. Study of \( f(k) \) which appears in the expression of \( \hat{R} \). Remember that \( f(k) = \sum_{p \mid k} \frac{1}{p} \).

Lemma 10. For an integer \( k \geq 1 \),

\[
f(k) \leq \frac{\ln k}{\ln 2}.
\]

Proof. We prove by recursion that

\[
f(k) \leq \frac{\ln k}{\ln 2}.
\]

For \( k = 1 \), it is obvious. For \( k = 2 \), \( f(k) = 1 \leq \frac{\ln 2}{\ln 2} \). Assume \( f(k) \leq \frac{\ln k}{\ln 2} \) holds for \( k \leq n \). Find an upper bound for \( f(n + 1) \).

If \( n + 1 \) is prime, then \( f(n + 1) = 1 \leq \frac{\ln n}{\ln 2} \). If \( n + 1 \) is not prime, then there exists \( p \leq n \), which divides \( n \). If \( p = 2 \) and \( 2^a \parallel n + 1 \),

\[
f(n + 1) = f\left(\frac{n + 1}{2^a} \cdot 2^a\right) = f\left(\frac{n + 1}{2^a}\right) + f(2) = 1 + f\left(\frac{n + 1}{2^a}\right) \leq \frac{\ln(n + 1)}{\ln 2} + 1 - \frac{\ln 2}{\ln 2}.
\]

If \( p > 2 \) and \( p^a \parallel n + 1 \),

\[
f(n + 1) = f\left(\frac{n + 1}{p^a} \cdot p^a\right) = f\left(\frac{n + 1}{p^a}\right) + f(p) = \frac{1}{p - 1} + f\left(\frac{n + 1}{p^a}\right) \leq \frac{\ln(n + 1)}{\ln 2} + 1 - \frac{\ln p}{\ln 2} - \frac{\ln p}{\ln 2} < 0 \text{ for } p > 2.
\]

\[\square\]

3. The method with \( m = 1 \)

Theorem 4. Let \( k \) be an integer, \( H \geq 1250 \), and \( H \geq k \). Assume GRH \( (k, H) \). Let \( C_1(k) \) defined by \( \eqref{C_1} \). Let \( x > 1 \). Write \( X = \sqrt{\frac{\ln x}{H}} \) and

\[
\varepsilon(x) = 2\sqrt{\frac{k \varphi(k)}{C_1(k) \sqrt{x}}} \left(1 + \frac{1}{2X} (15/16 + \ln(C_1(k)/\varphi(k)))\right) X^{3/4} \exp(-X).
\]

If \( \varepsilon(x) \leq 0.2 \) and \( X \geq \sqrt{2 \ln \left(\frac{k H}{\varepsilon(k)}\right)} \), then

\[
\max_{1 \leq y \leq x} |\psi(y; k, l) - y/\varphi(k)| \leq x \varepsilon(x)/\varphi(k).
\]

Proof. Take \( m = 1 \) in Theorem \( \ref{Thm4} \). Assuming \( X \geq \sqrt{2 \ln \left(\frac{k H}{\varepsilon(k)}\right)} \), then \( W_1 \geq H \).

In this situation, \( Y = W_1 \) and \( B_H < 2R(W_1)\phi_1(W_1) \). For \( y > 1 \), \( R(y)/\ln y \) is
decreasing; hence,

\[
\hat{B}_H < 2R(W_1)\phi_1(W_1) < 2\frac{R(H)}{\ln H} \phi_1(W_1) \ln W_1
\]

\[
= 2\frac{R(H)}{\ln H} \left( \frac{X}{\sqrt{2}} + \ln \left( \frac{C_1(k)}{k} \right) \right) \phi_1(W_1)
\]

\[
= 2\frac{R(H)}{\ln H} \left( \frac{X}{\sqrt{2}} + \ln \left( \frac{C_1(k)}{k} \right) \right) (k/C_1(k))^2 \exp(-2\sqrt{2}X).
\]

Inserting the upper bounds (12) and (13) into the bound for \(\hat{A}_H\) in Lemma 9,

\[
\hat{A}_H < 2 \left( \frac{k}{C_1(k)} \right) \left[ \sqrt{\frac{\pi}{4X}} \exp(-2X) \left( 1 + \frac{15}{16X} + \frac{105}{512X^2} \right) X^2/\pi 
\]

\[
+ \frac{1}{\pi} \ln \frac{C_1(k)}{2\pi} X \sqrt{\frac{\pi}{4X}} \exp(-2X) \left( 1 + \frac{3}{16X} \right) 
\]

\[
+ C_2 \frac{kX}{C_1(k)\sqrt{2}} \sqrt{\frac{\pi}{4\sqrt{2}X}} \exp(-2\sqrt{2}X) \left( 1 + \frac{3}{16X} \right) \right].
\]

Put

\[
F_1 := \frac{k}{\sqrt{\pi C_1(k)}} X^{3/2} \exp(-2X) \left[ 1 + \left( \frac{15}{16} + \ln \frac{C_1(k)}{2\pi} \right) \frac{1}{2X} \right]^2.
\]

In Lemma 11 below it is shown that

\[
\hat{A}_H + \hat{B}_H + (\hat{C}_H + \hat{D}_H + 3\hat{E}(H))/\sqrt{x} + \hat{R} \frac{2}{x\varphi(k)} < F_1.
\]

We must choose \(\delta\) to minimize

\[
\frac{A(1,\delta)}{2} \varphi(k) F_1 + \delta/2.
\]

Write \(f = \varphi(k) F_1\). As \(A_1(\delta) = (\delta^2 + 2\delta + 2)/\delta\), we must minimize \(g(\delta) = (\delta/2 + 1 + 1/\delta)f + \delta/2\). The minimum value here is at \(\delta = \sqrt{\frac{2f}{1+f}}\), and the value there is \(g(\sqrt{\frac{2f}{1+f}}) = f + \sqrt{2f}(1+f)\).

It is a simple matter to prove that for \(0 \leq f \leq 0.202\),

\[
f + \sqrt{2f}(1+f) < 2\sqrt{f}.
\]

As \(X \geq X_0 := \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right)\), then \(x_0 \geq \exp(122.5)\), and it is obvious that \(\delta\) meets the hypothesis \(0 < \delta < (x_0 - 2)/x_0\) in Theorem 3 since

\[
0 < \delta < \sqrt{2}\sqrt{f} < 0.6357 < \frac{x_0}{x_0 - 2}.
\]

\[
\square
\]

Lemma 11.

\[
\hat{A}_H + \hat{B}_H + (\hat{C}_H + \hat{D}_H + 3\hat{E}(H))/\sqrt{x} + \hat{R} \frac{2}{x\varphi(k)} < F_1.
\]
Proof. First we prove that $\tilde{A}_H + \tilde{B}_H < F_1$:

$$F_1 = k \frac{X^{3/2} e^{-2X}}{C_1(k) \sqrt{\pi}} \left( 1 + \frac{15}{16X} + \frac{105}{512X^2} + \ln \left( \frac{C_1(k)}{2\pi} \right) \left( \frac{1}{X} + \frac{3}{16X^2} \right) \right. \left. + \frac{k\pi}{C_1(k) \sqrt{2\pi}} \exp(-2(\sqrt{2} - 1)X)(1/X + 3/(16\sqrt{2}X^2)) \right).$$

$$\tilde{A}_H < k \frac{X^{3/2} e^{-2X}}{C_1(k) \sqrt{\pi}} \left( 1 + \frac{15}{16X} + \frac{105}{512X^2} + \ln \left( \frac{C_1(k)}{2\pi} \right) \left( \frac{1}{X} + \frac{3}{16X^2} \right) \right. \left. + \frac{k\pi}{C_1(k) \sqrt{2\pi}} \exp(-2(\sqrt{2} - 1)X)(1/X + 3/(16\sqrt{2}X^2)) \right),$$

$$\tilde{B}_H < k \frac{X^{3/2} e^{-2X}}{C_1(k) \sqrt{\pi}} \left( 1 + \frac{15}{16X} + \frac{105}{512X^2} + \ln \left( \frac{C_1(k)}{2\pi} \right) \left( \frac{1}{X} + \frac{3}{16X^2} \right) \right. \left. + \frac{k\pi}{C_1(k) \sqrt{2\pi}} \exp(-2(\sqrt{2} - 1)X)(1/X + 3/(16\sqrt{2}X^2)) \right),$$

This yields $F_1 - \tilde{A}_H - \tilde{B}_H > 0$ if

$$F_2 := \frac{1}{X^2} \left( \frac{15}{1024} + \frac{9}{32} \ln \left( \frac{C_1(k)}{2\pi} \right) + \frac{1}{4} \ln^2 \left( \frac{C_1(k)}{2\pi} \right) \right) + \frac{C_2\sqrt{k\pi}}{C_1(k)} \exp(-2(\sqrt{2} - 1)X) \frac{1}{\sqrt{2X}} \left( \left( \sqrt{2} \frac{3}{X^3} \right)^{1/2} + 2 \left( 1 + \frac{\ln k + C_3/C_2}{\ln H} \right) \left( 1 + \frac{\sqrt{2}}{X} \ln \frac{C_1(k)}{k} \right) \right).$$

This holds if we can show that

$$F_2 > \frac{C_2\sqrt{k\pi}}{C_1(k)} \exp(-2(\sqrt{2} - 1)X) \frac{1}{\sqrt{2X}} \cdot 16.9,$$

since $C_1(k) \leq 32\pi$, $H \geq 1250$, $X \geq \sqrt{2} \ln(1250/32\pi)$, and $k \leq H$.

It remains to be proved that

$$\frac{\sqrt{2}C_1(k)}{kC_2\sqrt{\pi} \cdot 16.9} (15/1024 + \cdots) > X^{3/2} \exp(-2(\sqrt{2} - 1)X).$$

But for $X \geq X_0 := \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right)$,

$$X^{3/2} \exp(-2(\sqrt{2} - 1)X) < X_0^{3/2} \left( \frac{kH}{C_1(k)} \right)^{-(1+a)} = \frac{1}{k} \cdot 2^{3/4} \left( \frac{C_1(k)}{H} \right)^{1+a} \left( \frac{\ln^{3/2}(kH/C_1(k))}{k^a} \right),$$

where $a = 2\sqrt{2}(\sqrt{2} - 1) - 1 \approx 0.17157$. The map $k \mapsto \frac{\ln^{3/2}(kH/C_1(k))}{k^a}$ reaches its maximum for $k = e^{\frac{C_1(k)}{H}}$. Hence

$$X^{3/2} \exp(-2(\sqrt{2} - 1)X) < \frac{C_1(k)}{kH} 2^{3/4} \left( \frac{3}{2a} \right)^{3/2} / e^{3/2}.$$
We must compare
\[ \frac{\sqrt{2}}{C_2 \sqrt{\pi} \cdot 16.9} \left( \frac{15}{1024} + \cdots \right) \text{ with } \frac{2^{3/4}(\frac{3}{2})^{3/2}}{He^{3/2}}. \]
Since \( C_1(k) \geq 9.14 \) (see the remark above (3)) and \( C_2 = 0.9185 \), it remains to be proved that
\[ 0.007976 > \frac{2^{3/4}(\frac{3}{2})^{3/2}}{He^{3/2}} (\approx 0.00776), \]
which is true since \( H \geq 1250 \).

We show below that the remaining terms \( (\hat{C}_H + \hat{D}_H + 3\hat{E}(H))/\sqrt{x} + \hat{R}/x \) are negligible.

\begin{itemize}
  \item We will find an upper bound for \( A(1, \delta) \frac{2(k)}{\sqrt{\varphi(k)}}(\hat{C}_H + \hat{D}_H) + \frac{3}{2} \frac{\varphi(k)}{\sqrt{\varphi(k)}} \hat{E}(H) + \hat{R}/x \).
\end{itemize}

We assume that \( X \geq \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right) \); hence, \( X \geq X_0 := \sqrt{2} \ln \left( \frac{1250}{55.65} \right) \approx 3.5644 \). It is straightforward but tedious to check that
\[
\text{Rest} := \hat{C}_H + \hat{D}_H + 3\hat{E}(H) + \frac{2\hat{R}}{\sqrt{\varphi(k)x}} \leq \left\{ \begin{array}{ll}
1250(\ln H \ln k)^2 & \text{if } k \neq 1, \\
1250(\ln H)^2 & \text{if } k = 1.
\end{array} \right.
\]

Let us consider the case \( k \neq 1 \). As \( X \geq \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right) \),
\[
\exp \left( \frac{X}{\sqrt{2}} \right) \geq \frac{kH}{C_1(k)}.
\]

This yields
\[
\text{Rest} \leq 1250(\ln H \ln k)^2 \leq 1250 \left( \frac{\ln H \ln k}{\sqrt{C_1(k)}} \right)^2 \exp(X \sqrt{2})
\]
\[
\leq 1250C_1^2(k) \frac{1}{e^2} \left( \frac{\ln 1250}{1250} \right)^2 \exp(X \sqrt{2})
\leq K \exp(X \sqrt{2}) \quad \text{because } C_1(k) \leq 32\pi,
\]
where \( K := 55.65 \). Now compare
\[
\frac{K \exp(X \sqrt{2})}{\sqrt{x}} = K \exp(X \sqrt{2} - RX^2/2)
\]
with the term involving \( 1/X^2 \) in \( F_1 \)
\[
1 \times \frac{k}{X^2 \sqrt{C_1(k)\pi}} X^{3/2} \exp(-2X).
\]

We may compute \( c \) such that
\[
K \exp(X \sqrt{2} - RX^2/2) \leq c \times \frac{1}{X^2} \times \frac{k}{C_1(k)\sqrt{\pi}} X^{3/2} \exp(-2X)
\]
\[ \iff c \geq K \sqrt{32\pi \pi} \exp(X \sqrt{2} - RX^2/2 + 2X) \frac{X^2}{X^{3/2}} \]
\[ \iff c \geq 0.7 \cdot 10^{-18} \quad \text{for } X \geq X_0. \]

Thus, the rest is negligible and absorbed by rounding up the constants. \( \square \)
4. The method with $m = 2$

Lemma 12. Let $A(m, \delta)$ be defined as in formula \[4\]. Write
$$R_m(\delta) = (1 + (1 + \delta)^{m+1})^m.$$ 
Then
$$A(m, \delta) \leq \frac{R_m(\delta)}{\delta^m}.$$ 

Proof. The proof appears in \[4\] p. 222].

Theorem 5. Let an integer $k \geq 1$. Remember that $R = 9.645908801$. Let $H \geq 1000$. Assume GRH $(k, H)$. Let $C_1(k)$ be defined by \[3\]. Let $X_0, X_1, X_2, X_3$ be such that
$$e^{X_0} = \frac{k \varphi(k)}{2\pi C_1(k)}; \quad e^{X_1} = 10 \varphi(k),$$
$$X_2 = kC_1(k)/(2\pi \varphi(k)), \quad X_3 = \frac{2k\pi e}{C_1(k)\varphi(k)}.$$ 
Let $X_4 := \max(10, X_0, X_1, X_2, X_3)$. Write
$$\varepsilon(X) = 3 \sqrt{\frac{k}{\varphi(k)C_1(k)}} X^{1/2} \exp(-X).$$ 
Then for all real $x$ such that $X = \sqrt{\ln x} \geq X_4$, we have
$$\max_{1 \leq y \leq x} |\psi(y; k, l) - y/\varphi(k)| < x\varepsilon \left( \frac{\ln x}{\sqrt{R}} \right),$$
$$\max_{1 \leq y \leq x} |\theta(y; k, l) - y/\varphi(k)| < x\varepsilon \left( \frac{\ln x}{\sqrt{R}} \right).$$

Corollary 2. With the notations and the hypothesis of Theorem \[5\] let $X_5 \geq X_4$ and $c := \varepsilon(X_5)$. For $x \geq \exp(RX_5^2)$, we have
$$|\psi(x; k, l) - x/\varphi(k)|, \quad |\theta(x; k, l) - x/\varphi(k)| < cx.$$ 

Proof. The idea is to judiciously split the integral into two parts, and bound each part optimally, using an $m = 0$ estimate in the first part and an $m = 2$ estimate in the second part.

We want to split the integral at $T$, where $T$ will optimally be chosen later. We take $T$ in the same form as $W_m$ (formula \[12\]):
$$T := \frac{C_1(k)}{k} \exp(\nu X),$$
where $\nu$ is a parameter.

Assume that $T \geq H$ and $1/\sqrt{m+1} \leq \nu \leq 1$. Hence $W_m \leq T \leq W_0$. This last hypothesis is needed to apply Corollary \[11\].

We use Theorem \[2\] and split the sums at $T$:
$$A(m, \delta) \sum_{\chi} \sum_{\rho \in \chi} \frac{x^{\beta-1}}{|\rho|\rho + 1} \cdots (\rho + m) + \left( 1 + \frac{\delta m}{2} \right) \sum_{\chi} \sum_{\rho \in \chi} \frac{x^{\beta-1}}{|\rho|} + \frac{\delta m}{2} + \frac{\tilde{R}}{x}.$$
Define

\[ \hat{A}_1 := \sum_{\chi} \sum_{\rho \in \rho_{\chi} \mid \gamma \leq T} \frac{x^{\beta - 1}}{|\rho|}, \]
\[ \hat{A}_2 := \sum_{\chi} \sum_{\rho \in \rho_{\chi} \mid \gamma > T} \frac{x^{\beta - 1}}{|\rho(\rho + 1) \cdots (\rho + m)|}. \]

Bounding the term \( \hat{A}_1 \), we get

\[ \hat{A}_1 = \frac{1}{x} \sum_{\chi} \left( \sum_{\rho \in \rho_{\chi} \mid \gamma \leq H} \frac{\sqrt{x}}{|\rho|} + \sum_{\rho \in \rho_{\chi} \mid H < \gamma \leq T} \frac{x^\beta}{|\rho|} \right) \]
\[ = \frac{1}{\sqrt{x}} \sum_{\chi} \sum_{\rho \in \rho_{\chi} \mid \gamma \leq H} \frac{1}{|\rho|} + \frac{1}{2x} \sum_{\chi} \left( \sum_{\rho \in \rho_{\chi} \mid H < \gamma \leq T} \frac{x^\beta}{|\rho|} + \sum_{\rho \in \rho_{\chi} \mid \gamma \leq T} \frac{x^\beta}{|\rho|} \right) \]
\[ \leq \frac{1}{\sqrt{x}} \varphi(k) \hat{E}(H) + \frac{1}{2x} \sum_{\chi} \left( \sum_{\rho \in \rho_{\chi} \mid H < \gamma \leq T} x \phi_0(\gamma) + \sqrt{x} \sum_{\rho \in \rho_{\chi} \mid \gamma \leq T} \frac{1}{|\gamma|} \right) \]
\[ \leq \varphi(k) \hat{E}(T)/\sqrt{x} + \frac{1}{2} \sum_{\chi} \sum_{\rho \in \rho_{\chi} \mid H \leq \gamma \leq T} \phi_0(\gamma). \]

Apply Corollary \( \# \) (\( j = 1, m = 0 \)) for the interval \([H, T]\) with \( \phi = \phi_0 \) and \( W = W_0 \)

\[ \sum_{\rho \in \rho_{\chi} \mid H \leq \gamma \leq T} \phi_0(\gamma) = \{1/\pi - q(T)\} \int_H^T \phi_0(y) \ln(ky/2\pi) dy + B_1(T, H, T). \]

Moreover, \( B_1(T, H, T) < 2R(T)\phi_0(T). \)

We want to find an upper bound for

\[ I_1 := \frac{1}{\pi} \int_H^T \phi_0(y) \ln \left( \frac{ky}{2\pi} \right) dy. \]

Write \( V'' = X^2/\ln \left( \frac{kT}{C_1(k)} \right) = X/\nu = Y'' + 2X - \nu X \), where \( Y'' := X(1 - \nu)^2/\nu. \)

Applying \( U'' = X^2/\ln \left( \frac{kH}{C_1(k)} \right) \) and \( \Gamma(\alpha, x) = \int_x^\infty e^{-u}u^{\alpha - 1} du. \) Now

\[ \int_H^T \ln \left( \frac{ky}{2\pi} \right) \phi_0(y) dy = \int_H^T \ln \left( \frac{ky}{2\pi} \right) \exp \left( -X^2/\ln \left( \frac{ky}{C_1(k)} \right) \right) dy \]
\[ = X^4 \{ \Gamma(-2, V'') - \Gamma(-2, U'') \}
\[ + X^2 \ln \left( \frac{C_1(k)}{2\pi} \right) \{ \Gamma(-1, V'') - \Gamma(-1, U'') \} \]
by making the change of variables \( y = \frac{C_1(k)}{k} \exp(X^2/u) \). Now if \( \alpha \leq 1 \) and \( x > 0 \),
then \( \Gamma(\alpha, x) \leq x^{\alpha-1} \int_x^\infty e^{-t} dt = x^{\alpha-1} e^{-x} \). Hence,

\[
\int_H^T \left( \ln \left( \frac{ky}{2\pi} \right) \phi_0(y) \right) dy \leq X^4 V'' - 3 e^{-V''} + X^2 \ln \left( \frac{C_1(k)}{2\pi} \right) V'' - 2 e^{-V''}.
\]

This yields

\[
I_1 \leq \frac{1}{\pi} X^2 \left( X^2 V'' - 3 + \ln \left( \frac{C_1(k)}{2\pi} \right) V'' - 2 \right) e^{-V''}
= \frac{1}{\pi} e^{-V''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) \left( \frac{X^4}{(X/\nu)^3} + \frac{dX^2}{(X/\nu)^2} \right)
= \frac{1}{\pi} e^{-V''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) XG_0,
\]

where \( d := \ln \left( \frac{C_1(k)}{2\pi} \right) \) and \( G_0 := \nu^2 (\nu + d/X) \). With the help of Corollary \( \mathbb{I} \) we write

\[
\tilde{A}_1 \leq \varphi(k) \tilde{E}(T)/\sqrt{\pi} + \frac{\varphi(k)}{2} \left\{ \frac{1}{\pi} e^{-V''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) XG_0 + 2R(T)\phi_0(T) \right\}.
\]

Bounding the term \( \tilde{A}_2 \), we get

\[
\tilde{A}_2 = \frac{1}{x} \sum \sum_{\rho \in \wp(\chi)} \frac{x^\beta}{\rho(\rho + 1) \cdots (\rho + m)}
= \frac{1}{2x} \sum \sum_{\rho \in \wp(\chi)} \frac{x^\beta}{\rho(\rho + 1) \cdots (\rho + m)} + \frac{x^\beta}{\rho(\rho + 1) \cdots (\rho + m)}
= \frac{1}{2x} \sum \sum_{\rho \in \wp(\chi)} \frac{x^\beta}{(\gamma)|\gamma|^{m+1} + \sum_{\rho \in \wp(\chi)} \frac{x^\beta}{(\gamma)|\gamma|^{m+1}}}
= \frac{1}{2x} \sum \sum_{\rho \in \wp(\chi)} \frac{x^\beta}{(\gamma)|\gamma|^{m+1} + \sum_{\rho \in \wp(\chi)} \frac{x^\beta}{(\gamma)|\gamma|^{m+1}}}
\]
by Lemma \( \mathbb{I} \).

By using Corollary \( \mathbb{I} \) \((j = 0)\) on \([U, V] = [T, \infty)\),

\[
\sum_{\rho \in \wp(\chi)} \phi_m(\gamma) \leq \{1/\pi + q(T)\} \int_T^\infty \phi_m(y) \ln \left( \frac{ky}{2\pi} \right) dy + B_0(T, T, \infty).
\]
We have

\[
B_0(T, T, \infty) < 2R(T)\phi_0(T).
\]

Moreover,

\[
\sum_{\rho \in \wp(\chi)} \frac{1}{(\gamma)|\gamma|^{m+1}} \leq \tilde{C}_T + \tilde{D}_T.
\]
Let us study more precisely

\[ I_2 := \int_T^{\infty} \phi_m(y) \ln \left( \frac{ky}{2\pi} \right) dy \]

\[ = \frac{z_m^2}{2m^2} \left( \frac{k}{C_1(k)} \right)^m \left( K_2(z_m, U_m) + \frac{2md}{z_m} K_1(z_m, U_m) \right), \]

where \( d = \ln \left( \frac{C_1(k)}{z_m^2} \right) \) and \( U_m = \frac{2m}{z_m} \ln \left( \frac{k}{C_1(k)} \right) = \nu \sqrt{m}. \) Now, by writing \( z = z_m \) and using Lemma 8,

\[ K_2(z, U') + \frac{2dm}{z} K_1(z, U') < (U' + 2/z + 2dm/z) Q_1(z, U') \leq \sqrt{m} \left( \nu + \frac{1+dm}{m}\nu \right) \frac{U'^2}{z(U'^2 - 1)} e^{-2(U' + 1/U')}. \]

But \( \frac{2}{T} (U' + 1/U') = X \sqrt{m}(\nu \sqrt{m} + 1/(\nu \sqrt{m})) = mX + X/\nu = mX + (Y'' + 2X - \nu X), \) where \( Y'' = X(1 - \nu)^2/\nu. \) Hence

\[ K_2(z, U') + \frac{2dm}{z} K_1(z, U') < G_1 e^{-Y''} \frac{m}{2(m-1)} X^{-1} e^{-2X} \left( \frac{kT}{C_1(k)} \right)^{(m-1)}, \]

where \( G_1 := \frac{m-1}{m} \frac{U'^2}{2(m-1)} (\nu + \frac{1+dm}{m}\nu) \) because

\[ e^{\nu X(m-1)} = \left( \frac{kT}{C_1(k)} \right)^{m-1} \]

and \( \frac{\sqrt{m}}{2} = \frac{1}{T} X^{-1}. \) This yields

\[ I_2 = \int_T^{\infty} \phi_m(y) \ln(ky/2\pi) dy < \frac{G_1 e^{-Y''}}{m-1} \frac{k}{C_1(k)} X e^{-2X} T^{-(m-1)} \]

Let \( G_2 := \frac{R_m(\delta)}{2\pi} (1 + \pi q(T)). \) So, by using Lemma 12,

\[ A(m, \delta) \frac{\varphi(k)}{2} (1/\pi + q(T)) \int_T^{\infty} \phi_m(y) \ln(ky/2\pi) dy < \left( \frac{2}{\delta} \right)^m \varphi(k) \frac{G_2}{\pi} \left( \frac{G_1 e^{-Y''}}{m-1} C_1(k) \right) X e^{-2X} T^{-(m-1)} \].

The results above yield

\[ (1 + m\delta/2) \hat{A_1} + A(m, \delta) \hat{A_2} \]

\[ < XG_2 e^{-2X} \varphi(k) \left( \frac{k}{C_1(k)} \right) \left\{ \frac{G_1}{m-1} X^{-(m-1)} \left( \frac{2}{\delta} \right)^m + G_0 T \right\} + r \]

because \( 1 + m\delta/2 < R_m(\delta)/2 \) \( < G_2, \) with

\[ r = \varphi(k)(1 + m\delta/2) R(T) \phi_0(T) + A(m, \delta) \varphi(k) R(T) \phi_m(T) \]

\[ + \varphi(k) \sqrt{x}(1 + m\delta/2) \bar{E}(T) + A(m, \delta)(\bar{C}_T + \bar{D}_T)/2). \]

Suppose \( G_0/G_1 \) were independent of \( \nu; \) then the expression between braces in (15) would be minimized for

\[ T = (G_1/G_0)^{1/m} \cdot \frac{2}{\delta}. \]
With this choice,
\[ \frac{G_1}{m - 1} T^{-(m-1)} \left( \frac{2}{\delta} \right)^m + G_0 T = \frac{m}{m - 1} G_1^{1/m} G_0^{1-1/m} 2^{-\delta}, \]
and we obtain \((G_2 > 1)\)
\[ \epsilon_1 := (1 + m\delta/2) \tilde{A}_1 + A(m, \delta) \tilde{A}_2 + \frac{1}{2} m \delta + \frac{\tilde{R}}{x} < \frac{1}{2} m G_2 \left\{ X e^{-2X} e^{-Y''} \frac{2k \varphi(k)}{\delta (m - 1) \pi C_1(k)} G_1^{1/m} G_0^{1-1/m} + \delta \right\} + r + \frac{\tilde{R}}{x}. \]
The expression between braces can be minimized by choosing
\[ \delta = \left\{ G_0^{1-1/m} G_1^{1/m} e^{-Y''} \frac{2k \varphi(k)}{(m - 1) \pi C_1(k)} \right\}^{1/2} X^{1/2} e^{-X}. \]
Hence, we write (by replacing the above value of \( \delta \) in (16))
\[ T = \left( \frac{G_1}{G_0} \right)^{1/2m} \left( \frac{2 C_1(k)}{k \varphi(k)} (m - 1) \pi e^{Y''} / G_0 \right)^{1/2} X^{-1/2} e^X \]
and
\[ \epsilon_1 < G_2 \left( G_0^{1-1/m} G_1^{1/m} e^{-Y''} \frac{2k \varphi(k)}{\pi C_1(k)} \right)^{1/2} \frac{m}{\sqrt{m - 1}} X^{1/2} e^{-X} + r + \frac{\tilde{R}}{x}. \]
The value \( m = 2 \) minimizes the expression \( \frac{m}{\sqrt{m-1}} \). For the remainder of the argument, we fix \( m = 2 \).

We now have two definitions for \( T \). On the one hand (equation (18)),
\[ T = \left( \frac{G_1}{G_0} \right)^{1/4} e^{Y''/2} \sqrt{\frac{2 \pi C_1(k)}{k \varphi(k)}} X^{-1/2} e^X \]
with \( Y'' = X(1 - \nu)^2 / \nu \), and on the other hand (equation (14))
\[ T = C_1(k) / k \exp(\nu X). \]
These two equations are compatible if and only if there exists \( \nu \) such that \( f(\nu) = 1 \), where
\[ f(\nu) = \frac{C_1(k) \varphi(k)}{2\pi k} \left( \frac{G_0^3}{G_1} \right)^{1/2} X e^{-X(1-\nu)^2/\nu} e^{-2X(1-\nu)}. \]
Here we have \( m = 2 \) and our assumption \( 1/\sqrt{m + 1} \leq \nu \leq 1 \) gives \( 1/\sqrt{3} \leq \nu \leq 1 \). Note that
\[ G_0 = \nu^2(\nu + d/X), \]
\[ G_1 = \frac{m - 1}{m} \frac{U'^2}{U'^2 - 1} \left( \nu + \frac{1 + dm}{mX} \right) = \frac{\nu^2}{2\nu^2 - 1} \left( \nu + \frac{1 + 2d}{2X} \right). \]
It is easy to check that on the interval \( 1/\sqrt{2} \leq \nu \leq 1 \), \( G_0^3/G_1 \) is increasing, and hence, \( f(\nu) \) is strictly increasing. Moreover, \( \lim_{\nu \to (1/\sqrt{2})^+} f(\nu) = 0 \) and \( f(1) > 1 \)
Write, for $X \geq X_3 := \frac{2\pi k}{C_1(k)}$, we have 

\[ H(\nu) := \frac{G_0^3}{G_1} = \frac{[\nu^2(\nu + d/X)]^3}{\nu^2 (\nu + \frac{1 + 2d}{2})} < (\nu + d/X)^2. \]

Write, for $X \geq X_3 := \frac{2\pi k}{C_1(k)}$, we obtain 

\[ \nu_0 = 1 - \frac{1}{2X} \ln \left( \frac{C_1(k) \varphi(k) X}{2\pi} \right). \]  

Let us study $H(\nu_0)$:

\[ H(\nu_0) < 1 \quad \text{if} \quad \nu_0 + d/X \leq 1, \]

equivalently

\[ 1 - \frac{1}{2X} \ln \left( \frac{C_1(k) \varphi(k) X}{2\pi} \right) + \frac{\ln (C_1(k)/2\pi)}{X} \leq 1, \]

which holds if $X \geq X_2 := \frac{kC_1(k)}{2\pi \varphi(k)}$.

As

\[ f(\nu) = \frac{C_1(k) \varphi(k)}{2\pi} \left( \frac{G_0^3}{G_1} \right)^{1/2} X \exp(-X(1-\nu)^2/\nu) \exp(-2X(1-\nu)), \]

replacing $\nu_0$ by (20), we obtain

\[ f(\nu_0) = \left( \frac{G_0^3}{G_1} \right)^{1/2} \exp \left( -\ln^2 \left( \frac{C_1(k) \varphi(k) X}{2\pi} \right) / (4\nu_0 X) \right). \]

Assume that $\nu_0 > 0$, then, for $X \geq X_2$, $f(\nu_0) < 1 = f(\nu)$ and hence $\nu_0 < \nu$. We will require $X \geq X_2$.

The assumption $T \geq H$ holds if $T \geq C_1(k)/k \exp(\nu_0 X) \geq H$. Using (20), rewrite $\frac{C_1(k)}{k} \exp(\nu_0 X) = \sqrt{2\pi C_1(k) \varphi(k)} e^{-\frac{1}{2} \ln X}. \quad X \geq X_0$. Let $X_0$ satisfy

\[ e^{X_0 - \frac{1}{2} \ln X_0} = \frac{k \varphi(k)}{2\pi C_1(k)}. \]

We have $T \geq H$ provided that $X \geq X_0$. We will require $X \geq X_0$.

For $X \geq X_3 := \frac{2\pi k}{C_1(k) \varphi(k)}$, $\nu_0$ is an increasing function of $X$. We will require that $X \geq \max(X_3, 10)$. Then since $C_1(k) \leq 32\pi$ and $X \geq 10$, we have

\[ \nu_0 > 0.7462413 \quad \text{and} \quad \nu_0 < \nu < 1. \]

The assumption $\nu > 1/\sqrt{2}$ is satisfied.

We want to evaluate

\[ K := G_2(\sqrt{G_0 G_1} e^{-Y''})^{1/2}, \]

which appears in (19). Again using $C_1(k) \leq 32\pi$ and $X \geq 10$, we find

\[ G_0 G_1 < (1 + d/X) \frac{\nu_0}{2\nu_0^2 - 1} \left( \nu_0 + \frac{1 + 2d}{2X} \right) < 8.995. \]

The following results will be needed in later computations.
1. Since $X > X_0$ and $\exp(X) / \sqrt[2]{X}$ is increasing for $X \geq 1/2$,
\[ \sqrt{\frac{k \varphi(k)}{2 \pi C_1(k)}} X^{1/2} \exp(-X) \leq \frac{1}{H}. \]

2. Since $G_0 G_1 < 9$,
\[ \delta = 2 \sqrt{G_0 G_1} \exp(-Y''/2) \sqrt{\frac{k \varphi(k)}{2 \pi C_1(k)}} X^{1/2} e^{-X} \]
\[ \leq 2 \sqrt{3} / H. \]

In particular, for $H \geq 1000$, we have $\delta \leq 0.00347$.

3. 
\[ G_2 = \frac{R_2(\delta)}{2^2} \left(1 + \pi q(T)\right) < (1 + 3.012 \cdot \delta / 2)^2 \left(1 + \pi q(T)\right), \]

because
\[ \frac{R_2(\delta)}{2^2} = \left\{ \frac{(1 + \delta)^3 + 1}{2} \right\}^2 = \left\{ 1 + \frac{1}{2} \delta (3 + 3 \delta + \delta^2) \right\}^2 < \left(1 + \frac{3.012}{2} \delta \right)^2 \]

since $1 + \delta + \delta^2 / 3 < 1.0035$.

4. Since $T \geq H$,
\[ q(T) = \frac{C_2}{T \ln(kT/2\pi)} \]
\[ \leq \frac{C_2}{H \ln(kH/2\pi)}. \]

But $\exp(-Y''/2) \leq 1$ and $H \geq 1000$, so this yields
\[ K < \left(8.995 \right)^{1/4} G_2 \]
\[ < \left(8.995 \right)^{1/4} \left(1 + \frac{\pi C_2}{1000 \ln(1000/(2\pi))} \right) \times \left(1 + \frac{3.012 \cdot 2 \sqrt{3}}{2 \cdot 1000} \right)^2 \]
\[ < 1.751. \]

Inserting this upper bound of $K$ (see formula (21) in [11]), we obtain
\[ \varepsilon_1 < 2 \sqrt{\frac{2}{\pi} K} \sqrt{\frac{k \varphi(k)}{C_1(k)}} X^{1/2} \exp(-X) + r + \frac{\tilde{R}}{x} \]
\[ < 2.7941 \sqrt{\frac{k \varphi(k)}{C_1(k)}} X^{1/2} \exp(-X) + r + \frac{\tilde{R}}{x}. \]

(22)

Now we want to bound $r$ and $\tilde{R}$.

• An upper bound for $\varphi(k)(1 + \delta) R(T) \phi_0(T)$ and $\varphi(k)A(2, \delta) R(T) \phi_2(T)$. Recall that
\[ R(T) = C_2 \ln(kT) + C_3, \]
\[ \phi_0(T) = \frac{1}{T} \exp \left(-X^2 / \ln(kT/C_1(k))\right), \]
\[ \phi_m(T) = \phi_0(T)T^{-m}. \]
Now
\[ \phi_0(T) = \frac{1}{T} \exp(-X^2/(\nu X)) = \frac{1}{T} \exp(-\frac{1}{\nu}X) \leq \frac{1}{T} \exp(-X) \]
and
\[ \frac{1}{T} = X^{1/2} \exp(-X) \sqrt{\frac{k\varphi(k)}{C_1(k)}} \left( \frac{G_0}{2\pi e^{y''}} \right)^{1/2} \left( \frac{G_0}{G_1} \right)^{1/4}, \]
hence
\[ R(T) \phi_0(T) \leq \frac{C_2 \ln(kT) + C_3}{T} \exp(-X) \]
\[ \leq \sqrt{X} e^{-X} \sqrt{\frac{k\varphi(k)}{C_1(k)}} \left[ (C_2 \ln(kT) + C_3) \left( \frac{G_0}{2\pi e^{y''}} \right)^{1/2} \left( \frac{G_0}{G_1} \right)^{1/4} e^{-X} \right]. \]

But
\[ G_0 \leq 1 + \frac{\ln(C_1(k)/2\pi)}{X}, \]
\[ \frac{G_0}{G_1} \leq 2\nu^2 - 1 < 1 \quad (m = 2), \]
\[ \exp(Y'') \geq 1, \]
\[ \ln(kT) = \nu X + \ln(C_1(k)) \leq X + \ln(C_1(k)) \leq X + \ln(32\pi). \]
So, since \( X \geq 10 \) and \( C_1(k) \leq 32\pi, \)
\[ (1 + \delta) \varphi(k) \left[ (C_2 \ln(kT) + C_3) \left( \frac{G_0}{2\pi e^{y''}} \right)^{1/2} \left( \frac{G_0}{G_1} \right)^{1/4} \exp(-X) \right] \]
\[ \leq \varphi(k) \left[ 1 + \frac{2\sqrt{3}}{1000} \right] \left[ (C_2(X + \ln 32\pi) + C_3) \sqrt{\frac{1 + \ln 16/10}{2\pi}} \exp(-X) \right] \]
\[ \leq 0.857 \varphi(k) X \exp(-X). \]

Furthermore, if \( X_1 \) is defined by \( \exp(X_1)/X_1 = 10 \varphi(k), \) and if we require that \( X \geq X_1, \) then this term is bounded by 0.0857. Hence, under the hypotheses on \( X \) in Theorem 5 an upper bound for \( \varphi(k)(1 + \delta)R(T)\phi_0(T) \) is
\[ 0.09 \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X). \]

Next, by (10)
\[ \delta T = 2 \sqrt{\frac{G_1}{G_0}}. \]

Hence, by Lemma [12]
\[ A(2, \delta)/T^2 \leq \frac{R_2(\delta)}{(\delta T)^2} \leq \frac{R_2(\delta)}{2^2} \frac{G_0}{G_1} \leq \frac{R_2(\delta)}{2^2} \]
and
\[ \varphi(k) A(2, \delta) R(T) \phi_2(T) \leq \varphi(k) \frac{R_2(\delta)}{2^2} R(T) \phi_0(T). \]
Using $\delta \leq 2\sqrt{3}/H \leq 2\sqrt{3}/1000$, we get $R_2(\delta)/2^2 \leq 1.0147$. Under the hypotheses on $X$ in Theorem 5, an upper bound for $\varphi(k)A(2, \delta)R(T)\phi_2(T)$ is therefore

$$
0.087 \cdot \frac{k\varphi(k)}{C_1(k)} X^{1/2} \exp(-X).
$$

The sum of the two terms can be bounded by

$$(23) \quad 0.2 \cdot \frac{k\varphi(k)}{C_1(k)} X^{1/2} \exp(-X).$$

- An upper bound for $(1 + \delta)\tilde{E}(T)\frac{\varphi(k)}{\sqrt{X}} + A(2, \delta)\frac{\varphi(k)}{2\sqrt{X}}(\tilde{C}_T + \tilde{D}_T) + \tilde{R}/x$.

For $f(k) = \sum_{p|k} \frac{1}{p-1}$ observe that (Lemma 10)

$$f(k) \leq \frac{\ln k}{\ln 2}.$$ 

We can explicitly rewrite for $m = 2$, $H \geq 1000$, and $C_1(k) \leq 32\pi$ the following expressions:

\begin{align*}
3\tilde{E}(T) &= 3 \left( \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln \left( \frac{k}{2\pi} \right) \ln T + C_2 \\
& \quad + 2 \left( \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right) \right), \\
\tilde{C}_T &= \frac{1}{2\pi T^2} \left( \ln \left( \frac{kT}{2\pi} \right) + 1/2 \right), \\
\tilde{D}_T &= (2C_2 \ln(kT) + 2C_3 + C_2/3)/T^3,
\end{align*}

$$\frac{\tilde{R}}{\varphi(k)\sqrt{X}} \leq \frac{[(f(k) + 0.5) \ln x + 4 \ln k + 13.4]}{\sqrt{x}}.$$

It is tedious but easy to check that the sum of the above quantities is less than

$$\begin{cases}
1000(\ln T \sqrt{\ln k})^2 & \text{for } k \neq 1, \\
1000 \ln^2 T & \text{for } k = 1.
\end{cases}$$

Now we want to find a number $c$ such that

$$A(2, \delta)\varphi(k)\frac{1000(\ln T \sqrt{\ln k})^2}{\sqrt{x}} \leq c \left( \frac{k\varphi(k)}{C_1(k)} \right)^{1/2} X^{1/2} \exp(-X)$$

with $X = \sqrt{\ln x}$. But $A(2, \delta) \leq \frac{R_2(\delta)}{2^2}$ and by (10), $T = \left( \frac{G_k}{G_0} \right)^{1/2} \frac{1}{2}$, so

$$A(2, \delta) \leq \frac{R_2(\delta)}{2^2} T^2 G_0 \frac{G_k}{G_1} G_1 \frac{G_0}{G_k} = \frac{R_2(\delta)}{2^2} T^2 \varphi(k)(\ln T \sqrt{\ln k})^2 \left( \frac{C_1(k)}{k\varphi(k)} \right)^{1/2} X^{-1/2} \exp(X - RX^2/2).$$

Moreover, $\frac{1}{\sqrt{x}} = \exp(-RX^2/2)$, hence

$$c \geq 1000 \frac{R_2(\delta)}{2^2} \frac{G_0}{G_1} T^2 \varphi(k)(\ln T \sqrt{\ln k})^2 \left( \frac{C_1(k)}{k\varphi(k)} \right)^{1/2} X^{-1/2} \exp(X - RX^2/2).$$

As $\frac{C_0}{G_1} < 1$, $T^2 = \frac{C_0^2(k)}{k^2} \exp(2\nu X) \leq \frac{C_0^2(k)}{k^2} \exp(2X)$, hence it suffices to take

$$c \geq 1000 \frac{R_2(\delta)}{2^2} C_0^2(k) \frac{\ln k}{k^2} (\ln(C_1(k)/k) + X)^2 \left( \frac{C_1(k)\varphi(k)}{kX} \right)^{1/2} \exp(3X - RX^2/2).$$
Since \( C_n \) finally, it suces to take

This would not have been possible if we had used only the results of \([3]\).

\[ C_1(k) = \frac{\ln^k}{k!} \leq 1, \quad \ln^k \leq 1, \quad \text{and} \quad R_2(\delta) \leq (1 + 3.012\delta/2)^2 \text{ with } \delta \leq \frac{2\sqrt{3}}{1000}. \]

So, finally, it suffices to take

\[ c \geq \frac{1000}{4} \left( 1 + \frac{3.012\sqrt{3}}{1000} \right)^2 C_1^2(k) (\ln C_1(k) + X)^2 \sqrt{C_1(k) X^{-1/2}} \exp(3X - Rx^2/2). \]

Since \( C_1(k) \leq 32\pi \) and \( X \geq 10 \), we can take

\[ c = 0.643 \cdot 10^{-187}. \]

In the case \( k = 1 \), we can replace the upper bound \( \ln^k \leq 1 \) by 1, and obtain the same result. Combining (22), (23), and (24), we obtain the result in Theorem 5:

\[ \frac{\ln^k}{k!} \geq 1, \quad \text{and} \quad \ln^k \leq 1. \]

Using \( x = \ln x \) for all \( X \) satisfying the conditions of the theorem,

\[ | \psi(x; k) - x/\varphi(x) | / x \leq 2.9941 \sqrt{\frac{k}{\varphi(k) C_1(k)}} X^{1/2} \exp(-X). \]

We also wish to allow \( \theta \) instead of \( \psi \), which can be done by recalling Theorem 13 of [3]:

\[ 0 \leq \psi(x; k) \leq \psi(x) \leq \theta(x) \leq 1.43\sqrt{x} \quad \text{for } x \geq 0. \]

Using \( X \geq 10 \), we find \( 1.43\sqrt{x}/x \leq d \cdot 3(k\varphi(k))/C_1(k) \cdot X^{1/2} \exp(-X) \), where \( d = 1.17 \cdot 10^{-204} \). This difference is absorbed by rounding up the constants. \( \blacklozenge \)

5. **Application for \( k = 3 \)**

Now we are able to compute \( x_0 \) and \( c \) such that, for \( x \geq x_0 \),

\[ | \theta(x; 3) - x/2 | < cx/\ln x. \]

This would not have been possible if we had used only the results of [3].

According to Theorem 5

\[ \varepsilon(X) = \frac{3}{2} \sqrt{\frac{6}{20.92}} X^{1/2} \exp(-X) \]

for \( k = 3 \).

To determine for which \( x \) this bound is valid, let us solve for the constants \( X_0, X_1, X_2, X_3 \) in Theorem 5. Noting that \( H_3 = 10000 \) by the table in Theorem 1, we need \( X_0 \) to satisfy

\[ \exp(X_0 - \frac{1}{2} \ln X_0) \geq 10000 \sqrt{\frac{6}{20.92}} \approx 2136.51. \]

\( X_0 \approx 8.76 \) works.

Find \( X_1 \) such that

\[ \exp(X_1 - \ln X_1) \geq 20. \]

\( X_1 \approx 4.5 \) works.

Compute the other two bounds: \( X_2 \approx 4.99, X_3 \approx 1.22 \). Thus we can take \( X = \max(10, X_0, X_1, X_2, X_3) = 10 \) in Theorem 5

- For \( \frac{\ln x}{x} \geq 10 \), write \( X = \sqrt{\frac{\ln x}{x}} \), then

  \[ \varepsilon(X) \ln x = RX^2 \varepsilon(X). \]

Find the value \( c \) such that

\[ \varepsilon(X) < c/\ln x. \]
For any $x$ such that $\sqrt{\ln x} \geq 10$, $c \leq R \cdot 10^2 \varepsilon(10) \leq 0.12$. Hence we have for $x \geq \exp(964.59 \cdots)$,

$$|\theta(x; 3, l) - x/2| \leq 0.12 \frac{x}{\ln x}.$$ 

We want to extend the above result for $x \leq \exp(964.59 \cdots)$. Olivier Ramaré has kindly computed some additional values supplementing Table 1 in [3]. We have

$$|\theta(x; 3, l) - x/2| < \tilde{c} \cdot x/2$$

with

\begin{align*}
\tilde{c} &= 0.0008464421 \text{ for } \ln x \geq 400 \quad (m = 3, \delta = 0.00042325), \\
\tilde{c} &= 0.0006048271 \text{ for } \ln x \geq 500 \quad (m = 3, \delta = 0.00030250), \\
\tilde{c} &= 0.0004190635 \text{ for } \ln x \geq 600 \quad (m = 2, \delta = 0.00027950).
\end{align*}

Hence,

- For $e^{600} \leq x \leq e^{964.59...}$
  $$c \leq 0.0004190635 \cdot 964.6/\varphi(3) \leq 0.203.$$ 
- For $e^{400} \leq x \leq e^{600}$
  $$c \leq 0.0008464421 \cdot 600/\varphi(3) \leq 0.254.$$ 

Using the computations of [3],

- For $10^{100} \leq x \leq e^{400}$
  $$c \leq 0.001310 \cdot 400/\varphi(3) \leq 0.262.$$ 
- For $10^{30} \leq x \leq 10^{100}$
  $$c \leq 0.001813 \cdot 100 \ln 10/\varphi(3) \leq 0.42/2 \leq 0.21.$$ 
- For $10^{13} \leq x \leq 10^{30}$
  $$c \leq 0.001951 \cdot 30 \ln 10/\varphi(3) \leq 0.14/2 \leq 0.07.$$ 
- For $10^{10} \leq x \leq 10^{13}$
  $$c \leq 0.002238 \cdot 13 \ln 10/\varphi(3) \leq 0.067/2 \leq 0.00335.$$ 
- For $4403 \leq x \leq 10^{10}$
  $$|\theta(x; 3, l) - x/2| < 2.072\sqrt{x} \quad \text{(Theorem 5.2.1 of Ramaré and Rumely [3])}.$$ 

We choose $c = 0.262$. We check that this bound is also valid for $1531 \leq x \leq 4403$.

**Theorem 6.** For $x \geq 1531$,

$$|\theta(x; 3, l) - x/2| \leq 0.262 \frac{x}{\ln x}.$$
6. Results assuming GRH($k, \infty$)

Assuming GRH($k, \infty$), we obtain more precise results. Under this hypothesis, one can show that function $\psi$ has the following asymptotic behaviour:

**Proposition 1** ([8, p. 294]). Assume GRH($k, \infty$). Then

$$\psi(x; k, l) = \frac{x}{\varphi(k)} + O(\sqrt{x} \ln^2 x).$$

**Theorem 7.** Let $x \geq 10^{10}$. Let $k$ be a positive integer. Assume GRH($k, \infty$).

1) If $k \leq \frac{1}{3} \ln x$, then

$$|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq 0.085 \sqrt{x} \ln^2 x.$$

2) If $k \leq 432$, then

$$|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq 0.061 \sqrt{x} \ln^2 x.$$

**Proof.** Let $x_0 = 10^{10}$. Applying Theorem 2 in the same way as Theorem 3 (assume that $T \geq 1$),

$$\frac{\varphi(k)}{x} |\psi(x; k, l) - \frac{x}{\varphi(k)}|$$

$$\leq A(m, \delta) \sum_{\chi \chi | \cdot \cdot T} \frac{x^{-1/2}}{|\rho(p+1) \cdot \cdot (p+m)|}$$

$$+ (1 + m\delta/2) \sum_{\chi \chi | \cdot \cdot T} \frac{x^{-1/2}}{|\rho|} + m\delta/2 + \tilde{R}/x$$

$$\leq A(m, \delta) \frac{1}{\sqrt{x}} \sum_{\chi \chi | \cdot \cdot T} \frac{1}{|\rho|^m+1} + (1 + m\delta/2) \frac{1}{\sqrt{x}} \sum_{\chi \chi | \cdot \cdot T} \frac{1}{|\rho|} + \frac{m\delta}{2} + \tilde{R}/x$$

$$\leq A(m, \delta) \frac{\varphi(k)}{\sqrt{x}} (\tilde{C}_T + \tilde{D}_T) + (1 + \frac{m\delta}{2}) \frac{\varphi(k)}{\sqrt{x}} \tilde{E}(T) + \frac{m\delta}{2} + \tilde{R}/x.$$

Take $m = 1$ and let

$$\varepsilon_k(x, T, \delta) := \frac{R_1(\delta) \varphi(k)}{\delta} \left( \frac{kT}{2\pi} \right) + 1 + \frac{\varphi(k)}{\sqrt{x}} \tilde{E}(T) + \frac{\delta}{2} + \tilde{R}/x,$$

where

$$\tilde{C}_T = \frac{1}{\pi T} \left( \ln \left( \frac{kT}{2\pi} \right) + 1 \right),$$

$$\tilde{D}_T = \frac{1}{T^2} \left( 2C_2 \ln(kT) + 2C_3 + C_2/2 \right),$$

$$\tilde{E}(T) = \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln(k/(2\pi)) \ln T + C_2 + 2 \left( \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right).$$

Choose

$$T = \frac{2R_1(\delta)}{\delta(2 + \delta)}.$$
to minimize in (25) the preponderant terms involving $T$. So
\[
R_1(\delta) (\tilde{C}_T + \tilde{D}_T) = \frac{2(2 + \delta)}{4\pi} \left[ \ln \left( \frac{k R_1(\delta)}{\pi \delta(2 + \delta)} \right) + 1 + \frac{\pi \delta(2 + \delta)}{2 R_1(\delta)} \left( 2 C_2 \ln \left( \frac{2k R_1(\delta)}{\delta(2 + \delta)} \right) + 2 C_3 + C_2/2 \right) \right],
\]
\[
(1 + \delta/2) \tilde{E}(T) = \frac{2 + \delta}{4\pi} \left[ \ln^2 \left( \frac{2 R_1(\delta)}{\delta(2 + \delta)} \right) + 2 \ln(k/(2\pi)) \ln \left( \frac{2 R_1(\delta)}{\delta(2 + \delta)} \right) + 2 \pi C_2 + 4 \pi \left( \frac{1}{\pi} \ln(k/(2\pi e)) + C_2 \ln k + C_3 \right) \right].
\]

With the choice of $T$, the main terms of $\varepsilon_k$ are
\[
\frac{\varphi(k)}{\sqrt{x}} \frac{1}{2\pi} \ln^2 \left( \frac{2 R_1(\delta)}{\delta(2 + \delta)} \right) + \frac{\delta}{2}.
\]

These terms are minimized by choosing
\[
(27) \quad \delta = \frac{\varphi(k) \ln x}{\pi \sqrt{x}}.
\]

Now, replacing (26) and (27) in (25), we only have a function of $x$ for fixed $k$:
\[
\varepsilon_k(x) := \varepsilon_k(x, T, \delta).
\]

We simplify expression (25):
\[
\frac{\varepsilon_k(x, T, \delta)}{\varphi(k)} \leq \tilde{\varepsilon}_k(x, T, \delta) := \frac{R_1(\delta)}{\delta} (\tilde{C}_T + \tilde{D}_T) / \sqrt{x} + (1 + \frac{\delta}{2}) \tilde{E}(T) / \sqrt{x} + \frac{\delta}{2} + \frac{\tilde{R}}{x \varphi(k)}.
\]

By choosing $T = \frac{2 R_1(\delta)}{\delta(2 + \delta)}$ and $\delta = \frac{\ln x}{\pi \sqrt{x}}$, $\tilde{\varepsilon}_k(x, T, \delta)$ became $\tilde{\varepsilon}_k(x)$.

Hence,
\[
\tilde{\varepsilon}_k(x) \sqrt{x} = \frac{2 + \delta}{4\pi} \left[ \ln^2 \left( \frac{2 \pi \sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) + 2 \ln \left( \frac{2 \pi \sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) + \ln \frac{x}{2 \pi \varphi(k)} \right] + \frac{2 \pi C_2 + 4 \pi \left( \frac{1}{\pi} \ln(k/(2\pi e)) + C_2 \ln k + C_3 \right)}{\varphi(k) \sqrt{x}}
\]

with
\[
A = 2 C_2 \ln \left( \frac{2 \pi \sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) + 2 C_3 + C_2/2.
\]

Let $\delta = \frac{\ln x}{\pi \sqrt{x}}$, and we have $R_1(\delta) = \frac{2 + 2 \delta + \delta^2}{2 + 3} = 1 + \frac{\delta^2 + \delta}{2 + \delta} \leq d_1 = 1 + \frac{\delta^2 + \delta}{2 + 3}$, because $x \geq x_0$ and $\frac{\delta}{2 + \delta} < 1$.

By direct computation, for all $k$ between 1 and 432 and $x \geq x_0$, of $\frac{\tilde{\varepsilon}_k(x) \sqrt{x}}{\varphi(k) \ln x}$, we find an upper bound 0.06012.
To obtain 1) in Theorem [7] we will study the sum in brackets for $1 \leq k \leq \frac{4}{5} \ln x$:

\[
\begin{align*}
\ldots &= \left[ \frac{1}{4} \ln^2 x + \ln x \left( \frac{2\pi d_1}{\ln x} \right) + \ln x \ln \left( \frac{2\pi d_1}{\ln x} \right) + 2 \ln \left( \frac{4 \ln x}{10 \pi} \right) \ln \left( \frac{2\pi d_1}{\ln x} \right) \\
&\quad + \ln \left( \frac{4 \ln x}{10 \pi} \right) \ln x + \frac{1}{2} \ln x + \ln(4d_1/5) + \ln x \ln(4d_1/5) + \ln \left( \frac{2\pi d_1}{\ln x} \right) \right] \\
&= \left[ \frac{1}{4} \ln^2 x + \ln x \ln \left( \frac{2\pi d_1}{\ln x} \right) + 1/2 + \ln(4 \ln x/(10 \pi)) \\
&\quad + \ln^2 \left( \frac{2\pi d_1}{\ln x} \right) + 2 \ln \left( \frac{4 \ln x}{10 \pi} \right) \ln \left( \frac{2\pi d_1}{\ln x} \right) + \ln(4d_1/5) + \ln \left( \frac{2\pi d_1}{\ln x} \right) \right].
\end{align*}
\]

We conclude that

\[
\lim_{x \to +\infty} \frac{\varepsilon_k(x) \sqrt{x}}{\ln^2 x} = \frac{1}{8\pi},
\]

which is the same asymptotic bound as Schoenfeld’s [7] for $\psi$.

The bound $\varepsilon_k(x) \sqrt{x}$ is an increasing function of $k$. Choose $k = \frac{4}{5} \ln x$. Now $\varepsilon_k(x) \sqrt{x} / \ln^2 x$ is a decreasing function of $x$ bounded by 0.0849229 for $x \geq x_0$. □

**Remark.** If we take $k = 1$ in Theorem [7] our upper bound is twice as bad as the result of Schoenfeld [7, p. 337]: for $x > 73.2$,

\[
|\psi(x) - x| \leq \frac{1}{8\pi} \sqrt{x} \ln^2 x.
\]

These differences are explained by:

- an exact computation of zeros with $\gamma \leq D \approx 158$ (the preponderant ones!) in the sum $\sum \frac{1}{|\gamma|}$;
- a better knowledge of $R(T)$ ($k$ fixed, $k = 1$).

**Corollary 3.** Assume GRH $(k, \infty)$. For all $k$ used in Lemma [3] and $x \geq 224$,

\[
\left| \psi(x; k, l) - \frac{x}{\varphi(k)} \right| \leq \frac{1}{4\pi} \sqrt{x} \ln^2 x.
\]

**Proof.** We use Theorem 5.2.1 of [3]: for all $k$ noted in Lemma [3] and $224 \leq x \leq 10^{10}$,

\[
|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq \sqrt{x}
\]

and $\sqrt{x} < \frac{1}{4\pi} \sqrt{x} \ln^2 x$ for $x \geq 35$. We conclude by Theorem [7]. □

### 7. Estimates for $\pi(x; 3, l)$

**Definition 1.** Let

\[
\pi(x; k, l) = \sum_{p \equiv l \mod k} 1
\]

be the number of primes smaller than $x$ which are congruent to $l$ modulo $k$.

Our aim is to have bounds for $\pi(x; 3, l)$. We show that

**Theorem 8.** For $l = 1$ or 2,

1. $\frac{\pi(x; 3, l)}{\ln x} < \pi(x; 3, l)$ for $x \geq 151$,
2. $\pi(x; 3, l) < 0.55 \frac{\pi(x)}{\ln x}$ for $x \geq 229869$.  

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From this, we can deduce that for all \( x \geq 151 \),
\[
\frac{x}{\ln x} < \pi(x)
\]
because
\[
\pi(x) = \pi(x; 3, 1) + \pi(x; 3, 2) + 1.
\]

7.1. **The upper bound.** First we give the proof of Theorem 8 (ii).

**Lemma 13.** Let \( I_n = \int_a^x \frac{dt}{\ln^n t} \). Then \( I_n = \frac{x}{\ln^n x} - \frac{a}{\ln^n a} + \alpha_I x + 1 \). Furthermore,
\[
\alpha_I = \frac{1}{(1 - x + a) / \ln^n (a - x)}.
\]

**Theorem 9** (Ramaré and Rumely [3]). For \( 1 \leq x \leq 10^{10} \), for all \( k \leq 72 \), for all \( l \) relatively prime with \( k \),
\[
\max_{1 \leq y \leq x} | \theta(y; k, l) - \frac{y}{\varphi(k)} | \leq 2.072\sqrt{x}.
\]
Furthermore, for \( x \geq 10^{10} \) and \( k = 3 \) or 4,
\[
| \theta(x; k, l) - \frac{x}{\varphi(k)} | \leq 0.002238 \frac{x}{\varphi(k)}.
\]

Write first
\[
\pi(x; k, l) = \frac{\theta(x; k, l)}{\ln(x)} - \frac{\theta(x_0; k, l)}{\ln(x_0)} + \int_{x_0}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt.
\]

Put \( x_0 := 10^5 \).

Preliminary computations : \( \theta(10^5, 3, 1) = 49753.417198 \cdots \)
\( \pi(10^5, 3, 1) = 4784. \)
\( \theta(10^5, 3, 2) = 49930.873458 \cdots \)
\( \pi(10^5, 3, 2) = 4807. \)

Put \( c_0 := \frac{1.002238}{2} \) and \( K = \max(\pi(10^5, 3, l) - \theta(10^5, 3, l)/\ln(10^5)) \approx 470. \)

- For \( 10^{20} \leq x \),
\[
\pi(x; k, l) - \pi(10^5; k, l) = \frac{\theta(x; k, l)}{\ln(x)} - \frac{\theta(10^5; k, l)}{\ln(10^5)} + \int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt.
\]

But
\[
\int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt = \int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt + \int_{10^5}^{\sqrt{x}} \frac{\theta(t; k, l)}{t \ln^2 t} dt + \int_{\sqrt{x}}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt
\]
and, by Theorem 9
\[
\int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt < M := \frac{1}{\varphi(k)} \cdot \int_{10^5}^{x} \frac{dt}{\ln^2 t} + 2.072 \cdot \int_{10^5}^{x} \frac{dt}{\sqrt{t} \ln^2 t}
\]
\[
\int_{10^5}^{\sqrt{x}} \frac{\theta(t; 3, l)}{t \ln^2 t} dt < c_0 \frac{\sqrt{x} - 10^{10}}{\ln^2 10^{10}}
\]
\[
\int_{\sqrt{x}}^{x} \frac{\theta(t; 3, l)}{t \ln^2 t} dt < c_0 \frac{x - \sqrt{x}}{\ln^2 \sqrt{x}}.
\]
We compute \( M = 10381055.54 \cdots \). Then
\[
\pi(x;3,l) < c_0 \frac{x}{\ln x} + K + M + c_0 \left( \frac{\sqrt{x} - 10^{10}}{\ln^{2/3} x} + \frac{x - \sqrt{x}}{\ln^2 x} \right)
\]
\[
< \frac{x}{\ln x} \left( c_0 + \left( K + M + c_0 \frac{600 - 10^{10}}{\ln^{2/3} x} \right) \frac{\ln 600}{10^{20}} \right)
\]
\[
< 0.545 \frac{x}{\ln x}.
\]

- For \( 10^{10} \leq x \leq 10^{20} \),
\[
\pi(x;3,l) < K + \int_{10^{10}}^{x} \frac{\theta(t;3,l)}{t \ln^2 t} \, dt + \int_{10^{10}}^{x} \frac{\theta(t;3,l)}{t \ln^2 t} \, dt + c_0 \frac{x}{\ln x}
\]
\[
< \frac{x}{\ln x} \left( c_0 + \frac{\ln x}{x} \left( K + M - 10^{10} c_0 \frac{1}{\ln^{2/3} x} \right) + c_0 \frac{\ln x}{10^{20}} \right)
\]
\[
< 0.5468 \frac{x}{\ln x}.
\]

- For \( 10^5 \leq x \leq 10^{10} \),
\[
\int_{10^5}^{x} \frac{\theta(t;3,l)}{t \ln^2 t} \, dt < \frac{1}{2} \int_{10^5}^{x} \frac{dt}{\ln^2 t} + 2.072 \int_{10^5}^{x} \frac{dt}{t \ln^2 t}
\]
\[
= \frac{1}{2} \left( \frac{x}{\ln^2 x} - \frac{10^5}{\ln^2 10^5} + 2 \int_{10^5}^{x} \frac{dt}{t \ln^2 t} \right) + 2.072 \int_{10^5}^{x} \frac{dt}{t \ln^2 t}.
\]

Now, \( \int_{a}^{b} \frac{dt}{\sqrt{t \ln^2 t}} = \left[ \frac{2 \sqrt{t}}{\ln^2 t} \right]_{a}^{b} + 4 \int_{a}^{b} \frac{dt}{\sqrt{t \ln^2 t}} \).

Therefore
\[
\pi(x;3,l) < \frac{1}{2} \frac{x}{\ln x} + 2.072 \frac{\sqrt{x}}{\ln x} + K
\]
\[
+ \frac{1}{2} \left( \frac{x}{\ln^2 x} - \frac{10^5}{\ln^2 10^5} + 2 \int_{10^5}^{x} \frac{dt}{t \ln^2 t} \right)
\]
\[
+ 2.072 \left( \frac{2 \sqrt{x}}{\ln^2 x} - \frac{2 \sqrt{10^5}}{\ln^2 10^5} + 4 \int_{10^5}^{x} \frac{dt}{t \ln^2 t} \right)
\]
\[
< 0.55 \frac{x}{\ln x} \text{ for } x \geq 10^5.
\]

7.2. The lower bound. Let \( KK = \min(\pi(10^5,3,l) - \theta(10^5,3,l)/\ln(10^5)) \approx 462 \) and \( c = 0.498881 = \frac{1-0.002238}{2} \).

- For \( 10^{10} \leq x \),
\[
\pi(x;3,l) < KK + \theta(x;3,l) + \int_{10^{10}}^{x} \frac{\theta(t;k,l)}{t \ln^2 t} \, dt
\]
\[
< \frac{cx}{\ln x}
\]

because
\[
KK > 0 \text{ and } \int_{10^{10}}^{\infty} \frac{\theta(t;k,l)}{t \ln^2 t} \, dt > 0.
\]

- For \( 10^5 \leq x \leq 10^{10} \).

Lemma 14 (McCurley [2]). For \( x \geq 91807 \) and \( c_2 = 0.49585 \), we have \( \theta(x;3,l) \geq c_2 x \).
Remark. This bound is better than the one given in Theorem 9 for \( x \approx 2.5 \cdot 10^5 \).

\[
\pi(x; 3, l) > KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt.
\]

Thus for any \( x_0, x_1 \) with \( 10^5 \leq x_0 < x_1 \),

\[
\pi(x; 3, l) > KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^{x_0} \frac{\theta(t; k, l)}{t \ln^2 t} dt \text{ for } x \geq x_0
\]

\[
> \frac{x}{\ln x} \left( c_2 + \left( KK + \int_{10^5}^{x_0} \frac{\theta(t)}{t \ln^2 t} dt \right) \ln x \right) \text{ for } x_0 \leq x \leq x_1.
\]

Using the previous remark, we find

\[
\int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt > c_2 \int_{10^5}^{x} \frac{dt}{\ln^2 t} \text{ if } 10^5 \leq x \leq 2.5 \cdot 10^5
\]

and

\[
> c_2 \int_{10^5}^{2.5 \cdot 10^5} \frac{dt}{\ln^2 t} + \int_{2.5 \cdot 10^5}^{x} \frac{t/2 - 2.072 \sqrt{t}}{t \ln^2 t} dt \text{ if } 2.5 \cdot 10^5 \leq x.
\]

We use this to make step by step computations with Maple:

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( x_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^5 )</td>
<td>( 2 \cdot 10^6 )</td>
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<tr>
<td>( 2 \cdot 10^6 )</td>
<td>( 3 \cdot 10^7 )</td>
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<td>( 3 \cdot 10^7 )</td>
<td>( 3 \cdot 10^8 )</td>
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<td>( 3 \cdot 10^8 )</td>
<td>( 3 \cdot 10^9 )</td>
</tr>
<tr>
<td>( 3 \cdot 10^9 )</td>
<td>( 10^{10} )</td>
</tr>
</tbody>
</table>

We conclude that \( \pi(x; 3, l) > 0.499 \frac{x}{\ln x} \) for \( 10^5 \leq x \leq 10^{10} \).

7.3. Small values. We now check whether \( 0.4988 \frac{x}{\ln x} < \pi(x; 3, l) < 0.55 \frac{x}{\ln x} \) for \( x < 6 \cdot 10^5 \). It is sufficient to prove that

\[
\pi(p; 3, l) < 0.55 \frac{p}{\ln p} \text{ for } p \equiv l \text{ mod } 3,
\]

and if

\[
0.4988 \frac{p}{\ln p} < \pi(p; 3, l) - 1 \text{ for } p \equiv l \text{ mod } 3.
\]

The highest value not satisfying the first inequality is \( p = 229849 \), and the highest value not satisfying the second is \( p = 151 \). Furthermore, \( \pi(229869; 3, l) \approx 10241 < 0.55 \frac{229869}{\ln 229869} \approx 10241.0075 \) and \( \pi(151; 3, l) \geq 16 > 0.4988 \frac{151}{\ln 151} \approx 15.01 \).

The conclusion is

\[
0.4988 \frac{x}{\ln x} > \pi(x; 3, l) > 0.55 \frac{x}{\ln x}.
\]

Remark. We cannot show that \( x/(2 \ln x) < \pi(x; 3, l) \) by using the formula \( \theta(x) < c \cdot x \). We have obtained other formulas (see Theorem 6) which we will use below.
7.4. More precise lower bound of \( \pi(x; 3, l) \). Now we will give the proof of Theorem [8](i).

Classically,

\[
\pi(x; 3, l) - \pi(10^5; 3, l) = \frac{\theta(x; 3, l)}{\ln(x)} - \frac{\theta(10^5; 3, l)}{\ln(10^5)} + \int_{10^5}^{x} \frac{\theta(t; 3, l)}{t \ln^2 t} dt.
\]

Now \( \theta(t; 3, l) > \frac{x}{\varphi(3)} (1 - \frac{\alpha}{\ln x}) \) with \( \alpha = \varphi(3) \cdot 0.262 \) by use of Theorem [5]. So we write

\[ KK = \min_k \left( \pi(10^5; 3, l) - \frac{\theta(10^5; 3, l)}{\ln(10^5)} \right), \]

\[ \pi(x; 3, l) > J(x, \alpha) = KK + \frac{x}{\varphi(k) \ln x} \left( 1 - \frac{\alpha}{\ln x} \right) + \frac{1}{\varphi(k)} \int_{10^5}^{x} \frac{1 - \alpha/\ln t}{t \ln^2 t} dt. \]

The derivative of \( J(x, \alpha) \) with respect to \( x \) equals

\[ \frac{1}{\varphi(k)} \left( 1 - \frac{\alpha}{\ln x} + \frac{\alpha}{\ln^3 x} \right). \]

Moreover, the derivative of \( \frac{1}{\varphi(k) \ln x} \) equals

\[ \frac{1}{\varphi(k)} \left( \frac{1}{\ln x} - \frac{1}{\ln^2 x} \right). \]

The inequality

\[ \frac{1}{\varphi(k)} \left( \frac{1}{\ln x} - \frac{1}{\ln^2 x} \right) < \frac{1}{\varphi(k)} \left( \frac{1 - \alpha/\ln x}{\ln x} + \frac{\alpha}{\ln^3 x} \right) \]

holds if \( \alpha - 1 < \alpha/\ln x \); this holds for all \( x > 1 \). The only thing to do is to find a value \( x_1 \) such that

\[ J(x_1, \alpha) > \frac{x_1}{\varphi(k) \ln x_1}. \]

For \( x_1 = 10^5 \), \( J(10^5, 0.524) \approx 4607.75 \) and \( \frac{10^5}{2 \ln 10^5} \approx 4342.94 \). We verify by computer that the inequality holds for \( x \leq 10^5 \) and \( l = 1 \) or \( 2 \). We conclude that

\[ \frac{x}{2 \ln x} < \pi(x; 3, l) \text{ for } x \geq 151. \]

References


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