

## ESTIMATES OF $\theta(x; k, l)$ FOR LARGE VALUES OF $x$

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ABSTRACT. We extend a result of Ramaré and Rumely, 1996, about the Chebyshev function  $\theta$  in arithmetic progressions. We find a map  $\varepsilon(x)$  such that  $|\theta(x; k, l) - x/\varphi(k)| < x\varepsilon(x)$  and  $\varepsilon(x) = O(\frac{1}{\ln^a x})$  ( $\forall a > 0$ ), whereas  $\varepsilon(x)$  is a constant. Now we are able to show that, for  $x \geq 1531$ ,

$$|\theta(x; 3, l) - x/2| < 0.262 \frac{x}{\ln x}$$

and, for  $x \geq 151$ ,

$$\pi(x; 3, l) > \frac{x}{2 \ln x}.$$

### 1. INTRODUCTION

Let  $R = 9.645908801$  and  $X = \sqrt{\frac{\ln x}{R}}$ . Rosser [6] and Schoenfeld [7, Th. 11 p. 342] showed that, for  $x \geq 101$ ,

$$|\theta(x) - x|, |\psi(x) - x| < x\varepsilon(x),$$

where

$$\varepsilon(x) = \sqrt{\frac{8}{17\pi}} X^{1/2} \exp(-X).$$

We adapt their work to the case of arithmetic progressions. Let us recall the usual notations for nonnegative real  $x$ :

$$\theta(x; k, l) = \sum_{\substack{p \equiv l \pmod{k} \\ p \leq x}} \ln p, \quad \text{where } p \text{ is a prime number,}$$

$$\psi(x; k, l) = \sum_{\substack{n \equiv l \pmod{k} \\ n \leq x}} \Lambda(n), \quad \text{where } \Lambda \text{ is Von Mangoldt's function,}$$

and  $\varphi$  is Euler's function. We show, for  $x \geq x_0(k)$  where  $x_0(k)$  can be easily computed, that

$$|\theta(x; k, l) - x/\varphi(k)|, |\psi(x; k, l) - x/\varphi(k)| < x\varepsilon(x),$$

where

$$\varepsilon(x) = 3 \sqrt{\frac{k}{\varphi(k)C_1(k)}} X^{1/2} \exp(-X)$$

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for an explicit constant  $C_1(k)$ . We apply the above results for  $k = 3$ . For small values, we use Ramaré and Rumely's results [3]. We show that for  $x \geq 1531$ ,

$$(1) \quad |\theta(x; 3, l) - x/2| < 0.262 \frac{x}{\ln x}.$$

If we assume that the Generalized Riemann Hypothesis is true, then we can show that, for  $x > 1$  and  $k \leq 432$ ,

$$|\psi(x; k, l) - x/\varphi(k)| < \frac{1}{4\pi} \sqrt{x} \ln^2 x.$$

Let us define, as usual,  $\pi(x)$  the number of primes not greater than  $x$ . In 1962, Rosser and Schoenfeld ([5, p. 69]) found a lower bound for  $\pi(x)$ :

$$(2) \quad \pi(x) > \frac{x}{\ln x} \quad \text{for } x \geq 17.$$

Letting

$$\pi(x; k, l) = \sum_{\substack{p \leq x, \\ p \equiv l \pmod{k}}} 1,$$

we show an analogous result in the case of arithmetic progression with  $k = 3$  and  $l = 1$  or  $2$ ,

$$\pi(x; 3, l) > \frac{x}{2 \ln x} \quad \text{for } x \geq 151.$$

This result, inferred from (1), implies (2) and cannot be proved with Ramaré and Rumely's results.

The method used for  $k = 3$  can also be applied for other fixed integers  $k$ .

## 2. PRELIMINARY LEMMAS

*Notations.* We will always denote by  $\rho$  a nontrivial zero of Dirichlet's function  $L$ , that is to say a zero such that  $0 < \Re \rho < 1$ . We write  $\rho = \beta + i\gamma$ . Let  $\wp(\chi)$  be the set of the zeros  $\rho$  of the function  $L(s, \chi)$ , with  $0 < \beta < 1$ .

For a positive real  $H$ , following Ramaré and Rumely, we say that GRH( $k, H$ ) holds<sup>1</sup> if, for all  $\chi$  modulo  $k$ , all the nontrivial zeros of  $L(s, \chi)$  with  $|\gamma| \leq H$  are such that  $\beta = 1/2$ .

As in Rosser and Schoenfeld (in [6, 7] where the case  $k = 1$  is studied), we must know the distribution of  $L(s, \chi)$ 's zeros; namely, find a real  $H$  such that GRH( $k, H$ ) is satisfied and is a zero-free region.

### 2.1. Zero-free region.

**Theorem 1** (Ramaré and Rumely [3]). *If  $\chi$  is a character with conductor  $k$ ,  $H \geq 1000$ , and  $\rho = \beta + i\gamma$  is a zero of  $L(s, \chi)$  with  $|\gamma| \geq H$ , then there exists a computable constant  $C_1(\chi, H)$  such that*

$$1 - \beta \geq \frac{1}{R \ln(k|\gamma|/C_1(\chi, H))}.$$

<sup>1</sup>Note that our GRH is an acronym for the usual Generalized Riemann Hypothesis.

**Examples.** Some examples, extracted from [3, p. 409], appear in the following table.

$k$	$H_k$	$C_1(\chi, H_k)$
1	545000000	38.31
3	10000	20.92
420	2500	56.59

*Proof.* See Theorem 3.6.3 of Ramaré and Rumely [3, p. 409]. □

*Remark.* For  $k \geq 1$  and  $H_k \geq 1000$ ,  $C_1(\chi, H) \geq C_1(\chi_0, 1000) \geq 9.14$ .

As  $C_1(\chi, H)$  could be large, we limit  $C_1(\chi, H)$  up to  $32\pi$  to make some computations. So we have in our hypothesis

$$9.14 \leq C_1(\chi, H) \leq 32\pi.$$

From now on,

$$(3) \quad C_1(k) = \min\left(\min_{\chi \bmod k} C_1(\chi, H_\chi), 32\pi\right).$$

## 2.2. GRH( $k, H$ ) and $N(T, \chi)$ .

**Lemma 1** (McCurley [1]). *Let  $C_2 = 0.9185$  and  $C_3 = 5.512$ . Write  $F(y, \chi) = \frac{y}{\pi} \ln\left(\frac{ky}{2\pi e}\right)$  and  $R(y, \chi) = C_2 \ln(ky) + C_3$ . If  $\chi$  is a character of Dirichlet with conductor  $k$ , if  $T \geq 1$  is a real number, and if  $N(T, \chi)$  denotes the number of zeros  $\beta + i\gamma$  of  $L(s, \chi)$  in the rectangle  $0 < \beta < 1$ ,  $|\gamma| \leq T$ , then*

$$|N(T, \chi) - F(T, \chi)| \leq R(T, \chi).$$

**Lemma 2** (deduced from [3, Theorem 2.1.1, p. 399] and [9]).

- GRH(1,  $H$ ) is true for  $H = 5.45 \times 10^8$ .
- GRH( $k, H$ ) is true for  $H = 10000$  and  $k \leq 13$ .
- GRH( $k, 2500$ ) is true for sets
  - $E_1 = \{k \leq 72\}$ ,
  - $E_2 = \{k \leq 112, k \text{ not prime}\}$ ,
  - $E_3 = \{116, 117, 120, 121, 124, 125, 128, 132, 140, 143, 144, 156, 163, 169, 180, 216, 243, 256, 360, 420, 432\}$ .

**2.3. Estimates of  $|\psi(x; k, l) - x/\varphi(k)|$  using properties of zeros of  $L(s, \chi)$ .** As in Ramaré and Rumely, we remove the zeros with  $\beta = 0$  and we consider only primitive  $L$ -series by adding small terms. Here we take the version stated in [3, Theorem 4.3.1] which is deduced from [1].

**Theorem 2** (McCurley [1]). *Let  $x > 2$  be a real number,  $m$  and  $k$  two positive integers,  $\delta$  a real number such that  $0 < \delta < \frac{x-2}{mx}$ , and  $T$  a positive real. Let*

$$(4) \quad A(m, \delta) = \frac{1}{\delta^m} \sum_{j=0}^m \binom{m}{j} (1 + j\delta)^{m+1}.$$

Assume  $GRH(k, 1)$ . Then

$$\begin{aligned} \frac{\varphi(k)}{x} \max_{1 \leq y \leq x} \left| \psi(y; k, l) - \frac{y}{\varphi(k)} \right| &< A(m, \delta) \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| > T}} \frac{x^{\beta-1}}{|\rho(\rho+1) \cdots (\rho+m)|} \\ &+ \left(1 + \frac{m\delta}{2}\right) \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq T}} \frac{x^{\beta-1}}{|\rho|} + \frac{m\delta}{2} + \tilde{R}/x, \end{aligned}$$

where  $\sum_{\chi}$  denotes the summation over all characters modulo  $k$ ,  $\tilde{R} = \varphi(k)[(f(k) + 0.5) \ln x + 4 \ln k + 13.4]$  and  $f(k) = \sum_{p|k} \frac{1}{p-1}$ .

**2.4. One more explicit form of estimates.** The next lemma can be found in [3] with the difference that the authors assumed  $GRH(k, H)$  but in fact they used only  $GRH(k, 1)$ . Since we must apply it with  $T > H$ , we repeat the proof.

**Lemma 3.** Let  $\chi$  be a character modulo  $k$ . Assume  $GRH(k, 1)$ . Then, for any  $T \geq 1$ , we have

$$\sum_{\substack{|\gamma| \leq T \\ \rho \in \varphi(\chi)}} \frac{1}{|\rho|} \leq \tilde{E}(T)$$

with  $\tilde{E}(T) = \frac{1}{2\pi} \ln^2(T) + \frac{\ln(\frac{k}{2\pi})}{\pi} \ln(T) + C_2 + 2 \left(\frac{1}{\pi} \ln\left(\frac{k}{2\pi e}\right) + C_2 \ln k + C_3\right)$ .

*Proof.* For  $|\gamma| \leq 1$ , we have  $GRH(k, 1)$  and so

$$\sum_{\substack{|\gamma| \leq 1 \\ \rho \in \varphi(\chi)}} \frac{1}{|\rho|} \leq \sum_{\substack{|\gamma| \leq 1 \\ \rho \in \varphi(\chi)}} \frac{1}{|1/2 + i\gamma|} \leq 2N(1, \chi).$$

For  $|\gamma| > 1$ ,

$$\sum_{\substack{1 < |\gamma| \leq T \\ \rho \in \varphi(\chi)}} \frac{1}{|\rho|} \leq \int_1^T \frac{dN(t, \chi)}{t} = \int_1^T \frac{N(t, \chi)}{t^2} dt + \frac{N(T, \chi)}{T} - \frac{N(1, \chi)}{1}.$$

Thus,

$$\sum_{\substack{|\gamma| \leq T \\ \rho \in \varphi(\chi)}} \frac{1}{|\rho|} \leq \int_1^T \frac{N(t, \chi)}{t^2} dt + \frac{N(T, \chi)}{T} + N(1, \chi).$$

We conclude by Lemma 1 that

$$\begin{aligned} \int_1^T \frac{N(t, \chi)}{t^2} dt &\leq \int_1^T \frac{F(t, \chi) + R(t, \chi)}{t^2} dt \\ &= \frac{1}{\pi} \int_1^T \frac{\ln(kt/(2\pi e))}{t} dt + C_2 \int_1^T \frac{\ln(kt)}{t^2} dt + C_3 \int_1^T \frac{1}{t^2} dt \\ &= \frac{1}{\pi} \left[ \frac{1}{2} \ln^2 \left( \frac{kT}{2\pi e} \right) \right]_1^T \\ &\quad + C_2 \left\{ \left[ -\frac{\ln(kt)}{t} \right]_1^T + \int_1^T \frac{1}{t^2} dt \right\} + C_3 [-1/t]_1^T \\ &= \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) \ln T + C_2 \left( -\frac{\ln(kT)}{T} + \ln k - \frac{1}{T} + 1 \right) \\ &\quad + C_3(1 - 1/T). \end{aligned}$$

In the same way, we have an upper bound of

$$\frac{N(T, \chi)}{T} \quad \text{with} \quad \frac{F(T, \chi) + R(T, \chi)}{T}$$

and

$$N(1, \chi) \quad \text{with} \quad F(1, \chi) + R(1, \chi).$$

Finally, we obtain

$$\begin{aligned} \sum_{\substack{|\gamma| \leq T \\ \rho \in \varphi(\chi)}} \frac{1}{|\rho|} &\leq \frac{1}{2\pi} \ln^2(T) + \frac{\ln\left(\frac{k}{2\pi\epsilon}\right)}{\pi} \ln(T) \\ &+ C_2 + 2 \left( \frac{1}{\pi} \ln\left(\frac{k}{2\pi}\right) + C_2 \ln k + C_3 \right) - \frac{C_2}{T}. \end{aligned}$$

□

Using the facts that

- if  $\rho$  is a zero of  $L(s, \chi)$  then  $\bar{\rho}$  is zero of  $L(s, \bar{\chi})$ ,
- these zeros are symmetrical with to the line  $\Re(z) = 1/2$ ,

we obtain Lemma 4 by examining the proof of [3, Lemma 4.1.3].

**Lemma 4** ([3]). *Let*

$$(5) \quad \phi_m(t) = \frac{1}{|t|^{m+1}} \exp\left(\frac{-\ln x}{R \ln(k|t|/C_1(k))}\right)$$

with  $R = 9.645908801$ . Let  $T \geq H$ . We have

$$\sum_{\substack{|\gamma| \geq T \\ \rho \in \varphi(\chi)}} \frac{x^\beta}{|\gamma|^{m+1}} + \sum_{\substack{|\gamma| \geq T \\ \rho \in \varphi(\bar{\chi})}} \frac{x^\beta}{|\gamma|^{m+1}} \leq x \sum_{\substack{|\gamma| \geq T \\ \rho \in \varphi(\chi)}} \phi_m(\gamma) + \sqrt{x} \sum_{\substack{|\gamma| \geq T \\ \rho \in \varphi(\chi)}} \frac{1}{|\gamma|^{m+1}}.$$

Let us rewrite Lemma 7 of [6] to adapt it to the new functions  $F(y, \chi)$  and  $R(y, \chi)$  which we use.

**Lemma 5.** *Write  $N(y) = N(y, \chi)$ ,  $F(y) = F(y, \chi)$ , and  $R(y) = R(y, \chi)$ . Let  $1 < U \leq V$  and  $\phi(y)$  be a positive and differentiable function for  $U \leq y \leq V$ . Let  $(W - y)\phi'(y) \geq 0$  for  $U < y < V$ , where  $W$  does not necessarily belong to  $[U, V]$ . Let  $Y$  be that one of the numbers  $U, V, W$  which is not numerically the least or greatest (or is the repeated one, if two among  $U, V, W$  are equal). Take  $j = 0$  or  $1$ , accordingly as  $W < V$  or  $W \geq V$ . Then*

$$\sum_{U < |\gamma| \leq V} \phi(|\gamma|) \leq \frac{1}{\pi} \int_U^V \phi(y) \ln\left(\frac{ky}{2\pi}\right) dy + (-1)^j C_2 \int_U^V \frac{\phi(y)}{y} dy + B_j(Y, U, V),$$

where

$$\begin{aligned} B_0(Y, U, V) &= 2R(Y)\phi(Y) + \{N(V) - F(V) - R(V)\}\phi(V) \\ &\quad - \{N(U) - F(U) + R(U)\}\phi(U), \\ B_1(Y, U, V) &= \{N(V) - F(V) + R(V)\}\phi(V) - \{N(U) - F(U) + R(U)\}\phi(U). \end{aligned}$$

*Proof.* We have

$$\begin{aligned} \sum_{U < |\gamma| \leq V} \phi(|\gamma|) &= \int_U^V \phi(y) dN(y) \\ &= - \int_U^V N(y) \phi'(y) dy + N(V)\phi(V) - N(U)\phi(U). \end{aligned}$$

- $j = 1$ . We have  $W > V$  and so  $Y = \min(V, W) = V$ . According to Theorem 1,  $N(y) \geq F(y) - R(y)$ .

$$\begin{aligned} \sum_{U < |\gamma| \leq V} \phi(|\gamma|) &\leq [(N(y) - F(y) + R(y))\phi(y)]_U^V + \frac{1}{\pi} \int_U^V \ln\left(\frac{ky}{2\pi}\right) \phi(y) dy \\ &\quad - \int_U^V R'(y)\phi(y) dy \end{aligned}$$

because  $F'(y) = \frac{1}{\pi} \left( \ln\left(\frac{ky}{2\pi e}\right) + 1 \right) = \frac{1}{\pi} \ln\left(\frac{ky}{2\pi}\right)$ . Moreover,

$$- \int_U^V R'(y)\phi(y) dy = -C_2 \int_U^V \frac{\phi(y)}{y} dy.$$

- $j = 0$ . We have  $V > W$ . Take  $Y = \max(U, W)$ . Split the integral at  $Y$ . Then  $-\phi'(y) \leq 0$  for  $y \in [U, Y]$  and  $-\phi'(y) \geq 0$  for  $y \in [Y, V]$ . Replacing  $N(y)$  by  $F(y) - R(y)$  in the first part and by  $F(y) + R(y)$  in the second part, we obtain

$$\begin{aligned} \sum_{U < |\gamma| \leq V} \phi(|\gamma|) &\leq \frac{1}{\pi} \int_U^V \ln\left(\frac{ky}{2\pi}\right) \phi(y) dy + \int_Y^V R'(y)\phi(y) dy - \int_U^Y R'(y)\phi(y) dy \\ &\quad + B_0(Y, U, V). \end{aligned}$$

Moreover,

$$\int_Y^V R'(y)\phi(y) dy \leq (-1)^j C_2 \int_U^V \frac{\phi(y)}{y} dy$$

and

$$- \int_U^Y R'(y)\phi(y) dy \leq 0.$$

□

We want to apply Lemma 5 with  $\phi = \phi_m$  defined by (5) and with  $W = W_m$  being the root of  $\phi'_m$ . Let

$$(6) \quad X = \sqrt{\frac{\ln x}{R}}$$

and, for  $m \geq 0$ ,

$$(7) \quad W_m = \frac{C_1(k)}{k} \exp(X/\sqrt{m+1}).$$

**Corollary 1** (Corollary from Lemma 5). *Under the hypothesis of Lemma 5, if moreover  $\frac{2\pi}{ke} \leq U$ , then*

$$\sum_{U < |\gamma| \leq V} \phi(|\gamma|) \leq \{1/\pi + (-1)^j q(Y)\} \int_U^V \phi(y) \ln(ky/2\pi) dy + B_j(Y, U, V),$$

where  $q(y) = \frac{C_2}{y \ln\left(\frac{ky}{2\pi}\right)}$ .

*Proof.* The map  $y \mapsto 1/(y \ln(ky/2\pi))$  is decreasing if  $y \geq 2\pi/(ke)$ .

- Case ( $j = 0$ ), then  $Y = \max(U, W)$ .

$$\sum_{U < |\gamma| \leq V} \phi(|\gamma|) < B_0(Y, U, V) + \frac{1}{\pi} \int_U^V \phi(y) \ln \left( \frac{ky}{2\pi} \right) dy + \int_Y^V R'(y) \phi(y) dy.$$

$$\begin{aligned} \int_Y^V R'(y) \phi(y) dy &= C_2 \int_Y^V \frac{\phi(y)}{y} dy = C_2 \int_Y^V \frac{\phi(y) \ln(ky/2\pi)}{y \ln(ky/2\pi)} dy \\ &\leq \frac{C_2}{Y \ln(kY/2\pi)} \int_Y^V \phi(y) \ln(ky/2\pi) dy. \end{aligned}$$

- Case ( $j = 1$ ), then  $Y = V$ .

$$-\int_U^V R'(y) \phi(y) dy \leq -\frac{C_2}{V \ln(kV/2\pi)} \int_U^V \phi(y) \ln(ky/2\pi) dy.$$

□

**Theorem 3.** Let  $k \geq 1$  an integer,  $H \geq 1000$  a real number. Assume  $GRH(k, H)$ . Let  $x_0 > 2$  be a real number,  $m$  a positive integer, and  $\delta$  a real number such that  $0 < \delta < (x_0 - 2)/(mx_0)$  and let  $Y$  be defined as in Lemma 5. We write

$$(8) \quad \tilde{A}_H = \frac{1}{\pi} \int_H^\infty \phi_m(y) \ln \left( \frac{ky}{2\pi} \right) dy + C_2 \int_H^\infty \frac{\phi_m(y)}{y} dy,$$

$$(9) \quad \tilde{B}_H = B_0(Y, H, \infty),$$

$$(10) \quad \tilde{C}_H = \frac{1}{m\pi H^m} \left( \ln \left( \frac{kH}{2\pi} \right) + 1/m \right),$$

$$(11) \quad \tilde{D}_H = \left( 2C_2 \ln(kH) + 2C_3 + \frac{C_2}{m+1} \right) / H^{m+1}.$$

Then for all  $x \geq x_0$ , we have

$$\begin{aligned} \frac{\varphi(k)}{x} \max_{1 \leq y \leq x} \left| \psi(y; k, l) - \frac{y}{\varphi(k)} \right| &\leq A(m, \delta) \frac{\varphi(k)}{2} \left( \tilde{A}_H + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H) / \sqrt{x} \right) \\ &\quad + \left( 1 + \frac{m\delta}{2} \right) \varphi(k) \tilde{E}(H) / \sqrt{x} + \frac{m\delta}{2} + \tilde{R}/x. \end{aligned}$$

*Remark.* We find a version of Theorem 4.3.2 of [3] where  $x_0$  is replaced by  $x$  in  $\tilde{A}$  and  $\tilde{B}$ .

*Proof.* According to Theorem 2,

$$\begin{aligned} \frac{\varphi(k)}{x} \max_{1 \leq y \leq x} \left| \psi(y; k, l) - \frac{y}{\varphi(k)} \right| &< A(m, \delta) \sum_x \sum_{\substack{\rho \in \varphi(x) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\rho(\rho+1) \cdots (\rho+m)|} \\ &\quad + \left( 1 + \frac{m\delta}{2} \right) \sum_x \sum_{\substack{\rho \in \varphi(x) \\ |\gamma| \leq H}} \frac{x^{\beta-1}}{|\rho|} + \frac{m\delta}{2} + \tilde{R}/x. \end{aligned}$$

We separately examine the different parts:

- We have

$$\sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\rho(\rho+1) \cdots (\rho+m)|} \leq \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\gamma|^{m+1}}.$$

By Lemma 4,

$$\begin{aligned} \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\gamma|^{m+1}} &= \sum_{\chi} \frac{1}{2} \left( \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\gamma|^{m+1}} + \sum_{\substack{\rho \in \varphi(\bar{\chi}) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\gamma|^{m+1}} \right) \\ &\leq \frac{1}{2} \sum_{\chi} \left( \sum_{\substack{|\gamma| \geq H \\ \rho \in \varphi(\chi)}} \phi_m(\gamma) + \frac{1}{\sqrt{x}} \sum_{\substack{|\gamma| \geq H \\ \rho \in \varphi(\chi)}} \frac{1}{|\gamma|^{m+1}} \right). \end{aligned}$$

Using Lemma 5 with  $U = H, V = \infty, \phi = \phi_m,$  and  $W = W_m,$

$$\sum_{\substack{|\gamma| \geq H \\ \rho \in \varphi(\chi)}} \phi_m(\gamma) \leq \tilde{A}_H + \tilde{B}_H.$$

Integration by parts gives

$$\sum_{\substack{|\gamma| \geq H \\ \rho \in \varphi(\chi)}} \frac{1}{|\gamma|^{m+1}} \leq \tilde{C}_H + \tilde{D}_H.$$

- By GRH( $k, H$ ) we have  $\beta = 1/2$  for all  $|\gamma| \leq H,$  and by Lemma 3,

$$\sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq H}} \frac{x^{\beta-1}}{|\rho|} \leq \tilde{E}(H)/\sqrt{x}.$$

□

**2.5. The leading term ( $\tilde{A}_H$ ).** To obtain an upper bound for the leading term, we proceed like Rosser and Schoenfeld with upper bounds on the integrals. The next three lemmas are issued directly from [6, p. 251-255].

**Lemma 6** (Functions of incomplete Bessel type). *Let*

$$K_{\nu}(z, u) = \frac{1}{2} \int_u^{\infty} t^{\nu-1} H^z(t) dt,$$

where  $z > 0, u \geq 0,$  and

$$H^z(t) = \{H(t)\}^z = \exp\left\{-\frac{z}{2}(t + 1/t)\right\}.$$

Further, write  $K_{\nu}(z, 0) = K_{\nu}(z).$  Then

$$(12) \quad K_1(z) \leq \sqrt{\frac{\pi}{2z}} \exp(-z) \left(1 + \frac{3}{8z}\right),$$

$$(13) \quad K_2(z) \leq \sqrt{\frac{\pi}{2z}} \exp(-z) \left(1 + \frac{15}{8z} + \frac{105}{128z^2}\right).$$

**Lemma 7.**

$$K_{\nu}(z, x) + K_{-\nu}(z, x) = K_{\nu}(z).$$

Hence,  $K_{\nu}(z, x) \leq K_{\nu}(z)$  ( $\nu \geq 0$ ).

**Lemma 8.** *Let*

$$Q_\nu(z, x) = \frac{x^{\nu+1}}{z(x^2 - 1)} \exp\{-z(x + 1/x)/2\}.$$

If  $z > 0$  and  $x > 1$ , then

$$K_1(z, x) < Q_1(z, x)$$

and

$$K_2(z, x) < (x + 2/z)Q_1(z, x).$$

The term  $\tilde{A}_H$  can be expressed using incomplete Bessel functions.

**Lemma 9.** *Let  $X$  be defined by (6). Let  $z_m = 2X\sqrt{m} = 2\sqrt{\frac{m \ln x}{R}}$  and  $U_m = \frac{2m}{z_m} \ln\left(\frac{kH}{C_1(k)}\right) = \sqrt{\frac{Rm}{\ln x}} \ln\left(\frac{kH}{C_1(k)}\right)$ .*

$$\begin{aligned} \tilde{A}_H &= \frac{2 \ln x}{\pi Rm} \left(\frac{k}{C_1(k)}\right)^m K_2(z_m, U_m) \\ &+ \frac{2}{\pi} \ln\left(\frac{C_1(k)}{2\pi}\right) \sqrt{\frac{\ln x}{Rm}} \left(\frac{k}{C_1(k)}\right)^m K_1(z_m, U_m) \\ &+ 2C_2 \sqrt{\frac{\ln x}{R(m+1)}} \left(\frac{k}{C_1(k)}\right)^{m+1} K_1(z_{m+1}, U_{m+1}). \end{aligned}$$

*Proof.* This is by straightforward algebraic manipulation; for example, we write

$$I = \int_H^\infty \frac{C_2}{y^{m+1}} \exp\left(\frac{-\ln x}{R \ln(ky/C_1(k))}\right) \frac{dy}{y}.$$

Changing variables:

$$\begin{aligned} t &= \sqrt{\frac{R(m+1)}{\ln x}} \ln\left(\frac{ky}{C_1(k)}\right), \\ dt &= \sqrt{\frac{R(m+1)}{\ln x}} \frac{dy}{y}. \end{aligned}$$

Now

$$\begin{aligned} \exp\left(\frac{-\ln x}{R \ln(ky/C_1(k))}\right) &= \exp\left(\frac{-\ln x}{Rt/\sqrt{\frac{R(m+1)}{\ln x}}}\right) \\ &= \exp\left(\sqrt{\frac{(m+1) \ln x}{R}} \frac{1}{t}\right) = \exp\left(\frac{-z_{m+1}}{2} \frac{1}{t}\right) \end{aligned}$$

and

$$\frac{1}{y^{m+1}} = \left(\frac{k}{C_1(k)}\right)^{m+1} \exp\left(-\frac{(m+1)t}{\sqrt{\frac{R(m+1)}{\ln x}}}\right) = \left(\frac{k}{C_1(k)}\right)^{m+1} \exp\left(-t \frac{z_{m+1}}{2}\right).$$

Consequently,

$$I = \int_{U_{m+1}}^\infty C_2 \sqrt{\frac{\ln x}{R(m+1)}} \left(\frac{k}{C_1(k)}\right)^{m+1} \exp\left(\frac{-z_{m+1}}{2}(t + 1/t)\right) dt.$$

□

2.6. **Study of  $f(k)$  which appears in the expression of  $\tilde{R}$ .** Remember that  $f(k) = \sum_{p|k} \frac{1}{p-1}$ .

**Lemma 10.** For an integer  $k \geq 1$ ,

$$f(k) \leq \frac{\ln k}{\ln 2}.$$

*Proof.* We prove by recursion that

$$f(k) \leq \frac{\ln k}{\ln 2}.$$

For  $k = 1$ , it is obvious. For  $k = 2$ ,  $f(k) = 1 \leq \frac{\ln 2}{\ln 2}$ . Assume  $f(k) \leq \frac{\ln k}{\ln 2}$  holds for  $k \leq n$ . Find an upper bound for  $f(n + 1)$ . If  $(n + 1)$  is prime, then  $f(n + 1) = 1/n \leq \ln n / \ln 2$ . If  $(n + 1)$  is not prime, then there exists  $p \leq n$ , which divides  $n$ . If  $p = 2$  and  $2^\alpha \parallel n + 1$ ,

$$\begin{aligned} f(n + 1) &= f\left(\frac{n + 1}{2^\alpha} \cdot 2^\alpha\right) = f\left(\frac{n + 1}{2^\alpha}\right) + f(2) \\ &= 1 + f\left(\frac{n + 1}{2^\alpha}\right) \leq \frac{\ln(n + 1)}{\ln 2} + 1 - \frac{\ln 2}{\ln 2} \\ &\leq \frac{\ln(n + 1)}{\ln 2}. \end{aligned}$$

If  $p > 2$  and  $p^\alpha \parallel n + 1$ ,

$$\begin{aligned} f(n + 1) &= f\left(\frac{n + 1}{p^\alpha} \cdot p^\alpha\right) = f\left(\frac{n + 1}{p^\alpha}\right) + f(p) \\ &= \frac{1}{p - 1} + f\left(\frac{n + 1}{p^\alpha}\right) \leq \frac{\ln(n + 1)}{\ln 2} + \frac{1}{p - 1} - \frac{\ln p}{\ln 2} \\ &\leq \frac{\ln(n + 1)}{\ln 2} \quad \text{because } \frac{1}{p - 1} - \frac{\ln p}{\ln 2} < 0 \text{ for } p > 2. \end{aligned}$$

□

### 3. THE METHOD WITH $m = 1$

**Theorem 4.** Let  $k$  be an integer,  $H \geq 1250$ , and  $H \geq k$ . Assume  $GRH(k, H)$ . Let  $C_1(k)$  defined by (3). Let  $x > 1$ . Write  $X = \sqrt{\frac{\ln x}{R}}$  and

$$\varepsilon(x) = 2\sqrt{\frac{k\varphi(k)}{C_1(k)\sqrt{\pi}}} \left(1 + \frac{1}{2X}(15/16 + \ln(C_1(k)/(2\pi)))\right) X^{3/4} \exp(-X).$$

If  $\varepsilon(x) \leq 0.2$  and  $X \geq \sqrt{2} \ln\left(\frac{kH}{C_1(k)}\right)$ , then

$$\max_{1 \leq y \leq x} |\psi(y; k, l) - y/\varphi(k)| \leq x\varepsilon(x)/\varphi(k).$$

*Proof.* Take  $m = 1$  in Theorem 3. Assuming  $X \geq \sqrt{2} \ln\left(\frac{kH}{C_1(k)}\right)$ , then  $W_1 \geq H$ . In this situation,  $Y = W_1$  and  $\tilde{B}_H < 2R(W_1)\phi_1(W_1)$ . For  $y > 1$ ,  $R(y)/\ln y$  is

decreasing; hence,

$$\begin{aligned}\tilde{B}_H &< 2R(W_1)\phi_1(W_1) < 2\frac{R(H)}{\ln H}\phi_1(W_1)\ln W_1 \\ &= 2\frac{R(H)}{\ln H}\left(\frac{X}{\sqrt{2}} + \ln\left(\frac{C_1(k)}{k}\right)\right)\phi_1(W_1) \\ &= 2\frac{R(H)}{\ln H}\left(\frac{X}{\sqrt{2}} + \ln\left(\frac{C_1(k)}{k}\right)\right)(k/C_1(k))^2\exp(-2\sqrt{2}X).\end{aligned}$$

Inserting the upper bounds (12) and (13) into the bound for  $\tilde{A}_H$  in Lemma 9,

$$\begin{aligned}\tilde{A}_H &< 2\left(\frac{k}{C_1(k)}\right)\left[\sqrt{\frac{\pi}{4X}}\exp(-2X)\left(1 + \frac{15}{16X} + \frac{105}{512X^2}\right)X^2/\pi\right. \\ &\quad \left.+ \frac{1}{\pi}\ln\frac{C_1(k)}{2\pi}X\sqrt{\frac{\pi}{4X}}\exp(-2X)\left(1 + \frac{3}{16X}\right)\right. \\ &\quad \left.+ C_2\frac{kX}{C_1(k)\sqrt{2}}\sqrt{\frac{\pi}{4\sqrt{2}X}}\exp(-2\sqrt{2}X)\left(1 + \frac{3}{16\sqrt{2}X}\right)\right].\end{aligned}$$

Put

$$F_1 := \frac{1}{\sqrt{\pi}}\frac{k}{C_1(k)}X^{3/2}\exp(-2X)\left[1 + \left(\frac{15}{16} + \ln\frac{C_1(k)}{2\pi}\right)\frac{1}{2X}\right]^2.$$

In Lemma 11 below it is shown that

$$\tilde{A}_H + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H))/\sqrt{x} + \tilde{R}\frac{2}{x\varphi(k)} < F_1.$$

We must choose  $\delta$  to minimize

$$\frac{A(1, \delta)}{2}\varphi(k)F_1 + \delta/2.$$

Write  $f = \varphi(k)F_1$ . As  $A_1(\delta) = (\delta^2 + 2\delta + 2)/\delta$ , we must minimize  $g(\delta) = (\delta/2 + 1 + 1/\delta)f + \delta/2$ . The minimum value here is at  $\delta = \sqrt{\frac{2f}{1+f}}$ , and the value there is  $g(\sqrt{\frac{2f}{1+f}}) = f + \sqrt{2f(1+f)}$ .

It is a simple matter to prove that for  $0 \leq f \leq 0.202$ ,

$$f + \sqrt{2f(1+f)} < 2\sqrt{f}.$$

As  $X \geq X_0 := \sqrt{2}\ln\left(\frac{kH}{C_1(k)}\right)$ , then  $x_0 \geq \exp(122.5)$ , and it is obvious that  $\delta$  meets the hypothesis  $0 < \delta < (x_0 - 2)/x_0$  in Theorem 3 since

$$0 < \delta < \sqrt{2}\sqrt{f} < 0.6357 < \frac{x_0}{x_0 - 2}.$$

□

**Lemma 11.**

$$\tilde{A}_H + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H))/\sqrt{x} + \tilde{R}\frac{2}{x\varphi(k)} < F_1.$$

*Proof.* First we prove that  $\tilde{A}_H + \tilde{B}_H < F_1$ :

$$\begin{aligned}
 F_1 &= \frac{k}{C_1(k)\sqrt{\pi}} X^{3/2} e^{-2X} \left( 1 + (15/16 + \ln(C_1(k)/2\pi))/X \right. \\
 &\quad \left. + (225/1024 + \frac{15}{32} \ln(C_1(k)/2\pi) + \frac{1}{4} \ln^2(C_1(k)/2\pi))/X^2 \right), \\
 \tilde{A}_H &< \frac{k}{C_1(k)\sqrt{\pi}} X^{3/2} e^{-2X} \left( 1 + \frac{15}{16X} + \frac{105}{512X^2} + \ln\left(\frac{C_1(k)}{2\pi}\right) \left(\frac{1}{X} + \frac{3}{16X^2}\right) \right. \\
 &\quad \left. + C_2 \frac{k\pi}{C_1(k)\sqrt{2\sqrt{2}}} \exp(-2(\sqrt{2}-1)X) (1/X + 3/(16\sqrt{2}X^2)) \right), \\
 \tilde{B}_H &< \frac{k}{\sqrt{\pi}C_1(k)} X^{3/2} \exp(-2X) \exp(-2(\sqrt{2}-1)X) \\
 &\quad \times \left[ \frac{2k\sqrt{\pi}}{C_1(k)\ln H} (C_2 \ln(kH) + C_3) \left( \frac{1}{\sqrt{2X}} + \frac{1}{X\sqrt{X}} \ln(C_1(k)/k) \right) \right].
 \end{aligned}$$

This yields  $F_1 - \tilde{A}_H - \tilde{B}_H > 0$  if

$$\begin{aligned}
 F_2 &:= \frac{1}{X^2} \left( \frac{15}{1024} + \frac{9}{32} \ln\left(\frac{C_1(k)}{2\pi}\right) + \frac{1}{4} \ln^2\left(\frac{C_1(k)}{2\pi}\right) \right) \\
 &> \frac{C_2\sqrt{\pi}k}{C_1(k)} \exp(-2(\sqrt{2}-1)X) \frac{1}{\sqrt{2X}} \\
 &\quad \times \left[ \sqrt{\frac{\pi}{2\sqrt{2}}} \left( \sqrt{\frac{2}{X}} \frac{3}{16X^{3/2}} \right) \right. \\
 &\quad \left. + 2 \left( 1 + \frac{\ln k + C_3/C_2}{\ln H} \right) \left( 1 + \frac{\sqrt{2}}{X} \ln \frac{C_1(k)}{k} \right) \right].
 \end{aligned}$$

This holds if we can show that

$$F_2 > \frac{C_2k\sqrt{\pi}}{C_1(k)} \exp(-2(\sqrt{2}-1)X) \frac{1}{\sqrt{2X}} \cdot 16.9,$$

since  $C_1(k) \leq 32\pi$ ,  $H \geq 1250$ ,  $X \geq \sqrt{2} \ln(1250/32\pi)$ , and  $k \leq H$ .

It remains to be proved that

$$\frac{\sqrt{2}C_1(k)}{kC_2\sqrt{\pi} \cdot 16.9} (15/1024 + \dots) > X^{3/2} \exp(-2(\sqrt{2}-1)X).$$

But for  $X \geq X_0 := \sqrt{2} \ln\left(\frac{kH}{C_1(k)}\right)$ ,

$$\begin{aligned}
 X^{3/2} \exp(-2(\sqrt{2}-1)X) &< X_0^{3/2} \left( \frac{kH}{C_1(k)} \right)^{-(1+a)} \\
 &= \frac{1}{k} \cdot 2^{3/4} \left( \frac{C_1(k)}{H} \right)^{1+a} \left( \frac{\ln^{3/2}(kH/C_1(k))}{k^a} \right),
 \end{aligned}$$

where  $a = 2\sqrt{2}(\sqrt{2}-1) - 1 \approx 0.17157$ . The map  $k \mapsto \frac{\ln^{3/2}(kH/C_1(k))}{k^a}$  reaches its maximum for  $k = e^{\frac{3}{2a} \frac{C_1(k)}{H}}$ . Hence

$$X^{3/2} \exp(-2(\sqrt{2}-1)X) < \frac{C_1(k)}{kH} 2^{3/4} \left( \frac{3}{2a} \right)^{3/2} / e^{3/2}.$$

We must compare

$$\frac{\sqrt{2}}{C_2\sqrt{\pi} \cdot 16.9} (15/1024 + \dots) \text{ with } \frac{2^{3/4}(\frac{3}{2a})^{3/2}}{He^{3/2}}.$$

Since  $C_1(k) \geq 9.14$  (see the remark above (3)) and  $C_2 = 0.9185$ , it remains to be proved that

$$0.007976 > \frac{2^{3/4}(\frac{3}{2a})^{3/2}}{He^{3/2}} (\approx 0.00776),$$

which is true since  $H \geq 1250$ .

We show below that the remaining terms  $(\tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H))/\sqrt{x} + \tilde{R}_{x\varphi(k)}^2$  are negligible.

- We will find an upper bound for  $A(1, \delta)^{\frac{\varphi(k)}{2}}(\tilde{C}_H + \tilde{D}_H) + \frac{3}{2}\varphi(k)\frac{\tilde{E}(H)}{\sqrt{x}} + \tilde{R}/x$ .

We assume that  $X \geq \sqrt{2} \ln\left(\frac{kH}{C_1(k)}\right)$ ; hence,  $X \geq X_0 := \sqrt{2} \ln\left(\frac{1250}{32\pi}\right) \approx 3.5644$ . It is straightforward but tedious to check that

$$\text{Rest} := \tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H) + \frac{2\tilde{R}}{\varphi(k)\sqrt{x}} \leq \begin{cases} 1250(\ln H \ln k)^2 & \text{if } k \neq 1, \\ 1250(\ln H)^2 & \text{if } k = 1. \end{cases}$$

Let us consider the case  $k \neq 1$ . As  $X \geq \sqrt{2} \ln\left(\frac{kH}{C_1(k)}\right)$ ,

$$\exp\left(\frac{X}{\sqrt{2}}\right) \geq \frac{kH}{C_1(k)}.$$

This yields

$$\begin{aligned} \text{Rest} &\leq 1250(\ln H \ln k)^2 \leq 1250 \frac{(\ln H \ln k)^2}{\left(\frac{kH}{C_1(k)}\right)^2} \exp(X\sqrt{2}) \\ &\leq 1250C_1^2(k) \frac{1}{e^2} \left(\frac{\ln 1250}{1250}\right)^2 \exp(X\sqrt{2}) \\ &\leq K \exp(X\sqrt{2}) \quad \text{because } C_1(k) \leq 32\pi, \end{aligned}$$

where  $K := 55.65$ . Now compare

$$\frac{K \exp(X\sqrt{2})}{\sqrt{x}} = K \exp(X\sqrt{2} - RX^2/2)$$

with the term involving  $1/X^2$  in  $F_1$

$$\frac{1}{X^2} \times \frac{k}{C_1(k)\sqrt{\pi}} X^{3/2} \exp(-2X).$$

We may compute  $c$  such that

$$\begin{aligned} K \exp(X\sqrt{2} - RX^2/2) &\leq c \times \frac{1}{X^2} \times \frac{k}{C_1(k)\sqrt{\pi}} X^{3/2} \exp(-2X) \\ \Leftrightarrow c &\geq K \sqrt{32\pi\sqrt{\pi}} \exp(X\sqrt{2} - RX^2/2 + 2X) \frac{X^2}{X^{3/2}} \\ \Leftrightarrow c &\geq 0.7 \cdot 10^{-18} \quad \text{for } X \geq X_0. \end{aligned}$$

Thus, the rest is negligible and absorbed by rounding up the constants. □

4. THE METHOD WITH  $m = 2$

**Lemma 12.** Let  $A(m, \delta)$  be defined as in formula (4). Write

$$R_m(\delta) = (1 + (1 + \delta)^{m+1})^m.$$

Then

$$A(m, \delta) \leq \frac{R_m(\delta)}{\delta^m}.$$

*Proof.* The proof appears in [4, p. 222]. □

**Theorem 5.** Let an integer  $k \geq 1$ . Remember that  $R = 9.645908801$ . Let  $H \geq 1000$ . Assume  $GRH(k, H)$ . Let  $C_1(k)$  be defined by (3). Let  $X_0, X_1, X_2$ , and  $X_3$  be such that

$$\frac{e^{X_0}}{\sqrt{X_0}} = H \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}, \quad \frac{e^{X_1}}{X_1} = 10\varphi(k),$$

$$X_2 = kC_1(k)/(2\pi\varphi(k)), \quad X_3 = \frac{2k\pi e}{C_1(k)\varphi(k)}.$$

Let  $X_4 := \max(10, X_0, X_1, X_2, X_3)$ . Write

$$\varepsilon(X) = 3\sqrt{\frac{k}{\varphi(k)C_1(k)}} X^{1/2} \exp(-X).$$

Then for all real  $x$  such that  $X = \sqrt{\frac{\ln x}{R}} \geq X_4$ , we have

$$\begin{aligned} \max_{1 \leq y \leq x} |\psi(y; k, l) - y/\varphi(k)| &< x\varepsilon \left( \sqrt{\frac{\ln x}{R}} \right), \\ \max_{1 \leq y \leq x} |\theta(y; k, l) - y/\varphi(k)| &< x\varepsilon \left( \sqrt{\frac{\ln x}{R}} \right). \end{aligned}$$

**Corollary 2.** With the notations and the hypothesis of Theorem 5, let  $X_5 \geq X_4$  and  $c := \varepsilon(X_5)$ . For  $x \geq \exp(RX_5^2)$ , we have

$$|\psi(x; k, l) - x/\varphi(k)|, \quad |\theta(x; k, l) - x/\varphi(k)| < cx.$$

*Proof.* The idea is to judiciously split the integral into two parts, and bound each part optimally, using an  $m = 0$  estimate in the first part and an  $m = 2$  estimate in the second part.

We want to split the integral at  $T$ , where  $T$  will optimally be chosen later. We take  $T$  in the same form as  $W_m$  (formula (7)):

$$(14) \quad T := \frac{C_1(k)}{k} \exp(\nu X),$$

where  $\nu$  is a parameter.

Assume that  $T \geq H$  and  $1/\sqrt{m+1} \leq \nu \leq 1$ . Hence  $W_m \leq T \leq W_0$ . This last hypothesis is needed to apply Corollary 1.

We use Theorem 2 and split the sums at  $T$ :

$$A(m, \delta) \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| > T}} \frac{x^{\beta-1}}{|\rho(\rho+1) \cdots (\rho+m)|} + \left(1 + \frac{m\delta}{2}\right) \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq T}} \frac{x^{\beta-1}}{|\rho|} + \frac{m\delta}{2} + \frac{\tilde{R}}{x}.$$

Define

$$\begin{aligned}\tilde{A}_1 &:= \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq T}} \frac{x^{\beta-1}}{|\rho|}, \\ \tilde{A}_2 &:= \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| > T}} \frac{x^{\beta-1}}{|\rho(\rho+1)\cdots(\rho+m)|}.\end{aligned}$$

Bounding the term  $\tilde{A}_1$ , we get

$$\begin{aligned}\tilde{A}_1 &= \sum_{\chi} \left( \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq H}} \frac{x^{\beta-1}}{|\rho|} + \sum_{\substack{\rho \in \varphi(\chi) \\ H < |\gamma| \leq T}} \frac{x^{\beta-1}}{|\rho|} \right) \\ &= \frac{1}{x} \sum_{\chi} \left( \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq H}} \frac{\sqrt{x}}{|\rho|} + \sum_{\substack{\rho \in \varphi(\chi) \\ H < |\gamma| \leq T}} \frac{x^{\beta}}{|\rho|} \right) \text{ by GRH}(k, H) \\ &= \frac{1}{\sqrt{x}} \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq H}} \frac{1}{|\rho|} + \frac{1}{2x} \sum_{\chi} \left( \sum_{\substack{\rho \in \varphi(\chi) \\ H < |\gamma| \leq T}} \frac{x^{\beta}}{|\rho|} + \sum_{\substack{\rho \in \varphi(\bar{\chi}) \\ H < |\gamma| \leq T}} \frac{x^{\beta}}{|\rho|} \right) \\ &\leq \frac{1}{\sqrt{x}} \varphi(k) \tilde{E}(H) + \frac{1}{2x} \sum_{\chi} \left( \sum_{\substack{\rho \in \varphi(\chi) \\ H \leq |\gamma| \leq T}} x \phi_0(\gamma) + \sqrt{x} \sum_{\substack{\rho \in \varphi(\bar{\chi}) \\ H \leq |\gamma| \leq T}} \frac{1}{|\gamma|} \right) \\ &\quad \text{by Lemmas 3 and 4} \\ &\leq \varphi(k) \tilde{E}(T) / \sqrt{x} + \frac{1}{2} \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ H \leq |\gamma| \leq T}} \phi_0(\gamma).\end{aligned}$$

Apply Corollary 1 ( $j = 1, m = 0$ ) for the interval  $[H, T]$  with  $\phi = \phi_0$  and  $W = W_0$

$$\sum_{\substack{\rho \in \varphi(\chi) \\ H \leq |\gamma| \leq T}} \phi_0(\gamma) = \{1/\pi - q(T)\} \int_H^T \phi_0(y) \ln(ky/2\pi) dy + B_1(T, H, T).$$

Moreover,  $B_1(T, H, T) < 2R(T)\phi_0(T)$ .

We want to find an upper bound for

$$I_1 := \frac{1}{\pi} \int_H^T \phi_0(y) \ln\left(\frac{ky}{2\pi}\right) dy.$$

Write  $V'' = X^2 / \ln\left(\frac{kT}{C_1(k)}\right) = X/\nu = Y'' + 2X - \nu X$ , where  $Y'' := X(1-\nu)^2/\nu$ .

Write  $U'' = X^2 / \ln\left(\frac{kH}{C_1(k)}\right)$  and  $\Gamma(\alpha, x) = \int_x^\infty e^{-u} u^{\alpha-1} du$ . Now

$$\begin{aligned}\int_H^T \ln\left(\frac{ky}{2\pi}\right) \phi_0(y) dy &= \int_H^T \ln\left(\frac{ky}{2\pi}\right) \exp\left(-X^2 / \ln\left(\frac{ky}{C_1(k)}\right)\right) \frac{dy}{y} \\ &= X^4 \{\Gamma(-2, V'') - \Gamma(-2, U'')\} \\ &\quad + X^2 \ln\left(\frac{C_1(k)}{2\pi}\right) \{\Gamma(-1, V'') - \Gamma(-1, U'')\}\end{aligned}$$

by making the change of variables  $y = \frac{C_1(k)}{k} \exp(X^2/u)$ . Now if  $\alpha \leq 1$  and  $x > 0$ , then  $\Gamma(\alpha, x) \leq x^{\alpha-1} \int_x^\infty e^{-t} dt = x^{\alpha-1} e^{-x}$ . Hence,

$$\int_H^T \ln\left(\frac{ky}{2\pi}\right) \phi_0(y) dy \leq X^4 V''^{-3} e^{-V''} + X^2 \ln\left(\frac{C_1(k)}{2\pi}\right) V''^{-2} e^{-V''}.$$

This yields

$$\begin{aligned} I_1 &\leq \frac{1}{\pi} X^2 \left( X^2 V''^{-3} + \ln\left(\frac{C_1(k)}{2\pi}\right) V''^{-2} \right) e^{-V''} \\ &= \frac{1}{\pi} e^{-Y''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) \left( \frac{X^4}{(X/\nu)^3} + \frac{dX^2}{(X/\nu)^2} \right) \\ &= \frac{1}{\pi} e^{-Y''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) XG_0, \end{aligned}$$

where  $d := \ln\left(\frac{C_1(k)}{2\pi}\right)$  and  $G_0 := \nu^2(\nu + d/X)$ . With the help of Corollary 1, we write

$$\tilde{A}_1 \leq \varphi(k) \tilde{E}(T) / \sqrt{x} + \frac{\varphi(k)}{2} \left\{ \frac{1}{\pi} e^{-Y''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) XG_0 + 2R(T) \phi_0(T) \right\}.$$

Bounding the term  $\tilde{A}_2$ , we get

$$\begin{aligned} \tilde{A}_2 &= \frac{1}{x} \sum_X \sum_{\substack{\rho \in \varphi(X) \\ |\gamma| > T}} \frac{x^\beta}{|\rho(\rho+1) \cdots (\rho+m)|} \\ &= \frac{1}{2x} \sum_X \left( \sum_{\substack{\rho \in \varphi(X) \\ |\gamma| > T}} \frac{x^\beta}{|\rho(\rho+1) \cdots (\rho+m)|} + \sum_{\substack{\rho \in \varphi(\bar{X}) \\ |\gamma| > T}} \frac{x^\beta}{|\rho(\rho+1) \cdots (\rho+m)|} \right) \\ &\leq \frac{1}{2x} \sum_X \left( \sum_{\substack{\rho \in \varphi(X) \\ |\gamma| > T}} \frac{x^\beta}{|\gamma|^{m+1}} + \sum_{\substack{\rho \in \varphi(\bar{X}) \\ |\gamma| > T}} \frac{x^\beta}{|\gamma|^{m+1}} \right) \\ &= \frac{1}{2x} \sum_X \left( x \sum_{\substack{\rho \in \varphi(X) \\ |\gamma| > T}} \phi_m(\gamma) + \sqrt{x} \sum_{\substack{\rho \in \varphi(X) \\ |\gamma| > T}} \frac{1}{|\gamma|^{m+1}} \right) \end{aligned}$$

by Lemma 4.

By using Corollary 1 ( $j = 0$ ) on  $[U, V] = [T, \infty)$ ,

$$\sum_{\substack{\rho \in \varphi(X) \\ |\gamma| > T}} \phi_m(\gamma) \leq \{1/\pi + q(T)\} \int_T^\infty \phi_m(y) \ln\left(\frac{ky}{2\pi}\right) dy + B_0(T, T, \infty).$$

We have

$$B_0(T, T, \infty) < 2R(T) \phi_m(T).$$

Moreover,

$$\sum_{\substack{\rho \in \varphi(X) \\ |\gamma| > T}} \frac{1}{|\gamma|^{m+1}} \leq \tilde{C}_T + \tilde{D}_T.$$

Let us study more precisely

$$\begin{aligned} I_2 &:= \int_T^\infty \phi_m(y) \ln\left(\frac{ky}{2\pi}\right) dy \\ &= \frac{z_m^2}{2m^2} \left(\frac{k}{C_1(k)}\right)^m \left(K_2(z_m, U_m) + \frac{2md}{z_m} K_1(z_m, U_m)\right), \end{aligned}$$

where  $d = \ln\left(\frac{C_1(k)}{2\pi}\right)$  and  $U' := U_m = \frac{2m}{z_m} \ln\left(\frac{kT}{C_1(k)}\right) = \nu\sqrt{m}$ . Now, by writing  $z = z_m$  and using Lemma 8,

$$\begin{aligned} K_2(z, U') + \frac{2dm}{z} K_1(z, U') &< (U' + 2/z + 2dm/z) Q_1(z, U') \\ &\leq \sqrt{m} \left(\nu + \frac{1+dm}{mX}\right) \frac{U'^2}{z(U'^2 - 1)} e^{-\frac{z}{2}(U'+1/U')}. \end{aligned}$$

But  $\frac{z}{2}(U' + 1/U') = X\sqrt{m}(\nu\sqrt{m} + 1/(\nu\sqrt{m})) = m\nu X + X/\nu = m\nu X + (Y'' + 2X - \nu X)$ , where  $Y'' = X(1 - \nu)^2/\nu$ . Hence

$$K_2(z, U') + \frac{2dm}{z} K_1(z, U') < G_1 e^{-Y''} \frac{m}{2(m-1)} X^{-1} e^{-2X} \left(\frac{kT}{C_1(k)}\right)^{-(m-1)},$$

where  $G_1 := \frac{m-1}{m} \frac{U'^2}{U'^2-1} \left(\nu + \frac{1+dm}{mX}\right)$  because

$$e^{\nu X(m-1)} = \left(\frac{kT}{C_1(k)}\right)^{m-1}$$

and  $\frac{\sqrt{m}}{z} = \frac{1}{2}X^{-1}$ . This yields

$$I_2 = \int_T^\infty \phi_m(y) \ln(ky/2\pi) dy < \frac{G_1 e^{-Y''}}{m-1} \frac{k}{C_1(k)} X e^{-2X} T^{-(m-1)}$$

Let  $G_2 := \frac{R_m(\delta)}{2^m} (1 + \pi q(T))$ . So, by using Lemma 12,

$$\begin{aligned} A(m, \delta) \frac{\varphi(k)}{2} (1/\pi + q(T)) \int_T^\infty \phi_m(y) \ln(ky/2\pi) dy \\ < \left(\frac{2}{\delta}\right)^m \frac{\varphi(k)}{2} \left\{ \frac{G_2}{\pi} \frac{k G_1 e^{-Y''}}{(m-1) C_1(k)} X e^{-2X} T^{-(m-1)} \right\}. \end{aligned}$$

The results above yield

$$(15) \quad \begin{aligned} &(1 + m\delta/2) \tilde{A}_1 + A(m, \delta) \tilde{A}_2 \\ &< \frac{X G_2 e^{-2X} e^{-Y''} \varphi(k)}{2\pi} \left(\frac{k}{C_1(k)}\right) \left\{ \frac{G_1}{m-1} T^{-(m-1)} \left(\frac{2}{\delta}\right)^m + G_0 T \right\} + r \end{aligned}$$

because  $1 + m\delta/2 < R_m(\delta)/2^m < G_2$ , with

$$\begin{aligned} r &= \varphi(k) (1 + m\delta/2) R(T) \phi_0(T) + A(m, \delta) \varphi(k) R(T) \phi_m(T) \\ &+ \frac{\varphi(k)}{\sqrt{x}} ((1 + m\delta/2) \tilde{E}(T) + A(m, \delta) (\tilde{C}_T + \tilde{D}_T)/2). \end{aligned}$$

Suppose  $G_0/G_1$  were independent of  $\nu$ ; then the expression between braces in (15) would be minimized for

$$(16) \quad T = (G_1/G_0)^{1/m} \cdot \frac{2}{\delta}.$$

With this choice,

$$\frac{G_1}{m-1} T^{-(m-1)} \left(\frac{2}{\delta}\right)^m + G_0 T = \frac{m}{m-1} G_1^{1/m} G_0^{1-1/m} \frac{2}{\delta},$$

and we obtain ( $G_2 > 1$ )

$$\begin{aligned} \varepsilon_1 &:= (1 + m\delta/2)\tilde{A}_1 + A(m, \delta)\tilde{A}_2 + \frac{1}{2}m\delta + \frac{\tilde{R}}{x} \\ &< \frac{1}{2}mG_2 \left\{ X e^{-2X} e^{-Y''} \frac{2k\varphi(k)}{\delta(m-1)\pi C_1(k)} G_1^{1/m} G_0^{1-1/m} + \delta \right\} + r + \frac{\tilde{R}}{x}. \end{aligned}$$

The expression between braces can be minimized by choosing

$$(17) \quad \delta = \left\{ G_0^{1-1/m} G_1^{1/m} e^{-Y''} \frac{2k\varphi(k)}{(m-1)\pi C_1(k)} \right\}^{1/2} X^{1/2} e^{-X}.$$

Hence, we write (by replacing the above value of  $\delta$  in (16))

$$(18) \quad T = \left(\frac{G_1}{G_0}\right)^{1/2m} \left(\frac{2C_1(k)}{k\varphi(k)}(m-1)\pi e^{Y''}/G_0\right)^{1/2} X^{-1/2} e^X$$

and

$$(19) \quad \varepsilon_1 < G_2 \left(G_0^{1-1/m} G_1^{1/m} e^{-Y''} \frac{2k\varphi(k)}{\pi C_1(k)}\right)^{1/2} \frac{m}{\sqrt{m-1}} X^{1/2} e^{-X} + r + \frac{\tilde{R}}{x}.$$

The value  $m = 2$  minimizes the expression  $\frac{m}{\sqrt{m-1}}$ . For the remainder of the argument, we fix  $m = 2$ .

We now have two definitions for  $T$ . On the one hand (equation (18)),

$$T = \left(\frac{G_1}{G_0^3}\right)^{1/4} e^{Y''/2} \sqrt{\frac{2\pi C_1(k)}{k\varphi(k)}} X^{-1/2} e^X$$

with  $Y'' = X(1 - \nu)^2/\nu$ , and on the other hand (equation (14))

$$T = \frac{C_1(k)}{k} \exp(\nu X).$$

These two equations are compatible if and only if there exists  $\nu$  such that  $f(\nu) = 1$ , where

$$f(\nu) = \frac{C_1(k)\varphi(k)}{2\pi k} \left(\frac{G_0^3}{G_1}\right)^{1/2} X e^{-X(1-\nu)^2/\nu} e^{-2X(1-\nu)}.$$

Here we have  $m = 2$  and our assumption  $1/\sqrt{m+1} \leq \nu \leq 1$  gives  $1/\sqrt{3} \leq \nu \leq 1$ . Note that

$$\begin{aligned} G_0 &= \nu^2(\nu + d/X), \\ G_1 &= \frac{m-1}{m} \frac{U'^2}{U'^2-1} \left(\nu + \frac{1+dm}{mX}\right) = \frac{\nu^2}{2\nu^2-1} \left(\nu + \frac{1+2d}{2X}\right). \end{aligned}$$

It is easy to check that on the interval  $1/\sqrt{2} \leq \nu \leq 1$ ,  $G_0^3/G_1$  is increasing, and hence,  $f(\nu)$  is strictly increasing. Moreover,  $\lim_{\nu \rightarrow (1/\sqrt{2})^+} f(\nu) = 0$  and  $f(1) > 1$

(for all  $X \geq \frac{2\pi k}{C_1(k)\varphi(k)}$ ). So there exists a unique  $\nu \in ]1/\sqrt{2}, 1[$  such that  $f(\nu) = 1$ . For  $1/\sqrt{2} < \nu < 1$ , we have ( $m = 2$ )

$$H(\nu) := \frac{G_0^3}{G_1} = \frac{[\nu^2(\nu + d/X)]^3}{\frac{\nu^2}{2\nu^2-1}(\nu + \frac{1+2d}{2X})} < (\nu + d/X)^2.$$

Write, for  $X \geq X_3 := \frac{2\pi ke}{C_1(k)\varphi(k)}$ ,

$$(20) \quad \nu_0 = 1 - \frac{1}{2X} \ln \left( \frac{C_1(k)\varphi(k)X}{2k\pi} \right).$$

Let us study  $H(\nu_0)$ :

$$\begin{aligned} H(\nu_0) < 1 & \quad \text{if} \quad \nu_0 + d/X \leq 1, \\ \text{equivalently} & \quad 1 - \frac{1}{2X} \ln \left( \frac{C_1(k)\varphi(k)X}{2k\pi} \right) + \frac{\ln(C_1(k)/2\pi)}{X} \leq 1, \\ \text{which holds if} & \quad X \geq X_2 := \frac{kC_1(k)}{2\pi\varphi(k)}. \end{aligned}$$

As

$$f(\nu) = \frac{C_1(k)\varphi(k)}{2k\pi} \left( \frac{G_0^3}{G_1} \right)^{1/2} X \exp(-X(1-\nu)^2/\nu) \exp(-2X(1-\nu)),$$

replacing  $\nu_0$  by (20), we obtain

$$f(\nu_0) = \left( \frac{G_0^3}{G_1} \right)^{1/2} \exp \left( -\ln^2 \left( \frac{C_1(k)\varphi(k)X}{2k\pi} \right) / (4\nu_0 X) \right).$$

Assume that  $\nu_0 > 0$ , then, for  $X \geq X_2$ ,  $f(\nu_0) < 1 = f(\nu)$  and hence  $\nu_0 < \nu$ . We will require  $X \geq X_2$ .

The assumption  $T \geq H$  holds if  $T \geq \frac{C_1(k)}{k} \exp(\nu_0 X) \geq H$ . Using (20), rewrite  $\frac{C_1(k)}{k} \exp(\nu_0 X) = \sqrt{\frac{2\pi C_1(k)}{k\varphi(k)}} e^{X - \frac{1}{2} \ln X}$ . Let  $X_0$  satisfy

$$e^{X_0 - \frac{1}{2} \ln X_0} = H \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}.$$

We have  $T \geq H$  provided that  $X \geq X_0$ . We will require  $X \geq X_0$ .

For  $X \geq X_3 = \frac{2\pi ke}{C_1(k)\varphi(k)}$ ,  $\nu_0$  is an increasing function of  $X$ . We will require that  $X \geq \max(X_3, 10)$ . Then since  $C_1(k) \leq 32\pi$  and  $X \geq 10$ , we have

$$\nu_0 > 0.7462413 \quad \text{and} \quad \nu_0 < \nu < 1.$$

The assumption  $\nu > 1/\sqrt{2}$  is satisfied.

We want to evaluate

$$(21) \quad K := G_2(\sqrt{G_0 G_1} e^{-Y''})^{1/2},$$

which appears in (19). Again using  $C_1(k) \leq 32\pi$  and  $X \geq 10$ , we find

$$\begin{aligned} G_0 G_1 & < (1 + d/X) \frac{\nu_0}{2\nu_0^2 - 1} \left( \nu_0 + \frac{1+2d}{2X} \right) \\ & < 8.995. \end{aligned}$$

The following results will be needed in later computations.

1. Since  $X \geq X_0$  and  $\exp(X)/\sqrt{X}$  is increasing for  $X \geq 1/2$ ,

$$\sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}} X^{1/2} \exp(-X) \leq \frac{1}{H}.$$

2. Since  $G_0 G_1 < 9$ ,

$$\begin{aligned} \delta &= 2 \sqrt[4]{G_0 G_1} \exp(-Y''/2) \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}} X^{1/2} e^{-X} \\ &\leq 2\sqrt{3}/H. \end{aligned}$$

In particular, for  $H \geq 1000$ , we have  $\delta \leq 0.00347$ .

- 3.

$$G_2 = \frac{R_2(\delta)}{2^2} (1 + \pi q(T)) < (1 + 3.012 \cdot \delta/2)^2 (1 + \pi q(T)),$$

because

$$\begin{aligned} \frac{R_2(\delta)}{2^2} &= \left\{ \frac{(1 + \delta)^3 + 1}{2} \right\}^2 \\ &= \left\{ 1 + \frac{1}{2} \delta (3 + 3\delta + \delta^2) \right\}^2 < \left( 1 + \frac{3.012}{2} \delta \right)^2 \end{aligned}$$

since  $1 + \delta + \delta^2/3 < 1.0035$ .

4. Since  $T \geq H$ ,

$$\begin{aligned} q(T) &= \frac{C_2}{T \ln(kT/2\pi)} \\ &\leq \frac{C_2}{H \ln(kH/2\pi)}. \end{aligned}$$

But  $\exp(-Y''/2) \leq 1$  and  $H \geq 1000$ , so this yields

$$\begin{aligned} K &< (8.995)^{1/4} G_2 \\ &< (8.995)^{1/4} \left( 1 + \frac{\pi C_2}{1000 \ln(1000/(2\pi))} \right) \times \left( 1 + \frac{3.012}{2} \frac{2\sqrt{3}}{1000} \right)^2 \\ &< 1.751. \end{aligned}$$

Inserting this upper bound of  $K$  (see formula (21) in (19), we obtain

$$\begin{aligned} \varepsilon_1 &< 2\sqrt{\frac{2}{\pi}} K \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X) + r + \frac{\tilde{R}}{x} \\ (22) \quad &< 2.7941 \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X) + r + \frac{\tilde{R}}{x}. \end{aligned}$$

Now we want to bound  $r$  and  $\frac{\tilde{R}}{x}$ .

- An upper bound for  $\varphi(k)(1 + \delta)R(T)\phi_0(T)$  and  $\varphi(k)A(2, \delta)R(T)\phi_2(T)$ . Recall that

$$\begin{aligned} R(T) &= C_2 \ln(kT) + C_3, \\ \phi_0(T) &= \frac{1}{T} \exp(-X^2/\ln(kT/C_1(k))), \\ \phi_m(T) &= \phi_0(T)T^{-m}. \end{aligned}$$

Now

$$\phi_0(T) = \frac{1}{T} \exp(-X^2/(\nu X)) = \frac{1}{T} \exp(-\frac{1}{\nu}X) \leq \frac{1}{T} \exp(-X)$$

and

$$\frac{1}{T} = X^{1/2} \exp(-X) \sqrt{\frac{k\varphi(k)}{C_1(k)}} \left(\frac{G_0}{2\pi e^{Y''}}\right)^{1/2} \left(\frac{G_0}{G_1}\right)^{1/4},$$

hence

$$\begin{aligned} R(T)\phi_0(T) &\leq \frac{C_2 \ln(kT) + C_3}{T} \exp(-X) \\ &\leq \sqrt{X} e^{-X} \sqrt{\frac{k\varphi(k)}{C_1(k)}} \left[ (C_2 \ln(kT) + C_3) \left(\frac{G_0}{2\pi e^{Y''}}\right)^{1/2} \left(\frac{G_0}{G_1}\right)^{1/4} e^{-X} \right]. \end{aligned}$$

But

$$\begin{aligned} G_0 &\leq 1 + \frac{\ln(C_1(k)/2\pi)}{X}, \\ \frac{G_0}{G_1} &\leq 2\nu^2 - 1 < 1 \quad (m = 2), \\ \exp(Y'') &\geq 1, \\ \ln(kT) &= \nu X + \ln(C_1(k)) \leq X + \ln(C_1(k)) \leq X + \ln(32\pi). \end{aligned}$$

So, since  $X \geq 10$  and  $C_1(k) \leq 32\pi$ ,

$$\begin{aligned} (1 + \delta)\varphi(k) &\left[ (C_2 \ln(kT) + C_3) \left(\frac{G_0}{2\pi e^{Y''}}\right)^{1/2} \left(\frac{G_0}{G_1}\right)^{1/4} \exp(-X) \right] \\ &\leq \varphi(k) \left(1 + \frac{2\sqrt{3}}{1000}\right) [C_2(X + \ln 32\pi) + C_3] \sqrt{\frac{1 + \ln 16/10}{2\pi}} \exp(-X) \\ &\leq 0.857\varphi(k)X \exp(-X). \end{aligned}$$

Furthermore, if  $X_1$  is defined by  $\exp(X_1)/X_1 = 10\varphi(k)$ , and if we require that  $X \geq X_1$ , then this term is bounded by 0.0857. Hence, under the hypotheses on  $X$  in Theorem 5, an upper bound for  $\varphi(k)(1 + \delta)R(T)\phi_0(T)$  is

$$0.09 \cdot \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X).$$

Next, by (16)

$$\delta T = 2\sqrt{\frac{G_1}{G_0}}.$$

Hence, by Lemma 12

$$A(2, \delta)/T^2 \leq \frac{R_2(\delta)}{(\delta T)^2} \leq \frac{R_2(\delta)}{2^2} \frac{G_0}{G_1} \leq \frac{R_2(\delta)}{2^2}$$

and

$$\varphi(k)A(2, \delta)R(T)\phi_2(T) \leq \varphi(k) \frac{R_2(\delta)}{2^2} R(T)\phi_0(T).$$

Using  $\delta \leq 2\sqrt{3}/H \leq 2\sqrt{3}/1000$ , we get  $R_2(\delta)/2^2 \leq 1.0147$ . Under the hypotheses on  $X$  in Theorem 5, an upper bound for  $\varphi(k)A(2, \delta)R(T)\phi_2(T)$  is therefore

$$0.087 \cdot \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X).$$

The sum of the two terms can be bounded by

$$(23) \quad 0.2 \cdot \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X).$$

- An upper bound for  $(1 + \delta)\tilde{E}(T)\frac{\varphi(k)}{\sqrt{x}} + A(2, \delta)\frac{\varphi(k)}{2\sqrt{x}}(\tilde{C}_T + \tilde{D}_T) + \tilde{R}/x$ .

For  $f(k) = \sum_{p|k} \frac{1}{p-1}$  observe that (Lemma 10)

$$f(k) \leq \frac{\ln k}{\ln 2}.$$

We can explicitly rewrite for  $m = 2$ ,  $H \geq 1000$ , and  $C_1(k) \leq 32\pi$  the following expressions:

$$\begin{aligned} 3\tilde{E}(T) &= 3 \left( \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln \left( \frac{k}{2\pi} \right) \ln T + C_2 \right. \\ &\quad \left. + 2 \left( \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right) \right), \\ \tilde{C}_T &= \frac{1}{2\pi T^2} \left( \ln \left( \frac{kT}{2\pi} \right) + 1/2 \right), \\ \tilde{D}_T &= (2C_2 \ln(kT) + 2C_3 + C_2/3)/T^3, \\ \frac{\tilde{R}}{\varphi(k)\sqrt{x}} &\leq [(f(k) + 0.5) \ln x + 4 \ln k + 13.4] / \sqrt{x}. \end{aligned}$$

It is tedious but easy to check that the sum of the above quantities is less than

$$\begin{cases} 1000(\ln T \sqrt{\ln k})^2 & \text{for } k \neq 1, \\ 1000 \ln^2 T & \text{for } k = 1. \end{cases}$$

Now we want to find a number  $c$  such that

$$A(2, \delta)\varphi(k) \frac{1000(\ln T \sqrt{\ln k})^2}{\sqrt{x}} \leq c \left( \frac{k\varphi(k)}{C_1(k)} \right)^{1/2} X^{1/2} \exp(-X)$$

with  $X = \sqrt{\frac{\ln x}{R}}$ . But  $A(2, \delta) \leq \frac{R_2(\delta)}{\delta^2}$  and by (16),  $T = \left( \frac{G_1}{G_0} \right)^{1/2} \frac{2}{\delta}$ , so

$$A(2, \delta) \leq \frac{R_2(\delta)}{2^2} T^2 \frac{G_0}{G_1}.$$

Moreover,  $\frac{1}{\sqrt{x}} = \exp(-RX^2/2)$ , hence

$$c \geq 1000 \frac{R_2(\delta)}{2^2} \frac{G_0}{G_1} T^2 \varphi(k) (\ln T \sqrt{\ln k})^2 \left( \frac{C_1(k)}{k\varphi(k)} \right)^{1/2} X^{-1/2} \exp(X - RX^2/2).$$

As  $\frac{G_0}{G_1} < 1$ ,  $T^2 = \frac{C_1^2(k)}{k^2} \exp(2\nu X) \leq \frac{C_1^2(k)}{k^2} \exp(2X)$ , hence it suffices to take

$$c \geq 1000 \frac{R_2(\delta)}{2^2} C_1^2(k) \frac{\ln k}{k^2} (\ln(C_1(k)/k) + X)^2 \left( \frac{C_1(k)\varphi(k)}{kX} \right)^{1/2} \exp(3X - RX^2/2).$$

But  $\frac{\varphi(k)}{k} \leq 1$ ,  $\frac{\ln k}{k^2} \leq 1$ , and  $R_2(\delta) \leq (1 + 3.012\delta/2)^2$  with  $\delta \leq \frac{2\sqrt{3}}{H} \leq \frac{2\sqrt{3}}{1000}$ . So, finally, it suffices to take

$$c \geq \frac{1000}{4} \left(1 + \frac{3.012\sqrt{3}}{1000}\right)^2 C_1^2(k) (\ln C_1(k) + X)^2 \sqrt{C_1(k)} X^{-1/2} \exp(3X - RX^2/2).$$

Since  $C_1(k) \leq 32\pi$  and  $X \geq 10$ , we can take

$$(24) \quad c = 0.643 \cdot 10^{-187}.$$

In the case  $k = 1$ , we can replace the upper bound  $\frac{\ln k}{k^2} \leq 1$  by 1, and obtain the same result. Combining (22), (23), and (24), we obtain the result in Theorem 5; more precisely, for all  $X$  satisfying the conditions of the theorem,

$$|\psi(x; k, l) - x/\varphi(k)|/x \leq 2.9941 \sqrt{\frac{k}{\varphi(k)C_1(k)}} X^{1/2} \exp(-X).$$

We also wish to allow  $\theta$  instead of  $\psi$ , which can be done by recalling Theorem 13 of [5]:

$$0 \leq \psi(x; k, l) - \theta(x; k, l) \leq \psi(x) - \theta(x) \leq 1.43\sqrt{x} \quad \text{for } x \geq 0.$$

Using  $X \geq 10$ , we find  $1.43\sqrt{x}/x \leq d \cdot 3(k\varphi(k))/C_1(k) X^{1/2} \exp(-X)$ , where  $d = 1.17 \cdot 10^{-204}$ . This difference is absorbed by rounding up the constants.  $\square$

## 5. APPLICATION FOR $k = 3$

Now we are able to compute  $x_0$  and  $c$  such that, for  $x \geq x_0$ ,

$$|\theta(x; 3, l) - x/2| < cx/\ln x.$$

This would not have been possible if we had used only the results of [3].

According to Theorem 5,

$$\varepsilon(X) = \frac{3}{2} \sqrt{\frac{6}{20.92}} X^{1/2} \exp(-X)$$

for  $k = 3$ .

To determine for which  $x$  this bound is valid, let us solve for the constants  $X_0$ ,  $X_1$ ,  $X_2$ ,  $X_3$  in Theorem 5. Noting that  $H_3 = 10000$  by the table in Theorem 1, we need  $X_0$  to satisfy

$$\exp(X_0 - \frac{1}{2} \ln X_0) \geq 10000 \sqrt{\frac{6}{2\pi \cdot 20.92}} \approx 2136.51.$$

$X_0 \approx 8.76$  works.

Find  $X_1$  such that

$$\exp(X_1 - \ln X_1) \geq 20.$$

$X_1 \approx 4.5$  works.

Compute the two other bounds:  $X_2 \approx 4.99$ ,  $X_3 \approx 1.22$ . Thus we can take  $X = \max(10, X_0, X_1, X_2, X_3) = 10$  in Theorem 5.

- For  $\sqrt{\frac{\ln x}{R}} \geq 10$ , write  $X = \sqrt{\frac{\ln x}{R}}$ , then

$$\varepsilon(X) \ln x = RX^2 \varepsilon(X).$$

Find the value  $c$  such that

$$\varepsilon(X) < c/\ln(x).$$

For any  $x$  such that  $\sqrt{\frac{\ln x}{R}} \geq 10$ ,  $c \leq R \cdot 10^2 \varepsilon(10) \leq 0.12$ . Hence we have for  $x \geq \exp(964.59 \dots)$ ,

$$|\theta(x; 3, l) - x/2| \leq 0.12 \frac{x}{\ln x}.$$

We want to extend the above result for  $x \leq \exp(964.59 \dots)$ . Olivier Ramaré has kindly computed some additional values supplementing Table 1 in [3]. We have

$$|\theta(x; 3, l) - x/2| < \tilde{c} \cdot x/2$$

with

$$\begin{aligned} \tilde{c} &= 0.0008464421 \text{ for } \ln x \geq 400 \quad (m = 3, \delta = 0.00042325), \\ \tilde{c} &= 0.0006048271 \text{ for } \ln x \geq 500 \quad (m = 3, \delta = 0.00030250), \\ \tilde{c} &= 0.0004190635 \text{ for } \ln x \geq 600 \quad (m = 2, \delta = 0.00027950). \end{aligned}$$

Hence,

- For  $e^{600} \leq x \leq e^{964.59 \dots}$

$$c \leq 0.0004190635 \cdot 964.6 / \varphi(3) \leq 0.203.$$

- For  $e^{400} \leq x \leq e^{600}$

$$c \leq 0.0008464421 \cdot 600 / \varphi(3) \leq 0.254.$$

Using the computations of [3],

- For  $10^{100} \leq x \leq e^{400}$

$$c \leq 0.001310 \cdot 400 / \varphi(3) \leq 0.262.$$

- For  $10^{30} \leq x \leq 10^{100}$

$$c \leq 0.001813 \cdot 100 \ln 10 / \varphi(3) \leq 0.42/2 \leq 0.21.$$

- For  $10^{13} \leq x \leq 10^{30}$

$$c \leq 0.001951 \cdot 30 \ln 10 / \varphi(3) \leq 0.14/2 \leq 0.07.$$

- For  $10^{10} \leq x \leq 10^{13}$

$$c \leq 0.002238 \cdot 13 \ln 10 / \varphi(3) \leq 0.067/2 \leq 0.00335.$$

- For  $4403 \leq x \leq 10^{10}$

$$|\theta(x; 3, l) - x/2| < 2.072\sqrt{x} \quad (\text{Theorem 5.2.1 of Ramaré and Rumely [3]})$$

We choose  $c = 0.262$ . We check that this bound is also valid for  $1531 \leq x \leq 4403$ .

**Theorem 6.** For  $x \geq 1531$ ,

$$|\theta(x; 3, l) - x/2| \leq 0.262 \frac{x}{\ln x}.$$

6. RESULTS ASSUMING  $\text{GRH}(k, \infty)$ 

Assuming  $\text{GRH}(k, \infty)$ , we obtain more precise results. Under this hypothesis, one can show that function  $\psi$  has the following asymptotic behaviour:

**Proposition 1** ([8, p. 294]). *Assume  $\text{GRH}(k, \infty)$ . Then*

$$\psi(x; k, l) = \frac{x}{\varphi(k)} + O(\sqrt{x} \ln^2 x).$$

**Theorem 7.** *Let  $x \geq 10^{10}$ . Let  $k$  be a positive integer. Assume  $\text{GRH}(k, \infty)$ .*

1) *If  $k \leq \frac{4}{5} \ln x$ , then*

$$|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq 0.085\sqrt{x} \ln^2 x.$$

2) *If  $k \leq 432$ , then*

$$|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq 0.061\sqrt{x} \ln^2 x.$$

*Proof.* Let  $x_0 = 10^{10}$ . Applying Theorem 2 in the same way as Theorem 3 (assume that  $T \geq 1$ ),

$$\begin{aligned} & \frac{\varphi(k)}{x} \left| \psi(x; k, l) - \frac{x}{\varphi(k)} \right| \\ & \leq A(m, \delta) \sum_x \sum_{|\gamma| > T} \frac{x^{-1/2}}{|\rho(\rho+1) \cdots (\rho+m)|} \\ & \quad + (1 + m\delta/2) \sum_x \sum_{|\gamma| \leq T} \frac{x^{-1/2}}{|\rho|} + m\delta/2 + \tilde{R}/x \\ & \leq A(m, \delta) \frac{1}{\sqrt{x}} \sum_x \sum_{|\gamma| > T} \frac{1}{|\gamma|^{m+1}} + \left(1 + \frac{m\delta}{2}\right) \frac{1}{\sqrt{x}} \sum_x \sum_{|\gamma| \leq T} \frac{1}{|\rho|} + \frac{m\delta}{2} + \frac{\tilde{R}}{x} \\ & \leq A(m, \delta) \frac{\varphi(k)}{\sqrt{x}} (\tilde{C}_T + \tilde{D}_T) + \left(1 + \frac{m\delta}{2}\right) \frac{\varphi(k)}{\sqrt{x}} \tilde{E}(T) + \frac{m\delta}{2} + \tilde{R}/x. \end{aligned}$$

Take  $m = 1$  and let

$$(25) \quad \varepsilon_k(x, T, \delta) := \frac{R_1(\delta)}{\delta} \frac{\varphi(k)}{\sqrt{x}} (\tilde{C}_T + \tilde{D}_T) + \left(1 + \frac{\delta}{2}\right) \frac{\varphi(k)}{\sqrt{x}} \tilde{E}(T) + \frac{\delta}{2} + \tilde{R}/x,$$

where

$$\begin{aligned} \tilde{C}_T &= \frac{1}{\pi T} \left( \ln \left( \frac{kT}{2\pi} \right) + 1 \right), \\ \tilde{D}_T &= \frac{1}{T^2} (2C_2 \ln(kT) + 2C_3 + C_2/2), \\ \tilde{E}(T) &= \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln(k/(2\pi)) \ln T + C_2 + 2 \left( \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right). \end{aligned}$$

Choose

$$(26) \quad T = \frac{2R_1(\delta)}{\delta(2 + \delta)}$$

to minimize in (25) the preponderant terms involving  $T$ . So

$$\begin{aligned} \frac{R_1(\delta)}{\delta}(\tilde{C}_T + \tilde{D}_T) &= \frac{2(2+\delta)}{4\pi} \left[ \ln \left( \frac{kR_1(\delta)}{\pi\delta(2+\delta)} \right) + 1 \right. \\ &\quad \left. + \frac{\pi\delta(2+\delta)}{2R_1(\delta)} \left( 2C_2 \ln \left( \frac{2kR_1(\delta)}{\delta(2+\delta)} \right) + 2C_3 + C_2/2 \right) \right], \\ (1+\delta/2)\tilde{E}(T) &= \frac{2+\delta}{4\pi} \left[ \ln^2 \left( \frac{2R_1(\delta)}{\delta(2+\delta)} \right) + 2 \ln(k/(2\pi)) \ln \left( \frac{2R_1(\delta)}{\delta(2+\delta)} \right) \right. \\ &\quad \left. + 2\pi C_2 + 4\pi \left( \frac{1}{\pi} \ln(k/(2\pi e)) + C_2 \ln k + C_3 \right) \right]. \end{aligned}$$

With the choice of  $T$ , the main terms of  $\varepsilon_k$  are

$$\frac{\varphi(k)}{\sqrt{x}} \frac{1}{2\pi} \ln^2 \left( \frac{2R_1(\delta)}{\delta(\delta+2)} \right) + \frac{\delta}{2}.$$

These terms are minimized by choosing

$$(27) \quad \delta = \frac{\varphi(k) \ln x}{\pi\sqrt{x}}.$$

Now, replacing (26) and (27) in (25), we only have a function of  $x$  for fixed  $k$ :

$$\varepsilon_k(x) := \varepsilon_k(x, T, \delta).$$

We simplify expression (25):

$$\begin{aligned} \frac{\varepsilon_k(x, T, \delta)}{\varphi(k)} &\leq \tilde{\varepsilon}_k(x, T, \delta) \\ &:= \frac{R_1(\delta)}{\delta}(\tilde{C}_T + \tilde{D}_T)/\sqrt{x} + (1 + \frac{\delta}{2})\tilde{E}(T)/\sqrt{x} + \frac{\delta}{2} + \frac{\tilde{R}}{x\varphi(k)}. \end{aligned}$$

By choosing  $T = \frac{2R_1(\delta)}{\delta(2+\delta)}$  and  $\delta = \frac{\ln x}{\pi\sqrt{x}}$ ,  $\tilde{\varepsilon}_k(x, T, \delta)$  became  $\tilde{\varepsilon}_k(x)$ .

Hence,

$$\begin{aligned} \tilde{\varepsilon}_k(x)\sqrt{x} &= \frac{2+\delta}{4\pi} \left[ \ln^2 \left( \frac{2\pi\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2+\delta} \right) + 2 \ln \left( \frac{k}{2\pi} \right) \ln \left( \frac{2\pi\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2+\delta} \right) \right. \\ &\quad \left. + 2 \ln \left( \frac{k\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2+\delta} \right) + \frac{\ln x}{\sqrt{x}} \frac{2+\delta}{R_1(\delta)}(A) \right] + \frac{\ln x}{2\pi\varphi(k)} + \frac{\tilde{R}}{\varphi(k)\sqrt{x}} \\ &\quad + \frac{2+\delta}{4\pi} (2 + 2\pi C_2 + 4\pi \left( \frac{1}{\pi} \ln(k/(2\pi e)) + C_2 \ln k + C_3 \right)) \end{aligned}$$

with

$$A = 2C_2 \ln \left( \frac{2k\pi\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2+\delta} \right) + 2C_3 + C_2/2.$$

Let  $\delta_1 = \frac{\ln x_0}{\pi\sqrt{x_0}}$ . But  $\frac{R_1(\delta)}{2+\delta} = \frac{2+2\delta+\delta^2}{2+\delta} = 1 + \frac{\delta^2+\delta}{2+\delta} \leq d_1 := 1 + \frac{\delta_1^2+\delta_1}{2+\delta_1}$  because  $x \geq x_0$  and  $\frac{2+\delta}{R_1(\delta)} < 1$ .

By direct computation, for all  $k$  between 1 and 432 and  $x \geq x_0$ , of  $\frac{\varepsilon_k(x)\sqrt{x}}{\varphi(k)\ln^2 x}$ , we find an upper bound 0.06012.

To obtain 1) in Theorem 7, we will study the sum in brackets for  $1 \leq k \leq \frac{4}{5} \ln x$ :

$$\begin{aligned} [\dots] &= \left[ \frac{1}{4} \ln^2 x + \ln^2 \left( \frac{2\pi d_1}{\ln x} \right) + \ln x \ln \left( \frac{2\pi d_1}{\ln x} \right) + 2 \ln \left( \frac{4 \ln x}{10\pi} \right) \ln \left( \frac{2\pi d_1}{\ln x} \right) \right. \\ &\quad \left. + \ln \left( \frac{4 \ln x}{10\pi} \right) \ln x + \frac{1}{2} \ln x + \ln(4d_1/5) + \frac{\ln x}{\sqrt{x}}(A) \right] \\ &= \left[ \frac{1}{4} \ln^2 x + \ln x \left( \ln \left( \frac{2\pi d_1}{\ln x} \right) + 1/2 + \ln(4 \ln x / (10\pi)) \right) \right. \\ &\quad \left. + \ln^2 \left( \frac{2\pi d_1}{\ln x} \right) + 2 \ln \left( \frac{4 \ln x}{10\pi} \right) \ln \left( \frac{2\pi d_1}{\ln x} \right) + \ln(4d_1/5) + \frac{\ln x}{\sqrt{x}}(A) \right]. \end{aligned}$$

We conclude that

$$\lim_{x \rightarrow +\infty} \frac{\varepsilon_k(x) \sqrt{x}}{\ln^2 x} = \frac{1}{8\pi},$$

which is the same asymptotic bound as Schoenfeld's [7] for  $\psi$ .

The bound  $\varepsilon_k(x) \sqrt{x}$  is an increasing function of  $k$ . Choose  $k = \frac{4}{5} \ln x$ . Now  $\varepsilon_k(x) \sqrt{x} / \ln^2 x$  is a decreasing function of  $x$  bounded by 0.0849229 for  $x \geq x_0$ .  $\square$

*Remark.* If we take  $k = 1$  in Theorem 7, our upper bound is twice as bad as the result of Schoenfeld [7, p. 337]: for  $x > 73.2$ ,

$$|\psi(x) - x| \leq \frac{1}{8\pi} \sqrt{x} \ln^2 x.$$

These differences are explained by:

- an exact computation of zeros with  $\gamma \leq D \approx 158$  (the preponderant ones!) in the sum  $\sum \frac{1}{|\rho|}$ ;
- a better knowledge of  $R(T)$  ( $k$  fixed,  $k = 1$ ).

**Corollary 3.** Assume  $GRH(k, \infty)$ . For all  $k$  used in Lemma 2 and  $x \geq 224$ ,

$$\left| \psi(x; k, l) - \frac{x}{\varphi(k)} \right| \leq \frac{1}{4\pi} \sqrt{x} \ln^2 x.$$

*Proof.* We use Theorem 5.2.1 of [3]: for all  $k$  noted in Lemma 2 and  $224 \leq x \leq 10^{10}$ ,

$$\left| \psi(x; k, l) - \frac{x}{\varphi(k)} \right| \leq \sqrt{x}$$

and  $\sqrt{x} < \frac{1}{4\pi} \sqrt{x} \ln^2 x$  for  $x \geq 35$ . We conclude by Theorem 7.  $\square$

## 7. ESTIMATES FOR $\pi(x; 3, l)$

**Definition 1.** Let

$$\pi(x; k, l) = \sum_{\substack{p \leq x \\ p \equiv l \pmod{k}}} 1$$

be the number of primes smaller than  $x$  which are congruent to  $l$  modulo  $k$ .

Our aim is to have bounds for  $\pi(x; 3, l)$ . We show that

**Theorem 8.** For  $l = 1$  or  $2$ ,

- $\frac{x}{2 \ln x} < \pi(x; 3, l)$  for  $x \geq 151$ ,
- $\pi(x; 3, l) < 0.55 \frac{x}{\ln x}$  for  $x \geq 229869$ .

From this, we can deduce that for all  $x \geq 151$ ,

$$\frac{x}{\ln x} < \pi(x)$$

because

$$\pi(x) = \pi(x; 3, 1) + \pi(x; 3, 2) + 1.$$

**7.1. The upper bound.** First we give the proof of Theorem 8 (ii).

**Lemma 13.** *Let  $I_n = \int_a^x \frac{dt}{\ln^n t}$ . Then  $I_n = \frac{x}{\ln^n x} - \frac{a}{\ln^n a} + nI_{n+1}$ . Furthermore,*

$$\text{for } a > e, \quad (x - a)/\ln^n(x) \leq I_n \leq (x - a)/\ln^n(a).$$

**Theorem 9** (Ramaré and Rumely [3]). *For  $1 \leq x \leq 10^{10}$ , for all  $k \leq 72$ , for all  $l$  relatively prime with  $k$ ,*

$$\max_{1 \leq y \leq x} \left| \theta(y; k, l) - \frac{y}{\varphi(k)} \right| \leq 2.072\sqrt{x}.$$

Furthermore, for  $x \geq 10^{10}$  and  $k = 3$  or  $4$ ,

$$\left| \theta(x; k, l) - \frac{x}{\varphi(k)} \right| \leq 0.002238 \frac{x}{\varphi(k)}.$$

Write first

$$\pi(x; k, l) - \pi(x_0; k, l) = \frac{\theta(x; k, l)}{\ln(x)} - \frac{\theta(x_0; k, l)}{\ln(x_0)} + \int_{x_0}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt.$$

Put  $x_0 := 10^5$ .

Preliminary computations :

$$\theta(10^5, 3, 1) = 49753.417198 \dots \quad \pi(10^5, 3, 1) = 4784.$$

$$\theta(10^5, 3, 2) = 49930.873458 \dots \quad \pi(10^5, 3, 2) = 4807.$$

Put  $c_0 := \frac{1.002238}{2}$  and  $K = \max_l (\pi(10^5, 3, l) - \theta(10^5, 3, l)/\ln(10^5)) \approx 470$ .

- For  $10^{20} \leq x$ ,

$$\pi(x; k, l) - \pi(10^5; k, l) = \frac{\theta(x; k, l)}{\ln(x)} - \frac{\theta(10^5; k, l)}{\ln(10^5)} + \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt.$$

But

$$\int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt = \int_{10^5}^{10^{10}} \frac{\theta(t; k, l)}{t \ln^2 t} dt + \int_{10^{10}}^{\sqrt{x}} \frac{\theta(t; k, l)}{t \ln^2 t} dt + \int_{\sqrt{x}}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt$$

and, by Theorem 9

$$\begin{aligned} \int_{10^5}^{10^{10}} \frac{\theta(t; k, l)}{t \ln^2 t} dt &< M := 1/\varphi(k) \cdot \int_{10^5}^{10^{10}} \frac{dt}{\ln^2 t} + 2.072 \cdot \int_{10^5}^{10^{10}} \frac{dt}{\sqrt{t} \ln^2 t} \\ \int_{10^{10}}^{\sqrt{x}} \frac{\theta(t; 3, l)}{t \ln^2 t} dt &< c_0 \frac{\sqrt{x} - 10^{10}}{\ln^2 10^{10}} \\ \int_{\sqrt{x}}^x \frac{\theta(t; 3, l)}{t \ln^2 t} dt &< c_0 \frac{x - \sqrt{x}}{\ln^2 \sqrt{x}}. \end{aligned}$$

We compute  $M = 10381055.54 \dots$ . Then

$$\begin{aligned} \pi(x; 3, l) &< c_0 \frac{x}{\ln x} + K + M + c_0 \left( \frac{\sqrt{x} - 10^{10}}{\ln^2 10^{10}} + \frac{x - \sqrt{x}}{\ln^2 \sqrt{x}} \right) \\ &< \frac{x}{\ln x} \left( c_0 + \left( K + M + c_0 \frac{10^{20} - 10^{10}}{\ln^2 10^{10}} \right) \frac{\ln 10^{20}}{10^{20}} \right) \\ &< 0.545 \frac{x}{\ln x}. \end{aligned}$$

- For  $10^{10} \leq x \leq 10^{20}$ ,

$$\begin{aligned} \pi(x; 3, l) &< K + \int_{10^5}^{10^{10}} \frac{\theta(t; 3, l)}{t \ln^2 t} dt + \int_{10^{10}}^x \frac{\theta(t; 3, l)}{t \ln^2 t} dt + c_0 \frac{x}{\ln x} \\ &< \frac{x}{\ln x} \left( c_0 + \frac{\ln x}{x} \left( K + M - 10^{10} \frac{c_0}{\ln^2 10^{10}} \right) + \frac{c_0}{\ln^2 10^{10}} \ln x \right) \\ &< 0.5468 \frac{x}{\ln x}. \end{aligned}$$

- For  $10^5 \leq x \leq 10^{10}$ ,

$$\begin{aligned} \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt &< \frac{1}{2} \int_{10^5}^x \frac{dt}{\ln^2 t} + 2.072 \int_{10^5}^x \frac{dt}{\sqrt{t} \ln^2 t} \\ &= \frac{1}{2} \left( \frac{x}{\ln^2 x} - \frac{10^5}{\ln^2 10^5} + 2 \int_{10^5}^x \frac{dt}{\ln^3 t} \right) + 2.072 \int_{10^5}^x \frac{dt}{\sqrt{t} \ln^2 t}. \end{aligned}$$

Now,  $\int_a^b \frac{dt}{\sqrt{t} \ln^2 t} = \left[ \frac{2\sqrt{t}}{\ln^2 t} \right]_a^b + 4 \int_a^b \frac{dt}{\sqrt{t} \ln^3 t}$ .

Therefore

$$\begin{aligned} \pi(x; 3, l) &< \frac{1}{2} \frac{x}{\ln x} + 2.072 \frac{\sqrt{x}}{\ln x} + K \\ &\quad + \frac{1}{2} \left( \frac{x}{\ln^2 x} - \frac{10^5}{\ln^2 10^5} + 2 \int_{10^5}^x \frac{dt}{\ln^3 t} \right) \\ &\quad + 2.072 \left( \frac{2\sqrt{x}}{\ln^2 x} - \frac{2\sqrt{10^5}}{\ln^2 10^5} + 4 \int_{10^5}^x \frac{dt}{\sqrt{t} \ln^3 t} \right) \\ &< 0.55 \frac{x}{\ln x} \quad \text{for } x \geq 6 \cdot 10^5. \end{aligned}$$

**7.2. The lower bound.** Let  $KK = \min_l(\pi(10^5, 3, l) - \theta(10^5, 3, l)/\ln(10^5)) \approx 462$  and  $c = 0.498881 = \frac{1-0.002238}{2}$ .

- For  $10^{10} \leq x$ ,

$$\begin{aligned} \pi(x; 3, l) &> KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt \\ &> \frac{cx}{\ln x} \end{aligned}$$

because

$$KK > 0 \quad \text{and} \quad \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt > 0.$$

- For  $10^5 \leq x \leq 10^{10}$ .

**Lemma 14** (McCurley [2]). *For  $x \geq 91807$  and  $c_2 = 0.49585$ , we have  $\theta(x; 3, l) \geq c_2 x$ .*

*Remark.* This bound is better than the one given in Theorem 9 for  $x \leq 2.5 \cdot 10^5$ .

$$\pi(x; 3, l) > KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt.$$

Thus for any  $x_0, x_1$  with  $10^5 \leq x_0 < x_1$ ,

$$\begin{aligned} \pi(x; 3, l) &> KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^{x_0} \frac{\theta(t; k, l)}{t \ln^2 t} dt \text{ for } x \geq x_0 \\ &> \frac{x}{\ln x} \left( c_2 + \left( KK + \int_{10^5}^{x_0} \frac{\theta(t)}{t \ln^2 t} dt \right) \frac{\ln x_1}{x_1} \right) \text{ for } x_0 \leq x \leq x_1. \end{aligned}$$

Using the previous remark, we find

$$\begin{aligned} \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt &> c_2 \int_{10^5}^x \frac{dt}{\ln^2 t} \text{ if } 10^5 \leq x \leq 2.5 \cdot 10^5 \\ \text{and} & \\ &> c_2 \int_{10^5}^{2.5 \cdot 10^5} \frac{dt}{\ln^2 t} + \int_{2.5 \cdot 10^5}^x \frac{t/2 - 2.072\sqrt{t}}{t \ln^2 t} dt \text{ if } 2.5 \cdot 10^5 \leq x. \end{aligned}$$

We use this to make step by step computations with Maple:

$x_0$	$x_1$
$10^5$	$2 \cdot 10^6$
$2 \cdot 10^6$	$3 \cdot 10^7$
$3 \cdot 10^7$	$3 \cdot 10^8$
$3 \cdot 10^8$	$3 \cdot 10^9$
$3 \cdot 10^9$	$10^{10}$

We conclude that  $\pi(x; 3, l) > 0.499 \frac{x}{\ln x}$  for  $10^5 \leq x \leq 10^{10}$ .

**7.3. Small values.** We now check whether  $0.49888 \frac{x}{\ln x} < \pi(x; 3, l) < 0.55 \frac{x}{\ln x}$  for  $x < 6 \cdot 10^5$ . It is sufficient to prove that

$$\pi(p; 3, l) < 0.55 \frac{p}{\ln p} \text{ for } p \equiv l \pmod{3},$$

and if

$$0.49888 \frac{p}{\ln p} < \pi(p; 3, l) - 1 \text{ for } p \equiv l \pmod{3}.$$

The highest value not satisfying the first inequality is  $p = 229849$ , and the highest value not satisfying the second is  $p = 151$ . Furthermore,  $\pi(229869; 3, l) \leq 10241 < 0.55 \frac{229869}{\ln 229869} \approx 10241.0075$  and  $\pi(151; 3, l) \geq 16 > 0.49888 \frac{151}{\ln 151} \approx 15.01$ .

The conclusion is

$$0.49888 \frac{x}{\ln x} \underset{x \geq 151}{<} \pi(x; 3, l) \underset{x \geq 229869}{<} 0.55 \frac{x}{\ln x}.$$

*Remark.* We cannot show that  $x/(2 \ln x) < \pi(x; 3, l)$  by using the formula  $\theta(x) < c \cdot x$ . We have obtained other formulas (see Theorem 6) which we will use below.

7.4. **More precise lower bound of  $\pi(x; 3, l)$ .** Now we will give the proof of Theorem 8(i).

Classically,

$$\pi(x; 3, l) - \pi(10^5; 3, l) = \frac{\theta(x; 3, l)}{\ln(x)} - \frac{\theta(10^5; 3, l)}{\ln(10^5)} + \int_{10^5}^x \frac{\theta(t; 3, l)}{t \ln^2 t} dt.$$

Now  $\theta(t; 3, l) > \frac{x}{\varphi(3)} \left(1 - \frac{\alpha}{\ln x}\right)$  with  $\alpha = \varphi(3) \cdot 0.262$  by use of Theorem 6. So we write

$$KK = \min_l \left( \pi(10^5; 3, l) - \frac{\theta(10^5; 3, l)}{\ln(10^5)} \right),$$

$$\pi(x; 3, l) > J(x, \alpha) = KK + \frac{x}{\varphi(k) \ln x} \left(1 - \frac{\alpha}{\ln x}\right) + \frac{1}{\varphi(k)} \int_{10^5}^x \frac{1 - \alpha/\ln t}{\ln^2 t} dt.$$

The derivative of  $J(x, \alpha)$  with respect to  $x$  equals

$$\frac{1}{\varphi(k)} \left( \frac{1 - \alpha/\ln x}{\ln x} + \frac{\alpha}{\ln^3 x} \right).$$

Moreover, the derivative of  $\frac{x}{\varphi(k) \ln x}$  equals

$$\frac{1}{\varphi(k)} \left( \frac{1}{\ln x} - \frac{1}{\ln^2 x} \right).$$

The inequality

$$\frac{1}{\varphi(k)} \left( \frac{1}{\ln x} - \frac{1}{\ln^2 x} \right) < \frac{1}{\varphi(k)} \left( \frac{1 - \alpha/\ln x}{\ln x} + \frac{\alpha}{\ln^3 x} \right)$$

holds if  $\alpha - 1 < \alpha/\ln x$ ; this holds for all  $x > 1$ . The only thing to do is to find a value  $x_1$  such that

$$J(x_1, \alpha) > \frac{x_1}{\varphi(k) \ln x_1}.$$

For  $x_1 = 10^5$ ,  $J(10^5, 0.524) \approx 4607.75$  and  $\frac{10^5}{2 \ln 10^5} \approx 4342.94$ . We verify by computer that the inequality holds for  $x \leq 10^5$  and  $l = 1$  or  $2$ . We conclude that

$$\frac{x}{2 \ln x} < \pi(x; 3, l) \text{ for } x \geq 151.$$

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