ESTIMATES OF $\theta(x; k, l)$ FOR LARGE VALUES OF $x$

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Abstract. We extend a result of Ramaré and Rumely, 1996, about the Chebyshev function $\theta$ in arithmetic progressions. We find a map $\varepsilon(x)$ such that $|\theta(x; k, l) - x/\varphi(k)| < x\varepsilon(x)$ and $\varepsilon(x) = O \left( \frac{\ln x}{\ln^a x} \right)$ (for all $a > 0$), whereas $\varepsilon(x)$ is a constant. Now we are able to show that, for $x \geq 1531$,

$$|\theta(x; 3, l) - x/2| < 0.262 \frac{x}{\ln x}$$

and, for $x \geq 151$,

$$\pi(x; 3, l) > \frac{x}{2\ln x}.$$ 

1. Introduction

Let $R = 9.645908801$ and $X = \sqrt{\frac{\ln x}{R}}$. Rosser [6] and Schoenfeld [7, Th. 11 p. 342] showed that, for $x \geq 101$,

$$|\theta(x) - x|, |\psi(x) - x| < x\varepsilon(x),$$

where

$$\varepsilon(x) = \sqrt{\frac{8}{17\pi}} X^{1/2} \exp(-X).$$

We adapt their work to the case of arithmetic progressions. Let us recall the usual notations for nonnegative real $x$:

$$\theta(x; k, l) = \sum_{\substack{p \equiv l \pmod{k} \\ p \leq x}} \ln p, \quad \text{where } p \text{ is a prime number},$$

$$\psi(x; k, l) = \sum_{\substack{n \equiv l \pmod{k} \\ n \leq x}} \Lambda(n), \quad \text{where } \Lambda \text{ is Von Mangold’s function},$$

and $\varphi$ is Euler’s function. We show, for $x \geq x_0(k)$ where $x_0(k)$ can be easily computed, that

$$|\theta(x; k, l) - x/\varphi(k)|, |\psi(x; k, l) - x/\varphi(k)| < x\varepsilon(x),$$

where

$$\varepsilon(x) = 3 \sqrt{\frac{k}{\varphi(k)C_1(k)}} X^{1/2} \exp(-X).$$

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for an explicit constant $C_1(k)$. We apply the above results for $k = 3$. For small values, we use Ramaré and Rumely’s results [3]. We show that for $x \geq 1531$,

$$|\theta(x; 3, l) - x/2| < 0.262 \frac{x}{\ln x}.\quad (1)$$

If we assume that the Generalized Riemann Hypothesis is true, then we can show that, for $x > 1$ and $k \leq 432$,

$$|\psi(x; k, l) - x/\varphi(k)| < \frac{1}{4\pi} \sqrt{x} \ln^2 x.$$

Let us define, as usual, $\pi(x)$ the number of primes not greater than $x$. In 1962, Rosser and Schoenfeld ([5, p. 69]) found a lower bound for $\pi(x)$:

$$\pi(x) > \frac{x}{\ln x} \quad \text{for } x \geq 17.\quad (2)$$

Letting

$$\pi(x; k, l) = \sum_{p \leq x, p \equiv 1 \mod k} 1,$$

we show an analogous result in the case of arithmetic progression with $k = 3$ and $l = 1$ or 2,

$$\pi(x; 3, l) > \frac{x}{2 \ln x} \quad \text{for } x \geq 151.$$

This result, inferred from (1), implies (2) and cannot be proved with Ramaré and Rumely’s results.

The method used for $k = 3$ can also be applied for other fixed integers $k$.

2. Preliminary Lemmas

Notations. We will always denote by $\rho$ a nontrivial zero of Dirichlet’s function $L$, that is to say a zero such that $0 < \Re \rho < 1$. We write $\rho = \beta + i\gamma$. Let $\varphi(\chi)$ be the set of the zeros $\rho$ of the function $L(s, \chi)$, with $0 < \beta < 1$.

For a positive real $H$, following Ramaré and Rumely, we say that GRH($k,H$) hold\(^\ddagger\) if, for all $\chi$ modulo $k$, all the nontrivial zeros of $L(s, \chi)$ with $|\gamma| \leq H$ are such that $\beta = 1/2$.

As in Rosser and Schoenfeld (in [6, 7] where the case $k = 1$ is studied), we must know the distribution of $L(s, \chi)$’s zeros; namely, find a real $H$ such that GRH($k,H$) is satisfied and is a zero-free region.

2.1. Zero-free region.

**Theorem 1** (Ramaré and Rumely [3]). If $\chi$ is a character with conductor $k$, $H \geq 1000$, and $\rho = \beta + i\gamma$ is a zero of $L(s, \chi)$ with $|\gamma| \geq H$, then there exists a computable constant $C_1(\chi, H)$ such that

$$1 - \beta \geq \frac{1}{R \ln(k|\gamma|/C_1(\chi, H))}.\quad (3)$$

\(^\ddagger\)Note that our GRH is an acronym for the usual Generalized Riemann Hypothesis.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$H_k$</th>
<th>$C_1(\chi, H_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5450000000</td>
<td>38.31</td>
</tr>
<tr>
<td>3</td>
<td>10000</td>
<td>20.92</td>
</tr>
<tr>
<td>420</td>
<td>2500</td>
<td>56.59</td>
</tr>
</tbody>
</table>

Proof. See Theorem 3.6.3 of Ramaré and Rumely [3, p. 409].

Remark. For $k \geq 1$ and $H_k \geq 1000$, $C_1(\chi, H) \geq C_1(\chi_0, 1000) \geq 9.14$.

As $C_1(\chi, H)$ could be large, we limit $C_1(\chi, H)$ up to $32\pi$ to make some computations. So we have in our hypothesis

$$9.14 \leq C_1(\chi, H) \leq 32\pi.$$ 

From now on,

$$C_1(k) = \min(\min_{\chi \mod k} C_1(\chi, H_\chi), 32\pi).$$

2.2. GRH($k, H$) and $N(T, \chi)$.

Lemma 1 (McCurley [1]). Let $C_2 = 0.9185$ and $C_3 = 5.512$. Write $F(y, \chi) = \frac{y}{\pi} \ln \left( \frac{ky}{2\pi e} \right)$ and $R(y, \chi) = C_2 \ln(ky) + C_3$. If $\chi$ is a character of Dirichlet with conductor $k$, if $T \geq 1$ is a real number, and if $N(T, \chi)$ denotes the number of zeros $\beta + i\gamma$ of $L(s, \chi)$ in the rectangle $0 < \beta < 1$, $|\gamma| \leq T$, then

$$|N(T, \chi) - F(T, \chi)| \leq R(T, \chi).$$

Lemma 2 (deduced from [3] Theorem 2.1.1, p. 399 and [3]).

- GRH($1, H$) is true for $H = 5.45 \times 10^8$.
- GRH($k, H$) is true for $H = 10000$ and $k \leq 13$.
- GRH($k, 2500$) is true for sets
  $$E_1 = \{k \leq 72\},$$
  $$E_2 = \{k \leq 112, k \text{ not prime}\},$$
  $$E_3 = \{116, 117, 120, 121, 124, 125, 128, 132, 140, 143, 144, 156, 163, 169, 180, 216, 243, 256, 360, 420, 432\}.$$

2.3. Estimates of $|\psi(x; k, l) - x/\varphi(k)|$ using properties of zeros of $L(s, \chi)$.

As in Ramaré and Rumely, we remove the zeros with $\beta = 0$ and we consider only primitive $L$-series by adding small terms. Here we take the version stated in [3] Theorem 4.3.1 which is deduced from [1].

Theorem 2 (McCurley [1]). Let $x > 2$ be a real number, $m$ and $k$ two positive integers, $\delta$ a real number such that $0 < \delta < \frac{x-2}{mx}$, and $T$ a positive real. Let

$$A(m, \delta) = \frac{1}{\delta^m} \sum_{j=0}^{m} \binom{m}{j} (1 + j\delta)^{m+1}. $$
Assume \( \text{GRH}(k, 1) \). Then
\[
\frac{\varphi(k)}{x} \max_{1 \leq y \leq x} \left| \psi(y; k, l) - \frac{y}{\varphi(k)} \right| < A(m, \delta) \sum_{\chi} \sum_{\rho \in \mu(k)} \frac{x^{\beta - 1}}{\rho(\rho + 1) \cdots (\rho + m)}
\]
\[
+ \left( 1 + \frac{m\delta}{2} \right) \sum_{\chi} \sum_{\rho \in \mu(k)} \frac{x^{\beta - 1}}{|\rho|} + \frac{m\delta}{2} + \tilde{R}/x,
\]
where \( \sum_{\chi} \) denotes the summation over all characters modulo \( k \), \( \tilde{R} = \varphi(k)((f(k) + 0.5) \ln x + 4 \ln k + 13.4) \) and \( f(k) = \sum_{p \leq k} \frac{1}{p-1} \).

2.4. One more explicit form of estimates. The next lemma can be found in [3] with the difference that the authors assumed \( \text{GRH}(k, H) \) but in fact they used only \( \text{GRH}(k, 1) \). Since we must apply it with \( T > H \), we repeat the proof.

Lemma 3. Let \( \chi \) be a character modulo \( k \). Assume \( \text{GRH}(k, 1) \). Then, for any \( T \geq 1 \), we have
\[
\sum_{\substack{|\gamma| \leq T \\ \rho \in \mu(\chi)}} \frac{1}{|\rho|} \leq \tilde{E}(T)
\]
with \( \tilde{E}(T) = \frac{1}{2\pi} \ln^2(T) + \frac{\ln(\frac{T}{\pi})}{T} \ln(T) + C_2 + \frac{2}{\pi^2} \ln \left( \frac{k}{2\pi e} \right) + C_3 \ln k + C_3 \).

Proof. For \( |\gamma| \leq 1 \), we have \( \text{GRH}(k, 1) \) and so
\[
\sum_{\substack{|\gamma| \leq 1 \\ \rho \in \mu(\chi)}} \frac{1}{|\rho|} \leq \sum_{\substack{|\gamma| \leq 1 \\ \rho \in \mu(\chi)}} \frac{1}{|1/2 + i\gamma|} \leq 2N(1, \chi).
\]
For \( |\gamma| > 1 \),
\[
\sum_{\substack{1 < |\gamma| \leq T \\ \rho \in \mu(\chi)}} \frac{1}{|\rho|} \leq \int_1^T \frac{dN(t, \chi)}{t} = \int_1^T \frac{N(t, \chi)}{t^2} dt + \frac{N(T, \chi)}{T} - \frac{N(1, \chi)}{1}.
\]
Thus,
\[
\sum_{\substack{|\gamma| \leq T \\ \rho \in \mu(\chi)}} \frac{1}{|\rho|} \leq \int_1^T \frac{N(t, \chi)}{t^2} dt + \frac{N(T, \chi)}{T} + N(1, \chi).
\]

We conclude by Lemma [4] that
\[
\int_1^T \frac{N(t, \chi)}{t^2} dt \leq \int_1^T F(t, \chi) + R(t, \chi) dt
\]
\[
= \frac{1}{\pi} \int_1^T \frac{\ln(kt/(2\pi e))}{t} dt + C_2 \int_1^T \frac{\ln(kt)}{t^2} dt + C_3 \int_1^T \frac{1}{t^2} dt
\]
\[
= \frac{1}{\pi} \left[ \frac{1}{2} \ln^2 \left( \frac{kT}{2\pi e} \right) \right]_1^T
\]
\[
+ C_2 \left\{ \left[ \frac{\ln(kt)}{t} \right]_1^T + \int_1^T \frac{1}{t^2} dt \right\} + C_3 [-1/t]_1^T
\]
\[
= \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) \ln T + C_2 \left( \frac{\ln(kT)}{T} + \ln k - \frac{1}{T} + 1 \right)
\]
\[
+ C_3 (1 - 1/T).
\]
In the same way, we have an upper bound of
\[
\frac{N(T, \chi)}{T} \quad \text{with} \quad \frac{F(T, \chi) + R(T, \chi)}{T}
\]
and
\[
N(1, \chi) \quad \text{with} \quad F(1, \chi) + R(1, \chi).
\]
Finally, we obtain
\[
\sum_{\rho \in \rho(\chi)} \frac{1}{|\rho|} \leq \frac{1}{2\pi} \ln^2(T) + \frac{\ln\left(\frac{k}{2\pi}\right)}{\pi} \ln(T)
\]
\[
+ C_2 + 2 \left(\frac{1}{\pi} \ln\left(\frac{k}{2\pi}\right) + C_2 \ln k + C_3\right) - \frac{C_2}{T}.
\]

Using the facts that

- if \( \rho \) is a zero of \( L(s, \chi) \) then \( \overline{\rho} \) is zero of \( L(s, \overline{\chi}) \),
- these zeros are symmetrical with to the line \( \Re(z) = 1/2 \),

we obtain Lemma 4 by examining the proof of [3, Lemma 4.1.3].

**Lemma 4** ([3]). Let

\[
\phi_m(t) = \frac{1}{|t|^{m+1}} \exp\left(-\frac{\ln x}{R \ln(k|t|/C_1(k))}\right)
\]

with \( R = 9.645908801 \). Let \( T \geq H \). We have
\[
\sum_{\rho \in \rho(\chi)} \frac{x^\beta}{|\gamma|^m} + \sum_{\rho \in \rho(\overline{\chi})} \frac{x^\beta}{|\gamma|^m} \leq x \sum_{\rho \in \rho(\chi)} \phi_m(\gamma) + \sqrt{x} \sum_{\rho \in \rho(\overline{\chi})} \frac{1}{|\gamma|^m}.
\]

Let us rewrite Lemma 7 of [3] to adapt it to the new functions \( F(y, \chi) \) and \( R(y, \chi) \) which we use.

**Lemma 5.** Write \( N(y) = N(y, \chi) \), \( F(y) = F(y, \chi) \), and \( R(y) = R(y, \chi) \). Let \( 1 < U \leq V \) and \( \phi(y) \) be a positive and differentiable function for \( U \leq y \leq V \). Let \( (W - y)\phi'(y) \geq 0 \) for \( U < y < V \), where \( W \) does not necessarily belong to \([U, V]\). Let \( Y \) be that one of the numbers \( U, V, W \) which is not numerically the least or greatest (or is the repeated one, if two among \( U, V, W \) are equal). Take \( j = 0 \) or \( 1 \), accordingly as \( W < V \) or \( W \geq V \). Then
\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) \leq \frac{1}{\pi} \int_U^V \phi(y) \ln\left(\frac{ky}{2\pi}\right) dy + (-1)^j C_2 \int_U^V \frac{\phi(y)}{y} dy + B_j(Y, U, V),
\]
where
\[
B_0(Y, U, V) = 2R(Y)\phi(Y) + \{N(V) - F(V) - R(V)\}\phi(V)
\]
\[
- \{N(U) - F(U) + R(U)\}\phi(U),
\]
\[
B_1(Y, U, V) = \{N(V) - F(V) + R(V)\}\phi(V) - \{N(U) - F(U) + R(U)\}\phi(U).
\]
Proof. We have

$$\sum_{U<|\gamma|\leq V} \phi(|\gamma|) = \int_{U}^{V} \phi(y) dN(y)$$

$$= - \int_{U}^{V} N(y)\phi'(y)dy + N(V)\phi(V) - N(U)\phi(U).$$

• $j = 1$. We have $W > V$ and so $Y = \min(V, W) = V$. According to Theorem 1, $N(y) \geq F(y) - R(y)$.

$$\sum_{U<|\gamma|\leq V} \phi(|\gamma|) \leq \left[ (N(y) - F(y) + R(y))\phi(y) \right]_{U}^{V} + \frac{1}{\pi} \int_{U}^{V} \ln \left( \frac{ky}{2\pi} \right) \phi(y) dy$$

$$- \int_{U}^{V} R'(y)\phi(y)dy$$

because $F'(y) = \frac{1}{\pi} \left( \ln \left( \frac{ky}{2\pi} \right) + 1 \right) = \frac{1}{\pi} \ln \left( \frac{ky}{2\pi} \right)$. Moreover,

$$- \int_{U}^{V} R'(y)\phi(y)dy = -C_2 \int_{Y}^{V} \frac{\phi(y)}{y} dy.$$

• $j = 0$. We have $V > W$. Take $Y = \max(U, W)$. Split the integral at $Y$. Then $-\phi'(y) \leq 0$ for $y \in [U, Y]$ and $-\phi'(y) \geq 0$ for $y \in [Y, V]$. Replacing $N(y)$ by $F(y) - R(y)$ in the first part and by $F(y) + R(y)$ in the second part, we obtain

$$\sum_{U<|\gamma|\leq V} \phi(|\gamma|) \leq \frac{1}{\pi} \int_{U}^{V} \ln \left( \frac{ky}{2\pi} \right) \phi(y) dy + \int_{Y}^{V} R'(y)\phi(y)dy - \int_{V}^{Y} R'(y)\phi(y)dy$$

$$+ B_0(Y, U, V).$$

Moreover,

$$\int_{Y}^{V} R'(y)\phi(y)dy \leq (-1)^j C_2 \int_{U}^{V} \frac{\phi(y)}{y} dy$$

and

$$- \int_{U}^{Y} R'(y)\phi(y)dy \leq 0.$$

We want to apply Lemma 5 with $\phi = \phi_m$ defined by (5) and with $W = W_m$ being the root of $\phi'_m$. Let

$$X = \sqrt{\frac{\ln x}{R}}$$

and, for $m \geq 0$,

$$W_m = \frac{C_1(k)}{k} \exp \left( \frac{X}{\sqrt{m + 1}} \right).$$

**Corollary 1** (Corollary from Lemma 5). Under the hypothesis of Lemma 5 if moreover $\frac{2k}{k} \leq U$, then

$$\sum_{U<|\gamma|\leq V} \phi(|\gamma|) \leq \left\{ 1/\pi + (-1)^j q(Y) \right\} \int_{U}^{V} \phi(y) \ln \left( ky/2\pi \right) dy + B_j(Y, U, V),$$

where $q(y) = \frac{C_2}{y \ln (\frac{ky}{2\pi})}$. 

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Proof. The map \( y \mapsto 1/(y \ln(ky/2\pi)) \) is decreasing if \( y > 2\pi/(ke) \).

- Case \((j = 0)\), then \( Y = \max(U, W) \).

\[
\sum_{U < |\gamma| \leq V} \phi(|\gamma|) < B_0(U, W, V) + \frac{1}{\pi} \int_U^V \phi(y) \ln \left( \frac{ky}{2\pi} \right) dy + \int_Y^V R'(y) \phi(y) dy.
\]

\[
\int_Y^V R'(y) \phi(y) dy = C_2 \int_Y^V \phi(y) \frac{dy}{y} = C_2 \int_Y^V \phi(y) \ln(\frac{ky}{2\pi}) \frac{dy}{y \ln(\frac{ky}{2\pi})} \leq \frac{C_2}{Y \ln(kY/2\pi)} \int_Y^V \phi(y) \ln(\frac{ky}{2\pi}) dy.
\]

- Case \((j = 1)\), then \( Y = V \).

\[
- \int_U^V R'(y) \phi(y) dy \leq - \frac{C_2}{V \ln(kV/2\pi)} \int_U^V \phi(y) \ln(\frac{ky}{2\pi}) dy.
\]

\[\square\]

**Theorem 3.** Let \( k \geq 1 \) an integer, \( H \geq 1000 \) a real number. Assume GRH\((k, H)\).

Let \( x_0 > 2 \) be a real number, \( m \) a positive integer, and \( \delta \) a real number such that \( 0 < \delta < (x_0 - 2)/(mx_0) \) and let \( Y \) be defined as in Lemma 5. We write

\[
\hat{A}_H = \frac{1}{\pi} \int_H^\infty \phi_m(y) \ln \left( \frac{ky}{2\pi} \right) dy + C_2 \int_H^\infty \frac{\phi_m(y)}{y} dy,
\]

\[
\hat{B}_H = B_0(Y, H, \infty),
\]

\[
\hat{C}_H = \frac{1}{m \pi H^m} \left( \ln \left( \frac{kH}{2\pi} \right) + 1/m \right),
\]

\[
\hat{D}_H = \left( 2C_2 \ln(kH) + 2C_3 + \frac{C_2}{m + 1} \right) / H^{m+1}.
\]

Then for all \( x \geq x_0 \), we have

\[
\frac{\varphi(k)}{x} \max_{1 \leq y \leq x} |\psi(y; k, l)| - \frac{y}{\varphi(k)} \leq A(m, \delta) \frac{\varphi(k)}{2} \left( \hat{A}_H + \hat{B}_H + (\hat{C}_H + \hat{D}_H) / \sqrt{x} \right) + \left( 1 + \frac{m\delta}{2} \right) \frac{\varphi(k)\hat{E}(H)}{\sqrt{x}} + \frac{m\delta}{2} + \hat{R}/x.
\]

**Remark.** We find a version of Theorem 4.3.2 of [3] where \( x_0 \) is replaced by \( x \) in \( \hat{A} \) and \( \hat{B} \).

**Proof.** According to Theorem 3, we have

\[
\frac{\varphi(k)}{x} \max_{1 \leq y \leq x} |\psi(y; k, l)| - \frac{y}{\varphi(k)} < A(m, \delta) \sum_{\chi \in \chi_{\psi(k)}} \sum_{\rho \in \rho(k) \delta > H} \frac{x^{\beta-1}}{|\rho + 1 \cdots (\rho + m)|} + \left( 1 + \frac{m\delta}{2} \right) \sum_{\chi \in \chi_{\psi(k)}} \sum_{\rho \in \rho(k) \delta > H} \frac{x^{\beta-1}}{|\rho|} + \frac{m\delta}{2} + \hat{R}/x.
\]
We separately examine the different parts:

- We have

\[
\sum_{\chi} \sum_{\rho \in \rho(\chi)} \frac{x^{\beta-1}}{|\rho(\rho + 1) \cdots (\rho + m)|} \leq \sum_{\chi} \sum_{\rho \in \rho(\chi)} \frac{x^{\beta-1}}{|\gamma|^{m+1}}.
\]

By Lemma 4,

\[
\sum_{\chi} \sum_{\rho \in \rho(\chi)} \frac{x^{\beta-1}}{|\gamma|^{m+1}} = \frac{1}{2} \left( \sum_{\rho \in \rho(\chi)} \frac{x^{\beta-1}}{|\gamma|^{m+1}} + \sum_{\rho \in \rho(\chi)} \frac{x^{\beta-1}}{|\gamma|^{m+1}} \right)
\leq \frac{1}{2} \sum_{\chi} \left( \sum_{\rho \in \rho(\chi)} \phi_{\gamma}(\rho) + \frac{1}{\sqrt{x}} \sum_{\rho \in \rho(\chi)} \frac{1}{|\gamma|^{m+1}} \right).
\]

Using Lemma 5 with \(U = H, V = 1, \phi = \phi_{m}, W = W_{m}, \)

\[
\sum_{\rho \in \rho(\chi)} \phi_{\gamma}(\rho) \leq \hat{A}_{H} + \hat{B}_{H}.
\]

Integration by parts gives

\[
\sum_{\rho \in \rho(\chi)} \frac{1}{|\gamma|^{m+1}} \leq \hat{C}_{H} + \hat{D}_{H}.
\]

- By GRH\((k, H)\) we have \(\beta = 1/2\) for all \(|\gamma| \leq H\), and by Lemma 3

\[
\sum_{\rho \in \rho(\chi)} \frac{x^{\beta-1}}{|\rho|} \leq \hat{E}(H)/\sqrt{x}.
\]

2.5. The leading term \((\hat{A}_{H})\). To obtain an upper bound for the leading term, we proceed like Rosser and Schoenfeld with upper bounds on the integrals. The next three lemmas are issued directly from [6, p. 251-255].

**Lemma 6 (Functions of incomplete Bessel type).** Let

\[
K_{\nu}(z, u) = \frac{1}{2} \int_{u}^{\infty} t^{\nu-1}H^{2}(t)dt,
\]

where \(z > 0, u \geq 0, \) and

\[
H^{2}(t) = \{H(t)\}^{2} = \exp\left\{-\frac{z}{2}(t + 1/t)\right\}.
\]

Further, write \(K_{\nu}(z, 0) = K_{\nu}(z)\). Then

\[
K_{1}(z) \leq \sqrt{\frac{\pi}{2z}} \exp(-z)\left(1 + \frac{3}{8z}\right),
\]

\[
K_{2}(z) \leq \sqrt{\frac{\pi}{2z}} \exp(-z)\left(1 + \frac{15}{8z} + \frac{105}{128z^{2}}\right).
\]

**Lemma 7.**

\[
K_{\nu}(z, x) + K_{-\nu}(z, x) = K_{\nu}(z).
\]

Hence, \(K_{\nu}(z, x) \leq K_{\nu}(z) (\nu \geq 0)\).
Lemma 8. Let
\[ Q_\nu(z,x) = \frac{x^{\nu+1}}{z(x^2 - 1)} \exp\{-z(x + 1)/2\}. \]
If \( z > 0 \) and \( x > 1 \), then
\[ K_1(z,x) < Q_1(z,x) \]
and
\[ K_2(z,x) < (x + 2/z)Q_1(z,x). \]

The term \( \tilde{A}_H \) can be expressed using incomplete Bessel functions.

Lemma 9. Let \( X \) be defined by (6). Let \( z_m = 2X\sqrt{m} = 2\sqrt{\frac{m\ln x}{R}} \) and \( U_m = \frac{2m}{z_m} \ln \left( \frac{kH}{C_1(k)} \right)^{m+1} \)
\[ \tilde{A}_H = \frac{2 \ln x}{\pi Rm} \left( \frac{k}{C_1(k)} \right)^m \left( \frac{k}{C_1(k)} \right)^m \exp \left( \frac{z_m + 1}{2} - \frac{t}{2} \right) \]
and
\[ K_m(z_m, U_m) \]
\[ K_{m+1}(z_m, U_{m+1}) \]

Proof. This is by straightforward algebraic manipulation; for example, we write
\[ I = \int_H^\infty \frac{C_2}{y^{m+1}} \exp \left( -\frac{\ln x}{R\ln(ky/C_1(k))} \right) dy. \]
Changing variables:
\[ t = \sqrt{\frac{R(m+1)}{\ln x}} \ln \left( \frac{ky}{C_1(k)} \right), \]
\[ dt = \sqrt{\frac{R(m+1)}{\ln x}} \frac{dy}{y}. \]
Now
\[ \exp \left( -\frac{\ln x}{R\ln(ky/C_1(k))} \right) = \exp \left( -\frac{\ln x}{R t/\sqrt{\frac{(m+1)\ln x}{R}}} \right) = \exp \left( \frac{-z_{m+1} + 1}{2} \right) \]
and
\[ \frac{1}{y^{m+1}} = \left( \frac{k}{C_1(k)} \right)^{m+1} \exp \left( -\frac{(m+1)t}{R(m+1)/\ln x} \right) = \left( \frac{k}{C_1(k)} \right)^{m+1} \exp \left( -\frac{z_{m+1}}{2} \right) \]
Consequently,
\[ I = \int_{U_{m+1}}^{\infty} C_2 \sqrt{\frac{\ln x}{R(m+1)}} \left( \frac{k}{C_1(k)} \right)^{m+1} \exp \left( \frac{-z_{m+1}}{2}(t + 1/t) \right). \]
2.6. Study of $f(k)$ which appears in the expression of $\tilde{R}$. Remember that
$f(k) = \sum_{p \mid k} \frac{1}{p}$.

Lemma 10. For an integer $k \geq 1$,

$$f(k) \leq \frac{\ln k}{\ln 2}.$$  

Proof. We prove by recursion that

$$f(k) \leq \frac{\ln k}{\ln 2}.$$  

For $k = 1$, it is obvious. For $k = 2$, $f(k) = 1 \leq \frac{\ln 2}{\ln 2}$. Assume $f(k) \leq \frac{\ln k}{\ln 2}$ holds for $k \leq n$. Find an upper bound for $f(n+1)$. If $(n+1)$ is prime, then $f(n+1) = 1/n \leq \ln n/\ln 2$. If $(n+1)$ is not prime, then there exists $p \leq n$, which divides $n$. If $p = 2$ and $2^\alpha \parallel n+1$,

$$f(n+1) = f\left(\frac{n+1}{2^\alpha} \cdot 2^\alpha\right) = f\left(\frac{n+1}{2^\alpha}\right) + f(2)$$

$$= 1 + f\left(\frac{n+1}{2^\alpha}\right) \leq \frac{\ln(n+1)}{\ln 2} + 1 - \frac{\ln 2}{\ln 2}$$

$$\leq \frac{\ln(n+1)}{\ln 2}.$$  

If $p > 2$ and $p^\alpha \parallel n+1$,

$$f(n+1) = f\left(\frac{n+1}{p^\alpha} \cdot p^\alpha\right) = f\left(\frac{n+1}{p^\alpha}\right) + f(p)$$

$$= \frac{1}{p-1} + f\left(\frac{n+1}{p^\alpha}\right) \leq \frac{\ln(n+1)}{\ln 2} + 1 - \frac{\ln p}{\ln 2}$$

$$\leq \frac{\ln(n+1)}{\ln 2} \text{ because } \frac{1}{p-1} - \frac{\ln p}{\ln 2} < 0 \text{ for } p > 2.$$

\[
\]

3. The method with $m = 1$

Theorem 4. Let $k$ be an integer, $H \geq 1250$, and $H \geq k$. Assume GRH$(k, H)$. Let $C_1(k)$ defined by (3). Let $x > 1$. Write $X = \sqrt{\frac{\ln x}{H}}$ and

$$\varepsilon(x) = 2\sqrt{\frac{k \varphi(k)}{C_1(k) \sqrt{x}} \left(1 + \frac{1}{2x} (15/16 + \ln(C_1(k)/(2\pi)))\right) X^{3/4} \exp(-X).$$

If $\varepsilon(x) \leq 0.2$ and $X \geq \sqrt{2} \ln \left(\frac{kH}{\varepsilon_1(k)}\right)$, then

$$\max_{1 \leq y \leq x} |\psi(y; k, 1) - y/\varphi(k)| \leq x \varepsilon(x)/\varphi(k).$$

Proof. Take $m = 1$ in Theorem 3. Assuming $X \geq \sqrt{2} \ln \left(\frac{kH}{C_1(k)}\right)$, then $W_1 \geq H$. In this situation, $Y = W_1$ and $\tilde{B}_H < 2R(W_1)\phi_1(W_1)$. For $y > 1$, $R(y)/\ln y$ is
ESTIMATES OF $\theta(x; k, l)$ FOR LARGE VALUES OF $x$

decreasing; hence,

$$
\tilde{B}_H < 2R(W_1)\phi_1(W_1) < 2\frac{R(H)}{\ln H}\phi_1(W_1)\ln W_1
= 2\frac{R(H)}{\ln H}\left(\frac{X}{\sqrt{2}} + \ln \left(\frac{C_1(k)}{k}\right)\right)\phi_1(W_1)
= 2\frac{R(H)}{\ln H}\left(\frac{X}{\sqrt{2}} + \ln \left(\frac{C_1(k)}{k}\right)\right)(k/C_1(k))^2 \exp(-2\sqrt{2}X).
$$

Inserting the upper bounds (12) and (13) into the bound for $\tilde{A}_H$ in Lemma 9,

$$
\tilde{A}_H < 2\left(k\frac{C_1(k)}{C_1(k)}\right)\left[\frac{\pi}{4X}\exp(-2X)\left(1 + \frac{15}{16X} + \frac{105}{512X^2}\right)X^2/\pi
+ \frac{1}{2}\ln \frac{C_1(k)}{2\pi}X\frac{\pi}{4X}\exp(-2X)\left(1 + \frac{3}{16X}\right)
+ C_2\frac{kX}{C_1(k)\sqrt{2}}\frac{\pi}{4\sqrt{2}X}\exp(-2\sqrt{2}X)\left(1 + \frac{3}{16X}\right)\right].
$$

Put

$$
F_1 := \frac{1}{\sqrt{\pi}}\frac{k}{C_1(k)}X^{3/2}\exp(-2X)\left[1 + \left(\frac{15}{16} + \ln \frac{C_1(k)}{2\pi}\right)\frac{1}{2X}\right]^2.
$$

In Lemma 11 below it is shown that

$$
\tilde{A}_H + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H))/\sqrt{x} + \tilde{R}\frac{2}{x\varphi(k)} < F_1.
$$

We must choose $\delta$ to minimize

$$
\frac{A(1, \delta)}{2}\varphi(k)F_1 + \delta/2.
$$

Write $f = \varphi(k)F_1$. As $A_1(\delta) = (\delta^2 + 2\delta + 2)/\delta$, we must minimize $g(\delta) = (\delta^2 + 2\delta + 2)/\delta + \delta/2$. The minimum value here is at $\delta = \sqrt{2f(1+f)}$, and the value there is $g(\sqrt{2f(1+f)}) = f + \sqrt{2f(1+f)}$.

It is a simple matter to prove that for $0 < f < 0.202$,

$$
f + \sqrt{2f(1+f)} < 2\sqrt{f}.
$$

As $X > X_0 := \sqrt{2}\ln \left(\frac{kH}{C_1(k)}\right)$, then $x_0 \geq \exp(122.5)$, and it is obvious that $\delta$ meets the hypothesis $0 < \delta < (x_0 - 2)/x_0$ in Theorem 3 since

$$
0 < \delta < \sqrt{2}\sqrt{f} < 0.6357 < \frac{x_0}{x_0 - 2}.
$$

Lemma 11.

$$
\tilde{A}_H + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H))/\sqrt{x} + \tilde{R}\frac{2}{x\varphi(k)} < F_1.
$$
Proof. First we prove that $\tilde{A}_H + \tilde{B}_H < F_1$:

$$F_1 = \frac{k}{C_1(k) \sqrt{\pi}} X^{3/2} e^{-2X} \left( 1 + (15/16 + \ln(C_1(k)/2\pi))/X \right.$$

$$\left. + (225/1024 + 15/32 \ln(C_1(k)/2\pi) + 1/4 \ln^2(C_1(k)/2\pi))/X^2 \right),$$

$$\tilde{A}_H < \frac{k}{C_1(k) \sqrt{\pi}} X^{3/2} e^{-2X} \left( 1 + \frac{15}{16} X + \frac{105}{512} X^2 + \ln \left( \frac{C_1(k)}{2\pi} \right) \left( \frac{1}{X} + \frac{3}{16X^2} \right) \right.$$

$$\left. + C_2 \frac{k \pi}{C_1(k) \sqrt{2 \pi}} \exp(-2(\sqrt{2} - 1)X)(1/X + 3/(16\sqrt{2}X^2)) \right),$$

$$\tilde{B}_H < \frac{k}{\sqrt{\pi} C_1(k)} X^{3/2} \exp(-2X) \exp(-2(\sqrt{2} - 1)X)$$

$$\times \left[ \frac{2k \sqrt{\pi}}{C_1(k) \ln H} (C_2 \ln(kH) + C_3) \left( \frac{1}{\sqrt{2X}} + \frac{1}{X \sqrt{X}} \ln(C_1(k)/k) \right) \right].$$

This yields $F_1 - \tilde{A}_H - \tilde{B}_H > 0$ if

$$F_2 := \frac{1}{X^2} \left( \frac{15}{1024} + \frac{9}{32} \ln \left( \frac{C_1(k)}{2\pi} \right) + \frac{1}{4} \ln^2 \left( \frac{C_1(k)}{2\pi} \right) \right)$$

$$\times \frac{C_2 \sqrt{\pi} k}{C_1(k)} \exp(-2(\sqrt{2} - 1)X) \frac{1}{\sqrt{2X}}$$

$$\times \left[ \sqrt{\frac{\pi}{2 \sqrt{2}}} \left( \sqrt{\frac{3}{2X 16X^{3/2}}} \right) \right.$$

$$\left. + 2 \left( 1 + \ln k + \frac{C_3/C_2}{\ln H} \right) \left( 1 + \frac{\sqrt{2} \ln C_1(k)}{k} \right) \right].$$

This holds if we can show that

$$F_2 > \frac{C_2 k \sqrt{\pi}}{C_1(k)} \exp(-2(\sqrt{2} - 1)X) \frac{1}{\sqrt{2X}} \cdot 16.9,$$

since $C_1(k) \leq 32\pi$, $H \geq 1250$, $X \geq \sqrt{2} \ln(1250/32\pi)$, and $k \leq H$.

It remains to be proved that

$$\frac{\sqrt{2} C_1(k)}{k C_2 \sqrt{\pi} \cdot 16.9} (15/1024 + \cdots) > X^{3/2} \exp(-2(\sqrt{2} - 1)X).$$

But for $X \geq X_0 := \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right)$,

$$X^{3/2} \exp(-2(\sqrt{2} - 1)X) < X_0^{3/2} \left( \frac{kH}{C_1(k)} \right)^{(1+a)}$$

$$= \frac{1}{k} \cdot 2^{3/4} \left( \frac{C_1(k)}{H} \right)^{1+a} \left( \frac{\ln^{3/2}(kH/C_1(k))}{k^a} \right),$$

where $a = 2\sqrt{2}(\sqrt{2} - 1) - 1 \approx 0.17157$. The map $k \mapsto \frac{\ln^{3/2}(kH/C_1(k))}{k^a}$ reaches its maximum for $k = e^{\frac{1}{2a} C_1(k)/H}$. Hence

$$X^{3/2} \exp(-2(\sqrt{2} - 1)X) < \frac{C_1(k)}{kH} 2^{3/4} \left( \frac{3}{2a} \right)^{3/2} / e^{3/2}.$$
We must compare
\[ \frac{\sqrt{2}}{C_2 \sqrt{\pi} \cdot 16.9} \left( 15/1024 + \cdots \right) \text{ with } \frac{2^{3/4} (\frac{2}{\pi})^{3/2}}{He^{3/2}}. \]

Since \( C_1(k) \geq 9.14 \) (see the remark above (3)) and \( C_2 = 0.9185 \), it remains to be proved that
\[ 0.007976 \geq \frac{2^{3/4} (\frac{2}{\pi})^{3/2}}{He^{3/2}} (\approx 0.00776), \]
which is true since \( H \geq 1250 \).

We show below that the remaining terms \((\hat{C}_H + \hat{D}_H + 3\hat{E}(H))/\sqrt{x} + \frac{2R}{\varphi(k)}\) are negligible.

\[ \bullet \] We will find an upper bound for \( A(1, \delta) \frac{2}{\varphi(k)}(\hat{C}_H + \hat{D}_H) + \frac{3}{2} \frac{\varphi(k)}{\sqrt{\pi}} \frac{\hat{E}(H)}{\sqrt{x}} + \frac{\hat{R}}{x}. \]

We assume that \( X \geq \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right) \); hence, \( X \geq X_0 := \sqrt{2} \ln \left( \frac{1250}{32 \pi} \right) \approx 3.5644 \). It is straightforward but tedious to check that

\[ \text{Rest} := \hat{C}_H + \hat{D}_H + 3\hat{E}(H) + \frac{2R}{\varphi(k) \sqrt{x}} \leq \begin{cases} 1250(\ln H \ln k)^2 & \text{if } k \neq 1, \\
1250(\ln H)^2 & \text{if } k = 1. \end{cases} \]

Let us consider the case \( k \neq 1 \). As \( X \geq \sqrt{2} \ln \left( \frac{kH}{C_1(k)} \right) \),
\[ \exp \left( \frac{X}{\sqrt{2}} \right) \geq \frac{kH}{C_1(k)}. \]

This yields
\[ \text{Rest} \leq 1250(\ln H \ln k)^2 \leq 1250 \left( \frac{\ln H \ln k}{C_1(k)} \right)^2 \exp(X \sqrt{2}) \]
\[ \leq 1250C_1^2(k) \frac{1}{e^2} \left( \frac{\ln 1250}{1250} \right)^2 \exp(X \sqrt{2}) \]
\[ \leq K \exp(X \sqrt{2}) \text{ because } C_1(k) \leq 32\pi, \]
where \( K := 55.65 \). Now compare
\[ \frac{K \exp(X \sqrt{2})}{\sqrt{x}} = K \exp(X \sqrt{2} - RX^2/2) \]
with the term involving \( 1/X^2 \) in \( F_1 \)
\[ \frac{1}{X^2} \times \frac{k}{C_1(k) \sqrt{\pi}} X^{3/2} \exp(-2X). \]

We may compute \( c \) such that
\[ K \exp(X \sqrt{2} - RX^2/2) \leq c \times \frac{1}{X^2} \times \frac{k}{C_1(k) \sqrt{\pi}} X^{3/2} \exp(-2X) \]
\[ \Leftrightarrow c \geq K \times 32\pi \sqrt{\pi} \exp(X \sqrt{2} - RX^2/2 + 2X) \times \frac{X^2}{X^{3/2}} \]
\[ \Leftrightarrow c \geq 0.7 \cdot 10^{-18} \text{ for } X \geq X_0. \]

Thus, the rest is negligible and absorbed by rounding up the constants. \( \square \)
4. The method with \( m = 2 \)

**Lemma 12.** Let \( A(m, \delta) \) be defined as in formula (4). Write \( R_m(\delta) = (1 + (1 + \delta)^{m+1})^m \).

Then \( A(m, \delta) \leq \frac{R_m(\delta)}{\delta^m} \).

**Proof.** The proof appears in [4, p. 222].

**Theorem 5.** Let an integer \( k \geq 1 \). Remember that \( R = 9.645908801 \). Assume GRH \((k, H)\). Let \( C_1(k) \) be defined by (3). Let \( X_0, X_1, X_2, \) and \( X_3 \) be such that

\[
\begin{align*}
\frac{e^{X_0}}{\sqrt{X_0}} &= H \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}, \\
\frac{e^{X_1}}{C_1(k)\varphi(k)} &= 10\varphi(k), \\
X_2 &= kC_1(k)/(2\pi\varphi(k)), \\
X_3 &= \frac{2k\pi e}{C_1(k)\varphi(k)}.
\end{align*}
\]

Let \( X_4 := \max(10, X_0, X_1, X_2, X_3) \). Write

\[
\varepsilon(X) = 3\sqrt{\frac{k}{\varphi(k)C_1(k)}}X^{1/2}\exp(-X).
\]

Then for all real \( x \) such that \( X = \sqrt{\frac{\ln x}{R}} \geq X_4 \), we have

\[
\begin{align*}
\max_{1 \leq y \leq x} |\psi(y; k, l) - y/\varphi(k)| &< x\varepsilon \left( \frac{\ln x}{\sqrt{R}} \right), \\
\max_{1 \leq y \leq x} |\theta(y; k, l) - y/\varphi(k)| &< x\varepsilon \left( \frac{\ln x}{\sqrt{R}} \right).
\end{align*}
\]

**Corollary 2.** With the notations and the hypothesis of Theorem 5 let \( X_5 \geq X_4 \) and \( c := \varepsilon(X_5) \). For \( x \geq \exp(RX_5^2) \), we have

\[
|\psi(x; k, l) - x/\varphi(k)|, \quad |\theta(x; k, l) - x/\varphi(k)| < cx.
\]

**Proof.** The idea is to judiciously split the integral into two parts, and bound each part optimally, using an \( m = 0 \) estimate in the first part and an \( m = 2 \) estimate in the second part.

We want to split the integral at \( T \), where \( T \) will optimally be chosen later. We take \( T \) in the same form as \( W_m \) (formula (11)):

\[
T := \frac{C_1(k)}{k} \exp(\nu X),
\]

where \( \nu \) is a parameter.

Assume that \( T \geq H \) and \( 1/\sqrt{m+1} \leq \nu \leq 1 \). Hence \( W_m \leq T \leq W_0 \). This last hypothesis is needed to apply Corollary [1].

We use Theorem [2] and split the sums at \( T \):

\[
A(m, \delta) \sum_{\chi} \sum_{\rho \in \chi(k)} \frac{x^{\beta-1}}{|\rho|^{(p+1)\cdots(p+m)}} + \left( 1 + \frac{m\delta}{2} \right) \sum_{\chi} \sum_{\rho \in \chi(k)} \frac{x^{\beta-1}}{|\rho|} + \frac{m\delta}{2} + \frac{\tilde{R}}{x}.
\]
Define

\[ \hat{A}_1 := \sum_{\chi} \sum_{\rho \in \rho(y)} \frac{x^{\beta - 1}}{|\rho|}, \]

\[ \hat{A}_2 := \sum_{\chi} \sum_{\rho \in \rho(y), |\gamma| > T} \frac{x^{\beta - 1}}{|\rho(\rho + 1) \cdots (\rho + m)|}. \]

Bounding the term \( \hat{A}_1 \), we get

\[
\hat{A}_1 = \sum_{\chi} \left( \sum_{\rho \in \rho(y) \gamma \leq H} \frac{x^{\beta - 1}}{|\rho|} + \sum_{\rho \in \rho(y), H < \gamma \leq T} \frac{x^{\beta - 1}}{|\rho|} \right)
= \frac{1}{x} \sum_{\chi} \left( \sum_{\rho \in \rho(y) \gamma \leq H} \sqrt{x} \frac{x^{\beta}}{|\rho|} + \sum_{\rho \in \rho(y), H < \gamma \leq T} \frac{x^{\beta}}{|\rho|} \right) \text{ by GRH}(k, H)
= \frac{1}{\sqrt{x}} \sum_{\chi} \sum_{\rho \in \rho(y) \gamma \leq H} \frac{1}{|\rho|} + \frac{1}{2x} \sum_{\chi} \left( \sum_{\rho \in \rho(y), H < \gamma \leq T} x\phi_0(\gamma) + \sqrt{x} \sum_{\rho \in \rho(y), H < \gamma \leq T} \frac{1}{|\gamma|} \right)
\leq \frac{1}{\sqrt{x}} \varphi(k) E(H) + \frac{1}{2 \sqrt{x}} \sum_{\chi} \left( \sum_{\rho \in \rho(y), H < \gamma \leq T} x\phi_0(\gamma) + \sqrt{x} \sum_{\rho \in \rho(y), H < \gamma \leq T} \frac{1}{|\gamma|} \right)
\]

by Lemmas 3 and 4

\[ \leq \varphi(k) E(T) / \sqrt{x} + \frac{1}{2} \sum_{\chi} \sum_{\rho \in \rho(y), H < \gamma \leq T} \phi_0(\gamma). \]

Apply Corollary 1 ( \( j = 1, m = 0 \) ) for the interval \([H, T]\) with \( \phi = \phi_0 \) and \( W = W_0 \)

\[ \sum_{\rho \in \rho(y) H < \gamma \leq T} \phi_0(\gamma) = \{1/\pi - q(T)\} \int_H^T \phi_0(y) \ln (ky/2\pi) dy + B_1(T, H, T). \]

Moreover, \( B_1(T, H, T) < 2R(T)\phi_0(T) \).

We want to find an upper bound for

\[ I_1 := \frac{1}{\pi} \int_H^T \phi_0(y) \ln \left( \frac{ky}{2\pi} \right) dy. \]

Write \( V'' = X^2 / \ln \left( \frac{kT}{C_1(k)} \right) = X/\nu = Y'' + 2Y - \nu X \), where \( Y'' := X(1 - \nu)^2 / \nu \).

Write \( U'' = X^2 / \ln \left( \frac{kH}{C_1(k)} \right) \) and \( \Gamma(\alpha, x) = \int_x^\infty e^{-u} u^{\alpha - 1} du \). Now

\[ \int_H^T \ln \left( \frac{ky}{2\pi} \right) \phi_0(y) dy = \int_H^T \ln \left( \frac{ky}{2\pi} \right) \exp \left( -X^2 / \ln \left( \frac{ky}{C_1(k)} \right) \right) dy \]

\[ = X^4 \{ \Gamma(-2, V'') - \Gamma(-2, U'') \}
+ X^2 \ln \left( \frac{C_1(k)}{2\pi} \right) \{ \Gamma(-1, V'') - \Gamma(-1, U'') \} \]
by making the change of variables \( y = \frac{C_1(k)}{k} \exp(X^2/u) \). Now if \( \alpha \leq 1 \) and \( x > 0 \), then \( \Gamma(\alpha, x) \leq x^{\alpha-1} \int_x^{\infty} e^{-t} dt = x^{\alpha-1} e^{-x} \). Hence,

\[
\int_T^H \ln \left( \frac{ky}{2\pi} \right) \phi_0(y) dy \leq X^4 V''^3 e^{-V''} + X^2 \ln \left( \frac{C_1(k)}{2\pi} \right) V'' - 2 e^{-V''}.
\]

This yields

\[
I_1 \leq \frac{1}{\pi} X^2 \left( X^2 V''^3 + \ln \left( \frac{C_1(k)}{2\pi} \right) V'' - 2 \right) e^{-V''}
\]

\[
= \frac{1}{\pi} e^{-V''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) \left( \frac{X^4}{(X/\nu)^3} + \frac{dX^2}{(X/\nu)^2} \right)
\]

\[
= \frac{1}{\pi} e^{-V''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) XG_0,
\]

where \( d := \ln \left( \frac{C_1(k)}{2\pi} \right) \) and \( G_0 := \nu^2 (\nu + d/X) \). With the help of Corollary 1, we write

\[
\tilde{A}_1 \leq \varphi(k) \tilde{E}(T)/\sqrt{X} + \varphi(k) \left\{ \frac{1}{\pi} e^{-V''} e^{-2X} \left( \frac{kT}{C_1(k)} \right) XG_0 + 2R(T)\phi_0(T) \right\}.
\]

Bounding the term \( \tilde{A}_2 \), we get

\[
\tilde{A}_2 = \frac{1}{x} \sum \sum r \in \mathbb{P}(\chi) |\rho(r + 1) \cdots (r + m)|
\]

\[
= \frac{1}{2x} \sum \sum r \in \mathbb{P}(\chi) \left( \frac{x^\beta}{|\gamma|^{m+1}} + \frac{x^\beta}{|\gamma|^{m+1}} \right)
\]

by Lemma 4.

By using Corollary 1 (\( j = 0 \)) on \([U, V] = [T, \infty)\),

\[
\sum_{r \in \mathbb{P}(\chi) \ |\gamma| > T} \phi_m(\gamma) \leq \{1/\pi + q(T)\} \int_T^{\infty} \phi_m(y) \ln \left( \frac{ky}{2\pi} \right) dy + B_0(T, T, \infty).
\]

We have

\[
B_0(T, T, \infty) < 2R(T)\phi_m(T).
\]

Moreover,

\[
\sum_{r \in \mathbb{P}(\chi) \ |\gamma| > T} \frac{1}{|\gamma|^{m+1}} \leq \tilde{C} + \tilde{D}.
\]
Let us study more precisely

\[ I_2 := \int_T^\infty \phi_m(y) \ln \left( \frac{ky}{2\pi} \right) dy \]

\[ = \frac{\zeta}{2m^2} \left( \frac{k}{C_1(k)} \right)^m \left( K_2(z_m, U_m) + \frac{2md}{z_m} K_1(z_m, U_m) \right), \]

where \( d = \ln \left( \frac{G(x)}{2\sqrt{3}} \right) \) and \( U' := U_m = \frac{2m}{z_m} \ln \left( \frac{kT}{C_1(k)} \right) = \nu \sqrt{m}. \) Now, by writing \( z = z_m \) and using Lemma 8,

\[ K_2(z, U') + \frac{2dm}{z} K_1(z, U') < (U' + 2z + 2dm/z)Q_1(z, U') \]

\[ \leq \sqrt{m} \left( \nu + \frac{1+dm}{mX} \right) \frac{U'^2}{z(U'^2-1)} e^{-\nu(U'+1/U')} \]

But \( \frac{1}{\nu}(U' + 1/U') = X \sqrt{m(\nu \sqrt{m} + 1/(\nu \sqrt{m}))} = mX + X/\nu = mX + (Y'' + 2X - \nu X), \) where \( Y'' = X(1-\nu)^2/\nu. \) Hence

\[ K_2(z, U') + \frac{2dm}{z} K_1(z, U') = G_1 e^{Y''} \frac{m}{2(m-1)} X^{-1} e^{-2X} \left( \frac{kT}{C_1(k)} \right)^{-(m-1)}, \]

where \( G_1 := \frac{m-1}{m} \frac{U'^2}{U'-1} \left( \nu + \frac{1+dm}{mX} \right) \) because

\[ e^{\nu X(m-1)} = \left( \frac{kT}{C_1(k)} \right)^{m-1} \]

and \( \frac{1}{\nu} = \frac{1}{2}X^{-1}. \) This yields

\[ I_2 = \int_T^\infty \phi_m(y) \ln(ky/2\pi) dy < \frac{G_1 e^{Y''}}{m-1} \frac{k}{C_1(k)} X e^{-2X T^{-(m-1)}} \]

Let \( G_2 := \frac{R_m(\delta)}{2\delta}(1 + \pi q(T)). \) So, by using Lemma 12,

\[ A(m, \delta) \frac{\varphi(k)}{2} \left( 1/\pi + q(T) \right) \int_T^\infty \phi_m(y) \ln(ky/2\pi) dy \]

\[ < \left( \frac{2}{\nu} \right)^{m-1} \frac{\varphi(k)}{2} \left( \frac{G_2}{\pi (m-1) C_1(k)} X e^{-2X T^{-(m-1)}} \right). \]

The results above yield

\[ (1 + m\delta/2)\tilde{A}_1 + A(m, \delta)\tilde{A}_2 \]

(15) \[ < \frac{XG_2 e^{-2X} e^{-Y''} \varphi(k)}{2\pi} \left( \frac{k}{C_1(k)} \right) \left( \frac{G_1}{m-1} T^{-(m-1)} \left( \frac{2}{\nu} \right)^m + G_0 T \right) \]

because \( 1 + m\delta/2 < R_m(\delta)/2 < G_2, \) with

\[ r = \varphi(k)(1 + m\delta/2)R(T)\phi_0(T) + A(m, \delta)\varphi(k)R(T)\phi_m(T) \]

\[ + \frac{\varphi(k)}{\sqrt{x}} ((1 + m\delta/2)\tilde{E}(T) + A(m, \delta)(\tilde{C}_T + \tilde{D}_T)/2). \]

Suppose \( G_0/G_1 \) were independent of \( \nu; \) then the expression between braces in (15) would be minimized for

\[ T = (G_1/G_0)^{1/m} \cdot \frac{2}{\delta}. \]
With this choice, 
\[ \frac{G_1}{m-1} T^{-(m-1)} \left( \frac{2}{\delta} \right)^m + G_0 T = \frac{m}{m-1} G_1^{1/m} G_0^{1-1/m} \frac{2}{\delta} \]
and we obtain \((G_2 > 1)\)
\[ \varepsilon_1 := (1 + m\delta/2) \bar{A}_1 + A(m, \delta) \bar{A}_2 + \frac{1}{2} m\delta + \frac{\bar{R}}{x} \]
\[ < \frac{1}{2} m G_2 \left\{ X e^{-2X} e^{-Y''} \frac{2k\varphi(k)}{(m-1)\pi C_1(k)} G_1^{1/m} G_0^{1-1/m} + \delta \right\} + r + \frac{\bar{R}}{x} . \]
The expression between braces can be minimized by choosing
\[ \delta = \left\{ G_0^{1-1/m} G_1^{1/m} e^{-Y''} \frac{2k\varphi(k)}{(m-1)\pi C_1(k)} \right\}^{1/2} X^{1/2} e^{-X} . \]
Hence, we write (by replacing the above value of \(\delta\) in \(16\))
\[ T = \left( \frac{G_1}{G_0} \right)^{1/2m} \left( \frac{2C_1(k)}{k\varphi(k)} \right)^{1/2} X^{-1/2} e^X \]
and
\[ \varepsilon_1 < G_2 \left( G_0^{1-1/m} G_1^{1/m} e^{-Y''} \frac{2k\varphi(k)}{\pi C_1(k)} \right)^{1/2} \frac{m}{\sqrt{m-1}} X^{1/2} e^{-X} + r + \frac{\bar{R}}{x} . \]
The value \(m = 2\) minimizes the expression \(\frac{m}{\sqrt{m-1}}\). For the remainder of the argument, we fix \(m = 2\).

We now have two definitions for \(T\). On the one hand (equation \(18\)),
\[ T = \left( \frac{G_1}{G_0} \right)^{1/4} e^{Y''/2} \sqrt{\frac{2\pi C_1(k)}{k\varphi(k)}} X^{-1/2} e^X \]
with \(Y'' = X(1-\nu)^2/\nu\), and on the other hand (equation \(14\))
\[ T = \frac{C_1(k)}{k} \exp(\nu X) . \]
These two equations are compatible if and only if there exists \(\nu\) such that \(f(\nu) = 1\),
where
\[ f(\nu) = \frac{C_1(k)\varphi(k)}{2\pi k} \left( \frac{G_0^3}{G_1^2} \right)^{1/2} X e^{-X(1-\nu^2)/\nu} e^{-2X(1-\nu)} . \]
Here we have \(m = 2\) and our assumption \(1/\sqrt{m+1} \leq \nu \leq 1\) gives \(1/\sqrt{3} \leq \nu \leq 1\). Note that
\[ G_0 = \nu^2(\nu + d/X) , \]
\[ G_1 = \frac{m-1}{m} \frac{U''}{U'^2-1} \left( \nu + \frac{1+dm}{mX} \right) = \frac{\nu^2}{2\nu^2-1} \left( \nu + \frac{1+2d}{2X} \right) . \]
It is easy to check that on the interval \(1/\sqrt{3} \leq \nu \leq 1\), \(G_0^3/G_1\) is increasing, and hence, \(f(\nu)\) is strictly increasing. Moreover, \(\lim_{\nu \to (1/\sqrt{3})^+} f(\nu) = 0\) and \(f(1) > 1\)
(for all \( X \geq \frac{2\pi k}{C_1(k)\varphi(k)} \)). So there exists a unique \( \nu \in [1/\sqrt{2}, 1] \) such that \( f(\nu) = 1 \).

For \( 1/\sqrt{2} < \nu < 1 \), we have \((m = 2)\)

\[
H(\nu) := \frac{G_0^3}{G_1} = \frac{[\nu^2(\nu + d/X)]^3}{\nu^2(\nu + 1 + 2dX)} < (\nu + d/X)^3.
\]

Write, for \( X \geq X_3 := \frac{2\pi k}{C_1(k)\varphi(k)} \),

\[
(20) \quad \nu_0 = 1 - \frac{1}{2X} \ln \left( \frac{C_1(k)\varphi(k)X}{2\pi k} \right).
\]

Let us study \( H(\nu_0) \):

\[
H(\nu_0) < 1 \quad \text{if} \quad \nu_0 + d/X \leq 1,
\]

equivalently

\[
1 - \frac{1}{2X} \ln \left( \frac{C_1(k)\varphi(k)X}{2\pi k} \right) + \ln \left( \frac{C_1(k)/2\pi}{X} \right) \leq 1,
\]

which holds if

\[
X \geq X_2 := \frac{kC_1(k)}{2\pi\varphi(k)}.
\]

As

\[
f(\nu) = \frac{C_1(k)\varphi(k)}{2\pi k} \left( \frac{G_0^3}{G_1} \right)^{1/2} X \exp(-X(1-\nu)^2/\nu) \exp(-2X(1-\nu)),
\]

replacing \( \nu_0 \) by \((20)\), we obtain

\[
f(\nu_0) = \left( \frac{G_0^3}{G_1} \right)^{1/2} \exp \left( -\ln^2 \left( \frac{C_1(k)\varphi(k)X}{2\pi k} \right) / (4\nu_0 X) \right).
\]

Assume that \( \nu_0 > 0 \), then, for \( X \geq X_2 \), \( f(\nu_0) < 1 = f(\nu) \) and hence \( \nu_0 < \nu \). We will require \( X \geq X_2 \).

The assumption \( T \geq H \) holds if \( T \geq \frac{C_1(k)}{k} \exp(\nu_0 X) \geq H \). Using \((20)\), rewrite

\[
\frac{C_1(k)}{k} \exp(\nu_0 X) = \sqrt{\frac{2\pi C_1(k)}{k\varphi(k)}} e^{-\frac{1}{2}\ln X}. \quad \text{Let } X_0 \text{ satisfy}
\]

\[
e^{X_0 - \frac{1}{2}\ln X_0} = H \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}.
\]

We have \( T \geq H \) provided that \( X \geq X_0 \). We will require \( X \geq X_0 \).

For \( X \geq X_3 := \frac{2\pi k}{C_1(k)\varphi(k)} \), \( \nu_0 \) is an increasing function of \( X \). We will require that \( X \geq \max(X_3, 10) \). Then since \( C_1(k) \leq 32\pi \) and \( X \geq 10 \), we have

\[
\nu_0 > 0.7462413 \quad \text{and} \quad \nu_0 < \nu < 1.
\]

The assumption \( \nu > 1/\sqrt{2} \) is satisfied.

We want to evaluate

\[
(21) \quad K := G_2(\sqrt{G_0G_1} e^{-Y''})^{1/2},
\]

which appears in \((19)\). Again using \( C_1(k) \leq 32\pi \) and \( X \geq 10 \), we find

\[
G_0G_1 < \left( 1 + d/X \right) \frac{\nu_0}{2\nu_0^2 - 1} \left( \nu_0 + \frac{1 + 2d}{2X} \right) < 8.995.
\]

The following results will be needed in later computations.
1. Since \( X \geq X_0 \) and \( \exp(X)/\sqrt{X} \) is increasing for \( X \geq 1/2 \),
\[
\sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}X^{1/2}\exp(-X) \leq \frac{1}{H}.
\]
2. Since \( G_0G_1 < 9 \),
\[
\delta = 2\sqrt{G_0G_1}\exp(-Y''/2)\sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}X^{1/2}e^{-X}
\leq 2\sqrt{3}/H.
\]
In particular, for \( H \geq 1000 \), we have \( \delta \leq 0.00347 \).
3. \( G_2 = \frac{R_2(\delta)}{2^2}(1 + \pi q(T)) < (1 + 3.012 \cdot \delta/2)^2(1 + \pi q(T)) \),

because
\[
\frac{R_2(\delta)}{2^2} = \left\{ \frac{(1 + \delta)^3 + 1}{2} \right\}^2
= \left\{ 1 + \frac{1}{2}\delta(3 + 3\delta + \delta^2) \right\}^2 < \left( 1 + \frac{3.012}{2} \delta \right)^2
\]
since \( 1 + \delta + \delta^2/3 < 1.0035 \).
4. Since \( T \geq H \),
\[
q(T) = \frac{C_2}{T \ln(kT/2\pi)}
\leq \frac{C_2}{H \ln(kH/2\pi)}.
\]
But \( \exp(-Y''/2) \leq 1 \) and \( H \geq 1000 \), so this yields
\[
\begin{align*}
K &< (8.995)^{1/4}G_2 \\
&< (8.995)^{1/4} \left( 1 + \frac{\pi C_2}{1000 \ln(1000/(2\pi))} \right) \times \left( 1 + \frac{3.012 \cdot 2\sqrt{3}}{2 \cdot 1000} \right)^2 \\
&< 1.751.
\end{align*}
\]
Inserting this upper bound of \( K \) (see formula (21) in [19]), we obtain
\[
\varepsilon_1 < 2\sqrt{\frac{2}{\pi}}K \sqrt{\frac{k\varphi(k)}{C_1(k)}}X^{1/2}\exp(-X) + r + \frac{\tilde{R}}{x}
\leq 2.7941 \sqrt{\frac{k\varphi(k)}{C_1(k)}}X^{1/2}\exp(-X) + r + \frac{\tilde{R}}{x}.
\]
(22)

Now we want to bound \( r \) and \( \tilde{R} \).
• An upper bound for \( \varphi(k)(1 + \delta)R(T)\phi_0(T) \) and \( \varphi(k)A(2, \delta)R(T)\phi_2(T) \). Recall that
\[
\begin{align*}
R(T) &= C_2 \ln(kT) + C_3, \\
\phi_0(T) &= \frac{1}{T} \exp\left( -X^2/\ln(kT/C_1(k)) \right), \\
\phi_m(T) &= \phi_0(T)T^{-m}.
\end{align*}
\]
Now
\[ \phi_0(T) = \frac{1}{T} \exp(-X^2/(\nu X)) = \frac{1}{T} \exp(-\frac{1}{\nu} X) \leq \frac{1}{T} \exp(-X) \]
and
\[ \frac{1}{T} = X^{1/2} \exp(-X) \sqrt{\frac{k\varphi(k)}{C_1(k)}} \left( \frac{G_0}{2\pi e^X} \right)^{1/2} \left( \frac{G_0}{G_1} \right)^{1/4}, \]
hence
\[ R(T)\phi_0(T) \leq \frac{C_2 \ln(kT) + C_3}{T} \exp(-X) \]
\[ \leq \sqrt{X} e^{-X} \sqrt{\frac{k\varphi(k)}{C_1(k)}} \left( C_2 \ln(kT) + C_3 \right) \left( \frac{G_0}{2\pi e^X} \right)^{1/2} \left( \frac{G_0}{G_1} \right)^{1/4}, \]
But
\[ G_0 \leq 1 + \frac{\ln(C_1(k)/2\pi)}{X}, \]
\[ \frac{G_0}{G_1} \leq 2\nu^2 - 1 < 1 \quad (m = 2), \]
\[ \exp(Y''') \geq 1, \]
\[ \ln(kT) = \nu X + \ln(C_1(k)) \leq X + \ln(C_1(k)) \leq X + \ln(32\pi). \]
So, since \( X \geq 10 \) and \( C_1(k) \leq 32\pi, \)
\[ (1 + \delta)\varphi(k) \left( C_2 \ln(kT) + C_3 \right) \left( \frac{G_0}{2\pi e^X} \right)^{1/2} \left( \frac{G_0}{G_1} \right)^{1/4} \exp(-X) \]
\[ \leq \varphi(k) \left( 1 + \frac{2\sqrt{3}}{1000} \right) \left[ C_2(X + \ln 32\pi) + C_3 \right] \sqrt{\frac{1 + \ln 16/10}{2\pi}} \exp(-X) \]
\[ \leq 0.857\varphi(k) X \exp(-X). \]
Furthermore, if \( X_1 \) is defined by \( \exp(X_1)/X_1 = 10\varphi(k), \) and if we require that \( X \geq X_1, \) then this term is bounded by 0.0857. Hence, under the hypotheses on \( X \) in Theorem 5 an upper bound for \( \varphi(k)(1 + \delta)R(T)\phi_0(T) \) is
\[ 0.09 \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X). \]
Next, by (10)
\[ \delta T = 2 \sqrt{\frac{G_1}{G_0}}. \]
Hence, by Lemma 12
\[ A(2, \delta)/T^2 \leq \frac{R_2(\delta)}{(\delta T)^2} \leq \frac{R_2(\delta) G_0}{2^2 G_1} \leq \frac{R_2(\delta)}{2^2} \]
and
\[ \varphi(k)A(2, \delta)R(T)\phi_2(T) \leq \varphi(k) \frac{R_2(\delta)}{2^2} R(T)\phi_0(T). \]
Using $\delta \leq 2\sqrt{3}/H \leq 2\sqrt{3}/1000$, we get \( R_2(\delta)/2^2 \leq 1.0147 \). Under the hypotheses on \( X \) in Theorem 5, an upper bound for \( \varphi(k)A(2, \delta)R(T)\phi_2(T) \) is therefore

\[
0.087 \cdot \frac{k\varphi(k)}{C_1(k)} X^{1/2} \exp(-X).
\]

The sum of the two terms can be bounded by

\[
0.2 \cdot \sqrt{\frac{k\varphi(k)}{C_1(k)} X^{1/2} \exp(-X)}.
\]

- An upper bound for \((1 + \delta)\tilde{E}(T)\frac{\varphi(k)}{\sqrt{X}} + A(2, \delta)\frac{\varphi(k)}{2\sqrt{X}}(\tilde{C}_T + \tilde{D}_T) + \tilde{R}/x\).

For \( f(k) = \sum_{p \mid k} \frac{1}{p-1} \) observe that (Lemma 10)

\[
f(k) \leq \frac{\ln k}{\ln 2}.
\]

We can explicitly rewrite for \( m = 2, H \geq 1000, \) and \( C_1(k) \leq 32\pi \) the following expressions:

\[
3\tilde{E}(T) = 3 \left( \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln \left( \frac{k}{2\pi} \right) \ln T + C_2 \right.
\]
\[
+ \left. 2 \left( \frac{1}{\pi} \ln \left( \frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right) \right),
\]

\[
\tilde{C}_T = \frac{1}{2\pi T^2} \left( \ln \left( \frac{kT}{2\pi} \right) + 1/2 \right),
\]

\[
\tilde{D}_T = (2C_2 \ln(kT) + 2C_3 + C_2/3)/T^3,
\]

\[
\frac{\tilde{R}}{\varphi(k)\sqrt{X}} \leq |(f(k) + 0.5) \ln x + 4 \ln k + 13.4|/\sqrt{x}.
\]

It is tedious but easy to check that the sum of the above quantities is less than

\[
\begin{cases} 
1000(\ln T \sqrt{\ln k})^2 & \text{for } k \neq 1, \\
1000 \ln^2 T & \text{for } k = 1.
\end{cases}
\]

Now we want to find a number \( c \) such that

\[
A(2, \delta)\varphi(k)\frac{1000(\ln T \sqrt{\ln k})^2}{\sqrt{x}} \leq c \left( \frac{k\varphi(k)}{C_1(k)} \right)^{1/2} X^{1/2} \exp(-X)
\]

with \( X = \sqrt{\ln \frac{x}{k}} \). But \( A(2, \delta) \leq \frac{R_2(\delta)}{2^2} \) and by (10), \( T = \left( \frac{G_0}{G_1} \right)^{1/2} \frac{2}{\pi} \), so

\[
A(2, \delta) \leq \frac{R_2(\delta)}{2^2} T^2 G_0 \frac{G_0}{G_1}.
\]

Moreover, \( \frac{1}{\sqrt{x}} = \exp(-RX^2/2) \), hence

\[
c \geq 1000 \frac{R_2(\delta)}{2^2} \frac{G_0}{G_1} T^2 \varphi(k)(\ln T \sqrt{\ln k})^2 \left( \frac{C_1(k)}{k\varphi(k)} \right)^{1/2} X^{-1/2} \exp(X - RX^2/2).
\]

As \( \frac{G_0}{G_1} < 1 \), \( T^2 = \frac{C_1^2(k)}{k^2} \exp(2\nu X) \leq \frac{C_1^2(k)}{k^2} \exp(2X) \), hence it suffices to take

\[
c \geq 1000 \frac{R_2(\delta)}{2^2} \frac{C_1^2(k)}{k^2} \ln \left( \frac{C_1(k)}{k} + 1 \right) X^2 \left( \frac{C_1(k)}{kX} \right)^{1/2} \exp(3X - RX^2/2).
\]
Since \( C \) finally, it suces to take

\[ \text{This would not have been possible if we had used only the results of [3].} \]

Using \( X \) more precisely, for all \( X \) satisfying the conditions of the theorem,

\[ \begin{align*}
| \psi(x; k, l) - x/\varphi(x) | / x & \leq 2.9941 \sqrt{\frac{k}{\varphi(k)C_1(k)}} X^{1/2} \exp(-X). \\
\text{We also wish to allow } \theta \text{ instead of } \psi, \text{ which can be done by recalling Theorem 13 of [5].} \\
0 \leq \psi(x; k, l) - \theta(x; k, l) \leq \psi(x) - \theta(x) & \leq 1.43\sqrt{x} \quad \text{for } x \geq 0. \\
\text{Using } X \geq 10, \text{ we find } 1.43\sqrt{x}/x & \leq d \cdot (k\varphi(k))/C_1(k) X^{1/2} \exp(-X), \text{ where } \\
d = 1.17 \cdot 10^{-204}. \text{ This difference is absorbed by rounding up the constants.}
\end{align*} \]

5. Application for \( k = 3 \)

Now we are able to compute \( x_0 \) and \( c \) such that, for \( x \geq x_0, \)

\[ \begin{align*}
| \theta(x; 3, l) - x/2 | & < cx/\ln x. \\
\text{This would not have been possible if we had used only the results of [3].}
\end{align*} \]

According to Theorem [5]

\[ \varepsilon(X) = 3 \sqrt{\frac{6}{20.92}} X^{1/2} \exp(-X) \]

for \( k = 3. \)

To determine for which \( X \) this bound is valid, let us solve for the constants \( X_0, X_1, X_2, X_3 \) in Theorem [5]. Noting that \( H_3 = 10000 \) by the table in Theorem [1] we need \( X_0 \) to satisfy

\[ \exp(X_0 - \frac{1}{2} \ln X_0) \geq 10000 \sqrt{\frac{6}{2\pi \cdot 20.92}} \approx 2136.51. \]

\( X_0 \approx 8.76 \) works.

Find \( X_1 \) such that

\[ \exp(X_1 - \ln X_1) \geq 20. \]

\( X_1 \approx 4.5 \) works.

Compute the two other bounds: \( X_2 \approx 4.99, X_3 \approx 1.22. \) Thus we can take

\( X = \max(10, X_0, X_1, X_2, X_3) = 10 \) in Theorem [5]

• For \( \sqrt{\ln x} \geq 10 \), write \( X = \sqrt{\ln x} \), then

\[ \varepsilon(X) \ln x = RX^2 \varepsilon(X). \]

Find the value \( c \) such that

\[ \varepsilon(X) < c/\ln(x). \]
For any $x$ such that $\sqrt{\frac{\ln x}{R}} \geq 10$, $c \leq R \cdot 10^2 \varepsilon(10) \leq 0.12$. Hence we have for $x \geq \exp(964.59 \cdots)$,

$$|\theta(x; 3, l) - x/2| \leq 0.12 \frac{x}{\ln x}.$$  

We want to extend the above result for $x \leq \exp(964.59 \cdots)$. Olivier Ramaré has kindly computed some additional values supplementing Table 1 in [3]. We have

$$|\theta(x; 3, l) - x/2| < \tilde{c} \cdot x/2$$

with

$$\tilde{c} = 0.0008464421 \text{ for } \ln x \geq 400 \quad (m = 3, \delta = 0.00042325),$$
$$\tilde{c} = 0.0006048271 \text{ for } \ln x \geq 500 \quad (m = 3, \delta = 0.00030250),$$
$$\tilde{c} = 0.0004190635 \text{ for } \ln x \geq 600 \quad (m = 2, \delta = 0.00027950).$$

Hence,

- For $e^{600} \leq x \leq e^{964.59 \cdots}$
  
  $$c \leq 0.0004190635 \cdot 964.6/\varphi(3) \leq 0.203.$$

- For $e^{400} \leq x \leq e^{600}$
  
  $$c \leq 0.0008464421 \cdot 600/\varphi(3) \leq 0.254.$$

Using the computations of [3],

- For $10^{100} \leq x \leq e^{400}$
  
  $$c \leq 0.001310 \cdot 400/\varphi(3) \leq 0.262.$$

- For $10^{30} \leq x \leq 10^{100}$
  
  $$c \leq 0.001813 \cdot 100 \ln 10/\varphi(3) \leq 0.42/2 \leq 0.21.$$

- For $10^{13} \leq x \leq 10^{30}$
  
  $$c \leq 0.001951 \cdot 30 \ln 10/\varphi(3) \leq 0.14/2 \leq 0.07.$$

- For $10^{10} \leq x \leq 10^{13}$
  
  $$c \leq 0.002238 \cdot 13 \ln 10/\varphi(3) \leq 0.067/2 \leq 0.00335.$$

- For $4403 \leq x \leq 10^{10}$
  
  $$|\theta(x; 3, l) - x/2| < 2.072 \sqrt{x} \quad \text{(Theorem 5.2.1 of Ramaré and Rumely [3])}$$

We choose $c = 0.262$. We check that this bound is also valid for $1531 \leq x \leq 4403$.

**Theorem 6.** For $x \geq 1531$,

$$|\theta(x; 3, l) - x/2| \leq 0.262 \frac{x}{\ln x}.$$
ESTIMATES OF $\theta(x; k, l)$ FOR LARGE VALUES OF $x$

6. Results assuming GRH($k, \infty$)

Assuming GRH($k, \infty$), we obtain more precise results. Under this hypothesis, one can show that function $\psi$ has the following asymptotic behaviour:

**Proposition 1** ([8] p. 294). Assume GRH($k, \infty$). Then

$$\psi(x; k, l) = \frac{x}{\varphi(k)} + O(\sqrt{x} \ln^2 x).$$

**Theorem 7.** Let $x \geq 10^{10}$. Let $k$ be a positive integer. Assume GRH($k, \infty$).

1) If $k \leq \frac{1}{5} \ln x$, then

$$|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq 0.085 \sqrt{x} \ln^2 x.$$ 

2) If $k \leq 432$, then

$$|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq 0.061 \sqrt{x} \ln^2 x.$$ 

**Proof.** Let $x_0 = 10^{10}$. Applying Theorem 2 in the same way as Theorem 3 (assume that $T \geq 1$),

$$\frac{\varphi(k)}{x} |\psi(x; k, l) - \frac{x}{\varphi(k)}|$$

$$\leq A(m, \delta) \sum_{\chi \mod \gamma > T} \frac{x^{-1/2}}{|\gamma| \cdot \rho(\rho+1) \cdots (\rho+m)}$$

$$+ (1 + m \delta/2) \sum_{\chi \mod \gamma \leq T} \frac{x^{-1/2}}{|\gamma|} + m \delta/2 + \tilde{R}/x$$

$$\leq A(m, \delta) \frac{1}{\sqrt{x}} \sum_{\chi \mod \gamma > T} \frac{1}{|\gamma|^{m+1}} + (1 + \frac{m \delta}{2}) \frac{1}{\sqrt{x}} \sum_{\chi \mod \gamma \leq T} \frac{1}{|\gamma|} + \frac{m \delta}{2} + \tilde{R}/x$$

$$\leq A(m, \delta) \frac{\varphi(k)}{\sqrt{x}} (\tilde{C}_T + \tilde{D}_T) + (1 + \frac{m \delta}{2}) \frac{\varphi(k)}{\sqrt{x}} \tilde{E}(T) + \frac{m \delta}{2} + \tilde{R}/x.$$ 

Take $m = 1$ and let

$$\varepsilon_k(x, T, \delta) := \frac{R_1(\delta)}{\delta} \frac{\varphi(k)}{\sqrt{x}} (\tilde{C}_T + \tilde{D}_T) + \left(1 + \frac{\delta}{2}\right) \frac{\varphi(k)}{\sqrt{x}} \tilde{E}(T) + \frac{\delta}{2} + \tilde{R}/x,$$

where

$$\tilde{C}_T = \frac{1}{\pi T} \left(\ln \left(\frac{kT}{2\pi}\right) + 1\right),$$

$$\tilde{D}_T = \frac{1}{T^2} \left(2C_2 \ln(kT) + 2C_3 + C_2/2\right),$$

$$\tilde{E}(T) = \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln(k/(2\pi)) \ln T + C_2 + 2 \left(\frac{1}{\pi} \ln \left(\frac{k}{2\pi e}\right) + C_2 \ln k + C_3\right).$$

Choose

$$T = \frac{2R_1(\delta)}{\delta(2+\delta)}$$
to minimize in \((25)\) the preponderant terms involving \(T\). So
\[
\frac{R_1(\delta)}{\delta} (\tilde{C}_T + \tilde{D}_T) = \frac{2(2 + \delta)}{4\pi} \left[ \ln \left( \frac{kR_1(\delta)}{\pi\delta(2 + \delta)} \right) + 1 \right.
+ \frac{\pi\delta(2 + \delta)}{2R_1(\delta)} \left( 2C_2 \ln \left( \frac{2kR_1(\delta)}{\delta(2 + \delta)} \right) + 2C_3 + \frac{C_2}{2} \right) \left. \right],
\]
\[(1 + \delta/2)\tilde{E}(T) = \frac{2 + \delta}{4\pi} \left[ \ln^2 \left( \frac{2R_1(\delta)}{\delta(2 + \delta)} \right) + 2\ln(k/(2\pi)) \ln \left( \frac{2R_1(\delta)}{\delta(2 + \delta)} \right) \right. \\
+ 2\pi C_2 + 4\pi \left( \frac{1}{\pi} \ln(k/(2\pi)) + C_2 \ln k + C_3 \right).\]

With the choice of \(T\), the main terms of \(\varepsilon_k\) are
\[\frac{\varphi(k)}{\sqrt{x}} \frac{1}{2\pi} \ln^2 \left( \frac{2R_1(\delta)}{\delta(2 + \delta)} \right) + \frac{\delta}{2}.\]

These terms are minimized by choosing
\[(27) \quad \delta = \frac{\varphi(k) \ln x}{\pi\sqrt{x}}.\]

Now, replacing \((26)\) and \((27)\) in \((25)\), we only have a function of \(x\) for fixed \(k\):
\[\varepsilon_k(x) := \varepsilon_k(x, T, \delta).\]

We simplify expression \((25)\):
\[
\frac{\varepsilon_k(x, T, \delta)}{\varphi(k)} \leq \tilde{\varepsilon}_k(x, T, \delta)
:= \frac{R_1(\delta)}{\delta} (\tilde{C}_T + \tilde{D}_T) / \sqrt{x} + (1 + \frac{\delta}{2})\tilde{E}(T) / \sqrt{x} + \frac{\delta}{2} + \frac{\tilde{R}}{x\varphi(k)}.\]

By choosing \(T = \frac{2R_1(\delta)}{\delta(2 + \delta)}\) and \(\delta = \frac{\ln x}{\pi\sqrt{x}}\), \(\tilde{\varepsilon}_k(x, T, \delta)\) became \(\tilde{\varepsilon}_k(x)\).

Hence,
\[
\tilde{\varepsilon}_k(x) \sqrt{x} = \frac{2 + \delta}{4\pi} \left[ \ln^2 \left( \frac{2\pi\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) + 2\ln \left( \frac{k}{2\pi} \right) \ln \left( \frac{2\pi\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) \right.
\left. + 2\ln \left( \frac{k\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) + \ln x \frac{2 + \delta}{\sqrt{x} \cdot R_1(\delta)} (A) \right] + \frac{\ln x}{2\pi\varphi(k)} + \frac{\tilde{R}}{\varphi(k) \sqrt{x}} \right.
\left. + \frac{2 + \delta}{4\pi} (2 + 2\pi C_2 + 4\pi \left( \frac{1}{\pi} \ln(k/(2\pi)) \right) + C_2 \ln k + C_3) \right)
\]

with
\[A = 2C_2 \ln \left( \frac{2k\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2 + \delta} \right) + 2C_3 + \frac{C_2}{2}.\]

Let \(\delta_1 = \frac{\ln x}{\pi\sqrt{x}}\) and \(\delta = \frac{2 + \delta + \delta^2}{2 + \delta} = 1 + \frac{\delta^2 + \delta}{2 + \delta} \leq d_1 := 1 + \frac{\delta^2 + \delta}{2 + \delta}\) because \(\delta \leq \frac{\ln x}{\pi\sqrt{x}}\).

By direct computation, for all \(k\) between 1 and 432 and \(x \geq x_0\), of \(\frac{\varepsilon_k(x) \sqrt{x}}{\varphi(k) \ln^n x}\), we find an upper bound 0.06012.
To obtain 1) in Theorem 7, we will study the sum in brackets for \(1 \leq k \leq \frac{4}{5} \ln x\):

\[
\left[ \cdots \right] = \left[ \frac{1}{4} \ln^2 x + \ln^2 \left( \frac{2\pi d_1}{\ln x} \right) + \ln x \ln \left( \frac{2\pi d_1}{\ln x} \right) + 2 \ln \left( \frac{4 \ln x}{10 \pi} \right) \ln \left( \frac{2\pi d_1}{\ln x} \right) 
+ \ln \left( \frac{4 \ln x}{10 \pi} \right) \ln x + \frac{1}{2} \ln x + \ln(4d_1/5) + \frac{\ln x}{\sqrt{x}}(A) \right]
+ \frac{1}{2} \ln x + \ln \left( \frac{2\pi d_1}{\ln x} \right) + 1/2 + \ln(4 \ln x/(10 \pi))
+ \ln^2 \left( \frac{2\pi d_1}{\ln x} \right) + 2 \ln \left( \frac{4 \ln x}{10 \pi} \right) \ln \left( \frac{2\pi d_1}{\ln x} \right) + \ln(4d_1/5) + \frac{\ln x}{\sqrt{x}}(A). \]

We conclude that

\[
\lim_{x \to \infty} \frac{\varepsilon_k(x) \sqrt{x}}{\ln^2 x} = \frac{1}{8\pi},
\]

which is the same asymptotic bound as Schoenfeld’s [7] for \(\psi\).

The bound \(\varepsilon_k(x) \sqrt{x}\) is an increasing function of \(k\). Choose \(k = \frac{4}{5} \ln x\). Now \(\varepsilon_k(x) \sqrt{x}/ \ln^2 x\) is a decreasing function of \(x\) bounded by 0.0849229 for \(x \geq x_0\). \(\square\)

**Remark.** If we take \(k = 1\) in Theorem 7 our upper bound is twice as bad as the result of Schoenfeld [7, p. 337]: for \(x > 73.2\),

\[
|\psi(x) - x| \leq \frac{1}{8\pi} \sqrt{x} \ln^2 x.
\]

These differences are explained by:

- an exact computation of zeros with \(\gamma \leq D \approx 158\) (the preponderant ones!) in the sum \(\sum \frac{1}{|\gamma|}\);
- a better knowledge of \(R(T)\) (\(k\) fixed, \(k = 1\)).

**Corollary 3.** Assume GRH\((k, \infty)\). For all \(k\) used in Lemma 7 and \(x \geq 224\),

\[
\left| \psi(x; k, l) - \frac{x}{\varphi(k)} \right| \leq \frac{1}{4\pi} \sqrt{x} \ln^2 x.
\]

**Proof.** We use Theorem 5.2.1 of [3]: for all \(k\) noted in Lemma 4 and \(224 \leq x \leq 10^{10}\),

\[
|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq \sqrt{x}
\]

and \(\sqrt{x} < \frac{1}{4\pi} \sqrt{x} \ln^2 x\) for \(x \geq 35\). We conclude by Theorem 7. \(\square\)

**7. Estimates for \(\pi(x; 3, l)\)**

**Definition 1.** Let

\[
\pi(x; k, l) = \sum_{\substack{p \leq x \mod k \leq x \text{ and } p \text{ prime}}} 1
\]

be the number of primes smaller than \(x\) which are congruent to \(l\) modulo \(k\).

Our aim is to have bounds for \(\pi(x; 3, l)\). We show that

**Theorem 8.** For \(l = 1\) or 2,

(i) \(\frac{x}{1.1x} < \pi(x; 3, l)\) for \(x \geq 151\),

(ii) \(\pi(x; 3, l) < 0.55 \frac{x}{1.6x}\) for \(x \geq 229869\).
From this, we can deduce that for all \( x \geq 151 \),
\[
\frac{x}{\ln x} < \pi(x)
\]
because
\[
\pi(x) = \pi(x; 3, 1) + \pi(x; 3, 2) + 1.
\]

7.1. The upper bound. First we give the proof of Theorem 9(ii).

Lemma 13. Let
\[
I_n = \int_a^x \frac{dt}{\ln^n t}.
\]
Then
\[
I_n = \frac{x}{\ln^n x} - \frac{a}{\ln^n a} + nI_{n+1}.
\]
Furthermore, for \( x \geq 10^{10} \), for all \( k \leq 72 \), for all \( l \) relatively prime with \( k \),
\[
\max_{1 \leq y \leq x} | \theta(y; k, l) - \frac{y}{\varphi(k)} | \leq 2.072 \sqrt{x}.
\]

Furthermore, for \( x \geq 10^{10} \) and \( k = 3 \) or \( 4 \),
\[
| \theta(x; k, l) - \frac{x}{\varphi(k)} | \leq 0.002238 \frac{x}{\varphi(k)}.
\]

Write first
\[
\pi(x; k, l) - \pi(x; 0; k, l) = \frac{\theta(x; k, l)}{\ln(x)} - \frac{\theta(x_0; k, l)}{\ln(x_0)} + \int_{x_0}^{x} \theta(t; k, l) \, \frac{dt}{t \ln^2 t}.
\]

Put \( x_0 := 10^5 \).

Preliminary computations :
\[
\theta(10^5, 3, 1) = 49753.417198 \cdots \quad \pi(10^5, 3, 1) = 4784.
\]
\[
\theta(10^5, 3, 2) = 49930.873458 \cdots \quad \pi(10^5, 3, 2) = 4807.
\]
Put \( c_0 := \frac{1.022238}{2} \) and \( K = \max_i (\pi(10^5, 3, l) - \theta(10^5, 3, l)/\ln(10^5)) \approx 470. \)

\bullet For \( 10^{20} \leq x \),
\[
\pi(x; k, l) - \pi(10^5; k, l) = \frac{\theta(x; k, l)}{\ln(x)} - \frac{\theta(10^5; k, l)}{\ln(10^5)} + \int_{10^5}^{x} \theta(t; k, l) \, \frac{dt}{t \ln^2 t}.
\]

But
\[
\int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} \, dt = \int_{10^5}^{10^{10}} \frac{\theta(t; k, l)}{t \ln^2 t} \, dt + \int_{10^{10}}^{\sqrt{x}} \frac{\theta(t; k, l)}{t \ln^2 t} \, dt + \int_{\sqrt{x}}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} \, dt
\]
and, by Theorem 9
\[
\int_{10^5}^{10^{10}} \frac{\theta(t; k, l)}{t \ln^2 t} \, dt < M := 1/\varphi(k) \cdot \int_{10^5}^{10^{10}} \frac{dt}{t \ln^2 t} + 2.072 \cdot \int_{10^5}^{10^{10}} \frac{dt}{\sqrt{t} \ln^2 t}.
\]
\[
\int_{10^{10}}^{\sqrt{x}} \frac{\theta(t; k, l)}{t \ln^2 t} \, dt < c_0 \frac{\sqrt{x} - 10^{10}}{\ln^2 10^{10}}.
\]
\[
\int_{\sqrt{x}}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} \, dt < c_0 \frac{x - \sqrt{x}}{\ln^2 \sqrt{x}}.
\]

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We compute $M = 10381055.54 \cdots$. Then

\[ \pi(x;3,l) < c_0 \frac{x}{\ln x} + K + M + c_0 \left( \frac{\sqrt{x} - 10^{10} \ln 10^{10} + x - \sqrt{x}}{\ln^2 x} \right) \]
\[ < \frac{x}{\ln x} \left( c_0 + \left( K + M + c_0 \frac{10^{20} - 10^{10} \ln 10^{20}}{\ln^2 10^{10}} \right) \right) \]
\[ < 0.545 \frac{x}{\ln x}. \]

- For $10^{10} \leq x \leq 10^{20}$,

\[ \pi(x;3,l) < K + \int_{10^{10}}^{x} \frac{\theta(t;3,l)}{t \ln^2 t} dt + \int_{10^{10}}^{x} \frac{\theta(t;3,l)}{t \ln^2 t} dt + c_0 \frac{x}{\ln x} \]
\[ < \frac{x}{\ln x} \left( c_0 + \frac{\ln x}{x} (K + M - 10^{10} \frac{c_0}{\ln^2 10^{10}} + c_0 \frac{c_0}{\ln^2 10^{10}} \ln x) \right) \]
\[ < 0.5468 \frac{x}{\ln x}. \]

- For $10^5 \leq x \leq 10^{10}$,

\[ \int_{10^5}^{x} \frac{\theta(t;3,l)}{t \ln^2 t} dt < \frac{1}{2} \int_{10^5}^{x} \frac{dt}{\ln^2 t} + 2.072 \int_{10^5}^{x} \frac{dt}{\sqrt{t} \ln^2 t} \]
\[ = \frac{1}{2} \left( \frac{x}{\ln^2 x} - \frac{10^5}{\ln^2 10^5} + 2 \int_{10^5}^{x} \frac{dt}{\ln^3 t} \right) + 2.072 \int_{10^5}^{x} \frac{dt}{\sqrt{t} \ln^2 t}. \]

Now, \[ \int_{a}^{b} \frac{dt}{\sqrt{t} \ln^2 t} = \left[ \frac{2 \sqrt{t}}{\ln^2 t} \right]_{a}^{b} + 4 \int_{a}^{b} \frac{dt}{\sqrt{t} \ln^2 t}. \]

Therefore, if \[ \pi(x;3,l) < \frac{1}{2} \frac{x}{\ln x} + 2.072 \frac{\sqrt{x}}{\ln x} + K \]
\[ + \frac{1}{2} \left( \frac{x}{\ln^2 x} - \frac{10^5}{\ln^2 10^5} + 2 \int_{10^5}^{x} \frac{dt}{\ln^3 t} \right) \]
\[ + 2.072 \left( \frac{2 \sqrt{x}}{\ln^2 x} - \frac{2 \sqrt{10^5}}{\ln^2 10^5} + 4 \int_{10^5}^{x} \frac{dt}{\sqrt{t} \ln^2 t} \right) \]
\[ < 0.55 \frac{x}{\ln x} \quad \text{for} \quad x \geq 6 \cdot 10^5. \]

7.2. The lower bound. Let \( KK = \min(\pi(10^5,3,l) - \theta(10^5,3,l)/\ln(10^5)) \approx 462 \)
and \( c = 0.498881 = \frac{1-0.999238}{2} \).

- For $10^{10} \leq x,$

\[ \pi(x;3,l) > KK + \frac{\theta(x;3,l)}{\ln x} + \int_{10^{10}}^{x} \frac{\theta(t;k,l)}{t \ln^2 t} dt \]
\[ > \frac{cx}{\ln x} \]

because

\[ KK > 0 \quad \text{and} \quad \int_{10^{10}}^{x} \frac{\theta(t;k,l)}{t \ln^2 t} dt > 0. \]

- For $10^5 \leq x \leq 10^{10}$.

Lemma 14 (McCurley [2]). For $x \geq 91807$ and $c_2 = 0.49585$, we have $\theta(x;3,l) \geq c_2 x$. 
Remark. This bound is better than the one given in Theorem 9 for $x \leq 2.5 \cdot 10^5$.

$$\pi(x; 3, l) > KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt.$$  

Thus for any $x_0, x_1$ with $10^5 \leq x_0 < x_1$,  

$$\pi(x; 3, l) > KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^{x_0} \frac{\theta(t; k, l)}{t \ln^2 t} dt \text{ for } x \geq x_0$$  

$$> \frac{x}{\ln x} \left( c_2 + \left( KK + \int_{10^5}^{x_0} \frac{\theta(t)}{t \ln^2 t} \frac{\ln x_1}{x_1} \right) \right) \text{ for } x_0 \leq x \leq x_1.$$  

Using the previous remark, we find

$$\int_{10^5}^{x} \frac{\theta(t; k, l)}{t \ln^2 t} dt > c_2 \int_{10^5}^{x} \frac{dt}{\ln^2 t} \text{ if } 10^5 \leq x \leq 2.5 \cdot 10^5$$

and

$$> c_2 \int_{10^5}^{2.5 \cdot 10^5} \frac{dt}{\ln^2 t} + \int_{2.5 \cdot 10^5}^{x} \frac{t/2 - 2.0722\sqrt{t}}{t \ln^2 t} dt \text{ if } 2.5 \cdot 10^5 \leq x.$$  

We use this to make step by step computations with Maple:

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$x_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^5$</td>
<td>$2 \cdot 10^6$</td>
</tr>
<tr>
<td>$2 \cdot 10^6$</td>
<td>$3 \cdot 10^7$</td>
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<tr>
<td>$3 \cdot 10^7$</td>
<td>$3 \cdot 10^8$</td>
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<tr>
<td>$3 \cdot 10^8$</td>
<td>$3 \cdot 10^9$</td>
</tr>
<tr>
<td>$3 \cdot 10^9$</td>
<td>$10^{10}$</td>
</tr>
</tbody>
</table>

We conclude that $\pi(x; 3, l) > 0.499 \frac{x}{\ln x}$ for $10^5 \leq x \leq 10^{10}$.

7.3. Small values. We now check whether $0.49888 \frac{x}{\ln x} < \pi(x; 3, l) < 0.55 \frac{x}{\ln x}$ for $x < 6 \cdot 10^5$. It is sufficient to prove that

$$\pi(p; 3, l) < 0.55 \frac{p}{\ln p} \text{ for } p \equiv l \mod 3,$$

and if

$$0.49888 \frac{p}{\ln p} < \pi(p; 3, l) - 1 \text{ for } p \equiv l \mod 3.$$  

The highest value not satisfying the first inequality is $p = 229849$, and the highest value not satisfying the second is $p = 151$. Furthermore, $\pi(229869; 3, l) \leq 10241 < \frac{229869}{\ln 229869} \approx 10241.0075$ and $\pi(151; 3, l) \geq 16 > 0.49888 \frac{151}{\ln 151} \approx 15.01$.

The conclusion is

$$0.49888 \frac{x}{\ln x} \leq \pi(x; 3, l) \leq 0.55 \frac{x}{\ln x} \text{ for } x > 229869.$$  

Remark. We cannot show that $x/(2 \ln x) < \pi(x; 3, l)$ by using the formula $\theta(x) < c \cdot x$. We have obtained other formulas (see Theorem 6) which we will use below.
7.4. More precise lower bound of $\pi(x;3, l)$. Now we will give the proof of Theorem 8(i).

Classically,

$$\pi(x;3,l) - \pi(10^5;3,l) = \frac{\theta(x;3,l)}{\ln(x)} - \frac{\theta(10^5;3,l)}{\ln(10^5)} + \int_{10^5}^{x} \frac{\theta(t;3,l)}{t \ln^2 t} dt.$$ 

Now $\theta(t;3,l) > \frac{x}{\varphi(3)} \left(1 - \frac{\alpha}{\ln x}\right)$ with $\alpha = \varphi(3) \cdot 0.262$ by use of Theorem 8. So we write

$$KK = \min_{l} \left( \pi(10^5;3,l) - \frac{\theta(10^5;3,l)}{\ln(10^5)} \right),$$

$$\pi(x;3,l) > J(x, \alpha) = KK + \frac{x}{\varphi(k) \ln x} \left(1 - \frac{\alpha}{\ln x}\right) + \frac{1}{\varphi(k)} \int_{10^5}^{x} \frac{1 - \alpha/\ln t}{\ln^2 t} dt.$$ 

The derivative of $J(x, \alpha)$ with respect to $x$ equals

$$\frac{1}{\varphi(k)} \left( \frac{1 - \alpha/\ln x}{\ln x} + \frac{\alpha}{\ln^3 x} \right).$$

Moreover, the derivative of $\frac{1}{\varphi(k) \ln x}$ equals

$$\frac{1}{\varphi(k)} \left( \frac{1}{\ln x} - \frac{1}{\ln^2 x} \right).$$

The inequality

$$\frac{1}{\varphi(k)} \left( \frac{1}{\ln x} - \frac{1}{\ln^2 x} \right) < \frac{1}{\varphi(k)} \left( \frac{1 - \alpha/\ln x}{\ln x} + \frac{\alpha}{\ln^3 x} \right)$$

holds if $\alpha - 1 < \alpha/\ln x$; this holds for all $x > 1$. The only thing to do is to find a value $x_1$ such that

$$J(x_1, \alpha) > \frac{x_1}{\varphi(k) \ln x_1}.$$ 

For $x_1 = 10^5$, $J(10^5, 0.524) \approx 4607.75$ and $\frac{10^5}{\ln 10^5} \approx 4342.94$. We verify by computer that the inequality holds for $x \leq 10^5$ and $l = 1$ or 2. We conclude that

$$\frac{x}{2 \ln x} < \pi(x;3,l) < x_1 \frac{1}{\varphi(k) \ln x_1}$$

for $x \geq 151$.

References


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