

## THUE'S THEOREM AND THE DIOPHANTINE EQUATION $x^2 - Dy^2 = \pm N$

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ABSTRACT. A constructive version of a theorem of Thue is used to provide representations of certain integers as  $x^2 - Dy^2$ , where  $D = 2, 3, 5, 6, 7$ .

### 1. INTRODUCTION

The idea of using Euclid's algorithm to construct solutions of  $p = x^2 + y^2$  goes back to Serret [9] and Hermite [5]. (Also see Wagon [12] and Brillhart [1].) The method easily extends to  $p = x^2 + ny^2$ ,  $n = 2, 3, 5$ . (See Wilker [13] for  $n = 5$ .)

Cornacchia [2, pp. 61–66] generalised the method to  $N = x^2 + ny^2$ ,  $n \geq 1$  and discussed the case  $n < 0$  [2, pp. 66–70]. (Also see Nitaĵ [8] and Hardy, Muskat and Williams [3], [4], Muskat [6], Williams [14], [15].)

It is not so well known that the Serret–Hermite method can be used to find explicit solutions of  $x^2 - Dy^2 = N$  when  $D > 1$  is small. Nagell [7, pp. 210–212] used a nonconstructive form of a theorem of Thue [10, p. 587] to deal with  $D = 2$  and 3, while a variant of Thue's theorem was also used in Uspensky and Heaslet [11, pp. 352–368] for  $D = 2, 3, 5$ .

In this paper we show how to obtain explicit representations of certain integers in the form  $x^2 - Dy^2$  for small  $D > 1$ , using a constructive version of Thue's theorem based on Euclid's algorithm. Amongst other things, if  $u^2 \equiv D \pmod{N}$ ,  $D \not\equiv 1 \pmod{N}$  is soluble and  $\gcd(D, N) = 1$ ,  $N$  odd, we show how to find the following representations:

$N = 8k \pm 1$	$N = x^2 - 2y^2$ $-N = x^2 - 2y^2$
$N = 12k + 1$	$N = x^2 - 3y^2$
$N = 12k - 1$	$-N = x^2 - 3y^2$
$N = 5k + 1$	$N = x^2 - 5y^2$
$N = 5k - 1$	$-N = x^2 - 5y^2$
$N = 24k + 1$ or $24k - 5$	$N = x^2 - 6y^2$
$N = 24k - 1$ or $24k + 5$	$-N = x^2 - 6y^2$
$N = 28k + 1, 28k + 9$ or $28k + 25$	$N = x^2 - 7y^2$
$N = 28k - 1, 28k - 9$ or $28k - 25$	$-N = x^2 - 7y^2$

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2. EUCLID'S ALGORITHM AND THUE'S THEOREM

*Euclid's algorithm.* Let  $a$  and  $b$  be natural numbers,  $a > b$ , where  $b$  does not divide  $a$ . Let  $r_0 = a$ ,  $r_1 = b$  and for  $1 \leq k \leq n$ ,  $r_{k-1} = r_k q_k + r_{k+1}$ , where  $0 < r_{k+1} < r_k$  and  $r_n = 0$ . Define sequences  $s_0, s_1, \dots, s_{n+1}$  and  $t_0, t_1, \dots, t_{n+1}$  by

$$s_0 = 1, s_1 = 0, t_0 = 0, t_1 = 1, t_{k-1} = t_k q_k + t_{k+1}, s_{k-1} = s_k q_k + s_{k+1},$$

for  $1 \leq k \leq n$ . Then the following are easily proved by induction:

- (i)  $s_k = (-1)^k |s_k|$ ,  $t_k = (-1)^{k+1} |t_k|$ ;
- (ii)  $0 = |s_1| < |s_2| < \dots < |s_{n+1}|$ ;
- (iii)  $1 = |t_1| < |t_2| < \dots < |t_{n+1}|$ ;
- (iv)  $a = |t_k| r_{k-1} + |t_{k-1}| r_k$  for  $1 \leq k \leq n + 1$ ;
- (v)  $r_k = s_k a + t_k b$  for  $1 \leq k \leq n + 1$ .

**Theorem 1** (Thue). *Let  $a$  and  $b$  be integers,  $a > b > 1$  with  $\gcd(a, b) = 1$ . Then the congruence  $bx \equiv y \pmod{a}$  has a solution in nonzero integers  $x$  and  $y$  satisfying  $|x| < \sqrt{a}$ ,  $|y| \leq \sqrt{a}$ .*

*Proof.* As  $r_n = \gcd(a, b) = 1$  and  $a > \sqrt{a} > 1$  and the remainders  $r_0, \dots, r_n$  in Euclid's algorithm decrease strictly to 1, there is a unique index  $k$  such that  $r_{k-1} > \sqrt{a} \geq r_k$ . Then the equation  $a = |t_k| r_{k-1} + |t_{k-1}| r_k$  gives  $a \geq |t_k| r_{k-1} > |t_k| \sqrt{a}$ . Hence  $|t_k| < \sqrt{a}$ .

Finally,  $r_k = s_k a + t_k b$ , so  $bt_k \equiv r_k \pmod{a}$  and we can take  $x = t_k, y = r_k$ .  $\square$

3. THE EQUATION  $x^2 - Dy^2 = \kappa N$  WITH SMALL  $\kappa$

Let  $N \geq 1$  be an odd integer,  $D > 1$  and not a perfect square. Then a necessary condition for solvability of the equation  $x^2 - Dy^2 = \kappa N$  with  $\gcd(x, y) = 1$  is that the congruence  $u^2 \equiv D \pmod{N}$  shall be soluble. From now on we assume this, together with  $\gcd(D, N) = 1$  and  $1 < u < N$ . Then the Jacobi symbol  $(\frac{D}{N}) = 1$ . We note that if  $N$  is prime, then  $(\frac{D}{N}) = 1$  also implies that  $u^2 \equiv D \pmod{N}$  is soluble.

If we take  $a = N$  and  $b = u$  in Euclid's algorithm, the integers  $r_k^2 - Dt_k^2$  decrease strictly for  $k = 0, \dots, n$ , from  $a^2$  to  $1 - Dt_n^2$  and are always multiples of  $N$ . For

$$r_k^2 - Dt_k^2 \equiv t_k^2 u^2 - Dt_k^2 \equiv t_k^2 (u^2 - D) \equiv 0 \pmod{N}.$$

If  $k$  is chosen so that  $r_{k-1} > \sqrt{N} > r_k$ , as in the proof of Thue's theorem, then as

$$(1) \quad N = r_{k-1} |t_k| + r_k |t_{k-1}| > r_{k-1} |t_k|,$$

we have  $|t_k| < \sqrt{N}$  and

$$(2) \quad -DN < r_k^2 - Dt_k^2 < N.$$

Hence  $r_k^2 - Dt_k^2 = -lN$ ,  $-1 < l < D$ . In fact  $1 \leq l < D$ . Hence

$$(3) \quad -DN < r_k^2 - Dt_k^2 \leq -N.$$

Also  $r_k^2 + lN = Dt_k^2$  and hence  $Dt_k^2 > lN$ . Hence

$$(4) \quad |t_k| > \sqrt{\frac{lN}{D}}.$$

From equation (1),  $N > r_{k-1} |t_k|$  and hence inequality (4) implies

$$(5) \quad r_{k-1} < \sqrt{\frac{DN}{l}}.$$

4. THE EQUATION  $x^2 - 2y^2 = \pm N$

The assumption  $(\frac{2}{N}) = 1$  is equivalent to  $N \equiv \pm 1 \pmod{8}$ . Also  $1 \leq l < 2$ , so  $l = 1$  and (3) gives  $r_k^2 - 2t_k^2 = -N$ . Hence from equation (5) with  $D = 2$ ,  $r_{k-1} < \sqrt{2N}$  and

$$-N = r_k^2 - 2t_k^2 < r_{k-1}^2 - 2t_{k-1}^2 < r_{k-1}^2 < 2N.$$

Hence  $r_{k-1}^2 - 2t_{k-1}^2 = N$ .

**Example.** Let  $N = 10000000033$ , a prime of the form  $8n + 1$ . Then  $u = 87196273$  gives  $k = 10$ ,  $r_{10} = 29015$ ,  $t_{10} = -73627$ ,  $r_9 = 118239$ ,  $t_9 = 44612$  and  $r_{10}^2 - 2t_{10}^2 = -N$ ,  $r_9^2 - 2t_9^2 = N$ .

*Remark.* We can express  $r_{k-1}$  and  $t_{k-1}$  in terms of  $r_k$  and  $t_k$ . The method is useful later for delineating cases when  $D = 5, 6, 7$ :

Using the identities

$$(6) \quad (r_k r_{k-1} - D t_k t_{k-1})^2 - D(t_k r_{k-1} - t_{k-1} r_k)^2 = (r_k^2 - D t_k^2)(r_{k-1}^2 - D t_{k-1}^2)$$

and

$$(7) \quad (-1)^k N = r_k t_{k-1} - r_{k-1} t_k,$$

we deduce that

$$(8) \quad r_k r_{k-1} - D t_k t_{k-1} = \epsilon N,$$

where  $\epsilon = \pm 1$ .

From equation (8), we see that  $\epsilon = 1$ , as  $t_k t_{k-1} < 0$ . Hence

$$(9) \quad r_k r_{k-1} + D T_k T_{k-1} = N,$$

where  $T_k = |t_k|$ . Then solving equations (7) and (9) with  $D = 2$  for  $r_{k-1}$  and  $T_{k-1}$  yields

$$r_{k-1} = -r_k + 2T_k, \quad T_{k-1} = T_k - r_k,$$

5. THE EQUATION  $x^2 - 3y^2 = \pm N$

The assumption  $(\frac{3}{N}) = 1$  is equivalent to  $N \equiv \pm 1 \pmod{12}$ . From equation (3), we have  $-3N < r_k^2 - 3t_k^2 \leq -N$ . Hence  $r_k^2 - 3t_k^2 = -2N$  or  $-N$ .

*Case 1.* Assume  $N \equiv 1 \pmod{12}$ . Then  $r_k^2 - 3t_k^2 = -N$  would imply the contradiction  $r_k^2 \equiv -1 \pmod{3}$ .

Hence  $r_k^2 - 3t_k^2 = -2N$  and inequality (5) implies  $r_{k-1} < \sqrt{\frac{3N}{2}}$ . Hence

$$-2N = r_k^2 - 3t_k^2 < r_{k-1}^2 - 3t_{k-1}^2 < r_{k-1}^2 < \frac{3N}{2}.$$

Consequently  $r_{k-1}^2 - 3t_{k-1}^2 = N$ .

We find  $2r_{k-1} = -r_k + 3T_k$  and  $2T_{k-1} = -r_k + T_k$ .

*Case 2.* Assume  $N \equiv -1 \pmod{12}$ . Then  $r_k^2 - 3t_k^2 = -2N$  would imply the contradiction  $r_k^2 \equiv 2 \pmod{3}$ . Hence  $r_k^2 - 3t_k^2 = -N$  and inequality (5) implies  $r_{k-1} < \sqrt{3N}$ . Hence

$$-N = r_k^2 - 3t_k^2 < r_{k-1}^2 - 3t_{k-1}^2 < r_{k-1}^2 < 3N.$$

Consequently  $r_{k-1}^2 - 3t_{k-1}^2 = N$  or  $2N$ . However  $r_{k-1}^2 - 3t_{k-1}^2 = N$  implies the contradiction  $r_{k-1}^2 \equiv -1 \pmod{3}$ . Hence  $r_{k-1}^2 - 3t_{k-1}^2 = 2N$ .

We find  $r_{k-1} = -r_k + 3T_k$  and  $T_{k-1} = -r_k + T_k$ .

6. THE EQUATION  $x^2 - 5y^2 = \pm N$

The assumption  $(\frac{5}{N}) = 1$  is equivalent to  $N \equiv \pm 1 \pmod{5}$ . Then from equation (3), we have  $-5N < r_k^2 - 5t_k^2 \leq -N$ . Hence  $r_k^2 - 5t_k^2 = -4N, -3N, -2N$  or  $-N$ .

We cannot have  $r_k^2 - 5t_k^2 = -3N$  as then  $(\frac{5}{3}) = 1$ . Neither can we have  $r_k^2 - 5t_k^2 = -2N$ , as  $N$  is odd.

*Case 1.* Assume  $N \equiv 1 \pmod{5}$ . Then  $r_k^2 - 5t_k^2 = -N$  would imply the contradiction  $r_k^2 \equiv -1 \pmod{5}$ . Hence  $r_k^2 - 5t_k^2 = -4N$ . Then  $r_k$  and  $t_k$  are both odd. Also inequality (5) implies  $r_{k-1} < \sqrt{\frac{5N}{4}}$ . Hence  $-N \leq r_{k-1}^2 - 5t_{k-1}^2 \leq N$ .

Then as in the remark above, we can show

- (i) if  $r_{k-1}^2 - 5t_{k-1}^2 = -N$ , then

$$4r_{k-1} = -3r_k + 5T_k, \quad 4T_{k-1} = -r_k + 3T_k$$

and hence  $r_k \equiv -T_k \pmod{4}$ ;

- (ii) if  $r_{k-1}^2 - 5t_{k-1}^2 = N$ , then

$$4r_{k-1} = -r_k + 5T_k, \quad 4T_{k-1} = -r_k + T_k$$

and hence  $r_k \equiv T_k \pmod{4}$ .

*Case 2.* Assume  $N \equiv -1 \pmod{5}$ . Then  $r_k^2 - 5t_k^2 = -4N$  would imply the contradiction  $r_k^2 \equiv 4 \pmod{5}$ . Hence  $r_k^2 - 5t_k^2 = -N$ . Then not both  $r_k$  and  $t_k$  are odd. Also inequality (5) implies  $r_{k-1} < \sqrt{5N}$  and we deduce that  $-N < r_{k-1}^2 - 5t_{k-1}^2 \leq 4N$ . Consequently  $r_{k-1}^2 - 5t_{k-1}^2 = N$  or  $4N$ .

Then as in the remark above, we can show

- (i) if  $r_{k-1}^2 - 5t_{k-1}^2 = N$ , then

$$r_{k-1} = -2r_k + 5T_k, \quad T_{k-1} = -r_k + 2T_k$$

and hence  $r_{k-1} \equiv -2r_k \pmod{5}$ ;

- (ii) if  $r_{k-1}^2 - 5t_{k-1}^2 = 4N$ , then

$$r_{k-1} = -r_k + 5T_k, \quad T_{k-1} = -r_k + T_k$$

and hence  $r_{k-1} \equiv -r_k \pmod{5}$ .

Here is a complete classification of the possible cases:

1.  $N = 5k + 1$ . Then  $r_k^2 - 5t_k^2 = -4N$ , while  $r_k$  and  $t_k$  are odd.
  - (i)  $r_k \equiv -T_k \pmod{4}$ . Then  $r_{k-1}^2 - 5t_{k-1}^2 = -N$ .
  - (ii)  $r_k \equiv T_k \pmod{4}$ . Then  $r_{k-1}^2 - 5t_{k-1}^2 = N$ .
2.  $N = 5k - 1$ . Then  $r_k^2 - 5t_k^2 = -N$ , while  $r_k$  and  $t_k$  are not both odd.
  - (i)  $r_{k-1} \equiv -2r_k \pmod{5}$ . Then  $r_{k-1}^2 - 5t_{k-1}^2 = N$ .
  - (ii)  $r_{k-1} \equiv -r_k \pmod{5}$ . Then  $r_{k-1}^2 - 5t_{k-1}^2 = 4N$ .

7. THE EQUATION  $x^2 - 6y^2 = \pm N$

The assumption  $(\frac{6}{N}) = 1$  is equivalent to  $N \equiv \pm 1 \pmod{24}$  or  $N \equiv \pm 5 \pmod{24}$ . Then from equation (3), we have  $-6N < r_k^2 - 6t_k^2 \leq -N$ . Hence  $r_k^2 - 6t_k^2 = -5N, -4N, -3N, -2N$  or  $-N$ . Only  $-4N$  is ruled out immediately and the other possibilities can occur.

As with the case  $D = 5$ , there is a complete classification of the possible cases:

1.  $N = 24k - 1$  or  $24k + 5$ .
  - (i)  $r_k \equiv 0 \pmod{3}$ . Then  $r_k^2 - 6t_k^2 = -3N$ ,  $r_{k-1}^2 - 6t_{k-1}^2 = -N$ .

- (ii)  $r_k \not\equiv 0 \pmod{3}$ . Then  $r_k^2 - 6t_k^2 = -N$ .
  - (a)  $r_{k-1} \equiv 0 \pmod{2}$ . Then  $r_{k-1}^2 - 6t_{k-1}^2 = 2N$ .
  - (b)  $r_{k-1} \equiv 1 \pmod{2}$ . Then  $r_{k-1}^2 - 6t_{k-1}^2 = 5N$ .
- 2.  $N = 24k + 1$  or  $24k - 5$ :
  - (i)  $r_k \equiv 0 \pmod{2}$ . Then  $r_k^2 - 6t_k^2 = -2N$ ,  $r_{k-1}^2 - 6t_{k-1}^2 = N$ .
  - (ii)  $r_k \equiv 1 \pmod{2}$ . Then  $r_k^2 - 6t_k^2 = -5N$ .
    - (a)  $r_k \equiv T_k \pmod{5}$ . Then  $r_{k-1}^2 - 6t_{k-1}^2 = N$ .
    - (b)  $r_k \equiv -T_k \pmod{5}$ . Then

$$r_{k-1}^2 - 6t_{k-1}^2 = -2N, \quad r_{k-2}^2 - 6t_{k-2}^2 = N.$$

8. THE EQUATION  $x^2 - 7y^2 = \pm N$

The assumption  $(\frac{7}{N}) = 1$  is equivalent to  $N \equiv 1, 3, 9, 19, 25, 27 \pmod{28}$ .  
 As with the case  $D = 6$ , there is a complete classification of the possible cases:

- 1.  $N = 28k + 1, 28k + 9$ , or  $28k + 25$ .
  - (i)  $r_k \equiv T_k \pmod{2}$ . Then  $r_k^2 - 7t_k^2 = -6N$ .
    - (a)  $r_k \equiv -T_k \pmod{6}$ . Then  $r_{k-1}^2 - 7t_{k-1}^2 = -3N$ .
      - (1)  $r_{k-1} \equiv -T_{k-1} \pmod{3}$ . Then  $r_{k-2}^2 - 7t_{k-2}^2 = N$ .
      - (2)  $r_{k-1} \equiv T_{k-1} \pmod{3}$ . Then  $r_{k-2}^2 - 7t_{k-2}^2 = 2N$ .
    - (b)  $r_k \equiv T_k \pmod{6}$ . Then  $r_{k-1}^2 - 7t_{k-1}^2 = N$ .
  - (ii)  $r_k \not\equiv T_k \pmod{2}$ . Then  $r_k^2 - 7t_k^2 = -3N$ .
    - (a)  $r_k \equiv -T_k \pmod{3}$ . Then  $r_{k-1}^2 - 7t_{k-1}^2 = N$ .
    - (b)  $r_k \equiv T_k \pmod{3}$ . Then  $r_{k-1}^2 - 7t_{k-1}^2 = 2N$ .
- 2.  $N = 28k + 3, 28k + 19$ , or  $28k + 27$ .
  - (i)  $r_k \equiv T_k \pmod{2}$ . Then  $r_k^2 - 7t_k^2 = -2N$ .
    - (a)  $r_{k-1} \equiv -T_{k-1} \pmod{3}$ . Then  $r_{k-1}^2 - 7t_{k-1}^2 = -N$ .
    - (b)  $r_{k-1} \equiv T_{k-1} \pmod{3}$ . Then  $r_{k-1}^2 - 7t_{k-1}^2 = 3N$ .
  - (ii)  $r_k \not\equiv T_k \pmod{2}$ . Then  $r_k^2 - 7t_k^2 = -N$ .
    - (a)  $r_{k-1} \equiv -T_{k-1} \pmod{3}$ . Then  $r_{k-1}^2 - 7t_{k-1}^2 = 3N$ .
    - (b)  $r_{k-1} \equiv T_{k-1} \pmod{3}$ . Then  $r_{k-1}^2 - 7t_{k-1}^2 = 6N$ .

In cases 1(a)(2) and 2(i), the equations  $r_{k-2}^2 - 7t_{k-2}^2 = 2N$  and  $r_k^2 - 7t_k^2 = -2N$  give rise to equations  $x^2 - 7y^2 = N, -N$ , respectively, if we write  $x + y\sqrt{7} = (r_{k-2} + t_{k-2}\sqrt{7})/(3 + \sqrt{7})$  and  $(r_k + t_k\sqrt{7})/(3 + \sqrt{7})$ , respectively. For if  $x + y\sqrt{7} = (r + t\sqrt{7})/(3 + \sqrt{7})$ , where  $r$  and  $t$  are odd, then  $x = \frac{3r-7t}{2}$  and  $y = \frac{3t+r}{2}$  are integers and  $x^2 - 7y^2 = (r^2 - 7t^2)/2$ .

We note that 1(a)(2) cannot occur unless  $N \equiv 0 \pmod{3}$  for we have

$$(10) \quad r_{k-1} = \frac{-5r_k + 7T_k}{6}, \quad T_{k-1} = \frac{-r_k + 5T_k}{6}$$

$$(11) \quad r_{k-2} = \frac{-r_{k-1} + 7T_{k-1}}{3}, \quad T_{k-2} = \frac{-r_{k-1} + T_{k-1}}{3}.$$

Then (10) implies  $r_{k-1} + T_{k-1} = -r_k + 2T_k \equiv -r_k - T_k \equiv 0 \pmod{3}$ . Also (11) implies  $r_{k-1} \equiv T_{k-1} \pmod{3}$ . Hence 3 divides  $r_{k-1}$  and  $T_{k-1}$ , and the equation  $r_{k-1}^2 - 7t_{k-1}^2 = -3N$  then implies 3 divides  $N$ .

**Example.**  $N = 57$ . The congruence  $u^2 \equiv 7 \pmod{57}$  has solutions  $u \equiv \pm 8, \pm 11 \pmod{57}$ . Then  $u = 8$  gives  $k = 2, r_1 = 8, t_1 = 1, r_2 = 1, t_2 = -7, r_k^2 - 7t_k^2 = -6N$  and  $r_{k-1}^2 - 7t_{k-1}^2 = N$ , while  $u = 11$  gives  $k = 2, r_1 = 11, t_1 = 1, r_2 = 2, t_2 = -5$  and  $r_k^2 - 7t_k^2 = -3N$  and  $r_{k-1}^2 - 7t_{k-1}^2 = 2N$ .

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