THUE’S THEOREM
AND THE DIOPHANTINE EQUATION \( x^2 - Dy^2 = \pm N \)

KEITH MATTHEWS

Abstract. A constructive version of a theorem of Thue is used to provide representations of certain integers as \( x^2 - Dy^2 \), where \( D = 2, 3, 5, 6, 7 \).

1. Introduction

The idea of using Euclid’s algorithm to construct solutions of \( p = x^2 + y^2 \) goes back to Serret [9] and Hermite [5]. (Also see Wagon [12] and Brillhart [1].) The method easily extends to \( p = x^2 + ny^2 \), \( n = 2, 3, 5 \). (See Wilker [13] for \( n = 5 \).)

Cornacchia [2, pp. 61–66] generalised the method to \( N = x^2 + ny^2 \), \( n \geq 1 \) and discussed the case \( n < 0 \) [2, pp. 66–70]. (Also see Nitaj [8] and Hardy, Muskat and Williams [3, 4], Muskat [6], Williams [14, 15].)

It is not so well known that the Serret–Hermite method can be used to find explicit solutions of \( x^2 - Dy^2 = N \) when \( D > 1 \) is small. Nagell [7, pp. 210–212] used a nonconstructive form of a theorem of Thue [10, p. 587] to deal with \( D = 2 \) and 3, while a variant of Thue’s theorem was also used in Uspensky and Heaslet [11, pp. 352–368] for \( D = 2, 3, 5 \).

In this paper we show how to obtain explicit representations of certain integers in the form \( x^2 - Dy^2 \) for small \( D > 1 \), using a constructive version of Thue’s theorem based on Euclid’s algorithm. Amongst other things, if \( u^2 \equiv D \) (mod \( N \)), \( D \neq 1 \) (mod \( N \)) is soluble and gcd\( (D, N) = 1 \), \( N \) odd, we show how to find the following representations:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( x^2 - 2y^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8k ± 1</td>
<td>( x^2 - 2y^2 )</td>
</tr>
<tr>
<td>12k + 1</td>
<td>( x^2 - 3y^2 )</td>
</tr>
<tr>
<td>12k - 1</td>
<td>( x^2 - 3y^2 )</td>
</tr>
<tr>
<td>5k + 1</td>
<td>( x^2 - 5y^2 )</td>
</tr>
<tr>
<td>5k - 1</td>
<td>( x^2 - 5y^2 )</td>
</tr>
<tr>
<td>24k + 1 or 24k - 5</td>
<td>( x^2 - 6y^2 )</td>
</tr>
<tr>
<td>24k - 1 or 24k + 5</td>
<td>( x^2 - 6y^2 )</td>
</tr>
<tr>
<td>28k + 1, 28k + 9 or 28k + 25</td>
<td>( x^2 - 7y^2 )</td>
</tr>
<tr>
<td>28k - 1, 28k - 9 or 28k - 25</td>
<td>( x^2 - 7y^2 )</td>
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</table>
2. Euclid’s algorithm and Thue’s theorem

Euclid’s algorithm. Let \( a \) and \( b \) be natural numbers, \( a > b \), where \( b \) does not divide \( a \). Let \( r_0 = a \), \( r_1 = b \) and for \( 1 \leq k \leq n \), \( r_{k-1} = r_kq_k + r_{k+1} \), where \( 0 < r_{k+1} < r_k \) and \( r_n = 0 \). Define sequences \( s_0, s_1, \ldots, s_{n+1} \) and \( t_0, t_1, \ldots, t_{n+1} \) by

\[
\begin{align*}
  s_0 &= 1, s_1 = 0, t_0 = 0, t_1 = 1, t_{k-1} = t_kq_k + t_{k+1}, s_{k-1} = s_kq_k + s_{k+1}, \\
  r_k &= s_k a + t_k b \text{ for } 1 \leq k \leq n.
\end{align*}
\]

for \( 1 \leq k \leq n \). Then the following are easily proved by induction:

(i) \( s_k = (-1)^k |s_k|, t_k = (-1)^{k+1} |t_k| \);
(ii) \( 0 = |s_1| < |s_2| < \cdots < |s_{n+1}| \);
(iii) \( 1 = |t_1| < |t_2| < \cdots < |t_{n+1}| \);
(iv) \( a = t_k r_{k-1} + t_{k-1} r_k \) for \( 1 \leq k \leq n + 1 \);
(v) \( r_k = s_k a + t_k b \) for \( 1 \leq k \leq n + 1 \).

**Theorem 1** (Thue). Let \( a \) and \( b \) be integers, \( a > b > 1 \) with \( \gcd(a, b) = 1 \). Then the congruence \( bx \equiv y \pmod{a} \) has a solution in nonzero integers \( x \) and \( y \) satisfying \( |x| < \sqrt{a}, |y| \leq \sqrt{a} \).

Proof. As \( r_n = \gcd(a, b) = 1 \) and \( a > \sqrt{a} > 1 \), and the remainders \( r_0, \ldots, r_n \) in Euclid’s algorithm decrease strictly to 1, there is a unique index \( k \) such that \( r_{k-1} > \sqrt{a} \). Then the equation \( a = t_k r_{k-1} + t_{k-1} r_k \) gives \( a \geq |t_k| r_{k-1} > |t_k| \sqrt{a} \).

Hence \( |t_k| < \sqrt{a} \).

Finally, \( r_k = s_k a + t_k b \), so \( b t_k \equiv r_k \pmod{a} \) and we can take \( x = t_k, y = r_k \). \( \square \)

3. The equation \( x^2 - Dy^2 = \kappa N \) with small \( \kappa \)

Let \( N \geq 1 \) be an odd integer, \( D > 1 \) and not a perfect square. Then a necessary condition for solvability of the equation \( x^2 - Dy^2 = \kappa N \) with \( \gcd(D, N) = 1 \) and \( 1 < u < N \) is that the congruence \( u^2 \equiv D \pmod{N} \) shall be soluble. From now on we assume this, together with \( \gcd(D, N) = 1 \) and \( 1 < u < N \). Then the Jacobi symbol \( \left( \frac{D}{N} \right) = 1 \).

We note that if \( N \) is prime, then \( \left( \frac{D}{N} \right) = 1 \) also implies that \( u^2 \equiv D \pmod{N} \) is soluble.

If we take \( a = N \) and \( b = u \) in Euclid’s algorithm, the integers \( r_k^2 - D t_k^2 \) decrease strictly for \( k = 0, \ldots, n \), from \( a^2 \) to \( 1 - D t_n^2 \) and are always multiples of \( N \). For

\[
  r_k^2 - D t_k^2 \equiv t_k^2(u^2 - D) \equiv \left( t_k \mod D \right) = 0 \quad \text{(mod \( N \))}.
\]

If \( k \) is chosen so that \( r_{k-1} > \sqrt{N} > r_k \), as in the proof of Thue’s theorem, then as

\[
N = r_{k-1} |t_k| + r_k |t_{k-1}| > r_{k-1} |t_k|,
\]

we have \( |t_k| < \sqrt{N} \) and

\[
-D N < r_k^2 - D t_k^2 < N.
\]

Hence \( r_k^2 - D t_k^2 = -l N \), \( -1 < l < D \). In fact \( 1 \leq l < D \). Hence

\[
-D N < r_k^2 - D t_k^2 \leq -N.
\]

Also \( r_k^2 + l N = D t_k^2 \) and hence \( D t_k^2 > l N \). Hence

\[
|t_k| > \sqrt{\frac{LN}{D}}.
\]

From equation (1), \( N > r_{k-1} |t_k| \) and hence inequality (4) implies

\[
r_{k-1} < \sqrt{\frac{DN}{l}}.
\]
4. The equation \( x^2 - 2y^2 = \pm N \)

The assumption \( \left( \frac{2}{N} \right) = 1 \) is equivalent to \( N \equiv \pm 1 \pmod{8} \). Also \( 1 \leq l < 2 \), so \( l = 1 \) and \( (3) \) gives \( r_k^2 - 2t_k^2 = -N \). Hence from equation \( (5) \) with \( D = 2 \),

\[
r_{k-1} < \sqrt{2N}
\]

and

\[
-N = r_k^2 - 2t_k^2 < r_{k-1}^2 - 2t_{k-1}^2 < r_{k-1}^2 < 2N.
\]

Hence \( r_{k-1}^2 - 2t_{k-1}^2 = N \).

**Example.** Let \( N = 10000000033 \), a prime of the form \( 8n + 1 \). Then \( u = 87196273 \) gives \( k = 10 \), \( r_{10} = 29015 \), \( t_{10} = -73627 \), \( r_9 = 118239 \), \( t_9 = 44612 \) and \( r_{10}^2 - 2t_{10}^2 = -N \), \( r_9^2 - 2t_9^2 = N \).

**Remark.** We can express \( r_{k-1} \) and \( t_{k-1} \) in terms of \( r_k \) and \( t_k \). The method is useful later for delineating cases when \( D = 5, 6, 7 \):

Using the identities

\[
(r_kr_{k-1} - Dtk_{k-1})^2 - D(t_kr_{k-1} - tk_{k-1}r_k)^2 = (r_k^2 - Dtk_k^2)(r_{k-1}^2 - Dt_{k-1}^2)
\]

and

\[
(-1)^kN = r_kr_{k-1} - r_{k-1}t_k,
\]

we deduce that

\[
r_kr_{k-1} - Dtk_{k-1} = \epsilon N,
\]

where \( \epsilon = \pm 1 \).

From equation \( (8) \), we see that \( \epsilon = 1 \), as \( t_kt_{k-1} < 0 \). Hence

\[
r_kr_{k-1} + Dtk_{k-1} = N,
\]

where \( T_k = |t_k| \). Then solving equations \( (7) \) and \( (9) \) with \( D = 2 \) for \( r_{k-1} \) and \( T_{k-1} \) yields

\[
r_{k-1} = -r_k + 2T_k, \quad T_{k-1} = T_k - r_k.
\]

5. The equation \( x^2 - 3y^2 = \pm N \)

The assumption \( \left( \frac{3}{N} \right) = 1 \) is equivalent to \( N \equiv \pm 1 \pmod{12} \). From equation \( (13) \), we have \(-3N < r_k^2 - 3t_k^2 \leq -N \). Hence \( r_k^2 - 3t_k^2 = -2N \) or \(-N \).

**Case 1.** Assume \( N \equiv 1 \pmod{12} \). Then \( r_k^2 - 3t_k^2 = -N \) would imply the contradiction \( r_k^2 \equiv -1 \pmod{3} \).

Hence \( r_k^2 - 3t_k^2 = -2N \) and inequality \( (13) \) implies \( r_{k-1} < \frac{3N}{2} \). Hence

\[
-2N = r_k^2 - 3t_k^2 < r_{k-1}^2 - 3t_{k-1}^2 < r_{k-1}^2 < \frac{3N}{2}.
\]

Consequently \( r_{k-1}^2 - 3t_{k-1}^2 = N \).

We find \( 2r_{k-1} = -r_k + 3T_k \) and \( 2T_{k-1} = -r_k + T_k \).

**Case 2.** Assume \( N \equiv -1 \pmod{12} \). Then \( r_k^2 - 3t_k^2 = -2N \) would imply the contradiction \( r_k^2 \equiv 2 \pmod{3} \). Hence \( r_k^2 - 3t_k^2 = -N \) and inequality \( (13) \) implies \( r_{k-1} < \sqrt{3N} \). Hence

\[
-N = r_k^2 - 3t_k^2 < r_{k-1}^2 - 3t_{k-1}^2 < r_{k-1}^2 < 3N.
\]

Consequently \( r_{k-1}^2 - 3t_{k-1}^2 = N \) or \( 2N \). However \( r_{k-1}^2 - 3t_{k-1}^2 = N \) implies the contradiction \( r_{k-1}^2 \equiv -1 \pmod{3} \). Hence \( r_{k-1}^2 - 3t_{k-1}^2 = 2N \).

We find \( r_{k-1} = -r_k + 3T_k \) and \( T_{k-1} = -r_k + T_k \).
6. The equation \( x^2 - 5y^2 = \pm N \)

The assumption \( \left( \frac{a}{N} \right) = 1 \) is equivalent to \( N \equiv \pm 1 \pmod{5} \). Then from equation (3), we have \(-5N < r_k^2 - 5t_k^2 \leq -N\). Hence \( r_k^2 - 5t_k^2 = -4N, -3N, -2N \) or \(-N\).

We cannot have \( r_k^2 - 5t_k^2 = -3N \) as then \( \left( \frac{a}{N} \right) = 1 \). Neither can we have \( r_k^2 - 5t_k^2 = -2N \), as \( N \) is odd.

Case 1. Assume \( N \equiv 1 \pmod{5} \). Then \( r_k^2 - 5t_k^2 = -N \) would imply the contradiction \( r_k^2 \equiv -1 \pmod{5} \). Hence \( r_k^2 - 5t_k^2 = -4N \). Then \( r_k \) and \( t_k \) are both odd. Also inequality (3) implies \( r_k-1 < \sqrt{\frac{5N}{4}} \). Hence \(-N \leq r_k^2 - 5t_k^2 \leq N \).

Then as in the remark above, we can show
(i) if \( r_k^2 - 5t_k^2 = -N \), then
\[
4r_k-1 = -3r_k + 5T_k, \quad 4T_k = -r_k + 3T_k
\]
and hence \( r_k \equiv -T_k \pmod{4} \);
(ii) if \( r_k^2 - 5t_k^2 = N \), then
\[
4r_k-1 = -r_k + 5T_k, \quad 4T_k = -r_k + T_k
\]
and hence \( r_k \equiv T_k \pmod{4} \).

Case 2. Assume \( N \equiv -1 \pmod{5} \). Then \( r_k^2 - 5t_k^2 = -4N \) would imply the contradiction \( r_k^2 \equiv 4 \pmod{5} \). Hence \( r_k^2 - 5t_k^2 = -N \). Then not both \( r_k \) and \( t_k \) are odd. Also inequality (3) implies \( r_k-1 < \sqrt{\frac{5N}{4}} \) and we deduce that \(-N < r_k^2 - 5t_k^2 \leq 4N \). Consequently \( r_k^2 - 5t_k^2 = N \) or \(-4N \).

Then as in the remark above, we can show
(i) if \( r_k^2 - 5t_k^2 = -N \), then
\[
r_k-1 = -2r_k + 5T_k, \quad T_k = -r_k + 2T_k
\]
and hence \( r_k \equiv -2r_k \pmod{5} \);
(ii) if \( r_k^2 - 5t_k^2 = 4N \), then
\[
r_k-1 = -r_k + 5T_k, \quad T_k = -r_k + T_k
\]
and hence \( r_k \equiv -r_k \pmod{5} \).

Here is a complete classification of the possible cases:

1. \( N = 5k + 1 \). Then \( r_k^2 - 5t_k^2 = -4N \), while \( r_k \) and \( t_k \) are odd.
   (i) \( r_k \equiv -T_k \pmod{4} \). Then \( r_k^2 - 5t_k^2 = -N \).
   (ii) \( r_k \equiv T_k \pmod{4} \). Then \( r_k^2 - 5t_k^2 = N \).

2. \( N = 5k - 1 \). Then \( r_k^2 - 5t_k^2 = -N \), while \( r_k \) and \( t_k \) are odd.
   (i) \( r_k-1 \equiv -2r_k \pmod{5} \). Then \( r_k^2 - 5t_k^2 = N \).
   (ii) \( r_k-1 \equiv -r_k \pmod{5} \). Then \( r_k^2 - 5t_k^2 = 4N \).

7. The equation \( x^2 - 6y^2 = \pm N \)

The assumption \( \left( \frac{a}{N} \right) = 1 \) is equivalent to \( N \equiv \pm 1 \pmod{24} \) or \( N \equiv \pm 5 \pmod{24} \). Then from equation (3), we have \(-6N < r_k^2 - 6t_k^2 \leq -N\). Hence \( r_k^2 - 6t_k^2 = -5N, -4N, -3N, -2N \) or \(-N\). Only \(-4N \) is ruled out immediately and the other possibilities can occur.

As with the case \( D = 5 \), there is a complete classification of the possible cases:

1. \( N = 24k - 1 \) or \( 24k + 5 \).
   (i) \( r_k \equiv 0 \pmod{3} \). Then \( r_k^2 - 6t_k^2 = -3N \), \( r_k^2 - 6t_k^2 = -N \).
Then (10) implies (11)

8. The equation $x^2 - 7y^2 = \pm N$

The assumption $(\frac{N}{T}) = 1$ is equivalent to $N \equiv 1, 3, 9, 19, 25, 27 \pmod{28}$.

As with the case $D = 6$, there is a complete classification of the possible cases:

1. $N = 28k + 1, 28k + 9, \text{ or } 28k + 25$.
   (i) $r_k \equiv T_k \pmod{2}$. Then $r_k^2 - 7t_k^2 = -6N$.
      (a) $r_k \equiv -T_k \pmod{6}$. Then $r_{k-1}^2 - 7t_{k-1}^2 = -3N$.
      (1) $r_{k-1} \equiv -T_{k-1} \pmod{3}$. Then $r_{k-2}^2 - 7t_{k-2}^2 = N$.
      (2) $r_{k-1} \equiv T_{k-1} \pmod{3}$. Then $r_{k-2}^2 - 7t_{k-2}^2 = 2N$.
      (b) $r_k \equiv T_k \pmod{6}$. Then $r_{k-1}^2 - 7t_{k-1}^2 = N$.
   (ii) $r_k \not\equiv T_k \pmod{2}$. Then $r_k^2 - 7t_k^2 = -3N$.
      (a) $r_k \equiv -T_k \pmod{3}$. Then $r_{k-1}^2 - 7t_{k-1}^2 = -N$.
      (b) $r_k \equiv T_k \pmod{3}$. Then $r_{k-1}^2 - 7t_{k-1}^2 = 2N$.

2. $N = 28k + 3, 28k + 19, \text{ or } 28k + 27$.
   (i) $r_k \equiv T_k \pmod{2}$. Then $r_k^2 - 7t_k^2 = -2N$.
      (a) $r_{k-1} \equiv -T_{k-1} \pmod{3}$. Then $r_{k-2}^2 - 7t_{k-2}^2 = -N$.
      (b) $r_{k-1} \equiv T_{k-1} \pmod{3}$. Then $r_{k-2}^2 - 7t_{k-2}^2 = 3N$.
   (ii) $r_k \not\equiv T_k \pmod{2}$. Then $r_k^2 - 7t_k^2 = -N$.
      (a) $r_{k-1} \equiv -T_{k-1} \pmod{3}$. Then $r_{k-2}^2 - 7t_{k-2}^2 = 3N$.
      (b) $r_{k-1} \equiv T_{k-1} \pmod{3}$. Then $r_{k-2}^2 - 7t_{k-2}^2 = 6N$.

In cases 1(a)(2) and 2(i), the equations $r_{k-2}^2 - 7t_{k-2}^2 = 2N$ and $r_k^2 - 7t_k^2 = -2N$ give rise to equations $x^2 - 7y^2 = N$, $-N$, respectively, if we write $x + y\sqrt{7} = (r_{k-2} + t_{k-2}\sqrt{7})/(3 + \sqrt{7})$ and $(r_k + t_k\sqrt{7})/(3 + \sqrt{7})$, respectively. For if $x + y\sqrt{7} = (r + t\sqrt{7})/(3 + \sqrt{7})$, where $r$ and $t$ are odd, then $x = \frac{3r - 7t}{2}$ and $y = \frac{3t - r}{2}$ are integers and $x^2 - 7y^2 = (r^2 - 7t^2)/2$.

We note that 1(a)(2) cannot occur unless $N \equiv 0 \pmod{3}$ for we have

\begin{align*}
(10) \quad & r_{k-1} = \frac{-5r_k + 7T_k}{6}, \quad T_{k-1} = \frac{-r_k + 5T_k}{6} \\
(11) \quad & r_{k-2} = \frac{-r_{k-1} + 7T_{k-1}}{3}, \quad T_{k-2} = \frac{-r_{k-1} + T_{k-1}}{3}.
\end{align*}

Then (10) implies $r_{k-1} + T_{k-1} = -r_k + 2T_k \equiv -r_k - T_k \equiv 0 \pmod{3}$. Also (11) implies $r_{k-1} \equiv T_{k-1} \pmod{3}$. Hence 3 divides $r_{k-1}$ and $T_{k-1}$, and the equation $r_{k-2}^2 - 7t_{k-2}^2 = -3N$ then implies 3 divides $N$.

**Example.** $N = 57$. The congruence $u^2 \equiv 7 \pmod{57}$ has solutions $u \equiv \pm 8, \pm 11 \pmod{57}$. Then $u = 8$ gives $k = 2, r_1 = 8, t_1 = 1, r_2 = 1, t_2 = -7, r_k^2 - 7t_k^2 = -6N$ and $r_{k-1}^2 - 7t_{k-1}^2 = N$, while $u = 11$ gives $k = 2, r_1 = 11, t_1 = 1, r_2 = 2, t_2 = -5$ and $r_k^2 - 7t_k^2 = -3N$ and $r_{k-1}^2 - 7t_{k-1}^2 = 2N$. 


ACKNOWLEDGMENTS

The author is grateful to Christina Miller for noticing the decreasing property of the integers $r_k^2 - Dt_k^2$.

The author is also grateful to Dr. Terence Jackson for his comments on an earlier draft of the paper.

The calculations were carried out with the author’s number theory calculator program CALC and a UNIX bc program thue, both available at http://www.maths.uq.edu.au/~krm/.

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Department of Mathematics, University of Queensland, Brisbane, Australia, 4072
E-mail address: krm@maths.uq.edu.au