COMPUTATION OF CLASS NUMBERS
OF QUADRATIC NUMBER FIELDS

STÉPHANE LOUBOUTIN

Abstract. We explain how one can dispense with the numerical computation
of approximations to the transcendental integral functions involved when com-
puting class numbers of quadratic number fields. We therefore end up with a
simpler and faster method for computing class numbers of quadratic number
fields. We also explain how to end up with a simpler and faster method for
computing relative class numbers of imaginary abelian number fields.

1. Introduction

Currently, the best available rigorous methods for computing class numbers of
quadratic number fields $k$ of discriminants $d_k$ are of complexity $O(|d_k|^{0.5+\epsilon})$. How-
ever, assuming suitable forms of the generalized Riemann hypothesis, one can de-
vise conditional but more efficient methods of lower complexity (see [MoWi] where
a conditional method of complexity $O(d_k^{0.2+\epsilon})$ for computing class numbers of real
quadratic fields is developed, and see [Coh] Sections 5.5 and 5.9] where the condi-
tional sub-exponential methods of McCurley and J. Buchmann for computing class
groups of quadratic fields are developed). These rigorous methods stem from the
analytic class number formulae (see [Coh, Sections 5.3.3 and 5.6.2], [MoWi], [ScWa]
and [WiBr]) and require the computation to sufficient accuracy of the transcenden-
tal integral functions

$$E(z) = \int_z^\infty e^{-x}x^{-1}dx \quad \text{(the exponential integral function)}$$

and

$$\text{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-x^2}dx \quad \text{(the complementary error function)}$$

by using the following

power series expansions (if $z$ is small) and continued fractional expansions (if $z$ is
large):

$$\int_z^\infty e^{-x^2}dx = \frac{1}{2} \sqrt{\pi} - \sum_{n \geq 0} \frac{(-1)^n z^{2n+1}}{n!(2n+1)} \quad \text{(1)}$$

$$= \frac{1}{2} e^{-z^2} \left( \frac{1}{z} + \frac{1}{z+1} + \frac{2}{z+1+1} + \frac{3}{z+1+1+1} + \cdots \right) \quad \text{(2)}$$

$$\int_z^\infty e^{-x}x^{-1}dx = -\gamma - \log(z) - \sum_{n \geq 1} \frac{(-1)^n z^n}{n \cdot n!} \quad \text{(3)}$$

$$= e^{-z} \left( \frac{1}{z} + \frac{1}{z+1} + \frac{2}{z+1+1} + \frac{3}{z+1+1+1} + \cdots \right) \quad \text{(4)}$$

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(here $\gamma = 0.577\cdots$ denotes Euler’s constant). In this paper we explain how one can dispense with these evaluations, thus greatly simplifying the implementation of these rigorous methods for computing class numbers of quadratic number fields, and making them faster but still of the same complexity $O(d_k^{0.5+\epsilon})$. In contrast with what is usually done, we will write all our class number formulae at $s = 0$. Indeed, values at $s = 0$ of $L$-functions associated with odd Dirichlet characters are algebraic numbers, and we explained in [Lou3] how useful this observation is for computing their exact values from the computation of their numerical approximations (see also [Lou5] where we use [Lou4] to generalize the method developed in [Lou3] for computing relative class numbers of nonabelian CM-fields).

Let us now set some of the notation we will be using throughout this paper. We let $\chi$ be a primitive Dirichlet character modulo $f > 1$. We set

$$\alpha = \sqrt{f}, \quad e_n = e^{-\pi n^2/f} = e^{-n^2\alpha^2} \quad \text{and} \quad \tau(\chi) = \sum_{x=1}^{f-1} \chi(x)e^{2\pi xi/f}.$$ 

For $t > 0$ real and $M \geq 0$ real, we set

$$B(t, M, f) = \sqrt{f\left(t\log(f/\pi) + M\right)/\pi} = \alpha^{-1}\sqrt{M - 2t\log\alpha}$$

and assume $f \neq 3$, which implies $f > \pi$, $0 < \alpha < 1$, $B(t, M, f) \geq \sqrt{f/\pi}$ and $e^{-m^2\alpha^2} \leq \alpha^2 e^{-M}$ for $m \geq B(t, M, f)$. Finally, we will use:

**Lemma 1.** Let $\alpha > 0$ be real and $g$ of class $C^2$ in the range $]0, +\infty[ \chi$ be given. Then for any positive rational integer $n \geq 1$ we have

$$\int_{\alpha n}^{(n+1)\alpha} g = \alpha g(\alpha(n+1)\alpha) + g(\alpha n) \frac{\theta\alpha^2}{8} \int_{\alpha n}^{(n+1)\alpha} |g''| \quad (|\theta| \leq 1).$$

**Proof.** Set $B_2(x) = x(x-1)/2$. Then the reader will check that we have

$$\int_{\alpha n}^{(n+1)\alpha} g(x)dx = \alpha g(\alpha(n+1)\alpha) + g(\alpha n) + \alpha^3 \int_0^1 B_2(x)g''(\alpha(x+n))dx.$$ 

Now, the bound $0 \leq |B_2(x)| \leq 1/8$ for $0 \leq u \leq 1$ yields the desired result. \hfill $\Box$

2. Imaginary abelian number fields

Let $\chi$ be a primitive odd Dirichlet character modulo $f > 3$. Set $W(\chi) = i^{-1}\tau(\chi)/\sqrt{f}$, which has absolute value equal to one. We can express $L(0, \chi)$ as the limit of rapidly absolutely convergent series (see [Dav] or [Lou5])

$$L(0, \chi) = \frac{1}{\sqrt{\pi}} \left( \frac{W(\chi)}{\alpha} \sum_{n \geq 1} \frac{\chi(n)}{n} e_n + 2 \sum_{n \geq 1} \chi(n) \int_{\alpha n}^{\infty} e^{-x^2}dx \right)$$

and

$$L(0, \chi) = \frac{1}{\sqrt{\pi}} \left( \frac{W(\chi)}{\alpha} \sum_{n \geq 1} \frac{\chi(n)}{n} e_n + 2 \sum_{n \geq 1} S_n(\chi) \int_{\alpha n}^{(n+1)\alpha} e^{-x^2}dx \right).$$
Proposition 2 (Compare with Proposition 7). Let $\chi$ be a primitive odd Dirichlet character of conductor $f$. For some $\theta$ satisfying $|\theta| \leq 1$, it holds that

$$L(0, \chi) = \frac{1}{\sqrt{\pi}} \frac{W(\chi)}{\alpha} \sum_{n \geq 1} \frac{\bar{\chi}(n)}{n} e_n + \alpha \sum_{n \geq 1} (e_n + e_{n+1}) S_n(\chi) + \frac{3\theta}{8f^{1/2}}.$$ 

Proof. Applying Lemma 1 to $g(x) = e^{-x^2}$ and noticing that $|S_n(\chi)| \leq n$, we obtain

$$(8) \quad 2 \sum_{n \geq 1} S_n(\chi) \int_{n\alpha}^{(n+1)\alpha} e^{-x^2} \, dx = \alpha \sum_{n \geq 1} (e_n + e_{n+1}) S_n(\chi) + \frac{\theta\alpha}{4} R'$$

with $|\theta| \leq 1$ and

$$R' = \alpha \sum_{n \geq 1} n \int_{n\alpha}^{(n+1)\alpha} |g''|$$

$$= \alpha \sum_{n \geq 1} \int_{n\alpha}^{\infty} |g''|$$

$$\leq \alpha \int_{0}^{\infty} \left( \int_{\alpha u}^{\infty} |g''(x)| \, dx \right) \, du = \int_{0}^{\infty} x |g''(x)| \, dx \overset{\text{def}}{=} R_{\text{odd}}$$

which, for $g(x) = e^{-x^2}$, yields

$$R_{\text{odd}} = \int_{0}^{\infty} x |4x^2 - 2e^{-x^2} | \, dx = (4/\sqrt{e}) - 1 = 1.426 \cdots \leq 3/2.$$ 

Using (7), (8) and (9), we obtain the desired result. \hfill \square

Proposition 3. Assume $f > 3$, $t > 0$ and $M \geq 1$. For any positive rational integer $m \geq B(t, M, f)$ it holds that

$$\left| \frac{W(\chi)}{\alpha} \sum_{n \geq m} \frac{\bar{\chi}(n)}{n} e_n \right| \leq \frac{1}{\alpha} \sum_{n \geq m} \frac{1}{n} e_n \leq \frac{1}{2} \alpha^{2t-1} e^{-M}$$

and

$$\left| \alpha \sum_{n \geq m} (e_n + e_{n+1}) S_n(\chi) \right| \leq \alpha \sum_{n \geq m} 2ne_n \leq \alpha^{2t-1} e^{-M}.$$ 

Proof. For $m \geq B(t, M, f)$ we have

$$\frac{1}{\alpha} \sum_{n \geq m} \frac{1}{n} e^{-n^2\alpha^2} \leq \frac{1}{\alpha} \int_{m}^{\infty} x e^{-x^2\alpha^2} \frac{dx}{x^2}$$

$$\leq \frac{1}{\alpha m^2} \int_{m}^{\infty} x e^{-x^2\alpha^2} \, dx = \frac{e^{-m^2\alpha^2}}{2m^2\alpha^3} \leq \frac{\alpha^{2t-1} e^{-M}}{2(M - 2t \log \alpha)} \leq \alpha^{2t-1} e^{-M/2}$$

and

$$\alpha \sum_{n \geq m} 2ne^{-n^2\alpha^2} \leq 2\alpha \int_{m}^{\infty} x e^{-m^2x^2} \, dx = \frac{1}{\alpha} e^{-m^2\alpha^2} \leq \alpha^{2t-1} e^{-M}.$$ \hfill \square

According to Propositions 2 and 3, we obtain
\textbf{Theorem 4} (Compare with Theorem 3 below). Let \( M \geq 1 \) be given, let \( \chi \) be a primitive odd Dirichlet character of conductor \( f \), let \( m \) be the least rational integer greater than or equal to \( B(\frac{1}{f}, M, f) = O(f^{0.5+\epsilon}) \) and set

\[
L_M(0, \chi) = \frac{1}{\sqrt{\pi}} \left( W(\chi) \alpha \sum_{n=1}^{m} \chi(n) \frac{n}{\alpha} e_n + \alpha \sum_{n=1}^{m} (e_n + e_{n+1}) S_n(\chi) \right),
\]

where \( \alpha = \sqrt{\pi/f} \), \( e_n = e^{-\pi^2 n^2} \) and \( S_n(\chi) \) is defined in \( \text{(5)} \). Then,

\[
|L(0, \chi) - L_M(0, \chi)| \leq \frac{3}{2\sqrt{\pi}} e^{-M} + \frac{3}{8\sqrt{f}}.
\]

\textbf{Remark 5.}

1. Of particular importance is the case where \( \chi \) is the primitive quadratic odd Dirichlet character of conductor \( f = |d_k| \) associated with an imaginary quadratic field \( k \) of discriminant \( d_k < -4 \) and class number \( h_k \). Then, \( \bar{\chi} = \chi \), \( W(\chi) = +1 \), \( h_k = L(0, \chi) \) and Theorem 4 provides us with a much more satisfactory result than \( \text{Lou1 \ Theorem 1} \).

2. In practice, we do not compute all the \( e_n = \exp(-\pi n^2/f) \)'s for \( 1 \leq n \leq m \) by using the exponential function. It is more efficient to compute the \( e_n \)'s inductively by setting \( f_n = \exp(-\pi(2n+1)/f) \), by computing \( f_0 = \exp(-\pi/f) \) and \( h = \exp(-2\pi/f) \) and by using the induction formulae \( f_{n+1} = h f_n \) and \( e_{n+1} = e_n f_n \). In this process, at each step \( n \), instead of performing the computation of \( \exp(-\pi n^2/f) \) we only perform two multiplications.

3. We explained in \( \text{Lou3} \) how to compute relative class numbers of imaginary abelian number fields of a given degree by computing numerical approximations to linear combinations with bounded coefficients of values at \( s = 0 \) of \( L \)-functions associated with odd primitive Dirichlet characters. Therefore, combining Theorem 4 and the method developed in \( \text{Lou3} \), we end up with an efficient method for computing relative class numbers of imaginary abelian number fields of a given degree. This method does not require us to compute approximations to transcendental integral functions.

4. We also explained in \( \text{Lou3} \) how to compute relative class numbers of CM-fields by computing numerical approximations to linear combinations with bounded coefficients of values at \( s = 0 \) of Heckes’s \( L \)-functions associated with characters on strict ray class groups.

Therefore, in order to extend our present method further, we would like to find a method (generalizing Proposition 2 and Proposition 3) which would enable us to dispense with the computation of numerical approximations to the complicated integral transcendental functions involved when computing numerical approximations to values at \( s = 0 \) of such Hecke’s \( L \)-functions (see \( \text{Lou2} \) and \( \text{Lou4} \)).

3. Real quadratic number fields

Let \( \chi \) be a primitive even Dirichlet character modulo \( f > 3 \). Set \( W(\chi) = \tau(\chi)/\sqrt{f} \), which has absolute value equal to one. Then \( L(0, \chi) = 0 \) and we can express the derivative \( L'(0, \chi) \) as the limit of rapidly absolutely convergent series (use \( \text{Dav} \)):

\[
L'(0, \chi) = \sum_{n \geq 1} \chi(n) \int_{\alpha n}^{\infty} e^{-x^2} \frac{dx}{x} + \frac{W(\chi)}{\alpha} \sum_{n \geq 1} \frac{\chi(n)}{n} \int_{\alpha n}^{\infty} e^{-x^2} \frac{dx}{x}.
\]

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and

\[ L'(0, \chi) = \sum_{n \geq 1} S_n(\chi) \int_{n\alpha}^{(n+1)\alpha} e^{-x^2} dx \frac{e^{-x^2} dx}{x} + \frac{W(\chi)}{\alpha} \sum_{n \geq 1} T_n(\chi) \int_{n\alpha}^{(n+1)\alpha} e^{-x^2} dx. \]

**Lemma 6.** For \( n \geq 1 \), set

\[ u_n = \frac{\epsilon_n}{n} + \frac{\epsilon_{n+1}}{n+1} + 2 \log\left(\frac{n+1}{n}\right) - \frac{1}{n} - \frac{1}{n+1}. \]

For any \( m \geq 1 \) we have

\[ \sum_{n \geq 1} S_n(\chi) \int_{n\alpha}^{(n+1)\alpha} e^{-x^2} dx = \frac{1}{2} \sum_{n \geq 1} u_n S_n(\chi) + \theta \alpha \frac{R''}{4} \]

for some \( \theta \) satisfying \(|\theta| \leq 1\), and

\[ \frac{1}{\alpha} \sum_{n \geq 1} T_n(\chi) \int_{n\alpha}^{(n+1)\alpha} e^{-x^2} dx = \frac{1}{2} \sum_{n \geq 1} (\epsilon_n + \epsilon_{n+1}) T_n(\chi) + \theta \alpha \frac{\log(e/\alpha)}{4} \]

for some \( \theta \) satisfying \(|\theta| \leq 1\).

**Proof.** Set \( g(x) = (e^{-x^2} - 1)/x \). Then \( xg''(x) = (4x^2 + 2)e^{-x^2} + 2(e^{-x^2} - 1)/x^2 \)

and according to Lemma 6 we obtain

\[ \sum_{n=1}^{m} S_n(\chi) \int_{n\alpha}^{(n+1)\alpha} e^{-x^2} dx \]

\[ = \sum_{n=1}^{m} S_n(\chi) \int_{n\alpha}^{(n+1)\alpha} g(x) dx + \sum_{n=1}^{m} S_n(\chi) \log\left(\frac{n+1}{n}\right) \]

\[ = \alpha \sum_{n=1}^{m} \frac{g((n+1)\alpha) + g(n\alpha)}{2} S_n(\chi) + \sum_{n=1}^{m} S_n(\chi) \log\left(\frac{n+1}{n}\right) + \frac{\theta \alpha}{8} R'' \]

\[ = \frac{1}{2} \sum_{n=1}^{m} u_n S_n(\chi) + \frac{\theta \alpha}{8} R'' \]

where, as in the proof of Proposition 2, we have

\[ R'' = \alpha \sum_{n=1}^{m} n \int_{n\alpha}^{(n+1)\alpha} |g''(x)| dx \leq \int_{0}^{\infty} x |g''(x)| dx \stackrel{\text{def}}{=} R_{\text{even}} = 2(\beta g'(\beta) - g(\beta)) \]

where \( \beta = 1.792641 \cdots \) is the only positive real zero of \( g'' \). Hence, \( R'' \leq R_{\text{even}} = 4(1 - (1 + \beta^2)e^{-\beta^2})/\beta = 1.853264 \cdots < 2 \).

Applying Lemma 6 to \( g(x) = e^{-x^2} \), we obtain

\[ \frac{1}{\alpha} \sum_{n \geq 1} T_n(\chi) \int_{n\alpha}^{(n+1)\alpha} e^{-x^2} dx = \frac{1}{2} \sum_{n \geq 1} (\epsilon_n + \epsilon_{n+1}) T_n(\chi) + \theta \alpha \frac{R''}{8} \]


with $|\theta| \leq 1$ and

$$R''' = \sum_{n \geq 1} \left( \sum_{k=1}^{n} \frac{1}{k} \int_{n\alpha}^{(n+1)\alpha} |g''| \right)$$

$$= \sum_{n \geq 1} \frac{1}{n} \int_{n\alpha}^{\infty} |g''|$$

$$\leq \int_{\alpha}^{\infty} |g''(x)| \, dx + \int_{1}^{\infty} \left( \int_{n\alpha}^{\infty} |g''(x)| \, dx \right) \frac{du}{u}$$

$$= \int_{\alpha}^{\infty} |g''(x)| \, dx + \int_{\alpha}^{\infty} |g''(x)| \log(x/\alpha) \, dx$$

$$\leq \int_{0}^{\infty} |g''(x)| \log(ex/\alpha) \, dx$$

$$\leq \sum_{n \geq 1} \frac{1}{n} \int_{n\alpha}^{\infty} e^{-x^2} \, dx + 2 \int_{1/\sqrt{\pi}}^{\infty} e^{-x^2} \, dx$$

$$\leq \frac{1}{2} - \frac{7}{4} \log \alpha \leq 2 \log(e/\alpha),$$

and using (16), we obtain the desired result. \hfill \Box

According to (12) and to Lemma 6 (where we take the limit as $m$ goes to infinity in (14)), we obtain

**Proposition 7** (Compare with Proposition 2). Let $\chi$ be a primitive even Dirichlet character of conductor $f > 1$. For some $\theta$ satisfying $|\theta| \leq 1$, it holds that

$$L'(0, \chi) = \frac{1}{2} \sum_{n \geq 1} u_n S_n(\chi) + \frac{W(\chi)}{2} \sum_{n \geq 1} (e_n + e_{n+1}) T_n(\chi) + \frac{\theta \alpha}{4} \log(e^2/\alpha).$$

**Proposition 8.** Assume $f > 3$, $t > 0$ and $M \geq 1$. For any positive rational integer $m \geq B(t, M, f)$ it holds that

$$\left| \sum_{n > m} S_n(\chi) \int_{n\alpha}^{(n+1)\alpha} e^{-x^2} \frac{dx}{x} \right| \leq \sum_{n > m} e_n \leq \frac{1}{2} \alpha^{2t-1} e^{-M}$$

and

$$\left| \frac{1}{2} \sum_{n > m} (e_n + e_{n+1}) T_n(\chi) \right| \leq \frac{1}{2} \alpha^{2t-1} e^{-M} \log(e/\alpha).$$

**Proof.** We have

$$\left| \sum_{n > m} S_n(\chi) \int_{n\alpha}^{(n+1)\alpha} e^{-x^2} \frac{dx}{x} \right| \leq \sum_{n > m} \int_{n\alpha}^{(n+1)\alpha} e^{-x^2} \frac{dx}{x}$$

$$\leq \sum_{n > m} e^{-n^2\alpha^2} \leq \frac{1}{m} \int_{m}^{\infty} xe^{-x^2} \, dx = \frac{e^{-m^2\alpha^2}}{2m\alpha^2} \leq \frac{\alpha^{2t-1} e^{-M}}{2\sqrt{M - 2t} \log \alpha}.$$
Since $M \geq 1$ and $t > 0$ imply $m \geq B(t, M, f) \geq 1/\alpha$ and since $x \mapsto \log(e^x)e^{-\alpha x^2}$ decreases in the range $x \geq 1/\alpha$, we obtain

$$R_m \leq \int_m^\infty \frac{\log(e^x)}{x} e^{-\alpha x^2} dx \leq \frac{\log(e^m)e^{-m^2\alpha^2}}{2m\alpha^2} \leq \frac{1}{2} \alpha^{-2t-1} e^{-M \log(e^u/\alpha)}$$

where $u = \sqrt{M - 2t \log \alpha} \geq 1$. Since $u \mapsto \frac{\log(e^u/\alpha)}{u}$ decreases in the range $u \geq \alpha$ and since we have $u \geq 1 > \alpha$, we obtain the desired result. 

According to (12), (14), (15), (17) and (18), we obtain

**Theorem 9** (Compare with Theorem 4). Let $M \geq 1$ be given, let $\chi$ be the primitive even Dirichlet character of conductor $f > 1$, let $m$ be the least rational integer greater than or equal to $B(\frac{1}{2}, M, f) = O(f^{0.5+\epsilon})$ and set

$$L_M'(0, \chi) = \frac{1}{2} \sum_{n=1}^{m} u_n S_n(\chi) + \frac{1}{2} \sum_{n=1}^{m} (e_n + e_{n+1}) T_n(\chi),$$

where $\alpha = \sqrt{\pi f}$, $e_n = e^{-n^2\alpha^2}$, $S_n(\chi)$ and $T_n(\chi)$ are defined in (15) and $u_n$ is defined in (13). Then,

$$|L'(0, \chi) - L_M'(0, \chi)| \leq \frac{\alpha + 2e^{-M}}{4} \log(e^2/\alpha).$$

**Remark 10.**

1. Of particular importance is the case where $\chi$ is the primitive quadratic even Dirichlet character of conductor $f = d_k$ associated with a real quadratic field $k$ of discriminant $d_k$, fundamental unit $e_k > 1$, and class number $h_k$. Then, $\bar{\chi} = \chi$, $W(\chi) = +1$, $h_k = L'(0, \chi)/\log e_k$ and Theorem 9 yields

$$|h_k - L_M'(0, \chi)/\log e_k| \leq \frac{5}{4} \sqrt{\frac{\pi}{d_k}} + \frac{5}{2} e^{-M}$$

(for $e_k \geq (\sqrt{d_k} - 4 + \sqrt{d_k})/2 \log(e^2/\alpha) \leq 5 \log e_k$). We refer the reader to [WiBr, Section 2] for the evaluation of the regulator $\log e_k$ of a real quadratic field $k$ by using an elementary algorithm of complexity $O(d_k^{0.5+\epsilon})$ based on the use of continued fractional expansions.

2. Here again, the second point of Remark 3 applies.

3. In contrast with Theorem 4 (see Point 2 of Remark 5), Theorem 9 cannot be used to compute class numbers of nonquadratic real abelian numbers fields of a given degree (for it is not known how to reduce their computation to the computation of numerical approximations to linear combinations with bounded coefficients of values at $s = 0$ of derivatives of $L$-functions associated with even characters (compare with [Lou3])).

**4. Numerical examples**

To assess the efficiency of the present technique for computing class numbers $h_k$ of quadratic number fields $k$ of large discriminants $d_k$, we programmed our formulas (10) and (13) with $M = 3$ in Kida’s language UBASIC, which allows fast arbitrary precision calculation on PC’s (the precision of real numbers in significant digits we used was equal to 28). Let us detail how much our method improves upon the previous ones based on the use of (9) and Point 1 of Remark 3 for $d_k < 0$, and of (11) and Point 1 of Remark 10 for $d_k > 0$ (see [Coh] and [WiBr]). We give all the
drawbacks and end up with a faster and easier to implement method for computing instead, by using (20) and point 2 of remark 5 we do not meet with any of these. instead of computing erfc(nα) with the indices for which we use (1) for computing approximations to erfc(nα) when disregarding the second point of remark 5, we cannot take advantage of the second point of our remark 5 while dealing with the indices for which we use (1) for computing approximations to erfc(nα) while using (22), (1) for z < 1.8 and (2) for nα = z ≥ 1.8 and, finally, T4 is the time required to compute h_k when using (22), (1) for nα = z < 1.8 and (2) for nα = z ≥ 1.8.

We present in tables 1 and 2 the results of applying these methods to compute class numbers of five imaginary quadratic fields with various size discriminants and of five real quadratic fields with various size discriminants and regulators. The computations were all carried out on a PC microcomputer with Pentium III, 333Mhz. Here, T1 is the time required to compute h_k when using (20) and the second point of remark 5, T2 is the time required to compute h_k when using (20) but when disregarding the second point of remark 5, T3 is the time required to compute h_k when using (22), (1) for nα = z < 1.8 and (2) for nα = z ≥ 1.8.

### Table 1. The imaginary quadratic case

<table>
<thead>
<tr>
<th>d_k &lt; 0</th>
<th>T1</th>
<th>T2</th>
<th>T3</th>
<th>T4</th>
<th>h_k</th>
</tr>
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<tbody>
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<td>4</td>
<td>10</td>
<td>31</td>
<td>24</td>
<td>38 272</td>
</tr>
<tr>
<td>5 - 10^{11}</td>
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<td>34</td>
<td>100</td>
<td>78</td>
<td>95 840</td>
</tr>
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<td>323</td>
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<td>5 - 10^{14}</td>
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<td>1224</td>
<td>3391</td>
<td>2619</td>
<td>3 312 448</td>
</tr>
</tbody>
</table>

(see theorem and point 1 of remark for which)

(21) $R_k(M) := |h_k - h_k(M)| \leq \frac{3}{2\sqrt{\pi}} e^{-M} + \frac{3}{8\sqrt{|d_k|}}$

and

(22) $h_k'(M) = \frac{1}{\sqrt{\pi}} \left( \frac{1}{\alpha_k} \sum_{n=1}^{m_k} \frac{\chi_k(n)}{n} e_n + \alpha_k \sum_{n=1}^{m_k} (e_n + e_{n+1}) S_n(\chi_k) \right)$

(see and point 1 of remark for which)

(23) $R_k'(M) := |h_k - h_k'(M)| \leq \frac{1}{\sqrt{\pi}} e^{-M}$

(notice that $2 \int_{-\infty}^{\infty} e^{-x^2} dx \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2x e^{-x^2} dx = e^{-X^2}/X$ and use proposition with $t = 0.5$). Notice that the number of terms in the truncated sums (20) and (22) are equal, and that the error terms $R_k$ and $R_k'$ are of the same quality. Now, the previously known rigorous method for computing $h_k$ consists in using (22) and the power series expansion (1) for small values of $z$ and continued fraction expansion (2) for large values of $z$ to compute approximations to $h_k'$. The main drawbacks of this method are (i) that we must carefully explain how many terms we have to consider in (1) and (2) to end up with good enough approximations to each erfc(nα), (ii) that computing erfc(nα) is slower than computing $e_n = \exp(-n^2\alpha_k^2)$ and (iii) that we cannot take advantage of the second point of our remark while dealing with the indices for which we use (1) for computing approximations to erfc(nα).

Instead, by using (20) and point 2 of remark 5 we do not meet with any of these drawbacks and end up with a faster and easier to implement method for computing $h_k$. We present in Tables 1 and 2 the results of applying these methods to compute class numbers of five imaginary quadratic fields with various size discriminants and of five real quadratic fields with various size discriminants and regulators. The computations were all carried out on a PC microcomputer with Pentium III, 333Mhz. Here, $T_1$ is the time required to compute $h_k$ when using (20) and the second point of remark 5, $T_2$ is the time required to compute $h_k$ when using (20) but when disregarding the second point of remark 5, $T_3$ is the time required to compute $h_k$ when using (22), (1) for $n\alpha_k = z < 1.8$ and (2) for $n\alpha_k = z ≥ 1.8$ and, finally, $T_4$ is the time required to compute $h_k$ when using (22), (1) for $n\alpha_k = z < 1.8$ and (2) for $n\alpha_k = z ≥ 1.8$. The imaginary quadratic case

Table 1.

<table>
<thead>
<tr>
<th>$d_k &lt; 0$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_3$</th>
<th>$T_4$</th>
<th>$h_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 - 10^{10}</td>
<td>4</td>
<td>10</td>
<td>31</td>
<td>24</td>
<td>38 272</td>
</tr>
<tr>
<td>5 - 10^{11}</td>
<td>12</td>
<td>34</td>
<td>100</td>
<td>78</td>
<td>95 840</td>
</tr>
<tr>
<td>5 - 10^{12}</td>
<td>41</td>
<td>112</td>
<td>323</td>
<td>251</td>
<td>506 880</td>
</tr>
<tr>
<td>5 - 10^{13}</td>
<td>138</td>
<td>376</td>
<td>1053</td>
<td>812</td>
<td>1 051 452</td>
</tr>
<tr>
<td>5 - 10^{14}</td>
<td>460</td>
<td>1224</td>
<td>3391</td>
<td>2619</td>
<td>3 312 448</td>
</tr>
</tbody>
</table>
Table 2. The real quadratic case

<table>
<thead>
<tr>
<th>$d_k &gt; 0$</th>
<th>$T_0$</th>
<th>$T_1$</th>
<th>$T_2$</th>
<th>$T_3$</th>
<th>$T_4$</th>
<th>$h_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{10} + 5$</td>
<td>0</td>
<td>8</td>
<td>15</td>
<td>51</td>
<td>41</td>
<td>1 134</td>
</tr>
<tr>
<td>$10^{11} + 21$</td>
<td>4</td>
<td>27</td>
<td>49</td>
<td>167</td>
<td>132</td>
<td>2</td>
</tr>
<tr>
<td>$10^{12} + 1$</td>
<td>0</td>
<td>90</td>
<td>161</td>
<td>545</td>
<td>427</td>
<td>50 280</td>
</tr>
<tr>
<td>$10^{13} + 1$</td>
<td>48</td>
<td>297</td>
<td>532</td>
<td>1763</td>
<td>1374</td>
<td>2</td>
</tr>
<tr>
<td>$10^{14} + 5$</td>
<td>0</td>
<td>959</td>
<td>1729</td>
<td>5706</td>
<td>4417</td>
<td>107 920</td>
</tr>
</tbody>
</table>

for $n\alpha_k = z \geq 1.8$ and the second point of Remark 5 to compute the $\exp(-(n\alpha_k)^2)$’s in (2) for the $n$’s for which $n\alpha_k = z \geq 1.8$. All these $T_i$ are expressed in seconds. Here, $T_0$ denotes the time required to compute $\log \epsilon_k$ by using continued fractions (see [WiBr]), and $T_1, T_2, T_3$ and $T_4$ are as in Table 1.

These Tables 1 and 2 clearly show that our method is significantly faster in practice than existing rigorous methods.

References


