MATHEMATICS OF COMPUTATION
Volume 71, Number 240, Pages 1597–1607
S 0025-5718(01)01399-0
Article electronically published on December 5, 2001

SOME NUMERICAL COMPUTATIONS CONCERNING SPINOR ZETA FUNCTIONS IN GENUS 2 AT THE CENTRAL POINT

WINFRIED KOHNEN AND MICHAEL KUSS

Abstract. We numerically compute the central critical values of odd quadratic character twists with respect to some small discriminants D of spinor zeta functions attached to Seigel–Hecke eigenforms F of genus 2 in the first few cases where F does not belong to the Maass space. As a result, in the cases considered we can numerically confirm a conjecture of Böcherer, according to which these central critical values should be proportional to the squares of certain finite sums of Fourier coefficients of F.

1. Introduction

In [3], Böcherer made an interesting conjecture concerning central critical values of odd quadratic character twists of spinor zeta functions attached to cuspidal Siegel–Hecke eigenforms of genus 2.

More precisely, let F be a nonzero cuspidal Hecke eigenform of even integral weight k w.r.t. the Siegel modular group $\Gamma_2 := \mathrm{Sp}_2(\mathbb{Z})$ and denote by $Z_F(s)$ ($\mathrm{Re}(s) \gg 0$) its spinor zeta function. Recall [2] that $Z_F(s)$ completed with appropriate Γ-factors has a meromorphic continuation to $\mathbb{C}$ and is invariant under $s \mapsto 2k-2-s$. Let $Z_F(s,\chi_D)$ ($\mathrm{Re}(s) \gg 0$) be the twist of $Z_F(s)$ by the quadratic character $\chi_D = \left(\frac{D}{\cdot}\right)$, where $D<0$ is a fundamental discriminant. Assume that $Z_F(s,\chi_D)$ enjoys similar analytic properties as $Z_F(s)$. Then according to [3], there should exist a constant $C_F>0$, depending only on F, such that

$$Z_F(k-1,\chi_D) = C_F|D|^{1-k}\Big(\sum_{\{T>0\,|\,\mathrm{discr}\,T=D\}/\Gamma_1} \frac{a(T)}{\varepsilon(T)}\Big)^2, \qquad (1)$$

where $a(T)$ (T a positive definite half-integral $(2,2)$-matrix) is the T-th Fourier coefficient of F, $\varepsilon(T) := \#\{U \in \Gamma_1 \mid T[U]=T\}$ (with $\Gamma_1 := \mathrm{SL}_2(\mathbb{Z})$, $T[U]=U^tTU$) is the order of the unit group of T and the summation in (1) extends over all T with discriminant equal to D, modulo the action $T \mapsto T[U]$ by Γ_1.

In [3], Böcherer proved his conjecture in the case where F is the Maass lift of a Hecke eigenform f of weight $2k-2$ w.r.t. Γ_1. The proof combines four inputs: i) the fact that $Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L(f,s)$, where $L(f,s)$ is the Hecke L-function of f [5]; ii) Waldspurger's theorem [13] on the relation between central critical values of quadratic twists of $L(f,s)$ and squares of Fourier coefficients of modular forms of half-integral weight; iii) the explicit description of the Maass

---

Received by the editor October 20, 1999 and, in revised form, January 3, 2001.
2000 *Mathematics Subject Classification.* Primary 11F46.

lift on the level of Fourier coefficients [2]; and finally iv) Dirichlet's classical class number formula.

Later on, Böcherer and Schulze-Pillot [4] proved an identity similar to (1) in the case of levels, where now F is the Yoshida lift of an elliptic cusp form.

Also in [3], a formula like (1) in the case where F is a Siegel– or Klingen–Eisenstein series was shown to be true.

The proof in all the above cases makes essential use of the fact that the spinor zeta function in question is a product of "known" L-series.

To the best of our knowledge, nothing regarding Böcherer's conjecture seems to be known in the case where F is a "true" Siegel modular form, i.e., is not a lift of an automorphic form on GL_2 (and so $Z_F(s)$ is not expected to split).

In the present paper, we would like to present some numerical data supporting the conjecture for small values of D in the first few "nontrivial" cases when F is of weight 20, 22, 24 resp. 26 and is not a Maass lift. It turns out that for those F and for $D = -3, -4, -7, -8$ identity (1) *numerically* is true at least up to 5 digits with some constant $C_F > 0$ independent of D (Thm., §4; numerical data are given in §5).

The first ingredient in the computation is a certain series representation (found by the first author many years ago) for central critical values of spinor zeta functions *supposing* "good" analytic properties of $Z_F(s, \chi_D)$ as required in the conjecture. We were kindly informed by D. Goldfeld that this series representation can also be derived from the more general work of Lavrik [10] when appropriately specialized. The formula for computing $Z_F(k-1, \chi_D)$ is given in §2.

Note that the holomorphic continuation of $Z_F(s, \chi_D)$ was proved in [6],[7] (using some round-about via Rankin–Dirichlet series) under the assumption that the first Fourier–Jacobi coefficient of F is nonzero. The latter condition is satisfied at least for all F with $k \leqslant 32$ according to Skoruppa [12]. The functional equation, however, was proved only very recently in [9].

The second main ingredient, which is entirely due to the second author, is the computation of the eigenvalues $\lambda_F(p)$ (p a prime $< 1\,000$) and $\lambda_F(p^2)$ (p a prime < 71) under the usual Hecke operators T_p resp. T_{p^2} of the F in question, following the method of Skoruppa [12] and an appropriate C++ computer program. This is presented in §3.

In §4, the results of §§2 and 3 are combined to calculate $Z_F(k-1, \chi_D)$ for the F and D in question with "good" accuracy. For an estimation of the error term we use the bounds for the eigenvalues of F implied by the Ramanujan–Petersson conjecture, for the latter cf. [14].

We finally remark that we have also numerically re-checked (1) using the identity given in §2 in case F is of weight 20, resp. 22, and is in the Maass space. We have not included the details here.

2. A series representation for central values of spinor zeta functions

Let $k \in 2\mathbb{N}$ and write $S_k(\Gamma_2)$ for the space of Siegel cusp forms of weight k w.r.t. Γ_2. If $F \in S_k(\Gamma_2)$ is a nonzero Hecke eigenform, we let

$$Z_F(s) = \prod_{p \text{ prime}} Z_{F,p}(p^{-s})^{-1} \qquad (\mathrm{Re}(s) \gg 0) \tag{2}$$

be the spinor zeta function of F, where

$$Z_{F,p}(X) = 1 - \lambda_F(p)X + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4})X^2 \\ - \lambda_F(p)p^{2k-3}X^3 + p^{4k-6}X^4$$

is the local spinor polynomial at p and $\lambda_F(p)$ resp. $\lambda_F(p^2)$ are the eigenvalues of F under the usual Hecke operator T_p resp. T_{p^2}.

According to Andrianov [2] the function

$$Z_F^\star(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$$

has a meromorphic continuation to $\mathbb{C}$ and is invariant under $s \mapsto 2k-2-s$. It is holomorphic everywhere if F is not contained in the Maass space (which is equivalent to saying $Z_F(s)$ is *not* of the form $Z_F(s) = \zeta(s-k+1)\zeta(s-k+2) \times L(f,s)$, where f is a normalized cuspidal Hecke eigenform of weight $2k-2$ w.r.t. Γ_1, and $L(f,s)$ is its associated Hecke L-function [5]).

If $D < 0$ is a fundamental discriminant, we define the twist of $Z_F(s)$ by χ_D as

$$Z_F(s,\chi_D) := \prod_{p \text{ prime}} Z_{F,p}(\chi_D(p)p^{-s})^{-1} \qquad (\mathrm{Re}(s) \gg 0). \tag{3}$$

We denote the n-th coefficient of the Dirichlet series $Z_F(s,\chi_D)$ by $\lambda_{F,D}(n)$.

We put

$$Z_F^\star(s,\chi_D) := \left(\tfrac{2\pi}{|D|}\right)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s,\chi_D) \qquad (\mathrm{Re}(s) \gg 0).$$

If F is in the Maass space, then by well-known properties of twists of $\zeta(s)$ and $L(f,s)$, $Z_F^\star(s,\chi_D)$ extends to an entire function, is of rapid decay for $\mathrm{Im}(s) \to \infty$ and is invariant under $s \mapsto 2k-2-s$. It is very natural to expect that the same holds for general F (cf. [3]). In fact, if F is not in the Maass space and the first Fourier–Jacobi coefficient of F is nonzero, this was proved in [6],[7],[9] (using the fact $\|\phi_1\|^2 Z_F(s) = D_F(s)$, where ϕ_1 is the first Fourier–Jacobi coefficient of F and $D_F(s)$ is a Rankin type Dirichlet series formed out of the Fourier–Jacobi coefficients of F introduced in [8]).

Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform such that $Z_F(s,\chi_D)$ has the above analytic properties. Using the integral transform

$$\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(s-k+2)y^{-s}\mathrm{d}s = 2y^{-\frac{k}{2}+1}K_{k-2}(2\sqrt{y}) \qquad (y>0,\ c>k-2),$$

where $K_{k-2}(y)$ denotes the modified Bessel function of order $k-2$, we have for $y > 0$ and $c \gg 0$

$$\begin{aligned} &\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z_F^\star(s,\chi_D)y^{-s}\mathrm{d}s \\ &= \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\tfrac{2\pi}{|D|}\right)^{-2s}\Gamma(s)\Gamma(s-k+2)\sum_{n=1}^{\infty}\lambda_{F,D}(n)n^{-s}y^{-s}\mathrm{d}s \\ &= \sum_{n=1}^{\infty}\lambda_{F,D}(n)\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(s)\Gamma(s-k+2)\left(\tfrac{4\pi^2 ny}{D^2}\right)^{-s}\mathrm{d}s \\ &= y^{-\frac{k}{2}+1}f_{F,D}(y), \end{aligned} \tag{4}$$

where

$$f_{F,D}(y) = 2\big(\tfrac{4\pi^2}{D^2}\big)^{-\frac{k}{2}+1} \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-\frac{k}{2}+1} K_{k-2}\big(\tfrac{4\pi\sqrt{ny}}{|D|}\big).$$

Since $Z_F^\star(s,\chi_D)$ is holomorphic and of rapid decay for $\operatorname{Im} s \to \infty$, we may shift the path of integration in (4) to the line $c = k-1$. We replace y by $\frac{1}{y}$ and apply the functional equation of $Z_F^\star(s,\chi_D)$ to obtain

$$\begin{aligned} &y^{\frac{k}{2}-1} f_{F,D}\big(\tfrac{1}{y}\big) \\ &\quad = \tfrac{1}{2\pi i}\int_{k-1-i\infty}^{k-1+i\infty} Z_F^\star(s,\chi_D) y^s \mathrm{d}s = \tfrac{1}{2\pi i}\int_{k-1-i\infty}^{k-1+i\infty} Z_F^\star(2k-2-s,\chi_D) y^s \mathrm{d}s \\ &\quad = \tfrac{1}{2\pi i}\int_{k-1-i\infty}^{k-1+i\infty} Z_F^\star(s,\chi_D) y^{2k-2-s} \mathrm{d}s = y^{\frac{3}{2}k-1} f_{F,D}(y), \end{aligned}$$

i.e., the function $f_{F,D}(y)$ satisfies the functional equation $f_{F,D}(\frac{1}{y}) = y^k f_{F,D}(y)$.

Using the usual splitting trick and the formula

$$2\int_0^\infty K_{k-2}(2\sqrt{y}) y^{s-\frac{k}{2}} \mathrm{d}y = \Gamma(s)\Gamma(s-k+2) \qquad (\operatorname{Re}(s) > k-2),$$

we conclude for $\operatorname{Re}(s) \gg 0$ that

(5)

$$\begin{aligned} Z_F^\star(s,\chi_D) &= 2\big(\tfrac{2\pi}{|D|}\big)^{-2s} \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-s} \int_0^\infty K_{k-2}(2\sqrt{y}) y^{s-\frac{k}{2}} \mathrm{d}y \\ &= 2\big(\tfrac{2\pi}{|D|}\big)^{-2s} \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-s} \big(\tfrac{4\pi^2 n}{D^2}\big)^{s-\frac{k}{2}+1} \int_0^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{ny}}{|D|}\big) y^{s-\frac{k}{2}} \mathrm{d}y \\ &= \int_0^\infty f_{F,D}(y) y^{s-\frac{k}{2}} \mathrm{d}y = \int_1^\infty f_{F,D}(y)\big(y^{\frac{3}{2}k-2-s} + y^{s-\frac{k}{2}}\big) \mathrm{d}y. \end{aligned}$$

As $f_{F,D}(y)$ is of exponential decay for $y \to \infty$, the right hand side of (5) has a holomorphic continuation to the whole complex plane, and (5) is valid for all $s \in \mathbb{C}$.

Setting $s = k-1$ in (5), we get the formulas

$$\begin{aligned} Z_F^\star(k-1,\chi_D) &= 4\big(\tfrac{4\pi^2}{D^2}\big)^{-\frac{k}{2}+1} \int_1^\infty \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-\frac{k}{2}+1} K_{k-2}\big(\tfrac{4\pi\sqrt{ny}}{|D|}\big) y^{\frac{k}{2}-1} \mathrm{d}y \\ &= 4\big(\tfrac{4\pi^2}{D^2}\big)^{-\frac{k}{2}+1} \sum_{n=1}^{\infty} \int_1^\infty \lambda_{F,D}(n) n^{-\frac{k}{2}+1} K_{k-2}\big(\tfrac{4\pi\sqrt{ny}}{|D|}\big) y^{\frac{k}{2}-1} \mathrm{d}y \qquad (6) \\ &= 4\big(\tfrac{4\pi^2}{D^2}\big)^{-\frac{k}{2}+1} \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-k+1} \int_n^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\frac{k}{2}-1} \mathrm{d}y. \end{aligned}$$

Hence

$$Z_F(k-1,\chi_D) = \tfrac{4(2\pi)^k}{|D|^k (k-2)!} \sum_{n=1}^{\infty} \lambda_{F,D}(n) n^{-k+1} \int_n^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\frac{k}{2}-1} \mathrm{d}y, \qquad (7)$$

where the exponential decay of $K_{k-2}(y)$ for $y \to \infty$ justifies the interchange of summation and integration in (6).

3. Numerical computations

Let $M_k(\Gamma_1)$ be the space of elliptic modular forms of weight k w.r.t. Γ_1 and $S_k(\Gamma_1)$ be the subspace of cusp forms in $M_k(\Gamma_1)$. For $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$, write $q = \exp(2\pi i\tau)$, and let

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

be the Ramanujan Δ-function in $S_{12}(\Gamma_1)$ and

$$E_{2k} = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n \qquad (k \in \mathbb{Z},\ k \geqslant 2,\ \sigma_{2k-1}(n) = \textstyle\sum_{d|n} d^{2k-1},$$
$$B_{2k} = 2k\text{th Bernoulli number})$$

be the normalized Eisenstein series in $M_{2k}(\Gamma_1)$.

If $J_{k,1}^{\mathrm{cusp}}$ denotes the space of Jacobi cusp forms on Γ_1 of index 1 and weight k, the Maass space [11] is the image of the Hecke equivariant embedding $V : J_{k,1}^{\mathrm{cusp}} \hookrightarrow S_k(\Gamma_2)$ defined by

$$\begin{aligned}\phi = &\sum_{\substack{D,r\in\mathbb{Z},D<0\\ D\equiv r^2 \bmod 4}} C_\phi(D)q^{(r^2-D)/4}\zeta^r\\ \longmapsto &\sum_{\substack{n,r,m\in\mathbb{Z}\\ r^2-4mn<0,\\ n,m>0}} a(n,r,m)q^n\zeta^r q'^m,\end{aligned}$$

where

$$a(n,r,m) := \sum_{d|(n,r,m)} d^{k-1}C_\phi\left(\frac{r^2-4mn}{d^2}\right)$$

and

$$\zeta = \exp(2\pi i z)(z \in \mathbb{C}), \qquad q' = \exp(2\pi i\tau')(\tau' \in \mathbb{C}, \mathrm{Im}(\tau') > 0).$$

By ϕ_{10} resp. ϕ_{12} we denote the Jacobi cusp forms in the one-dimensional spaces $J_{10,1}^{\mathrm{cusp}}$, resp. $J_{12,1}^{\mathrm{cusp}}$, normalized to $C(-3) = 1$.

The first cuspidal Hecke eigenforms for genus 2 that do not belong to the Maass space appear in weight 20, 22, 24, resp. 26, and are denoted $\Upsilon_{20}, \ldots, \Upsilon_{26b}$ in [12]. In [12], Skoruppa gives explicit formulas for them (involving the forms $V(\phi)$, where ϕ are appropriate Jacobi forms) and calculates some of their Fourier coefficients. Note that there is a misprint in the formula for Υ_{22}; the corrected formula is

$$\begin{aligned}\Upsilon_{22} = &-2^5\cdot 3\cdot 5\cdot 7\cdot 1423\cdot V(\phi_{10})V(\phi_{12})\\ &+ V(-\tfrac{5}{2\cdot 3}\phi_{12}E_{10} + \tfrac{11}{2\cdot 3}\phi_{10}E_6^2 + 2^4\cdot 3\cdot 61\cdot\phi_{10}\Delta).\end{aligned}$$

To compute the coefficients of the relevant Jacobi forms ϕ, we proceed slightly differently from [12] and try to avoid multiplication of Jacobi modular forms with elliptic modular forms. More precisely, the operator $\mathcal{D}_{2\nu}$ is defined by

$$\mathcal{D}_{2\nu}\phi := \sum_{n=0}^{\infty}\Big(\sum_{r} p_{2\nu}^{(k-1)}(r,nm)c(n,r)\Big)q^n \qquad (\nu\in\mathbb{Z},\ \nu\geqslant 0)$$

where $\phi=\sum_{n,r}c(n,r)q^n\zeta^r\in J_{k,m}^{\mathrm{cusp}}$ and

$$\frac{(k-\nu-2)!}{(2\nu)!(k-2)!}\,p_{2\nu}^{(k-1)}(r,n) = \text{coefficient of } t^{2\nu} \text{ in } (1-rt+nt^2)^{-k+1}$$

maps $J_{k,1}^{\mathrm{cusp}}$ to $S_{k+2\nu}(\Gamma_1)$ [5].

We consider the system of equations $\{\mathcal{D}_{2\nu}(f)=g_{f,\nu}\}$ where f is one of the Jacobi forms

$$\phi_{10},\phi_{10}E_4,\phi_{10}E_6,\phi_{10}E_{10},\phi_{10}E_{14},\phi_{10}E_{16},\phi_{10}\Delta,\phi_{10}E_6^2,\phi_{10}E_8^2,\phi_{10}\Delta E_4,$$
$$\phi_{12},\phi_{12}E_8,\phi_{12}E_{10},\phi_{12}E_6^2,\phi_{12}\Delta,\phi_{12}E_{14},$$

$\nu\in\{0,2,4\}$ and $g_{f,\nu}$ is the corresponding elliptic modular form which is determined by its first coefficients, e.g., we have

$$\begin{aligned}&\mathcal{D}_0(\phi_{10})=0, &&\mathcal{D}_2(\phi_{10})=20\Delta, &&\mathcal{D}_4(\phi_{10})=0,\\ &\mathcal{D}_0(\phi_{12})=12\Delta, &&\mathcal{D}_2(\phi_{12})=0, &&\mathcal{D}_4(\phi_{12})=196\Delta E_4.\end{aligned}$$

We solve the system recursively for the Fourier coefficients of the Jacobi forms. (To start the recursion the first Fourier coefficients of ϕ_{10}, ϕ_{12} are taken from [5].) This method needs only $O(|D|^{\frac{3}{2}})$ operations to calculate a complete table of Fourier coefficients up to a "large" discriminant. Hence it is less "expensive" than the usual multiplication of Jacobi forms and elliptic modular forms ($O(|D|^2)$).

Proceeding in this way and using a C++ computer program, we computed the Fourier coefficients $C(D)$ of the Jacobi forms in question for $|D|\leqslant 3\,000\,000$. Then we are able to compute any Fourier coefficient $a(n,r,m)$ of $\Upsilon_{20},\dots,\Upsilon_{26b}$ with discriminant $4mn-r^2\leqslant 3\,000\,000$.

In [12] Skoruppa calculates the eigenvalues $\lambda_F(p)$, $\lambda_F(p^2)$ (p prime) of a Hecke eigenform

$$F=\sum_{\substack{r,n,m\in\mathbb{Z},\\ r^2-4mn<0,\\ n,m>0}} a(n,r,m)q^n\zeta^r q'^m \in S_k(\Gamma_2)$$

by means of the formulas

$$\lambda_F(p)a(1,1,1)=a(p,p,p)+p^{k-2}\big(1+\big(\tfrac{p}{3}\big)\big)a(1,1,1)$$

and

$$\begin{aligned}\lambda_F(p^2)&a(1,1,1)\\ &=\Big[\lambda_F(p)^2-\lambda_F(p)p^{k-2}\big(1+\big(\tfrac{p}{3}\big)\big)-p^{2k-3}+p^{2k-4}\big(\big(\tfrac{p}{3}\big)+\big(\tfrac{p}{3}\big)^2\big)\Big]a(1,1,1)\\ &\quad -p^{k-2}a(1,p,p^2)-p^{k-2}\sum_{\substack{\nu \bmod p,\\ 1+\nu+\nu^2\not\equiv 0 \bmod p}} a(1+\nu+\nu^2,p(1+2\nu),p^2),\end{aligned}$$

which are based on Andrianov's results in [2].

Using another C++ computer program, we computed the eigenvalues $\lambda_F(p)$ for $p < 1\,000$ prime and $\lambda_F(p^2)$ for $p < 71$ prime of $F = \Upsilon_{20}, \dots, \Upsilon_{26b}$ from the above formulas.

4. Summing up

By (7) we have

$$Z_F(k-1, \chi_D) = \sum_{n=1}^{\infty} \lambda_{F,D}(n) g_D(n),$$

where

$$g_D(n) = \tfrac{4(2\pi)^k}{|D|^k (k-2)!}\, n^{-k+1} \int_n^{\infty} K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\frac{k}{2}-1} \mathrm{d}y. \tag{8}$$

Now $g_D(n)$ is of exponential decay for $n \to \infty$ and $\lambda_{F,D}(n)$ is of polynomial growth. Thus for a numerical approximation of $Z_F(k-1, \chi_D)$ it is important to calculate as many terms as possible in the sum for *small* n (say $n \leqslant N$ for some N — we will later choose $N = 4000$), while for *large* n ($n > N$) the total sum of *all* terms with $n > N$ is rather small. Hence we approximate $Z_F(k-1, \chi_D)$ by

$$\mathcal{Z}_{F,D}(k-1) = \sum_{\substack{1 \leqslant n \leqslant N \\ n \text{ has no prime} \\ \text{divisor} > 1\,000}} \lambda_{F,D}(n) g_D(n),$$

where the values of $\lambda_{F,D}(n)$ can be calculated from the Euler product of $Z_{F,D}(s, \chi_D)$ for $n < 71^2$.

Suppose there are positive constants C_1, C_2, α, β such that the estimates $|\lambda_F(p)| \leqslant C_1 \cdot p^{\alpha}$ (p prime) and $|\lambda_F(n)| \leqslant C_2 \cdot n^{\beta}$ ($n > N$) hold. Then the error term

$$\varepsilon(F, D) = Z_{F,D}(k-1) - \mathcal{Z}_{F,D}(k-1)$$

can be estimated by

$$\begin{aligned} |\varepsilon(F,D)| \leqslant & \sum_{\substack{p > 1\,000 \\ p \text{ prime} \\ 1 \leqslant \nu \leqslant N/p}} |\lambda_{F,D}(\nu p)| g_D(\nu p) \\ & + \sum_{n > N} |\lambda_{F,D}(\nu n)| \int_n^{\infty} K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\frac{k}{2}-1} \mathrm{d}y. \end{aligned} \tag{9}$$

Suppose now that $N < 1007^2$. Then clearly for the first sum $\sum_1$ in the above equation we have the estimate

$$\begin{aligned} \sum\nolimits_1 &= \sum_{\substack{p > 1\,000 \\ p \text{ prime} \\ 1 \leqslant \nu \leqslant N/p}} \big|\lambda_{F,D}(\nu)\lambda_F(p)\big|\, g_D(\nu p) \\ &\leqslant C_1 \sum_{\substack{p > 1\,000 \\ p \text{ prime} \\ 1 \leqslant \nu \leqslant N/p}} |\lambda_{F,D}(\nu)| p^{\alpha} g_D(\nu p). \end{aligned}$$

The second sum $\sum_2$ in (9) satisfies

$$
\begin{aligned}
\tfrac{|D|^k(k-2)!}{4(2\pi)^k}\sum\nolimits_2 &= \sum_{n>N} |\lambda_{F,D}(n)| n^{-k+1} \int_n^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\frac{k}{2}-1} \mathrm{d}y \\
&\leqslant C_2 \sum_{n>N} n^{\beta-k+1} \int_n^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\frac{k}{2}-1} \mathrm{d}y \\
&\leqslant C_2 \sum_{n>N} \int_n^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\beta-\frac{k}{2}} \mathrm{d}y \\
&\leqslant C_2 \sum_{n>N} \sum_{m\geqslant n} \int_m^{m+1} K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\beta-\frac{k}{2}} \mathrm{d}y \\
&\leqslant C_2 \sum_{m>N} (m-N) \int_m^{m+1} K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\beta-\frac{k}{2}} \mathrm{d}y \\
&\leqslant C_2 \int_{N+1}^\infty K_{k-2}\big(\tfrac{4\pi\sqrt{y}}{|D|}\big) y^{\beta-\frac{k}{2}} (y-N) \mathrm{d}y.
\end{aligned}
$$

For the estimation of the dominating term $\sum_1$ in $\varepsilon(F,D)$ we use the result of Weissauer [14] that any eigenform $F \in S_k(\Gamma_2)$ which does not belong to the Maass space fulfills the Ramanujan–Petersson conjecture (i.e., all complex roots of $Z_{F,p}$ have absolute value $p^{\frac{3}{2}-k}$). Thus we have to choose $C_1 = 4$, $\alpha = k - \frac{3}{2}$ to obtain the *best* estimate for $\sum_1$ possible by our methods.

The contribution of $\sum_2$ to $\varepsilon(F,D)$ is absorbed by $\sum_1$ if N is large enough, so we do not have to use the optimal estimate for $\lambda_F(n)$. One obtains a very crude (but simple and for our purpose sufficient) estimate for $\lambda_F(n)$ from the Ramanujan–Petersson conjecture if one uses $\sigma_0(n) \leqslant n$, namely

$$
|\lambda_F(n)| \leqslant \sum_{d|n} \sigma_0(d)\sigma_0(\tfrac{n}{d}) n^{k-\frac{3}{2}} \leqslant \sum_{d|n} n^{k-\frac{1}{2}} = \sigma_0(n) n^{k-\frac{1}{2}} \leqslant n^{k+\frac{1}{2}}.
$$

Thus we set $C_2 = 1$ and $\beta = k + \frac{1}{2}$.

We choose $N = 4\,000$ (then $\sum_2$ is dominated by $\sum_1$ for the D in question) and calculate the numerical approximations of $Z_F(k-1, \chi_D)$ and the corresponding error terms using Mathematica. From (1) we computed the constants C_F for $F = \Upsilon_{20}$, $\dots$, Υ_{26b} and $D = -3, -4, -7, -8$. The numerical results have been checked using Maple.

We obtain

Theorem. *For $F = \Upsilon_{20}, \dots , \Upsilon_{26b}$ there are constants C_F such that equation* (1) *(i.e., Böcherer's conjecture) holds for $D = -3, -, 4-7, -8$ numerically up to* 5 *digits.*

5. Numerical data

Table 1. Approximate constants C_F for $F = \Upsilon_{20}, \Upsilon_{22}$

D	$C_{\Upsilon_{20}}$	$C_{\Upsilon_{22}}$
-3	$2.067215202868\,8 \cdot 10^{11} \pm 0.5 \cdot 10^{-2}$	$1.305668526829\,0 \cdot 10^{12} \pm 0.5 \cdot 10^{-1}$
-4	$2.067215202868\,8 \cdot 10^{11} \pm 0.5 \cdot 10^{-2}$	$1.305668526829\,0 \cdot 10^{12} \pm 0.5 \cdot 10^{-1}$
-7	$2.067215202\,9206 \cdot 10^{11} \pm 2.9 \cdot 10^{1}$	$1.3056685268\,295 \cdot 10^{12} \pm 1.1 \cdot 10^{1}$
-8	$2.067215202\,8644 \cdot 10^{11} \pm 3.1 \cdot 10^{1}$	$1.305668\,5179067 \cdot 10^{12} \pm 1.1 \cdot 10^{6}$

Table 2. Approximate constants C_F for $F = \Upsilon_{24a}, \Upsilon_{24b}$

D	$C_{\Upsilon_{24a}}$	$C_{\Upsilon_{24b}}$
-3	$1.095337244519\,4 \cdot 10^{13} \pm 0.5 \cdot 10^{0}$	$6.138805283929\,6 \cdot 10^{11} \pm 0.5 \cdot 10^{-2}$
-4	$1.095337244519\,4 \cdot 10^{13} \pm 0.5 \cdot 10^{0}$	$6.138805283929\,6 \cdot 10^{11} \pm 0.5 \cdot 10^{-2}$
-7	$1.0953372445\,111 \cdot 10^{13} \pm 8.7 \cdot 10^{2}$	$6.1388052\,891963 \cdot 10^{11} \pm 9.3 \cdot 10^{3}$
-8	$1.09533724\,45386 \cdot 10^{13} \pm 2.6 \cdot 10^{4}$	$6.1388\,034612038 \cdot 10^{11} \pm 1.3 \cdot 10^{6}$

Table 3. Approximate constants C_F for $F = \Upsilon_{26a}, \Upsilon_{26b}$

D	$C_{\Upsilon_{26a}}$	$C_{\Upsilon_{26b}}$
-3	$9.615528574589\,1 \cdot 10^{13} \pm 0.5 \cdot 10^{0}$	$6.232883950541\,7 \cdot 10^{12} \pm 0.5 \cdot 10^{-1}$
-4	$9.615528574589\,1 \cdot 10^{13} \pm 0.5 \cdot 10^{0}$	$6.232883950541\,7 \cdot 10^{12} \pm 0.5 \cdot 10^{-1}$
-7	$9.61552857\,46522 \cdot 10^{13} \pm 1.1 \cdot 10^{4}$	$6.232883950\,5729 \cdot 10^{12} \pm 2.3 \cdot 10^{2}$
-8	$9.615528\,5333968 \cdot 10^{13} \pm 8.2 \cdot 10^{6}$	$6.232883\,9821394 \cdot 10^{12} \pm 1.6 \cdot 10^{5}$

Table 4. The first Fourier coefficients of $\Upsilon_{20}, \ldots, \Upsilon_{26b}$

D	n, r, m	Υ_{20}	Υ_{22}	Υ_{24a}	Υ_{24b}	Υ_{26a}	Υ_{26b}
-3	1, 1, 1	1	1	1	3	1	3
-4	1, 0, 1	4	−12	−16	76	−8	124
-7	1, 1, 2	56	1 344	4 408	−616	−7 456	51 632
-8	1, 0, 2	2 616	216	44 256	−2 904	15 216	−109 752
-11	1, 1, 3	−55 077	409 779	−1 147 701	2 122 593	−1 180 509	7 299 177
-12	1, 0, 3	408 832	468 448	−378 272	11 995 968	3 505 408	−39 833 376
-12	2, 2, 2	−840 960	−2 215 680	−795 324	18 309 504	9 218 340	495 227 520

TABLE 5. The first eigenvalues of Υ_{20}

n	$\lambda(n)$
2	−840 960
3	346 935 960
5	−5 232 247 240 500
7	2 617 414 076 964 400
11	1 427 823 701 421 564 744
13	−83 773 835 478 688 698 980
17	14 156 088 476 175 218 899 620
19	146 957 560 176 221 097 673 720
23	−7 159 245 922 546 757 692 913 520
29	1 055 528 218 470 800 414 110 149 180
31	4 031 470 549 468 367 403 585 068 224
37	−154 882 657 977 740 251 483 442 365 940
41	1 126 683 124 934 949 617 518 831 346 964
43	74 572 686 686 194 644 813 168 430 600
47	−13 773 335 595 379 978 013 820 602 730 720
53	29 292 488 702 536 161 643 591 933 657 260
59	521 943 213 201 995 351 655 113 144 025 960
61	896 978 197 899 858 751 399 574 623 768 444
67	−2 921 787 486 641 381 474 027 809 454 434 280
2^2	248 256 200 704
3^2	−452 051 040 393 665 991
5^2	−94 655 785 156 653 029 446 859 375
7^2	−5 501 629 950 184 780 949 434 983 315 951
11^2	−126 258 221 861 417 704 499 584 077 355 164 268 151
13^2	2 528 254 555 352 510 520 887 488 261 241 887 242 369
17^2	262 144 933 510 286 336 089 464 293 262 250 165 947 750 889
19^2	−283 417 759 450 334 375 466 210 009 895 464 677 379 295 086 759
23^2	127 862 428 522 278 879 932 688 110 084 314 434 400 497 569 566 129
29^2	408 550 299 154 535 330 723 926 336 201 059 419 422 405 306 949 883 361
31^2	−9 417 686 481 892 622 568 784 061 821 415 683 057 728 289 096 885 473 471
37^2	4 270 657 975 661 931 417 960 508 434 757 260 969 748 219 593 839 247 065 169
41^2	129 620 395 091 878 626 890 240 343 719 327 738 119 688 391 311 944 613 269 369
43^2	−2 118 391 905 744 174 698 890 014 439 813 915 105 652 042 393 393 982 400 772 151
47^2	10 717 867 956 150 312 430 187 083 192 735 560 357 439 349 298 395 760 667 696 609
53^2	−6 359 983 052 359 692 969 866 068 986 893 310 598 482 880 773 029 488 944 413 754 191
59^2	159 291 906 542 794 821 742 879 348 124 552 646 753 906 149 121 778 952 350 318 431 721
61^2	−653 805 853 261 332 407 170 328 486 766 159 640 869 797 840 457 778 124 369 821 593 951
67^2	25 254 882 862 606 589 034 647 035 623 760 404 781 292 970 925 413 106 240 956 567 868 089

References

[1] M. Abramowitz, I. Stegun: *Pocketbook of mathematical functions*. Verlag Harri Deutsch, (1984). MR **85j:**00005b

[2] A. Andrianov: *Euler products corresponding to Siegel modular forms of genus 2*. Russ. Math. Surveys 29, No.3, 45-116 (1974). MR **55:**5540

[3] S. Böcherer: *Bemerkungen über die Dirichletreihen von Koecher und Maass*. Math. Gottingensis, Schriftenr. d. Sonderforschungsbereichs Geom. Anal. 68, (1986).

[4] S. Böcherer, R. Schulze-Pillot: *The Dirichlet series of Koecher and Maass and modular forms of weight 3/2*. Math. Z. 209, No.2, 273-287 (1992). MR **93b:**11053

[5] M. Eichler, D. Zagier: *The theory of Jacobi forms*. Progress in Mathematics, Vol. 55. Boston-Basel-Stuttgart: Birkhäuser (1985). MR **86j:**11043

[6] W. Kohnen: *On character twists of certain Dirichlet series*. Mem. Fac. Sci. Kyushu Univ., vol. 47, 103-117 (1993). MR **94c:**11044

[7] W. Kohnen, J. Sengupta, A. Krieg: *Characteristic twists of a Dirichlet series for Siegel cusp forms*. Manuscripta Math. 87, 489-499 (1995). MR **96f:**11071

[8] W. Kohnen, N.-P. Skoruppa *A certain Dirichlet series attached to Siegel modular forms of degree two*. Invent. Math. 95, 541-558 (1989). MR **90b:**11050

[9] M. Kuß: *Die getwistete Spinor Zeta Funktion und die Böcherer Vermutung*. Dissertation. (2000)

[10] A.F. Lavrik: *Functional equations of Dirichlet functions*. Soviet Math. Dokl. 7, 1471-1473 (1966). MR **34:**4464

[11] H. Maass: *Ueber eine Spezialschar von Modulformen zweiten Grades. I, II, III* Invent. Math. 52, 95-104 (1979), Invent. Math. 53, 249-253, 255-265 (1979). MR **80f:**10031; MR **81a:**11037; MR **81a:**11038

[12] N.P. Skoruppa: *Computations of Siegel modular forms of genus two*. Math. Comput. 58, 381-398 (1992). MR **92e:**11041

[13] J.L. Waldspurger: *Sur les coefficients de Fourier des formes modulaires de poids demi-entier*. J. Math. Pures Appl. (9) 60, 375-484 (1981). MR **83h:**10061

[14] R. Weissauer: *The Ramanujan conjecture for genus two Siegel modular forms (an application of the trace formula)*. Preprint, Mannheim (1993)

Universität Heidelberg, Mathematisches Institut, Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany

E-mail address: `winfried@mathi.uni-heidelberg.de`

Universität Heidelberg, Mathematisches Institut, Im Neuenheimer Feld 288, D-69120 Heidelberg, Germany

E-mail address: `michael.kuss@urz.uni-heidelberg.de`