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FULL-WAVE ANALYSIS OF DIELECTRIC WAVEGUIDES AT A GIVEN FREQUENCY

L. VARDAPETYAN AND L. DEMKOWICZ

ABSTRACT. New variational formulation to compute propagation constants is proposed. Based on it, vector finite element method is proved to exclude spurious modes provided finite elements possess discrete compactness property. Convergence analysis is conducted in the framework of collectively compact operators. Reported theoretical results apply to a wide class of vector finite elements including two families of Nedelec and their generalization, the *hp*-edge elements. Numerical experiments fully support theoretical estimates for convergence rates.

1. Introduction

Vector finite elements have been successfully used in the analysis and design of electromagnetic resonators. Standard Nedelec elements of [26] have been tested in practice to deliver solutions free of spurious modes. These elements can be readily applied to waveguides for the computation of cutoff frequencies ω for modes with a given propagation constant β. The case with a given real-valued β has been extensively studied, see for example [7, 17]. In this paper, we are interested in the more physically relevant case, when ω is given but β is unknown. It differs significantly from the well-studied one. In general, even nonlossy waveguides yield nonselfadjoint eigenvalue problems with possibly complex-valued β and the corresponding evanescent modes. To the best of the authors' knowledge, this problem has been addressed only by the engineering community (for a review see [31]), with no comprehensive convergence analysis. Moreover, an existing approach to this problem allows for an infinite dimensional subspace of spurious modes corresponding to propagation constant $\beta = 0$, see [23, 22]. These spurious modes may pollute numerical TEM-like solutions, especially as $\omega \to 0$, if no extra care is taken in the numerical scheme. Our goal has been to develop a variational formulation applicable for quasistatic regimes and to provide a comprehensive mathematical analysis of a suitable finite element discretization with convergence rate estimates. The proposed variational formulation of the appropriate eigenvalue problem (although less memory-efficient than in [23, 22]) is uniformly stable as $\omega \to 0$, does not suffer from spurious modes[1] and, leading to the spectral analysis of a compact operator, greatly simplifies the

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[1]One might argue that now $\beta = \infty$, not $\beta = 0$ as in [23, 22], has infinite multiplicity and spurious eigenmodes. However, the fields corresponding to $\beta = \infty$ are completely eliminated from actual computations.

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mathematical analysis. As a practical advantage, zero is not an eigenvalue and all (noninfinite) eigenvalues have finite multiplicities. In this paper, we develop a new formulation, prove it to be free of spurious modes, and study convergence properties of vector finite element methods based on this formulation. The convergence results apply to any finite elements which have discrete compactness and appropriate approximability properties, see [18, 8]. This includes two families of Nedelec elements [26, 27] and their generalization, the *hp*-adaptive elements of [14, 33]. The rates of convergence relate to the interpolation error estimates. We present convergence rates for Nedelec elements [26, 27] and show that numerical experiments confirm the theoretical estimates. We refer to [32] for additional numerical examples.

2. Problem setup

Let us consider a closed waveguide defined by a right cylinder with cross section $\Omega \subset \mathbb{R}^2$. The waveguide is filled with inhomogeneous media. Real-valued functions ϵ and μ describe the electromagnetic properties of the waveguide. As in [17], we assume that the functions ϵ and μ are piecewise Lipschitz continuous and have no variation along the waveguide. Ω is a bounded, Lipschitz, simply connected polyhedral domain with boundary Γ.

We are interested in finding solutions to Maxwell's equations which propagate along the source-free waveguide. The general ansatz for such fields is given by

$$
\begin{aligned}
\mathcal{E}(\boldsymbol{x}, x_3, t) &= (\boldsymbol{E}(\boldsymbol{x}), E_3(\boldsymbol{x}))e^{\jmath(\omega t \mp \beta x_3)},\\
\mathcal{H}(\boldsymbol{x}, x_3, t) &= (\boldsymbol{H}(\boldsymbol{x}), H_3(\boldsymbol{x}))e^{\jmath(\omega t \mp \beta x_3)},
\end{aligned}
\tag{2.1}
$$

where $\boldsymbol{x} \in \Omega$ and the x_3-axis is along the waveguide. The positive number ω denotes frequency, and β is the constant of propagation. $\boldsymbol{E}$ and $\boldsymbol{H}$ are electric and magnetic field components in the plane of the cross section, and E_3 and H_3 are electric and magnetic components along the waveguide. With ansatz (2.1), the second order 3D Maxwell equations expressed in terms of electric field $(\boldsymbol{E}, E_3)$ alone reduce to three 2D equations:

$$
\left\{
\begin{aligned}
&\boldsymbol{\nabla} \times (\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}) - \omega^2 \epsilon \boldsymbol{E} + \frac{\beta^2}{\mu}\boldsymbol{E} - \frac{\jmath\beta}{\mu}\nabla E_3 = 0,\\
&\boldsymbol{\nabla} \circ (\frac{1}{\mu}\nabla E_3) + \omega^2 \epsilon E_3 + \jmath\beta \boldsymbol{\nabla} \circ (\frac{1}{\mu}\boldsymbol{E}) = 0,\\
&\boldsymbol{\nabla} \circ (\epsilon \boldsymbol{E}) - \jmath\beta\epsilon E_3 = 0.
\end{aligned}
\right.
\tag{2.2}
$$

For simplicity, perfect electric conductor boundary conditions are imposed on Γ:

$$
\boldsymbol{E} \times \boldsymbol{n} = 0, \quad E_3 = 0,
\tag{2.3}
$$

where $\boldsymbol{n}$ is the outward unit normal on Γ. Since no sources are given, (2.2) is an eigenvalue problem. Either ω or β is assumed to be known, and the goal is to find all possible pairs which consist of the other missing constant β (or ω) and the corresponding electric field $(\boldsymbol{E}, E_3)$ that solve (2.2) and satisfy (2.3). The case with a given real-valued β has been examined in [7, 17]. We are interested in the more physically relevant case, when ω is given, but β is unknown.

3. Variational formulation

As mentioned above, our goal is to find all pairs $(\beta, (\boldsymbol{E}, E_3))$ which satisfy equations (2.2) and boundary conditions (2.3) for a given frequency $\omega > 0$.

Before we construct our variational formulation, let us develop a functional setting suitable for the problem at hand. Let $\mathcal{H}$ denote the Hilbert space $\boldsymbol{L}^2_\epsilon(\Omega) \times L^2_\epsilon(\Omega)$, equipped with the norm

$$\|(\boldsymbol{E}, p)\|_{\mathcal{H}} = (\|\boldsymbol{E}\|^2_\epsilon + \|p\|^2_\epsilon)^{\frac{1}{2}}, \tag{3.1}$$

where $\|\cdot\|_\epsilon$ is the ϵ-weighted L^2-norm:

$$\|p\|_\epsilon = \left(\int_\Omega \epsilon \mid p \mid^2 d\boldsymbol{x}\right)^{\frac{1}{2}}. \tag{3.2}$$

We also introduce the Hilbert space $\boldsymbol{X} = \boldsymbol{W} \times V$, where $\boldsymbol{W}$ and V are Hilbert spaces defined below. $\boldsymbol{W}$ will provide us with vector-valued test and trial functions:

$$\boldsymbol{W} \stackrel{\text{def}}{=} \{\boldsymbol{E} \in \boldsymbol{L}^2_\epsilon(\Omega) \;:\; \boldsymbol{\nabla} \times \boldsymbol{E} \in \boldsymbol{L}^2(\Omega),\; \boldsymbol{n} \times \boldsymbol{E} = \boldsymbol{0} \text{ on } \Gamma\} \tag{3.3}$$

equipped with the norm

$$\|\boldsymbol{E}\|_{\boldsymbol{W}} = (\|\boldsymbol{E}\|^2_\epsilon + \|\boldsymbol{\nabla} \times \boldsymbol{E}\|^2)^{\frac{1}{2}}. \tag{3.4}$$

And V will provide us with scalar-valued test and trial functions:

$$V \stackrel{\text{def}}{=} \{p \in H^1(\Omega) \;:\; p = 0 \text{ on } \Gamma\} \tag{3.5}$$

equipped with the norm

$$\|p\|_V = (\nabla p, \nabla p)_\epsilon. \tag{3.6}$$

The norm on $\boldsymbol{X}$ is simply

$$\|(\boldsymbol{F}, q)\|_{\boldsymbol{X}} = (\|\boldsymbol{F}\|^2_{\boldsymbol{W}} + \|q\|^2_V)^{\frac{1}{2}}. \tag{3.7}$$

The boundary condition in (3.3) is understood in the sense of the generalized Green formula [16], and the boundary condition in (3.5) in the sense of traces.

Using the density of infinitely differentiable functions of finite support in $\mathcal{H}$ as well as in $\boldsymbol{X}$, we may conclude that $\boldsymbol{X}$ is dense in $\mathcal{H}$.

$\boldsymbol{W}_0$ is a subspace of $\boldsymbol{W}$:

$$\boldsymbol{W}_0 \stackrel{\text{def}}{=} \{\boldsymbol{E} \in \boldsymbol{W} \;:\; \boldsymbol{\nabla} \times \boldsymbol{E} = \boldsymbol{0}\}. \tag{3.8}$$

We also note that, in the context of our setup, the gradient operator maps V onto $\boldsymbol{W}_0$:

$$\nabla V = \boldsymbol{W}_0. \tag{3.9}$$

If we multiply $(2.2)_1$ by a vector test function $\boldsymbol{F}$, and $(2.2)_2$ and $(2.2)_3$ by a scalar test function q, integrate by parts and apply proper boundary conditions, we can recast initial system (2.2) into the following variational form:

Find pairs $(\beta, (\boldsymbol{E}, E_3)) \in (\mathbb{C}, \boldsymbol{X})$ such that $\forall (\boldsymbol{F}, q) \in \boldsymbol{X}$:

$$\left\{ \begin{array}{l} (\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}, \boldsymbol{\nabla} \times \boldsymbol{F}) - \omega^2(\epsilon\boldsymbol{E}, \boldsymbol{F}) + \beta^2(\frac{1}{\mu}\boldsymbol{E}, \boldsymbol{F}) - \jmath\beta(\frac{1}{\mu}\nabla E_3, \boldsymbol{F}) = 0, \\ -(\frac{1}{\mu}\nabla E_3, \nabla q) + \omega^2(\epsilon E_3, q) - \jmath\beta(\frac{1}{\mu}\boldsymbol{E}, \nabla q) = 0, \\ -(\epsilon\boldsymbol{E}, \nabla q) - \jmath\beta(\epsilon E_3, q) = 0. \end{array} \right. \tag{3.10}$$

First, we consider the special case when $\beta = 0$. The problem (3.10) then splits into two problems:

For a given $\omega \in R$, find $\boldsymbol{E} \in \boldsymbol{W}$ such that $\forall(\boldsymbol{F}, q) \in \boldsymbol{W} \times V$

$$\begin{cases} (\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}, \boldsymbol{\nabla} \times \boldsymbol{F}) - \omega^2(\epsilon \boldsymbol{E}, \boldsymbol{F}) = 0, \\ (\epsilon \boldsymbol{E}, \nabla q) = 0; \end{cases} \tag{3.11}$$

For a given $\omega \in R$, find $E_3 \in V$ such that $\forall q \in V$

$$(\frac{1}{\mu}\nabla E_3, \nabla q) - \omega^2(\epsilon E_3, q) = 0. \tag{3.12}$$

Solutions of (3.11) and (3.12) taken as a pair satisfy the original system.

Now let us recall two theorems which play important roles in our analysis. Details can be found in [24].

Theorem 1. *With our assumptions on the domain, the spectrum of the following variational eigenvalue problem for the curl-curl operator:*

$$\begin{cases} \boldsymbol{E} \in \boldsymbol{W},\ \nu \in \mathbb{C}, \\ \displaystyle\int_\Omega \frac{1}{\mu}(\boldsymbol{\nabla} \times \boldsymbol{E}) \circ (\boldsymbol{\nabla} \times \bar{\boldsymbol{F}})\, d\boldsymbol{x} = \nu \int_\Omega \epsilon \boldsymbol{E} \circ \bar{\boldsymbol{F}}\, d\boldsymbol{x} \quad \forall \boldsymbol{F} \in \boldsymbol{W}, \\ \displaystyle\int_\Omega \epsilon \boldsymbol{E} \circ \nabla \bar{q}\, d\boldsymbol{x} = 0 \quad \forall q \in V, \end{cases} \tag{3.13}$$

is a pure point spectrum on the positive real axis, extending to infinity:

$$\{0 < \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n \leq \cdots < +\infty\}.$$

Theorem 2. *With our assumptions on the domain, the spectrum of the following variational eigenvalue problem for the div-grad operator:*

$$\begin{cases} p \in V,\ \eta \in \mathbb{C}, \\ \displaystyle\int_\Omega \frac{1}{\mu}(\nabla p) \circ (\nabla \bar{q})\, d\boldsymbol{x} = \eta \int_\Omega \epsilon p \bar{q}\, d\boldsymbol{x} \quad \forall q \in V, \end{cases} \tag{3.14}$$

is a pure point spectrum on the positive real axis, extending to infinity:

$$\{0 < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_n \leq \cdots < +\infty\}.$$

It follows from the theorems above that $\beta = 0$ has a nontrivial eigenmode $(\boldsymbol{E}, E_3)$ associated with it if and only if ω^2 is an eigenvalue of (3.13) or (3.14). Thus, the multiplicity of $\beta = 0$ is equal to the sum of multiplicities of ω^2 that is an eigenvalue of (3.13) or (3.14). Therefore, zero as an eigenvalue of (3.10) may have only a finite multiplicity.

Let us now narrow our task and seek only those solutions of (3.10) that correspond to a nonzero propagating constant β. In other words, we assume that

$$\bullet \quad \boldsymbol{R}(\omega) : \text{Frequency } \omega \notin \{\eta_i\}_{i=1}^{\infty} \cup \{\nu_j\}_{j=1}^{\infty}\,. \tag{3.15}$$

Clearly, all propagating modes are among these solutions. We note that the constant β enters system (3.10) in both the first and second powers. However, if the component E_3 is rescaled as $E_3^{\text{new}} = \jmath\beta E_3$ or $E_3^{\text{new}} = \dfrac{1}{\jmath\beta} E_3$, then only β^2 is present in the modified system.

Moreover, since $\beta \neq 0$, the equations in (3.10) are not all linearly independent. Indeed, if $\boldsymbol{F} = \nabla q$, then $(3.10)_1$ yields

$$-\omega^2(\epsilon\boldsymbol{E}, \nabla q) + \beta^2(\frac{1}{\mu}\boldsymbol{E}, \nabla q) - \jmath\beta(\frac{1}{\mu}\nabla E_3, \nabla q) = 0. \tag{3.16}$$

Now we can express the first term of (3.16) via $(\epsilon E_3, q)$ of $(3.10)_3$ to get

$$\jmath\beta\left[\omega^2(\epsilon E_3, q) - \jmath\beta(\frac{1}{\mu}\boldsymbol{E}, \nabla q) - (\frac{1}{\mu}\nabla E_3, \nabla q)\right] = 0, \tag{3.17}$$

which yields $(3.10)_2$ multiplied by $\jmath\beta$. Similarly, the difference of $(3.10)_2$ multiplied by $\jmath\beta$ and (3.16) gives $(3.10)_3$ multiplied by ω^2.

Thus, we can actually consider not all three but only two equations of the variational system (3.10). The question is: which equation to discard? And how to rescale E_3? Following the logic of our previous papers [14] and [33], we keep $(3.10)_3$, which explicitly involves a weak divergence condition on $\boldsymbol{E}$, and turn it into a constraint on $(\boldsymbol{E}, E_3)$ by rescaling E_3 as follows:

$$E_3^{\text{new}} = \jmath\beta E_3. \tag{3.18}$$

To limit our notation, E_3 will represent E_3^{new} for the remainder of this paper.

With the proposed simplifications, formulation (3.10) reduces to:

For a given $\omega > 0$, find all pairs $(\beta, (\boldsymbol{E}, E_3)) \in \mathbb{C} \times \boldsymbol{X}$, $\beta \neq 0$, such that $\forall(\boldsymbol{F}, q) \in \boldsymbol{X}$

$$\begin{cases} (\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}, \boldsymbol{\nabla} \times \boldsymbol{F}) - \omega^2(\epsilon\boldsymbol{E}, \boldsymbol{F}) - (\frac{1}{\mu}\nabla E_3, \boldsymbol{F}) = -\beta^2(\frac{1}{\mu}\boldsymbol{E}, \boldsymbol{F}), \\ -(\epsilon\boldsymbol{E}, \nabla q) - (\epsilon E_3, q) = 0. \end{cases} \tag{3.19}$$

We stress that for nonzero β the new formulation is equivalent to the original one. Later on, we will show that the new formulation does not allow nontrivial solutions if $\beta = 0$.

Remark 1. In [23, 22], Lee et al. kept $(3.10)_1$ and $(3.10)_2$ and used a transformation equivalent to $E_3 = \jmath\beta E_3^{\text{new}}$ to get a symmetric system. When discretized, this formulation leads to a generalized eigenvalue problem with a singular matrix, whose spurious solutions that correspond to $\beta = 0$ can present considerable complications, especially at low frequencies.

4. Analysis of the continuous problem

In order to discuss the properties of the problem at hand, let us consider a sesquilinear form $\boldsymbol{a}_\omega$ defined on $\boldsymbol{X} \times \boldsymbol{X}$ corresponding to the left hand side of (3.19):

$$\begin{aligned} &\boldsymbol{a}_\omega : \boldsymbol{X} \times \boldsymbol{X} \longrightarrow \mathbb{C}, \\ &\boldsymbol{a}_\omega((\boldsymbol{E}, E_3), (\boldsymbol{F}, q)) = (\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}, \boldsymbol{\nabla} \times \boldsymbol{F}) - \omega^2(\epsilon\boldsymbol{E}, \boldsymbol{F}) - (\frac{1}{\mu}\nabla E_3, \boldsymbol{F}) \\ &\quad +\omega^2((\epsilon\boldsymbol{E}, \nabla q) + (\epsilon E_3, q)). \end{aligned} \tag{4.1}$$

Clearly, the form $\boldsymbol{a}_\omega$ is bounded, i.e., there exists $C > 0$ such that

$$|\ \boldsymbol{a}_\omega((\boldsymbol{E}, E_3), (\boldsymbol{F}, q))\ | \leq C\|(\boldsymbol{E}, E_3)\|_{\boldsymbol{X}}\|(\boldsymbol{F}, q)\|_{\boldsymbol{X}} \quad \forall(\boldsymbol{E}, E_3), (\boldsymbol{F}, q) \in \boldsymbol{X}. \tag{4.2}$$

If we also consider the bounded symmetric projection operator $\mathbb{B} : \boldsymbol{X} \longrightarrow \boldsymbol{X}$ defined as

$$\mathbb{B}\,(\boldsymbol{F}, q) = (\frac{-1}{\epsilon\mu}\boldsymbol{F}, 0), \tag{4.3}$$

then, with the new notation, (3.19) becomes:

For a given $\omega > 0$, find all pairs $(\beta, (\boldsymbol{E}, E_3)) \in \mathbb{C} \times \boldsymbol{X}$, $\beta \neq 0$, such that $\forall(\boldsymbol{F}, q) \in \boldsymbol{X}$

$$\boldsymbol{a}_\omega((\boldsymbol{E}, E_3), (\boldsymbol{F}, q)) = \beta^2(\mathbb{B}(\boldsymbol{E}, E_3), (\boldsymbol{F}, q))_\epsilon. \tag{4.4}$$

We show now that, with the assumption $\boldsymbol{R}(\omega)$ on ω, the linear operator L : $\boldsymbol{X} \longrightarrow \boldsymbol{X'}$ defined by the form a_ω,

$$\langle L(\boldsymbol{E}, E_3), (\boldsymbol{F}, q)\rangle := \boldsymbol{a}_\omega((\boldsymbol{E}, E_3), (\boldsymbol{F}, q)) \quad \forall(\boldsymbol{F}, q) \in \boldsymbol{X} \tag{4.5}$$

is an isomorphism. First, it follows from (4.2) that L is a continuous operator:

$$\|L\|_{\mathcal{L}(\boldsymbol{X},\boldsymbol{X'})} \leq C. \tag{4.6}$$

Now, let us confirm that for any $(\boldsymbol{J}, Q) \in \boldsymbol{X'}$ there exists a unique pair $(\boldsymbol{E}, E_3) \in \boldsymbol{X}$ such that

$$\boldsymbol{a}_\omega(\boldsymbol{E}, E_3), (\boldsymbol{F}, q)) = \langle(\boldsymbol{J}, Q), (\boldsymbol{F}, q)\rangle, \quad \forall(\boldsymbol{F}, q) \in \boldsymbol{X}. \tag{4.7}$$

The explicit form of the system (4.7) follows:

Find $(\boldsymbol{E}, E_3) \in \boldsymbol{X}$, such that $\forall(\boldsymbol{F}, q) \in \boldsymbol{X}$:

$$\begin{cases} (\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}, \boldsymbol{\nabla} \times \boldsymbol{F}) - \omega^2(\epsilon\boldsymbol{E}, \boldsymbol{F}) - (\frac{1}{\mu}\nabla E_3, \boldsymbol{F}) = \langle \boldsymbol{J}, \boldsymbol{F}\rangle_1, \\ \omega^2\,[(\epsilon\boldsymbol{E}, \nabla q) + (\epsilon E_3, q)] = \langle Q, q\rangle_2; \end{cases} \tag{4.8}$$

here $\langle\cdot,\cdot\rangle_1$ denotes the duality pairing on $\boldsymbol{W}$ and $\langle\cdot,\cdot\rangle_2$ is the duality pairing on V. We solve the above system in two steps. First, we solve for E_3 and then for $\boldsymbol{E}$ with E_3 shifted to the source side.

Let us consider $(4.8)_1$ with $F = \boldsymbol{\nabla} q$ and add that equation to $(4.8)_2$:

$$-(\frac{1}{\mu}\boldsymbol{\nabla} E_3, \boldsymbol{\nabla} q) + \omega^2(\epsilon E_3, q) = \langle Q, q\rangle_2 + \langle \boldsymbol{J}, \boldsymbol{\nabla} q\rangle_1, \qquad \forall q \in V. \tag{4.9}$$

Since $\boldsymbol{J} \in \boldsymbol{W'}$,

$$\mid \langle \boldsymbol{J}, \boldsymbol{\nabla} q\rangle_1 \mid \leq \|\boldsymbol{J}\|_{\boldsymbol{W'}}\|\boldsymbol{\nabla} q\|_{\boldsymbol{W}} = \|\boldsymbol{J}\|_{\boldsymbol{W'}}\|q\|_V, \quad \forall q \in V. \tag{4.10}$$

Thus, we can define $\hat{J} \in V'$ by

$$\langle \hat{J}, q\rangle_2 \;:=\; \langle \boldsymbol{J}, \boldsymbol{\nabla} q\rangle_1, \quad \forall q \in V. \tag{4.11}$$

It immediately follows from (4.10) that

$$\|\hat{J}\|_{V'} \leq \|\boldsymbol{J}\|_{\boldsymbol{W'}}. \tag{4.12}$$

Thus, $(\hat{J} + Q) \in V'$ acts as the source in (4.9):

$$-(\frac{1}{\mu}\nabla E_3, \nabla q) + \omega^2(\epsilon E_3, q) = \langle \hat{J} + Q, q\rangle_1, \qquad \forall q \in V. \tag{4.13}$$

Since ω^2 is not an eigenvalue of $\boldsymbol{\nabla}\frac{1}{\mu}\boldsymbol{\nabla}$, a unique solution to (4.13) exists and is bounded in V by $\|\hat{J}+Q\|_{V'}$. Utilizing (4.12), we get that

$$\|E_3\|_V \leq \frac{C_1}{\gamma_1}(\|Q\|^2_{V'} + \|\boldsymbol{J}\|^2_{\boldsymbol{W}'})^{\frac{1}{2}}, \tag{4.14}$$

where $C_1 > 0$ is some constant and γ_1 is the stability constant for the Helmholtz equation expressed via the eigenvalues of the operator $\boldsymbol{\nabla}\frac{1}{\mu}\nabla$ as (see [11])

$$\gamma_1 = \min_{i=1\ldots\infty}\left\{\frac{\mid \eta_i - \omega^2 \mid}{\eta_i}\right\}. \tag{4.15}$$

Now, let us use (4.8) to solve for $\boldsymbol{E}$. We move the terms with E_3 to the right-hand side:

$$\begin{cases} (\frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{E}, \boldsymbol{\nabla}\times\boldsymbol{F}) - \omega^2(\epsilon\boldsymbol{E},\boldsymbol{F}) = (\frac{1}{\mu}\boldsymbol{\nabla}E_3,\boldsymbol{F}) + \langle\boldsymbol{J},\boldsymbol{F}\rangle_1 \quad \forall\boldsymbol{F}\in\boldsymbol{W}, \\ \omega^2(\epsilon\boldsymbol{E},\nabla q) = -\omega^2(\epsilon E_3, q) + \langle Q, q\rangle_2 \quad \forall q\in V. \end{cases} \tag{4.16}$$

We notice that since ω is positive, equation $(4.16)_2$ is redundant. Indeed, let us use $\boldsymbol{F} = \nabla q$ in $(4.16)_1$. And since E_3 solves (4.9), we can further simplify $(4.16)_1$ to get

$$\omega^2(\epsilon\boldsymbol{E},\nabla q) + \omega^2(\epsilon E_3, q) - \langle Q, q\rangle_2 = 0. \tag{4.17}$$

Therefore equation $(4.16)_2$ acts as a constraint. In order to resolve it, we introduce in $(4.16)_1$ a Lagrange multiplier $p\in V$. After dividing both sides of $(4.16)_2$ by ω^2 we have

$$\begin{cases} (\frac{1}{\mu}\boldsymbol{\nabla}\times\boldsymbol{E}, \boldsymbol{\nabla}\times\boldsymbol{F}) - \omega^2(\epsilon\boldsymbol{E},\boldsymbol{F}) - (\epsilon\nabla p,\boldsymbol{F}) \\ \qquad = (\frac{1}{\mu}\boldsymbol{\nabla}E_3,\boldsymbol{F}) + \langle\boldsymbol{J},\boldsymbol{F}\rangle_1, \quad \forall\boldsymbol{F}\in\boldsymbol{W}, \\ (\epsilon\boldsymbol{E},\nabla q) = -(\epsilon E_3, q) + \langle\frac{Q}{\omega^2}, q\rangle_2, \quad \forall q\in V. \end{cases} \tag{4.18}$$

Formulations (4.16) and (4.18) are equivalent. Indeed, using (4.9) we can reduce a linear combination of $(4.18)_1$ with $\boldsymbol{F} = \nabla p$ and $(4.18)_2$ with $q = p$ to give

$$(\epsilon\nabla p, \nabla p) = 0. \tag{4.19}$$

Therefore, as an element of V, $p = 0$. So, we can solve (4.18) instead of (4.16).

We also notice that, by the Cauchy-Schwarz and Poincaré inequalities,

$$\mid (\epsilon E_3, q) \mid \leq C\|E_3\|_\epsilon\|q\|_V, \tag{4.20}$$

for some $C > 0$. Therefore we can define $\hat{E}_3 \in V'$ by

$$\langle\hat{E}_3, q\rangle_2 = (\epsilon E_3, q) \quad \forall q\in V. \tag{4.21}$$

And by the Poincaré inequality, it follows from (4.20) that

$$\|\hat{E}_3\|_{V'} \leq C\|E_3\|_V. \tag{4.22}$$

Similarly, using the Cauchy-Schwarz inequality and the natural assumption that ϵ and μ are strictly positive and bounded on Ω, we can find some $C_2 > 0$ so that

$$\mid (\frac{1}{\mu}\boldsymbol{\nabla}E_3, \boldsymbol{F}) \mid \leq C_2\|E_3\|_V\|\boldsymbol{F}\|_{\boldsymbol{W}}. \tag{4.23}$$

Thus we can define $\boldsymbol{G} \in \boldsymbol{W}'$ by

$$\langle \boldsymbol{G}, \boldsymbol{F} \rangle_1 \; = \; (\frac{1}{\mu}\boldsymbol{\nabla} E_3, \boldsymbol{F}) \quad \forall \boldsymbol{F} \in \boldsymbol{W}. \tag{4.24}$$

It follows from (4.23) that

$$\| \boldsymbol{G} \|_{\boldsymbol{W}'} \leq C_2 \| E_3 \|_V. \tag{4.25}$$

With the newly introduced $\hat{E}_3$ and $\boldsymbol{G}$ system (4.18) reads as follows:

$$\left\{ \begin{array}{l} (\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}, \boldsymbol{\nabla} \times \boldsymbol{F}) - \omega^2(\epsilon \boldsymbol{E}, \boldsymbol{F}) - (\epsilon \nabla p, \boldsymbol{F}) = \langle \boldsymbol{G} + \boldsymbol{J}, \boldsymbol{F} \rangle_1, \quad \forall \boldsymbol{F} \in \boldsymbol{W}, \\ (\epsilon \boldsymbol{E}, \nabla q) = \langle (\frac{Q}{\omega^2} - \hat{E}_3), q \rangle_2, \quad \forall q \in V. \end{array} \right. \tag{4.26}$$

Since ω^2 is not in the spectrum of $\boldsymbol{\nabla} \times \frac{1}{\mu}\boldsymbol{\nabla}\times$, the bounded sesquilinear form associated with the system (4.26) satisfies the $\inf - \sup$ conditions; see for example [14] and [33], where the stability constant is computed. Therefore, a unique solution to this system exists and is bounded by $\|(\boldsymbol{G} + \boldsymbol{J}), (\frac{Q}{\omega^2} - \hat{E}_3)\|_{\boldsymbol{X}'}$. Using the estimates for E_3, $\hat{E}_3$, and $\boldsymbol{G}$ in (4.14), (4.22), (4.25), the triangle inequality, and some elementary algebra, we can get an estimate on $\boldsymbol{E}$:

$$\|\boldsymbol{E}\|_{\boldsymbol{W}} \leq \frac{C_3}{\gamma_1\gamma_2}(\|\boldsymbol{J}\|^2_{\boldsymbol{W}'} + \|Q\|^2_{V'})^{\frac{1}{2}}. \tag{4.27}$$

Here $C_3 > 0$ is a constant, γ_1 is as in (4.15) and γ_2 is a stability constant, [14]:

$$\gamma_2 = \min\left\{ \frac{1}{1+\omega^2}, \min_{j=1\ldots\infty} \left\{ \frac{|\,\nu_j - \omega^2\,|}{\nu_j} \right\} \right\}. \tag{4.28}$$

Therefore we have demonstrated that the continuous operator L is bijective, and the estimates in (4.14) and (4.27) show that L^{-1}, its inverse, is bounded. So, by definition, L is an isomorphism, which in turn, means that form $\boldsymbol{a}_\omega$ satisfies the $\inf - \sup$ conditions, see e.g., [5, 28]. We formulate our findings in a theorem.

Theorem 3. *With the assumption on ω as in* (3.15)*, the continuous form $\boldsymbol{a}_\omega$ satisfies the following conditions*:

There exists $C > 0$ such that

$$\sup_{(\boldsymbol{F},q)\in \boldsymbol{X}} \frac{|\,\boldsymbol{a}_\omega((\boldsymbol{E}, E_3), (\boldsymbol{F}, q))\,|}{\|(\boldsymbol{F}, q)\|_{\boldsymbol{X}}} \geq C\gamma_1\gamma_2 \|(\boldsymbol{E}, E_3)\|_{\boldsymbol{X}} \qquad \forall(\boldsymbol{E}, E_3) \in \boldsymbol{X}, \tag{4.29}$$

and

$$\forall(\boldsymbol{F}, q) \in \boldsymbol{X},\ \textit{such that } (\boldsymbol{F}, q) \neq 0, \qquad \sup_{(\boldsymbol{E},E_3)\in \boldsymbol{X}} |\,\boldsymbol{a}_\omega((\boldsymbol{E}, E_3), (\boldsymbol{F}, q))\,| > 0. \tag{4.30}$$

The constant in (4.29), $\alpha := C\gamma_1\gamma_2$, is referred to as the $\inf - \sup$ constant.

We note that the following conditions are automatically satisfied, see [5]:

$$\sup_{(\boldsymbol{E},E_3)\in \boldsymbol{X}} \frac{|\,\boldsymbol{a}_\omega((\boldsymbol{E}, E_3), (\boldsymbol{F}, q))\,|}{\|(\boldsymbol{F}, q)\|_{\boldsymbol{X}}} \geq \alpha \|(\boldsymbol{E}, E_3)\|_{\boldsymbol{X}} \qquad \forall(\boldsymbol{F}, q) \in \boldsymbol{X} \tag{4.31}$$

and

$$\forall(\boldsymbol{E}, E_3) \in \boldsymbol{X}, \text{ such that } (\boldsymbol{E}, E_3) \neq 0, \tag{4.32}$$
$$\sup_{(\boldsymbol{F},q)\in\boldsymbol{X}} \mid \boldsymbol{a}_\omega((\boldsymbol{E}, E_3), (\boldsymbol{F}, q)) \mid > 0.$$

We notice that (4.32) implies that $\beta = 0$ cannot be in the spectrum of $\boldsymbol{a}_\omega$.

Spectral properties of the problem. As shown in [4] and [5], (4.1), (4.29), (4.30) imply that there are unique bounded solution operators $\hat{\boldsymbol{T}} : \mathcal{H} \longrightarrow \boldsymbol{X}$ and $\hat{\boldsymbol{T}}^* : \mathcal{H} \longrightarrow \boldsymbol{X}$ satisfying

$$\boldsymbol{a}_\omega(\hat{\boldsymbol{T}}(\boldsymbol{J}, Q), (\boldsymbol{F}, q)) = ((\boldsymbol{J}, Q), (\boldsymbol{F}, q))_{\mathcal{H}}, \qquad (\boldsymbol{J}, Q) \in \mathcal{H},\ (\boldsymbol{F}, q) \in \boldsymbol{X}, \tag{4.33}$$

$$\boldsymbol{a}_\omega((\boldsymbol{F}, q), \hat{\boldsymbol{T}}^*(\boldsymbol{J}, Q)) = ((\boldsymbol{F}, q), (\boldsymbol{J}, Q))_{\mathcal{H}}, \qquad (\boldsymbol{F}, q) \in \boldsymbol{X},\ (\boldsymbol{J}, Q) \in \mathcal{H}. \tag{4.34}$$

$\hat{\boldsymbol{T}}^*$ is the $\mathcal{H}$-adjoint of operator $\hat{\boldsymbol{T}}$. Furthermore, (4.1), (4.29) imply that

$$\|\hat{\boldsymbol{T}}(\boldsymbol{J}, Q)\|_{\boldsymbol{X}} \leq \frac{C}{\alpha}\|(\boldsymbol{J}, Q)\|_\epsilon \qquad \forall(\boldsymbol{J}, Q) \in \mathcal{H}. \tag{4.35}$$

Theorem 4. *$\hat{\boldsymbol{T}} : \mathcal{H} \longrightarrow \mathcal{H}$ is a compact operator.*

Proof. First, (4.35) confirms that $\hat{\boldsymbol{T}} : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded operator. By Rellich's lemma the space V is compactly embedded in $L^2(\Omega)$. We note that (4.35) alone is not sufficient to finish the proof, because $\boldsymbol{X}$ is not compactly embedded in $\mathcal{H}$ since $\boldsymbol{W} = H_0(\mathrm{curl}, \Omega)$ is not compact in $\boldsymbol{L}^2(\Omega)$. To overcome this difficulty, we notice that, for a given $(\boldsymbol{J}, Q) \in \mathcal{H}$, the pair $(\boldsymbol{E}, E_3) = \hat{\boldsymbol{T}}(\boldsymbol{J}, Q)$ must satisfy $(4.18)_2$ adjusted as

$$(\epsilon\boldsymbol{E}, \boldsymbol{\nabla} q) = -(\epsilon E_3, q) + \frac{1}{\omega^2}(\epsilon Q, q), \quad \forall q \in V. \tag{4.36}$$

Since $E_3 \in V$ and $Q \in L^2(\Omega)$, using Green's formula in (4.36) we may conclude that $\mathrm{div}(\epsilon\boldsymbol{E}) \in L^2(\Omega)$, and there is $C > 0$ such that

$$\|\boldsymbol{\nabla} \circ (\epsilon\boldsymbol{E})\|_2 \leq C\|(\boldsymbol{J}, Q)\|_{\mathcal{H}}. \tag{4.37}$$

Therefore, $\boldsymbol{E} \in H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}_\epsilon, \Omega)$, which is indeed compactly embedded in $\boldsymbol{L}^2(\Omega)$, see [24, 34]. Thus, we can show that $\boldsymbol{T}$ maps an $\mathcal{H}$-bounded set into an $\mathcal{H}$-compact one. Therefore, $\hat{\boldsymbol{T}} : \mathcal{H} \longrightarrow \mathcal{H}$ is a compact operator. □

$\hat{\boldsymbol{T}}^*$ is also a compact operator on $\mathcal{H}$ as the adjoint of a compact operator, see [1]. It follows from (4.33) that $\boldsymbol{T} = \hat{\boldsymbol{T}}\mathbb{B}$, with $\mathbb{B}$ defined in (4.3), solves

$$\boldsymbol{a}_\omega(T(\boldsymbol{E}, E_3), (\boldsymbol{F}, q)) = (\mathbb{B}(\boldsymbol{E}, E_3), (\boldsymbol{F}, q))_{\mathcal{H}} \quad \forall(\boldsymbol{E}, E_3) \in \mathcal{H},\ \forall(\boldsymbol{F}, q) \in \boldsymbol{X}. \tag{4.38}$$

Considered on $\mathcal{H}$, $\boldsymbol{T}$ is a compact operator as the product of a bounded operator with a compact one. Using (4.38), we can rewrite (4.4) as

$$\boldsymbol{a}_\omega((1 - \beta^2\boldsymbol{T})(\boldsymbol{E}, E_3), (\boldsymbol{F}, q)) = 0 \quad \forall(\boldsymbol{E}, E_3) \in \mathcal{H},\ \forall(\boldsymbol{F}, q) \in \boldsymbol{X}. \tag{4.39}$$

It follows immediately from (4.32) and (4.39) that the pair $(\beta, (\boldsymbol{E}, E_3)) \in \mathbb{C} \times \boldsymbol{X}$ solves (4.4) if and only if it solves the eigenvalue problem with the compact operator $\boldsymbol{T}$

$$\beta^2\boldsymbol{T}(\boldsymbol{E}, E_3) = (\boldsymbol{E}, E_3). \tag{4.40}$$

Thus, (4.40) means that $\lambda \neq 0$ is an eigenvalue of $\boldsymbol{T}$ if and only if λ^{-1} is an eigenvalue of $\boldsymbol{a}_\omega$. So, by studying the spectral properties of (4.40) we can learn all we need to know about the spectral properties of $\boldsymbol{a}_\omega$.

As in [6], we introduce one more bounded operator[2]

$$\boldsymbol{T}_* = \hat{\boldsymbol{T}}^* \mathbb{B}. \tag{4.41}$$

$\boldsymbol{T}_*$ is compact on $\mathcal{H}$ as the composition of a bounded operator with a compact one. Since $\mathbb{B}$ is symmetric, we have

$$\boldsymbol{a}_\omega((\boldsymbol{F},q), \boldsymbol{T}_*(\boldsymbol{E},E_3) = (\mathbb{B}(\boldsymbol{F},q),(\boldsymbol{E},E_3))_{\mathcal{H}}, \qquad \forall(\boldsymbol{F},q) \in \boldsymbol{X}, \quad \forall(\boldsymbol{E},E_3) \in \mathcal{H}. \tag{4.42}$$

Pairs $(\beta,(\boldsymbol{E},E_3)) \in \mathbb{C} \times \boldsymbol{X}$ for the eigenvalue problem with T_*

$$\bar{\beta}^2 \boldsymbol{T}_*(\boldsymbol{E},E_3) = (\boldsymbol{E},E_3) \tag{4.43}$$

correspond to the "so-called" adjoint eigenpairs of (4.4), see [6],

$$\boldsymbol{a}_\omega((\boldsymbol{F},q),(\boldsymbol{E},E_3)) = \beta^2(\mathbb{B}(\boldsymbol{F},q),(\boldsymbol{E},E_3))_{\mathcal{H}}, \qquad \forall(\boldsymbol{F},q) \in \boldsymbol{X}. \tag{4.44}$$

The introduction of the adjoint eigenproblem (4.44) facilitates the convergence analysis for eigenvalues of the original problem (4.4).

For a short survey of spectral theory for compact operators, see [6] and the references therein. Since the eigenvalue problems with symmetric and nonsymmetric compact operators are rather different, we have decided to review the more important notions for our particular case. We denote by $\rho(\boldsymbol{T}_\omega)$ the resolvent set of $\boldsymbol{T}_\omega$, defined as

$$\rho(\boldsymbol{T}_\omega) = \left\{ z : z \in \mathbb{C}, \quad (z - \boldsymbol{T}_\omega)^{-1} \text{ exists and it is bounded in } \mathcal{H} \right\} \tag{4.45}$$

and by $\sigma(\boldsymbol{T}_\omega)$ the spectrum of $\boldsymbol{T}_\omega$, the set defined as

$$\sigma(\boldsymbol{T}_\omega) = \mathbb{C} \setminus \rho(\boldsymbol{T}_\omega). \tag{4.46}$$

$\sigma(\boldsymbol{T}_\omega)$ is countable with no nonzero limit points. Since the space $\mathcal{H}$ is infinite dimensional, zero is in $\sigma(\boldsymbol{T}_\omega)$ as a limiting point. Nonzero elements of $\sigma(\boldsymbol{T}_\omega)$ are eigenvalues.

Let $\lambda \in \sigma(\boldsymbol{T}_\omega)$ be nonzero. We define the ascent of $(\lambda - \boldsymbol{T}_\omega)$ as the smallest integer κ which satisfies

$$\mathrm{Ker}((\lambda - \boldsymbol{T}_\omega)^\kappa) = \mathrm{Ker}((\lambda - \boldsymbol{T}_\omega)^{\kappa+1}), \tag{4.47}$$

where Ker denotes the null space. The dimension of $\mathrm{Ker}((\lambda - \boldsymbol{T}_\omega)^\kappa)$ is finite and the algebraic multiplicity of λ is defined as

$$m(\lambda) = \dim(\mathrm{Ker}((\lambda - \boldsymbol{T}_\omega)^\kappa)). \tag{4.48}$$

The vectors in $\mathrm{Ker}((\lambda - \boldsymbol{T}_\omega)^\kappa)$ are called generalized eigenvectors of $\boldsymbol{T}_\omega$ corresponding to λ.

The vectors in $\mathrm{Ker}(\lambda - \boldsymbol{T}_\omega)$ are called eigenvectors of $\boldsymbol{T}_\omega$ corresponding to λ.

The geometric multiplicity of λ is defined as

$$g(\lambda) = \dim(\mathrm{Ker}(\lambda - T_\omega)). \tag{4.49}$$

It is clear that $g(\lambda) \leq m(\lambda)$.

[2]If the operators $\hat{\boldsymbol{T}}^*$ and $\mathbb{B}$ commute, then $\boldsymbol{T}_*$ is the $\mathcal{H}$-adjoint of $\boldsymbol{T}$. As shown in the Appendix, in general, this is not the case.

The sesquilinear form $\boldsymbol{a}_\omega$ is nonsymmetric, so the operator $\boldsymbol{T}_\omega$ is not self-adjoint and the elements of $\sigma(\boldsymbol{T}_\omega)$ are not necessarily real, and neither eigenvectors nor generalized eigenvectors are expected to form an orthogonal basis in $\mathcal{H}$.

5. Analysis of the discretized problem

In this section we take Ω as a bounded polyhedral domain and consider a family of discretizations $\{\mathcal{T}_h\}_{h>0}$ by triangles or quadrilaterals. The discretization parameter h usually denotes the maximum diameter of finite elements $K \in \mathcal{T}_h$.

We obtain a discretized version of eigenvalue problem (4.4) by taking test and trial functions not from $\boldsymbol{X}$ but from its subspace $\boldsymbol{X}_h = \boldsymbol{W}_h \times V_h$. We assume that the subspaces $\boldsymbol{W}_h$ and V_h satisfy the following requirements:

- $(\boldsymbol{R}1)$: $\boldsymbol{W}_h$ and V_h are compatible, in the sense that

$$\boldsymbol{\nabla} V_h = \{\boldsymbol{E}_h \in \boldsymbol{W}_h : \boldsymbol{\nabla} \times \boldsymbol{E}_h = 0\}\,. \tag{5.1}$$

- $(\boldsymbol{R}2)$: $\boldsymbol{W}_h$ is discretely compact. This means, see [18], that any $\boldsymbol{W}$-bounded sequence $\{\boldsymbol{E}_h\}_{h>0}$ such that $\boldsymbol{E}_h \in \boldsymbol{W}_h$ and satisfying $(\epsilon \boldsymbol{E}_h, \boldsymbol{\nabla} q_h) = 0$, $\forall q_h \in V_h$, $\forall h$, contains a subsequence which converges weakly in $\boldsymbol{W}$ and strongly in $\boldsymbol{L}^2(\Omega)$ to an element $\boldsymbol{E} \in \boldsymbol{W}$.
- $(\boldsymbol{R}3)$: $\boldsymbol{W}_h \times V_h$ approximizes $\boldsymbol{W} \times V$ well:

$$\lim_{h \longrightarrow 0} \inf_{(\boldsymbol{F}_h, q_h) \in \boldsymbol{X}_h} \|(\boldsymbol{E}, p) - (\boldsymbol{F}_h, q_h)\|_{\boldsymbol{X}} = 0, \quad \forall (\boldsymbol{E}, p) \in \boldsymbol{X}\,. \tag{5.2}$$

Notice that, in order to satisfy requirement $(\boldsymbol{R}1)$, vector functions should be mapped from the master element into the mesh as gradients, see [16, 12, 33] for details.

Two families of edge elements have been shown to meet $(\boldsymbol{R}2)$: the widely used edge elements of Nedelec defined in [26, 27] and the recently introduced hp-adaptive generalization of Nedelec elements constructed in [14] and improved in [33], see also [12, 30]. These elements allow local refinement in h and enrichment in p. About a decade ago, Kikuchi proved the discrete compactness property for the Nedelec edge elements of lowest order defined on simplexes, see [19]. In [17], Joly extended that result to the case of nonconstant ϵ and μ. Recently, Kikuchi has modified his proof to include the lowest order elements defined on quadrilaterals as well, see [20, 21]. In [25], Monk et al., in the context of Maxwell's equations in $\mathbb{R}^3$ proved that the discrete compactness holds for Nedelec elements of any fixed order p. Some restrictive assumptions on the mesh were necessary in order for an inverse inequality to be applied. The results of that work remain valid for problems in two dimensions as well. In [13], Demkowicz et al. relaxed restrictions on the mesh. Moreover, the results were generalized to include the new family of hp-adaptive edge elements. The proof uses discrete compactness of the Nedelec elements of lowest order.

Remark 2. In [8], Boffi has linked $(\boldsymbol{R}1)$ and $(\boldsymbol{R}2)$ to the commuting property of the "de Rham complex" diagram which involves the spaces H^1_0, $\boldsymbol{H}_0(\mathrm{curl})$, $H_0(\mathrm{div})$, $L^2 \setminus \mathbb{R}$ and their corresponding discretizations. Boffi has shown that this commuting property, combined with the uniform convergence of a particular projection operator, is in fact equivalent to the discrete compactness property. That the de Rham complex commutes for the standard edge elements has been known for a while, see [26, 3]. In [12], Demkowicz et al. have confirmed that it commutes for the hp-adaptive elements as well.

In conjunction with the appropriately chosen conforming nodal approximation of $H^1(\Omega)$, both families satisfy ($\boldsymbol{R}1$) and ($\boldsymbol{R}3$) and can be employed in our analysis. Thus, the discretized version of eigenvalue problem (4.4) reads as follows.

For a given $\omega > 0$ find $(\beta_h, (\boldsymbol{E}_h, E_{3h})) \in (\mathbb{C}, \boldsymbol{X}_h)$ such that $\forall(\boldsymbol{F}_h, q_h) \in \boldsymbol{X}_h$

$$\boldsymbol{a}_\omega((\boldsymbol{E}_h, E_{3h}), (\boldsymbol{F}_h, q_h)) = \beta_h^2(\mathbb{B}(\boldsymbol{E}_h, E_{3h}), (\boldsymbol{F}_h, q_h))_{\mathcal{H}}. \tag{5.3}$$

We note that, for a fixed $h > 0$, utilizing ($\boldsymbol{R}1$), we can repeat the steps of the argument leading to the proof of the continuous $\inf - \sup$ conditions (4.29), (4.30) and obtain the discretized $\inf - \sup$ constant

$$\alpha_h = C\gamma_{1h}\gamma_{2h}, \tag{5.4}$$

where $C > 0$ is some constant and

$$\gamma_{1h} = \min_{i=1\ldots\infty} \left\{ \frac{\mid \eta_{ih} - \omega^2 \mid}{\eta_{ih}} \right\}, \tag{5.5}$$

with η_{ih} being the eigenvalues of the discretized eigenproblem on $\boldsymbol{\nabla}\frac{1}{\mu}\boldsymbol{\nabla}$ (3.14) considered on V_h, and

$$\gamma_{2h} = \min\left\{ \min_{j=1\ldots\infty} \left\{ \frac{\mid \nu_{jh} - \omega^2 \mid}{\nu_{jh}} \right\}, \frac{1}{1+\omega^2} \right\}, \tag{5.6}$$

with ν_{jh} being eigenvalues of the discretized eigenproblem on $\boldsymbol{\nabla} \times \frac{1}{\mu}\boldsymbol{\nabla}\times$ considered on $\boldsymbol{W}_h$.

By the classical theory of H^1-conforming discretizations of the Laplace operator we get

$$\lim_{h\to 0} \eta_{ih} = \eta_i, \quad \forall i \geq 1, \tag{5.7}$$

and, using the recent results in [25, 13, 9, 15], applicable to our problem, owing to ($\boldsymbol{R}2$), we may conclude that

$$\lim_{h\to 0} \nu_{jh} = \nu_j, \quad \forall j \geq 1. \tag{5.8}$$

Since all η_i and ν_j are positive, η_{ih}^{-1} and ν_{jh}^{-1} remain bounded as $h \to 0$. Therefore, as a direct consequence of Theorem 3 and (5.4)-(5.8) we have

Theorem 5. *With the assumption on ω as in* (3.15)*, there is $h_0 > 0$ such that $\forall h > h_0$ the form $\boldsymbol{a}_\omega$ satisfies the following conditions:*

$$\begin{aligned} &\exists\alpha_0 > 0, \ \textit{such that } \forall\alpha_h \geq \alpha_0 \\ &\sup_{(\boldsymbol{F}_h,q_h)\in\boldsymbol{X}_h} \frac{\mid \boldsymbol{a}_\omega((\boldsymbol{E}_h, E_{3h}), (\boldsymbol{F}_h, q_h)) \mid}{\|(\boldsymbol{F}_h, q_h)\|_{\boldsymbol{X}}} \\ &\qquad \geq \alpha_0\|(\boldsymbol{E}_h, E_{3h})\|_{\boldsymbol{X}} \quad \forall(\boldsymbol{E}_h, E_{3h}) \in \boldsymbol{X}_h \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} &\forall(\boldsymbol{F}_h, q_h) \in \boldsymbol{X}_h, \ \textit{such that } (\boldsymbol{F}_h, q_h) \neq 0 \\ &\sup_{(\boldsymbol{E}_h,E_{3h})\in\boldsymbol{X}_h} \mid \boldsymbol{a}_\omega((\boldsymbol{E}_h, E_{3h}), (\boldsymbol{F}_h, q_h)) \mid > 0. \end{aligned} \tag{5.10}$$

Inequality (5.9) is called the discrete inf − sup condition. We also automatically get

(5.11)
$$\sup_{(\boldsymbol{E}_h, E_{3h})\in \boldsymbol{X}_h} \frac{|\ \boldsymbol{a}_\omega((\boldsymbol{E}_h, E_{3h}), (\boldsymbol{F}_h, q_h))\ |}{\|(\boldsymbol{F}_h, q_h)\|_{\boldsymbol{X}}} \geq \alpha_0 \|(\boldsymbol{E}_h, E_{3h})\|_{\boldsymbol{X}} \qquad \forall (\boldsymbol{F}_h, q_h) \in \boldsymbol{X}_h$$

and

(5.12)
$$\forall (\boldsymbol{E}_h, E_{3h}) \in \boldsymbol{X}_h,\ \text{such that } (\boldsymbol{E}_h, E_{3h}) \neq 0,$$
$$\sup_{(\boldsymbol{F}_h, q_h)\in \boldsymbol{X}_h} |\ \boldsymbol{a}_\omega((\boldsymbol{E}_h, E_{3h}), (\boldsymbol{F}_h, q_h))\ | > 0.$$

We notice that (5.12) implies that $\beta_h = 0$ is not in the spectrum of (5.3). Just as for the continuous problem, (5.9) and (5.10) allow us to introduce a pair of compact operators, $\hat{\boldsymbol{T}}_h$ and its $\mathcal{H}$-adjoint ${\hat{\boldsymbol{T}}_h}^*$, defined uniquely by

(5.13)
$$\boldsymbol{a}_\omega(\hat{\boldsymbol{T}}_h(\boldsymbol{J}, Q), (\boldsymbol{F}_h, q_h)) = ((\boldsymbol{J}, Q), (\boldsymbol{F}_h, q_h))_{\mathcal{H}}, \quad \forall (\boldsymbol{J}, Q) \in \mathcal{H}, \quad \forall (\boldsymbol{F}_h, q_h) \in \boldsymbol{X}_h$$

and

(5.14)
$$\boldsymbol{a}_\omega((\boldsymbol{F}_h, q_h), {\hat{\boldsymbol{T}}_h}^*(\boldsymbol{J}, Q)) = ((\boldsymbol{F}_h, q_h), (\boldsymbol{J}, Q))_{\mathcal{H}}, \quad \forall (\boldsymbol{F}_h, q_h) \in \boldsymbol{X}_h, \quad \forall (\boldsymbol{J}, Q) \in \mathcal{H}.$$

The operators $\hat{\boldsymbol{T}}_h$ and $\hat{\boldsymbol{T}}^*{}_h$ are compact since they both have finite range. And, as on the continuous level, using (5.13) and (5.12) we can show that (5.3) is equivalent to

$$(\boldsymbol{E}_h, E_{3h}) = \beta_h^2 \boldsymbol{T}_h(\boldsymbol{E}_h, E_{3h}), \tag{5.15}$$

where $\boldsymbol{T}_h : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\boldsymbol{T}_h = \hat{\boldsymbol{T}}_h \mathbb{B} \tag{5.16}$$

and $\mathbb{B}$ comes from (4.3). Since any finite element method produces eigenvalues and eigenvectors which correspond to $\boldsymbol{T}_h$, we need to analyze how well the spectral characteristics of $\boldsymbol{T}_h$ reflect the spectral characteristics of $\boldsymbol{T}$, as $h \to 0$. We address this issue in the next section.

6. Convergence analysis

The convergence analysis, performed in the framework of collectively compact operators as in [17, 25], relies on the following theorem (slightly adjusted to fit our case) by Chatelin, see [10].

Theorem 6. *Let us consider a family of operators $\{\boldsymbol{T}_h\}_{0<h<h_0}$ with the corresponding $\mathcal{H}$-adjoints $\{\boldsymbol{T}_h^*\}_{0<h<h_0}$ converging pointwise to compact operators $\boldsymbol{T}$ and $\boldsymbol{T}^*$, resp., i.e.,*

$$\lim_{h\to 0} \|\boldsymbol{T}_h(\boldsymbol{J}, Q) - \boldsymbol{T}(\boldsymbol{J}, Q)\|_{\mathcal{H}} = 0 \quad \forall (\boldsymbol{J}, Q) \in \mathcal{H}, \tag{6.1}$$

$$\lim_{h\to 0} \|\boldsymbol{T}_h^*(\boldsymbol{J}, Q) - \boldsymbol{T}^*(\boldsymbol{J}, Q)\|_{\mathcal{H}} = 0 \quad \forall (\boldsymbol{J}, Q) \in \mathcal{H}. \tag{6.2}$$

Moreover, assume that the family $\{\boldsymbol{T}_h\}_{0<h<h_0}$ is collectively compact, i.e., the set

$$K = \{\boldsymbol{T}_h(\boldsymbol{J}, Q) : \|(\boldsymbol{J}, Q)\|_{\mathcal{H}} \leq 1\}_{0<h<h_0} \tag{6.3}$$

is sequentially compact in $\mathcal{H}$. Then $\boldsymbol{T}_h \to \boldsymbol{T}$ in the norm

$$\lim_{h\to 0} \|\boldsymbol{T}_h - \boldsymbol{T}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} = 0. \tag{6.4}$$

Proof. Let us suppose that assertion (6.4) is false; then there exist a sequence $\{(\boldsymbol{J}_h, Q_h) : \|(\boldsymbol{J}_h, Q_h)\|_{\mathcal{H}} \leq 1\}_{0<h<h_0}$ and δ such that

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{J}_h, Q_h)\|_{\mathcal{H}} \geq \delta, \tag{6.5}$$

and, since the set K is sequentially compact and $\boldsymbol{T}$ is a compact operator, there exists $(F, q) \in \mathcal{H}$ such that

$$\lim_{h\to 0} (\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{J}_h, Q_h) = (\boldsymbol{F}, q), \qquad \|(\boldsymbol{F}, q)\|_{\mathcal{H}} \geq \delta. \tag{6.6}$$

Now let $(\boldsymbol{F}^*, q^*) \in \mathcal{H}$ be scaled to yield

$$\|(\boldsymbol{F}, q)\|_{\mathcal{H}} = ((\boldsymbol{F}^*, q^*), (\boldsymbol{F}, q))_{\mathcal{H}}. \tag{6.7}$$

Then

$$\begin{aligned} \|(\boldsymbol{F}, q)\|_{\mathcal{H}} &= \lim_{h\to 0} ((\boldsymbol{F}^*, q^*), (\boldsymbol{T} - \boldsymbol{T}_h)(\boldsymbol{J}_h, Q_h))_{\mathcal{H}} \\ &= \lim_{h\to 0} ((\boldsymbol{T}^* - \boldsymbol{T}_h^*)(\boldsymbol{F}^*, q^*), (\boldsymbol{J}_h, Q_h))_{\mathcal{H}} = 0, \end{aligned} \tag{6.8}$$

since we assumed pointwise convergence of $\boldsymbol{T}_h^*$. Clearly, (6.8) contradicts $\|(\boldsymbol{F}, q)\|_{\mathcal{H}} \geq \delta$. Therefore, (6.4) must be true. □

Thus, if we confirm properties (6.1)-(6.3), all of Osborn's theorems on convergence in [29] become applicable.

6.1. **Pointwise convergence of $\{\boldsymbol{T}_h\}_{0<h<h_0}$ and $\{\boldsymbol{T}_h^*\}_{0<h<h_0}$.** We note that since $\mathbb{B}$ is bounded it is sufficient to verify pointwise convergence of $\left\{\hat{\boldsymbol{T}}_h\right\}_{0<h<h_0}$ and $\left\{\hat{\boldsymbol{T}}_h^*\right\}_{0<h<h_0}$.

Using the standard convergence theory of the finite element method and (5.9), we can show that, see [5],

$$\|\hat{\boldsymbol{T}}(\boldsymbol{J}, Q) - \hat{\boldsymbol{T}}_h(\boldsymbol{J}, Q)\|_{\boldsymbol{X}} \leq (1 + \frac{C}{\alpha_0}) \inf_{(\boldsymbol{F}_h, q_h)\in \boldsymbol{X}_h} \|\hat{\boldsymbol{T}}(\boldsymbol{J}, Q) - (\boldsymbol{F}_h, q_h)\|_{\boldsymbol{X}}. \tag{6.9}$$

Now we recall that $\boldsymbol{X}_h$ satisfies "approximability requirement" ($\boldsymbol{R}3$). Therefore it follows from (6.9) that

$$\lim_{h\to 0} \|\hat{\boldsymbol{T}}(\boldsymbol{J}, Q) - \hat{\boldsymbol{T}}_h(\boldsymbol{J}, Q)\|_{\boldsymbol{X}} = 0, \quad \forall (\boldsymbol{J}, Q) \in \mathcal{H}. \tag{6.10}$$

And since the $\boldsymbol{X}$-norm is stronger that the $\mathcal{H}$-norm, we confirm pointwise convergence of $\hat{\boldsymbol{T}}_h$ in $\mathcal{H}$:

$$\lim_{h\to 0} \|\hat{\boldsymbol{T}}(\boldsymbol{J}, Q) - \hat{\boldsymbol{T}}_h(\boldsymbol{J}, Q)\|_{\mathcal{H}} = 0, \quad \forall (\boldsymbol{J}, Q) \in \mathcal{H}. \tag{6.11}$$

Applying the same argument and (5.11), we verify pointwise convergence of $\hat{\boldsymbol{T}}_h^*$ in $\mathcal{H}$:

$$\lim_{h\to 0} \|\hat{\boldsymbol{T}}^*(\boldsymbol{J}, Q) - \hat{\boldsymbol{T}}_h^*(\boldsymbol{J}, Q)\|_{\mathcal{H}} = 0, \quad \forall (\boldsymbol{J}, Q) \in \mathcal{H}. \tag{6.12}$$

6.2. **Collective compactness of $\{\boldsymbol{T}_h\}_{0<h<h_0}$.** Let us consider a closed unit ball in $\mathcal{H}$,

$$S = \{(\boldsymbol{J}, Q) \in \mathcal{H} : \|(\boldsymbol{J}, Q)\|_{\mathcal{H}} \leq 1\}. \tag{6.13}$$

The set K defined in (6.3) can be described as

$$K = \bigcup_{0<h<h_0} \{\boldsymbol{T}_h(\boldsymbol{J}, Q) : (\boldsymbol{J}, Q) \in S\}. \tag{6.14}$$

We show that any sequence $\{(\boldsymbol{E}_h, E_{3h})\}_{0<h<h_0} \in K$ contains a subsequence converging strongly in $\mathcal{H}$ in three steps.

Step 1. From (6.10), the Banach-Steinhaus Theorem, and the boundedness of $\mathbb{B}$, we may conclude that

$$\|\boldsymbol{T}_h\|_{\mathcal{L}(\mathcal{H},\boldsymbol{X})} \leq C, \qquad 0 < h < h_0, \tag{6.15}$$

where C is some positive constant. Therefore,

$$\|(\boldsymbol{E}_h, E_{3h})\|_{\boldsymbol{X}} \leq C, \qquad \forall (\boldsymbol{E}_h, E_{3h}) \in K. \tag{6.16}$$

This means that

$$\|\boldsymbol{E}_h\|_{\boldsymbol{W}} \leq C \quad \text{ and } \quad \|E_{3h}\|_V \leq C. \tag{6.17}$$

We recall now that V is compactly embedded in $L^2_\epsilon(\Omega)$. Consequently, any sequence $\{(\boldsymbol{E}_h, E_{3h})\} \in K$ contains a subsequence (with elements still marked by h) such that

$$\lim_{h\to 0} \|E_{3h} - E_3\|_\epsilon = 0, \tag{6.18}$$

where E_3 is some element of V. Let us consider this subsequence in further detail. Since $(\boldsymbol{E}_h, E_{3h})$ are in K, then

$$(\boldsymbol{E}_h, E_{3h}) = \boldsymbol{T}_h(\boldsymbol{J}, Q) \tag{6.19}$$

for some $(\boldsymbol{J}, Q) \in \mathcal{H}$. It follows from (4.3) and (4.4) that all $(\boldsymbol{E}_h, E_{3h})$ must satisfy

$$\begin{cases} (\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}_h, \boldsymbol{\nabla} \times \boldsymbol{F}_h) - \omega^2(\epsilon\boldsymbol{E}_h, \boldsymbol{F}_h) \\ \qquad = (\frac{1}{\mu}\boldsymbol{\nabla} E_{3h}, \boldsymbol{F}_h) - (\frac{1}{\mu}\boldsymbol{J}, \boldsymbol{F}_h) \qquad \forall \boldsymbol{F}_h \in \boldsymbol{W}_h \\ (\epsilon\boldsymbol{E}_h, \boldsymbol{\nabla} q_h) = -(\epsilon E_{3h}, q_h) \qquad \forall q_h \in V_h \end{cases} \tag{6.20}$$

where E_{3h} is known as the solution to

$$-(\frac{1}{\mu}\boldsymbol{\nabla} E_{3h}, \boldsymbol{\nabla} q_h) + \omega^2(\epsilon E_{3h}, q_h) = -(\frac{1}{\mu}\boldsymbol{J}, \boldsymbol{\nabla} q_h) \qquad \forall q_h \in V_h. \tag{6.21}$$

Step 2. To facilitate further analysis, we would like to split $\boldsymbol{E}_h$ into the curl-free part and the discrete divergence-free part. We can accomplish this task by using $(6.20)_2$.

First, we note that the function $g_h \in V_h$ is uniquely defined as the solution to

$$(\epsilon\boldsymbol{\nabla} g_h, \boldsymbol{\nabla} q_h) = -(\epsilon E_{3h}, q_h), \quad \forall q_h \in V_h, \tag{6.22}$$

and E_{3h} is already known.

Now, if we introduce $\boldsymbol{w}_h \in \boldsymbol{W}_h$ as

$$\boldsymbol{w}_h = \boldsymbol{E}_h - \boldsymbol{\nabla} g_h, \tag{6.23}$$

then it follows from $(6.20)_2$ and (6.22) that $\boldsymbol{w}_h$ is indeed discrete divergence free:

$$(\epsilon \boldsymbol{w}_h, \boldsymbol{\nabla} q_h) = 0, \qquad \forall q_h \in V_h. \tag{6.24}$$

Thus, by (6.23) we have the needed decomposition

$$\boldsymbol{E}_h = \boldsymbol{\nabla} g_h + \boldsymbol{w}_h \tag{6.25}$$

We also consider an auxiliary problem on $\hat{g}_h$ derived from (6.22) with E_{3h} on the right-hand side replaced by E_3:

$$(\epsilon \boldsymbol{\nabla} \hat{g}_h, \boldsymbol{\nabla} q_h) = -(\epsilon E_3, q_h), \quad \forall q_h \in V_h. \tag{6.26}$$

By subtracting (6.26) from (6.22), taking $q_h = g_h - \hat{g}_h$ and applying the Poincaré inequality, we can show that there is a $C > 0$ such that

$$\|g_h - \hat{g}_h\|_V \leq C \|E_3 - E_{3h}\|_\epsilon \quad \forall h > 0. \tag{6.27}$$

Using the standard theory of H^1-conforming approximation for elliptic equations, we can deduce that there exists a $g \in V$ such that $\hat{g}_h$, the solution to (6.26), satisfies

$$\lim_{h \to 0} \|g - \hat{g}_h\|_V = 0. \tag{6.28}$$

From (6.27), combined with (6.18) and (6.28), using the triangle inequality, we readily get

$$\lim_{h \to 0} \|\boldsymbol{\nabla} g - \boldsymbol{\nabla} g_h\|_\epsilon = 0. \tag{6.29}$$

And, since a converging sequence is always bounded,

$$\|\boldsymbol{\nabla} g_h\|_\epsilon \leq C, \qquad 0 < h < h_0, \tag{6.30}$$

where C is some positive constant, we have shown that the curl-free part of $\boldsymbol{E}_h$ converges in $\boldsymbol{L}^2_\epsilon(\Omega)$. The discrete compactness property $(\boldsymbol{R}2)$ is necessary to make the remaining part of $\boldsymbol{E}_h$ converge as well.

Step 3. By its definition, $\boldsymbol{w}_h$ is expressed as $\boldsymbol{w}_h = \boldsymbol{E}_h - \boldsymbol{\nabla} g_h$. Moreover, we have established separate bounds on $\boldsymbol{E}_h$ and $\boldsymbol{\nabla} g_h$ in (6.17) and (6.30). Therefore, by the triangle inequality,

$$\|\boldsymbol{w}_h\|_\epsilon \leq \|\boldsymbol{w}_h\|_{\boldsymbol{W}} < C, \qquad 0 < h < h_0, \tag{6.31}$$

where C is some positive constant. Now, since the spaces $\boldsymbol{W}_h$ satisfy $(\boldsymbol{R}2)$, there exists a subsequence of $\{(\boldsymbol{E}_h, E_{3h})\}$ (still marked by h) whose elements possess discrete divergence free parts $\boldsymbol{w}_h$ converging strongly in $\boldsymbol{L}^2_\epsilon(\Omega)$:

$$\lim_{h \to 0} \|\boldsymbol{w}_h - \boldsymbol{w}\|_\epsilon = 0, \tag{6.32}$$

where $\boldsymbol{w}$ is some element in $\boldsymbol{W}$. It follows from (6.18), (6.29), (6.32), and decomposition (6.25) that this subsequence converges strongly in $\mathcal{H}$:

$$\lim_{h \to 0} \|(\boldsymbol{E}_h, E_{3h}) - (\boldsymbol{E}, E_3)\|_{\mathcal{H}} = 0, \tag{6.33}$$

where $\boldsymbol{E} = \boldsymbol{w} + \boldsymbol{\nabla} g$ of (6.32) and (6.28), and E_3 comes from (6.18). Thus, the set K is sequentially compact, and collective compactness has been confirmed. Therefore, by theorem 6 we get convergence in the norm: $\lim_{h \to 0} \|\boldsymbol{T} - \boldsymbol{T}_h\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} = 0$.

6.3. **Convergence estimates.** Let λ be a nonzero eigenvalue of (4.4) with algebraic multiplicity m. This also means that λ^{-1} is an eigenvalue of $\boldsymbol{T}$, with equal multiplicity. And let us assume that the ascent of $\lambda^{-1} - \boldsymbol{T}$ is κ. We denote by $\boldsymbol{E}$ the spectral projection (see e.g., [6]) associated with λ^{-1} and $\boldsymbol{T}$, and by $R(\boldsymbol{E})$ its range. $\boldsymbol{E}^*$ denotes the spectral projection associated with $\bar{\lambda}^{-1}$ and $\boldsymbol{T}^*$, $R(\boldsymbol{E}^*)$ being the range of $\boldsymbol{E}^*$.

Like Babuška and Osborn in [6], we introduce the following two sets related to the eigenproblems (4.4) and (4.44) on $\boldsymbol{a}_\omega$:

$$S = S(\lambda) = \{(\boldsymbol{E}, E_3) : (\boldsymbol{E}, E_3) \text{ is a generalized eigenvector of (4.4)} \\ \text{corresponding to } \lambda, \|(\boldsymbol{E}, E_3)\|_{\mathcal{H}} = 1\}, \tag{6.34}$$

$$S^* = S^*(\lambda) = \{(\boldsymbol{E}^*, E_3^*) : (\boldsymbol{E}^*, E_3^*) \text{ is a generalized adjoint eigenvector of (4.4)} \\ \text{corresponding to } \lambda, \|(\boldsymbol{E}^*, E_3^*)\|_{\mathcal{H}} = 1\}, \tag{6.35}$$

and since such $(\boldsymbol{E}, E_3)$ and $(\boldsymbol{E}^*, E_3^*)$ are in $\boldsymbol{X}$, let us also define

$$\varepsilon_h = \varepsilon_h(\lambda) = \sup_{(\boldsymbol{E},E_3)\in S(\lambda)} \inf_{(\boldsymbol{F}_h,q_h)\in \boldsymbol{X}_h} \|(\boldsymbol{E}, E_3) - (\boldsymbol{F}_h, q_h)\|_{\boldsymbol{X}}, \tag{6.36}$$

and

$$\varepsilon_h^* = \varepsilon_h^*(\lambda) = \sup_{(\boldsymbol{E}^*,E_3^*)\in S^*(\lambda)} \inf_{(\boldsymbol{F}_h,q_h)\in \boldsymbol{X}_h} \|(\boldsymbol{E}^*, E_3^*) - (\boldsymbol{F}_h, q_h)\|_{\boldsymbol{X}}. \tag{6.37}$$

In [29] Osborn proves

Theorem 7. *Consider a disc centered at λ^{-1}, of radius $\epsilon > 0$ such that it contains no other eigenvalue of $\boldsymbol{T}$. Then there is an $\hat{h} > 0$ such that, for any $0 < h < \hat{h}$, this disc contains exactly m eigenvalues $\lambda_{i,h}^{-1}$, $i = 1, \ldots, m$ (counted with multiplicity), of $\boldsymbol{T}_h$, and there is a constant $C > 0$ such that for any $i = 1, \ldots, m$*

$$\frac{|\lambda^{-1} - \lambda_{i,h}^{-1}|^\kappa}{C} \le \sum_{j,l=1}^{m} |((\boldsymbol{T} - \boldsymbol{T}_h)\phi_j, \phi_l^{'})_{\mathcal{H}}| \\ + \|(\boldsymbol{T} - \boldsymbol{T}_h)|_{R(\boldsymbol{E})}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \|(\boldsymbol{T}^* - \boldsymbol{T}_h^*)|_{R(\boldsymbol{E}^*)}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}. \tag{6.38}$$

Here $\{\phi_j\}_{j=1}^m$ is any basis for $R(\boldsymbol{E})$ and $\{\phi_j^{'}\}_{j=1}^m$ is the dual basis with respect to the $\mathcal{H}$-inner product.

It is clear that since λ is nonzero, the estimate (6.38) also applies to $|\lambda - \lambda_{i,h}|^\kappa$ (with a different C).

Remark 3. Osborn has shown that vectors $\{\phi_j^{'}\}_{j=1}^m$ can be extended to all of $\mathcal{H}$ as the generalized eigenvectors of $\boldsymbol{T}^*$ corresponding to $\bar{\lambda}^{-1}$.

In [6], Babuška and Osborn use Theorem 7 to derive convergence estimates which involve ε_h and ε_h^* defined in (6.36) and (6.37), which are evidently closely related to the original eigenvalue problem (4.4). Their approach has to be adjusted to the current case because we did not demonstrate compactness of $\boldsymbol{T}$ on the space $\boldsymbol{X}$, where the isomorphism L of (4.5), corresponding to the form $\boldsymbol{a}_\omega$ is defined. We consider this matter in full detail in the Appendix. There we show that

$$\sum_{j,l=1}^{m} \mid ((\boldsymbol{T}-\boldsymbol{T}_h)\phi_j, \phi_l^{'})_{\mathcal{H}} \mid + \|(\boldsymbol{T}-\boldsymbol{T}_h)|_{R(\boldsymbol{E})}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \|(\boldsymbol{T}^*-\boldsymbol{T}_h^*)|_{R(\boldsymbol{E}^*)}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq \frac{C}{\alpha_0}\varepsilon_h\varepsilon_h^*, \tag{6.39}$$

where C is some positive constant. Thus, we arrive at our final convergence result for the eigenvalue problem on $\boldsymbol{a}_\omega$:

$$\mid \lambda - \lambda_{i,h} \mid^{\kappa} = \mathcal{O}(\varepsilon_h \varepsilon_h^*), \tag{6.40}$$

which shows that the rate of convergence depends upon the ascent κ and the interpolation error estimates for the eigenspaces.

6.4. **Convergence rates for Nedelec elements.** The estimates of convergence rates are necessary for the analysis and design of successful adaptive schemes. Since the hp-edge elements [14, 33] generalize Nedelec elements [26, 27], we consider the Nedelec elements first, with the intention of using these results for further study of the hp-convergence mechanism and hp-adaptivity. To get the rates of convergence for a propagation constant β, we need to know the regularity of the eigenmodes $(\boldsymbol{E}, E_3)$ and the adjoint eigenmodes $(\boldsymbol{E}^*, E_3^*)$ which correspond to β^2, see [6]. Let us assume that all the eigenmodes and the adjoint eigenmodes for β^2 of multiplicity m and ascent κ are such that

$$\begin{array}{lll} \boldsymbol{E} \in \boldsymbol{H}^r(\Omega), & \boldsymbol{\nabla} \times \boldsymbol{E} \in \boldsymbol{H}^r(\Omega), & E_3 \in H^{r+1}(\Omega), \\ \boldsymbol{E}^* \in \boldsymbol{H}^r(\Omega), & \boldsymbol{\nabla} \times \boldsymbol{E}^* \in \boldsymbol{H}^r(\Omega), & E_3^* \in H^{r+1}(\Omega), \end{array} \tag{6.41}$$

where $\boldsymbol{H}^r$ is Sobolev space of order $r > 0$. If H^1-conforming scalar elements of order p are used in shape-regular affine meshes with $N = N(h)$ degrees of freedom, then, slightly adjusting the proofs given in [2] to the 2D case, we can bound the interpolation error and, consequently, estimate the eigenvalue rate of convergence as follows:

$$\begin{aligned} &\|(\boldsymbol{E}, E) - (\boldsymbol{E}_h, E_{3,h})\|_{\boldsymbol{X}} \leq \mathcal{O}(N^{-\frac{s}{2}}), \\ &\mid \beta - \beta_{i,h} \mid^{\kappa} \leq \mathcal{O}(N^{-s}), \end{aligned} \tag{6.42}$$

where $s = \min(p, r)$, provided that the compatible Nedelec triangles or quads of the first family [26] are used. In general, if the compatible Nedelec triangles of the second family [27] are used, then $s = \min(p-1, r)$. However, if we solve for $(\boldsymbol{E}, E_3)$ and the modes corresponding to β^2 are all TM or TEM, or if we solve for $(\boldsymbol{H}, H_3)$ and all the modes are TE or TEM, then the triangles of both families offer the same rate of convergence.

7. Sample numerical examples

We confirm convergence rates (6.42) by applying our method to homogeneously loaded waveguides with known exact solutions derived from eigenfunctions for the Laplace operator with Dirichlet or Neumann boundary conditions. For all considered cases, solution operators are such that ascent $\kappa = 1$, and all eigenmodes and the corresponding adjoint eigenmodes have the same regularity. To confirm (6.42), we implement Nedelec quads of [26] and Nedelec triangles of [27] on uniform meshes. We use 2Dhp90_EM, a FE package for electromagnetics [30], designed to support the hp-edge elements which generalize these quads and triangles of Nedelec.

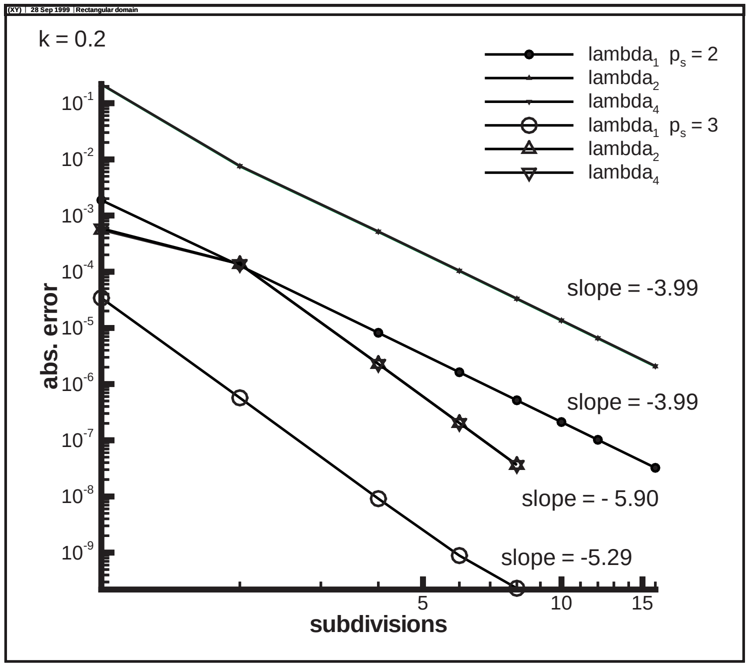

FIGURE 1. Dielectric rectangle: h-refinement by quads

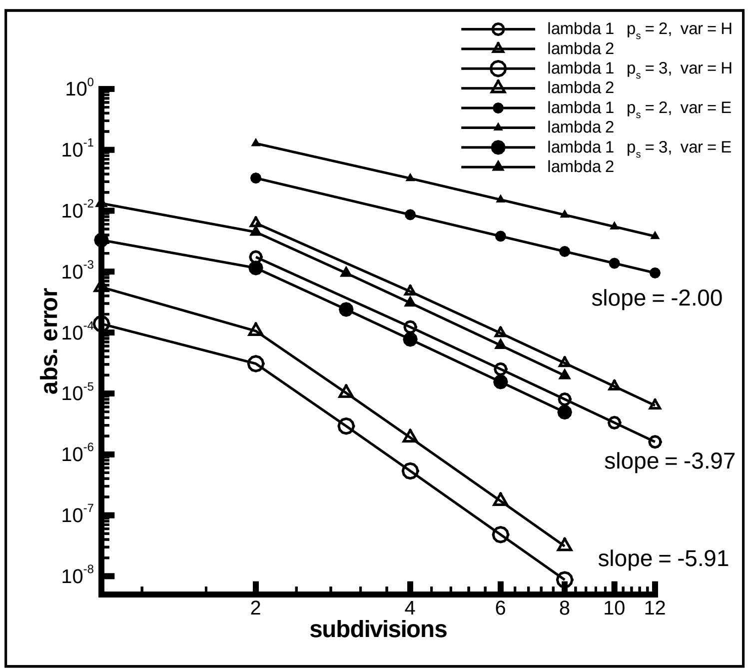

FIGURE 2. Dielectric rectangle: h-refinement by triangles

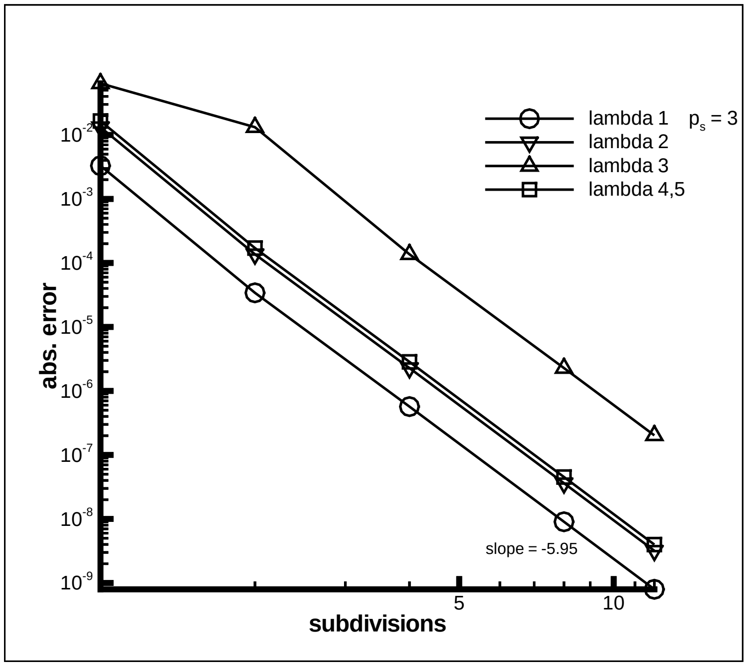

FIGURE 3. Conducting rectangle: h-refinement by quads

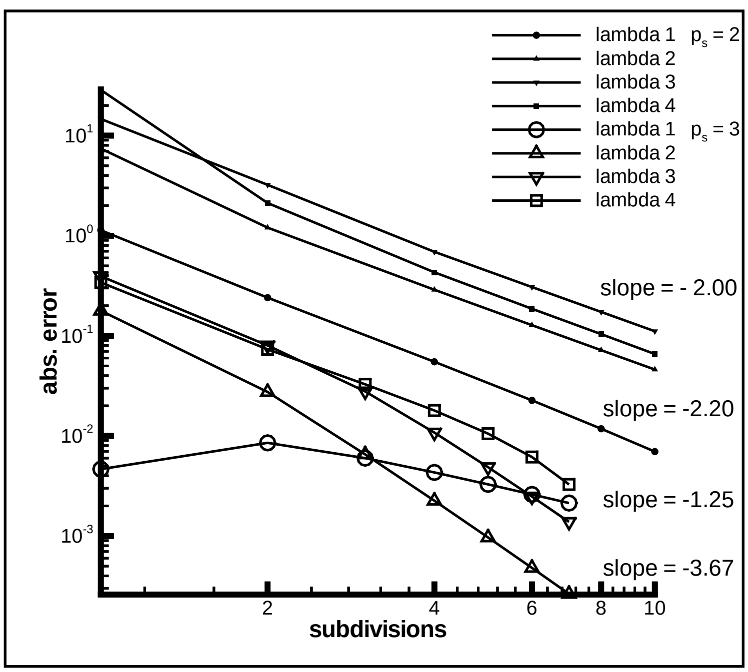

FIGURE 4. Dielectric sector: h-refinement by isoparametric triangles

For the plots in Figures 1 and 2, the cross section Ω is taken as a 1×2 rectangle at $k = 0.2$ with $\sigma = 0$. The slopes are in full agreement with (6.42).

In Figure 2, the eigenvalues λ_1 and λ_2 correspond to the TE-modes. This implies that $\boldsymbol{\nabla} \times \boldsymbol{H} = 0$. Therefore, as predicted by theory and now seen in the plots, solving Maxwell's equations for $(\boldsymbol{H}, H_3)$, not for $(\boldsymbol{E}, E_3)$, provides for higher rates of convergence.

For the plots in Figure 3, the cross section Ω is taken as a $1\text{mm} \times 2\text{mm}$ rectangle at 100KHz with $\epsilon_r = 1$, $\mu_r = 1$, $\sigma = 5.8 \cdot 10^4 \text{S/mm}$. Although the theoretical results apply only to nonlossy waveguides, the rates of convergence are as expected.

For an example of domains with singular eigenmodes, we take the cross section Ω as a $3/2\pi$-sector of unit radius. The eigenmode with the lowest regularity of $r = 2/3 - \delta$, $\delta > 0$, corresponds to λ_1, the smallest eigenvalue. Since Ω is not a polygon, a map isoparametric with respect to the scalar element is used to model curved elements. The rates of convergence given in Figure 4 are as expected in all plots but one: for λ_1 the rate is higher then expected if $p_s = 2$.

Appendix: On convergence estimates

We note that the results of [6] cannot be directly applied to the problem at hand because we did not demonstrate compactness of $\boldsymbol{T}$ on the space $\boldsymbol{X}$, where the isomorphism L of (4.5) corresponding to the form $\boldsymbol{a}_\omega$ is defined.

Let us simplify our notation and denote the pairs $(\boldsymbol{E}, E_3)$ and $(\boldsymbol{F}, q)$ we have been working with by the single letters u and v. Thus, in this simplified notation (4.33) and (4.34) take on the form

$$\boldsymbol{a}_\omega(\hat{\boldsymbol{T}}u, v) = (u, v)_{\mathcal{H}}, \qquad \forall u \in \mathcal{H}, \quad \forall v \in \boldsymbol{X}, \tag{A.1}$$

$$\boldsymbol{a}_\omega(u, \hat{\boldsymbol{T}}^* v) = (u, v)_{\mathcal{H}}, \qquad \forall u \in \boldsymbol{X}, \quad \forall v \in \mathcal{H}. \tag{A.2}$$

Similarly, (4.38) with $\boldsymbol{T}$ and (4.42) with $\boldsymbol{T}_*$ become

$$\boldsymbol{a}_\omega(\boldsymbol{T}u, v) = (\mathbb{B}u, v)_{\mathcal{H}}, \qquad \forall u \in \mathcal{H}, \quad \forall v \in \boldsymbol{X}, \tag{A.3}$$

$$\boldsymbol{a}_\omega(u, \boldsymbol{T}_* v) = (\mathbb{B}u, v)_{\mathcal{H}}, \qquad \forall u \in \boldsymbol{X}, \quad \forall v \in \mathcal{H}. \tag{A.4}$$

We recall that the convergence estimate of Theorem 7 involves $\boldsymbol{T}$ and its adjoint $\boldsymbol{T}^*$, not the operator $\boldsymbol{T}_*$. Next, we investigate how the spectral properties of $\boldsymbol{T}^*$ relate to those of $\boldsymbol{T}_*$. As a result of Theorem 3, we can define a bounded operator $A : \boldsymbol{X} \to \boldsymbol{X}$ by

$$\boldsymbol{a}_\omega(u, v) = [Au, v]_{\boldsymbol{X}}, \qquad \forall u \in \boldsymbol{X}, \quad \forall v \in \boldsymbol{X}, \tag{A.5}$$

where $[\cdot, \cdot]_{\boldsymbol{X}}$ is the inner product on $\boldsymbol{X}$. The adjoint of A with respect to this inner product, denoted by A', can be defined by

$$\boldsymbol{a}_\omega(u, v) = [u, A'v]_{\boldsymbol{X}}, \qquad \forall u \in \boldsymbol{X}, \quad \forall v \in \boldsymbol{X}. \tag{A.6}$$

Theorem 3 also implies that both A^{-1} and $(A')^{-1}$ exist as bounded operators on $\boldsymbol{X}$.

Now, we note that the $[\cdot, \cdot]_{\boldsymbol{X}}$ -inner product, as a sesquilinear form on $\boldsymbol{X}$, satisfies the "$\inf - \sup$" conditions. Therefore, we can define a bounded operator $M : \mathcal{H} \longrightarrow \boldsymbol{X}$ as

$$[u, Mv]_{\boldsymbol{X}} = (u, v)_{\mathcal{H}}, \qquad \forall u \in \boldsymbol{X}, \quad \forall v \in \mathcal{H}. \tag{A.7}$$

By the definition of A' in (A.6) and by the identity $A'\,(A')^{-1} = Id$ on $\boldsymbol{X}$, we can write (A.7) as

$$(u,v)_{\mathcal{H}} = [u, Mv]_{\boldsymbol{X}} = [u, A'(A')^{-1}\, M\, v]_{\boldsymbol{X}} = \boldsymbol{a}_\omega(u, (A')^{-1} M\, v) \tag{A.8}$$

to get the following identity:

$$(u,v)_{\mathcal{H}} = \boldsymbol{a}_\omega(u, (A')^{-1} M\, v), \quad \forall u \in \boldsymbol{X},\ \forall v \in \mathcal{H}. \tag{A.9}$$

Let us show now that

$$\begin{array}{rcl} \sigma(\boldsymbol{T}^*) & \subset & \sigma(\boldsymbol{T}_*), \\ (A')^{-1} M\ \mathrm{Ker}(\boldsymbol{T}^* - \bar{\lambda})^j & \subset & \mathrm{Ker}(\boldsymbol{T}_* - \bar{\lambda})^j, \qquad \forall \bar{\lambda} \in \sigma(\boldsymbol{T}^*). \end{array} \tag{A.10}$$

We take any $\bar{\lambda} \in \sigma(\boldsymbol{T}^*)$ and any generalized eigenvector v^* associated with $\bar{\lambda}$ so that $v^* \in \mathrm{Ker}(\boldsymbol{T}^* - \bar{\lambda})^j$:

$$(\boldsymbol{T}^* - \bar{\lambda})^j v^* = 0, \tag{A.11}$$

and obviously

$$(u, (\boldsymbol{T}^* - \bar{\lambda})^j v^*)_{\mathcal{H}} = 0, \quad \forall u \in \boldsymbol{X}. \tag{A.12}$$

Since $\boldsymbol{T}^*$ is the $\mathcal{H}$-adjoint of $\boldsymbol{T}$, (A.12) yields

$$((\boldsymbol{T} - \lambda)^j u, v^*)_{\mathcal{H}} = 0, \quad \forall u \in \boldsymbol{X}. \tag{A.13}$$

Now, by (A.9), (A.13) implies that

$$\boldsymbol{a}_\omega((\boldsymbol{T} - \lambda)^j u, (A')^{-1} M v^*) = 0, \quad \forall u \in \boldsymbol{X}. \tag{A.14}$$

Using the identity (which can be verified by induction to hold on $\boldsymbol{X} \times \boldsymbol{X}$)

$$\boldsymbol{a}_\omega((\boldsymbol{T} - \lambda)^s y, z) = \boldsymbol{a}_\omega(y, (\boldsymbol{T}_* - \bar{\lambda})^s z), \quad s = 0, 1, 2, \ldots, \tag{A.15}$$

we can conclude that (A.14) yields

$$\boldsymbol{a}_\omega(u, (\boldsymbol{T}_* - \bar{\lambda})^j\, (A')^{-1} M\, v^*) = 0, \quad \forall u \in \boldsymbol{X}. \tag{A.16}$$

Since $\boldsymbol{a}_\omega$ satisfies the "inf − sup" conditions on $\boldsymbol{X}$, it follows from (A.16) and (A.11) that

$$(\boldsymbol{T}_* - \bar{\lambda})^j\, (A')^{-1} M\, v^* = 0, \quad \forall v^* \in \mathrm{Ker}(\boldsymbol{T}^* - \bar{\lambda})^j. \tag{A.17}$$

Thus, (A.17) confirms both inclusions in (A.10).

Let λ be a nonzero eigenvalue of (4.4) with algebraic multiplicity m. This also means that λ^{-1} is an eigenvalue of $\boldsymbol{T}$, with equal multiplicity.

Let us use the result (6.38) of Theorem 7 to bound $|\,\lambda^{-1} - \lambda_{i,h}^{-1}\,|^\kappa$ by ε_h and ε_h^* introduced in (6.36) and (6.37).

We consider $\|(\boldsymbol{T} - \boldsymbol{T}_h)|_{R(\boldsymbol{E})}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}$ first. From [4] and [5] and (4.2), (4.29), (4.30), (4.3), (4.38), (5.9), (5.13), and (5.16) we have, for any $u \in \mathcal{H}$,

$$\|(\boldsymbol{T} - \boldsymbol{T}_h)u\|_{\mathcal{H}} \;\le\; \|(\boldsymbol{T} - \boldsymbol{T}_h)u\|_{\boldsymbol{X}} \;\le\; \left(1 + \frac{C}{\alpha_0}\right) \inf_{\chi_h \in \boldsymbol{X}_h} \|\boldsymbol{T}u - \chi_h\|_{\boldsymbol{X}}. \tag{A.18}$$

Recalling that the range of the spectral projection $R(\boldsymbol{E})$ is invariant with respect to $\boldsymbol{T}$ (see e.g., [6]), for any $u \in R(\boldsymbol{E})$ we obtain

(A.19)

$$\inf_{\chi_h \in \boldsymbol{X}_h} \|\boldsymbol{T}u - \chi_h\|_{\boldsymbol{X}} = \inf_{\chi_h \in \boldsymbol{X}_h} \left\| \frac{\boldsymbol{T}u}{\|\boldsymbol{T}u\|_{\mathcal{H}}} - \chi_h \right\|_{\boldsymbol{X}} \|\boldsymbol{T}u\|_{\mathcal{H}} \;\le\; \varepsilon_h \|\boldsymbol{T}\|_{\mathcal{L}(H,H)} \|u\|_{\mathcal{H}}.$$

Therefore, combining (A.18) and (A.19) and using the definition of operator norm, we get

$$\|(\boldsymbol{T}-\boldsymbol{T}_h)\,|_{R(\boldsymbol{E})}\,\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} = \sup_{u\in S(\lambda)} \|(\boldsymbol{T}-\boldsymbol{T}_h)u\| \le \frac{C}{\alpha_0}\varepsilon_h, \tag{A.20}$$

where C is some positive constant independent of h.

For any $u \in \mathcal{H}$ with $\|u\|_{\mathcal{H}} = 1$ and for any $v^* \in R(\boldsymbol{E}^*)$ with $\|v^*\|_{\mathcal{H}} = 1$, we have by (A.9)

$$\begin{aligned} ((\boldsymbol{T}-\boldsymbol{T}_h)u, v^*)_{\mathcal{H}} &= \boldsymbol{a}_\omega\left((\boldsymbol{T}-\boldsymbol{T}_h)u, (A')^{-1}Mv^*\right) \\ &= \boldsymbol{a}_\omega\left((\boldsymbol{T}-\boldsymbol{T}_h)u, (A')^{-1}Mv^* - \chi_h\right) \\ &\le C\|(\boldsymbol{T}-\boldsymbol{T}_h)u\|_{\boldsymbol{X}}\|\,(A')^{-1}Mv^* - \chi_h\|_{\boldsymbol{X}}, \qquad \forall \chi_h \in \boldsymbol{X}_h. \end{aligned} \tag{A.21}$$

In (A.21), the appearance of any $\chi_h \in \boldsymbol{X}_h$ is justified by (4.38), (5.13) and (5.16).

Let us demonstrate that $(A')^{-1}M$ is a bounded operator on $\mathcal{H}$. Indeed, since $(A')^{-1} \in \mathcal{L}(\boldsymbol{X},\boldsymbol{X})$, and $M : \mathcal{H} \longrightarrow \boldsymbol{X}$ is bounded, the following chain of inequalities holds for any $u \in \mathcal{H}$:

$$\begin{aligned} \|\,(A')^{-1}Mu\|_{\mathcal{H}} \le \|\,(A')^{-1}Mu\|_{\boldsymbol{X}} &\le \|\,(A')^{-1}\,\|_{\mathcal{L}(\boldsymbol{X},\boldsymbol{X})}\|Mu\|_{\boldsymbol{X}} \\ &\le \|\,(A')^{-1}\,\|_{\mathcal{L}(\boldsymbol{X},\boldsymbol{X})}\|M\|_{\mathcal{L}(\mathcal{H},\boldsymbol{X})}\|u\|_{\mathcal{H}}. \end{aligned} \tag{A.22}$$

Therefore,

$$\|(A')^{-1}M\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \le \|\,(A')^{-1}\,\|_{\mathcal{L}(\boldsymbol{X},\boldsymbol{X})}\|M\|_{\mathcal{L}(\mathcal{H},\boldsymbol{X})}. \tag{A.23}$$

We know from (A.17) that $(A')^{-1}M$ maps

$$R(\boldsymbol{E}^*) = \mathrm{Ker}((\bar{\lambda}^{-1} - T^*)^\kappa) \quad \text{into} \quad \mathrm{Ker}((\bar{\lambda}^{-1} - T_*)^\kappa),$$

with κ being the ascent for $\bar{\lambda}^{-1}$. Therefore, just like in (A.19) we can obtain

$$\inf_{\chi_h\in\boldsymbol{X}_h} \|\,(A')^{-1}Mv^* - \chi_h\|_{\boldsymbol{X}} \;\le\; \varepsilon_h^*\|\,(A')^{-1}M\|_{\mathcal{L}(H,H)}\|v^*\|_{\mathcal{H}}. \tag{A.24}$$

Consequently, we get for (A.21)

$$|\,((\boldsymbol{T}-\boldsymbol{T}_h)u, v^*)_{\mathcal{H}}\,| \le C\|(\boldsymbol{T}-\boldsymbol{T}_h)u\|_{\boldsymbol{X}}\|\,(A')^{-1}M\|_{\mathcal{L}(H,H)}\varepsilon_h^* \tag{A.25}$$

where C is a positive constant. From (A.25) it is immediate that, for any $v^* \in R(\boldsymbol{E}^*)$, $\|v^*\|_{\mathcal{H}} = 1$,

$$\begin{aligned} \|(\boldsymbol{T}^* - \boldsymbol{T}_h^*)v^*\|_{\mathcal{H}} &= \sup_{u\in\mathcal{H}} \frac{|\,(u, (\boldsymbol{T}^* - \boldsymbol{T}_h^*)v^*)_{\mathcal{H}}\,|}{\|u\|_{\mathcal{H}}} \\ &\le C\|\,\boldsymbol{T}-\boldsymbol{T}_h\|_{\mathcal{L}(\mathcal{H},\boldsymbol{X})}\|\,(A')^{-1}M\|_{\mathcal{L}(H,H)}\varepsilon_h^*. \end{aligned} \tag{A.26}$$

We recall that $\lim_{h\to 0}\|\,\boldsymbol{T}_h - \boldsymbol{T}\|_{\mathcal{L}(H,H)} = 0$. This implies that $\|\,\boldsymbol{T}-\boldsymbol{T}_h\|_{\mathcal{L}(\mathcal{H},\boldsymbol{X})}$ is uniformly bounded with respect to h. Thus, we can obtain from (A.26)

$$\|(\boldsymbol{T}^* - \boldsymbol{T}_h^*)\,|_{R(\boldsymbol{E}^*)}\,\|_{\mathcal{L}(H,H)} \le \frac{C}{\alpha_0}\varepsilon_h^*, \tag{A.27}$$

where C is some positive constant. By using (A.25), (A.18) and (A.19) we may bound $|\left((\boldsymbol{T}-\boldsymbol{T}_h)\phi_i, \phi_j^*\right)|$, where $\phi_i \in R(\boldsymbol{E})$, $\phi_i^* \in R(\boldsymbol{E}^*)$, and $\|\phi_i\|_{\mathcal{H}} = \|\phi_j^*\|_{\mathcal{H}} = 1$, as follows:

$$
\text{(A.28)} \qquad \begin{aligned} \left| \left((\boldsymbol{T} - \boldsymbol{T}_h)\phi_i, \phi_j^* \right)_{\mathcal{H}} \right| &\leq C_1 \|(\boldsymbol{T} - \boldsymbol{T}_h)\phi_i\|_{\boldsymbol{X}} \| \left(A'\right)^{-1} M \|_{\mathcal{L}(H,H)} \varepsilon_h^* \\ &\leq \frac{C_2}{\alpha_0} \|\boldsymbol{T}\|_{\mathcal{L}(H,H)} \| \left(A'\right)^{-1} M \|_{\mathcal{L}(H,H)} \varepsilon_h \varepsilon_h^*, \end{aligned}
$$

where C_1, C_2 are some positive constants.

Estimates (A.20), (A.27) and (A.28) indicate that indeed

$$
\text{(A.29)} \qquad \begin{aligned} &\sum_{j,l=1}^{m} | ((\boldsymbol{T} - \boldsymbol{T}_h)\phi_j, \phi_l^{'})_{\mathcal{H}} | \\ &\qquad + \|(\boldsymbol{T} - \boldsymbol{T}_h)|_{R(\boldsymbol{E})}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \|(\boldsymbol{T}^* - \boldsymbol{T}_h^*)|_{R(\boldsymbol{E}^*)}\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq \frac{C}{\alpha_0} \varepsilon_h \varepsilon_h^*. \end{aligned}
$$

Remark 4. Theorems 8.1–8.4 dealing with the convergence estimates for eigenvalues and eigenvectors stated in [6] by Babuška and Osborn are all valid for our case due to the estimates (A.20), (A.27) and (A.28).

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The Texas Institute for Computational and Applied Mathematics, The University of Texas at Austin, Taylor Hall 2.400, Austin, Texas 78712
E-mail address: `leonv@research.bell-labs.com`

The Texas Institute for Computational and Applied Mathematics, The University of Texas at Austin, Taylor Hall 2.400, Austin, Texas 78712
E-mail address: `leszek@ticam.utexas.edu`