CORRIGENDA AND ADDITION TO
“COMPUTER VERIFICATION
OF THE ANKENY-ARTIN-CHOWLA CONJECTURE
FOR ALL PRIMES LESS THAN 100,000,000,000”

A. J. VAN DER POORTEN, H. J. J. TE RIELE, AND H. C. WILLIAMS

Abstract. An error in the program for verifying the Ankeny-Artin-Chowla (AAC) conjecture is reported. As a result, in the case of primes \( p \) which are \( \equiv 5 \) mod 8, the AAC conjecture has been verified using a different multiple of the regulator of the quadratic field \( \mathbb{Q}(\sqrt{p}) \) than was meant. However, since any multiple of this regulator is suitable for this purpose, provided that it is smaller than \( 8p \), the main result that the AAC conjecture is true for all the primes \( \equiv 1 \) mod 4 which are < \( 10^{11} \), remains valid.

As an addition, we have verified the AAC conjecture for all the primes \( \equiv 1 \) mod 4 between \( 10^{11} \) and \( 2 \times 10^{11} \), with the corrected program.

1. Corrigenda

Let \( p \) be a prime \( \equiv 1 \) mod 4, and consider the Dirichlet \( L \)-function
\[
L(1, \chi_p) = \prod_q (1 - \chi_p(q)/q)^{-1},
\]
where the product is taken over all the primes \( q \), and the character \( \chi_p(q) \) is the same as the Kronecker symbol \( (p/q) \). This coincides with the Legendre symbol when \( q \) is an odd prime. When \( q = 2 \), the Kronecker symbol \( (p/2) \) is only defined when \( p \) is 0 or 1 mod 4. We have
\[
(p/2) = 1 \text{ when } p \equiv 1 \text{ mod 4}, \quad (p/2) = -1 \text{ when } p \equiv 5 \text{ mod 8}
\]
and
\[
(p/2) = 0 \text{ when } p \equiv 0 \text{ mod 4}
\]
(but the latter case does not apply here). In [1, p. 1314], an approximation \( S(T, p) \) of the function \( \log L(1, \chi_p) \) is used, given by
\[
S(T, p) = \sum_{q < 2T - 1} w(q) \log \left( \frac{q}{q - \chi_p(q)} \right),
\]
where \( T \) is some parameter which varies between 500 and 5000, \( w(q) = 1 \) when \( q < T \), and \( w(q) \) is a more complicated function of \( q \) and \( T \) when \( T \leq q < 2T - 1 \), explicitly given in [1, p. 1314].

Received by the editor June 19, 2002.
2000 Mathematics Subject Classification. Primary 11A55, 11J70, 11Y40, 11Y65, 11R11.
Key words and phrases. Periodic continued fractions, function field.
In the computer program written by one of the authors (HJJtR), a mistake was made with the implementation of the Kronecker symbol \((p/q)\). Namely, this was assumed to coincide with the Legendre symbol not only for \(q > 2\) (which is correct), but also for \(q = 2\), which is correct for primes \(p \equiv 1 \mod 8\) but not for primes \(p \equiv 5 \mod 8\): in that case, we have \((p/2) = 1\) for the Legendre symbol and \((p/2) = -1\) for the Kronecker symbol. Hence, for \(q = 2\), and \(p \equiv 5 \mod 8\), for the value of

\[
w(q) \log \left( \frac{q}{q - \chi_p(q)} \right) = w(q) \log \left( \frac{q}{q - (p/q)} \right)
\]

we took \(\log(2/(2 - 1)) = \log 2\), where we should have taken \(\log(2/(2 - (-1))) = \log(2/3)\). As a consequence, in the approximation \(\exp(S(T,p))\) of \(L(1, \chi_p)\) we used the factor \(\exp(\log(2)) = 2\), where we should have taken \(\exp(\log(2/3)) = 2/3\). This means that in our computation of an estimate \(E\) for a multiple of \(R_2\) according to the formula

\[
E = \frac{\sqrt{p}}{\log 4} \exp(S(T,p))
\]

we computed a value of \(E\) which was too big by a factor of 3 in those cases where \(p \equiv 5 \mod 8\). As a consequence, the integral multiple of \(R_2\), computed from \(E\) with Algorithm 5.4 of [1] p. 1322, was also too big by a factor of 3. However, in the verification of the AAC conjecture with Algorithm 6.1 of [1] p. 1324, any multiple of \(R_2\) is suitable, as long as it is smaller than \(8p\). So the programming mistake was fortunate enough not to influence the main result of [1]!

**Corrections** in [1] as a result of the incorrect implementation of the Kronecker symbol: In Table 2.1, the values of \(E\) and \(E/R_2\) should be divided by 3 for the second and third prime (which are 5 mod 8). The detailed example given in Section 8 of [1] provides a correct proof of the AAC conjecture for \(p = 97843343893\), but a correct implementation of the Kronecker symbol would have given a value of \(E\) which is 1/3 of the value of \(E\) given there. The other computational results in Section 8 would have changed correspondingly.

**Additional correction**: The values of \(\xi(kR_2)\) and \(\eta(kR_2)\) given on page 1327 of [1] should be interchanged.

### 2. Addition

With our corrected program, we have extended the verification of the AAC conjecture to all the primes \(\equiv 1 \mod 4\) between \(10^{11}\) and \(2 \times 10^{11}\). No exceptions were found. The computations were carried out on one CPU of CWI’s SGI Origin 2000 computer, and took 950 CPU hours.

### Reference


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Centre for Number Theory Research, Macquarie University, Sydney, New South Wales 2109, Australia
E-mail address: alf@math.mq.edu.au

CWI, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands
E-mail address: herman@cwi.nl

Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4
E-mail address: williams@math.ucalgary.ca