THE CLASS NUMBER ONE PROBLEM FOR SOME NON-ABELIAN NORMAL CM-FIELDS OF DEGREE 48

KU-YOUNG CHANG AND SOUN-HI KWON

Abstract. We prove that there is precisely one normal CM-field of degree 48 with class number one which has a normal CM-subfield of degree 16: the narrow Hilbert class field of $\mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$ with $\theta^3 - \theta^2 - 5\theta - 1 = 0$.

1. Introduction

According to [O] and [H], there exist only finitely many normal CM-fields with class number one, and their degrees are less than or equal to 436. All imaginary abelian number fields with class number one are known in [Y]: their degrees are less than or equal to 24. All normal CM-fields of degree less than 48 with class number one are known by many authors ([LO1], [LO2], [Le], [LLO], [LP1], [LP2], [LOO], [Lou3], [P], [YPK], [PsK], [CK2], and [CK3]). In the following table we sum up the numbers of the non-abelian normal CM-fields $N$ with class number one according to their degrees.

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In this paper we study the non-abelian normal CM-fields that contain a normal CM-subfield of degree 16, and will prove the following:

**Theorem 1.** There exists one and only one normal CM-field $N$ of degree 48 with class number one which has a normal CM-subfield of degree 16: the narrow Hilbert class field of the real dihedral number field $K_{12} = \mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$ of degree 12 with $\theta^3 - \theta^2 - 5\theta - 1 = 0$, narrow class number 4 and class number 2. The extension $N/K_{12}$ is cyclic quartic, $d_N = d_{K_{12}}^4 = 2^{32} \cdot 5^{24} \cdot 101^{24}$, the maximal totally real subfield of $N$ is the Hilbert class field of $K_{12}$, and the Galois group Gal($N/\mathbb{Q}$) is isomorphic to the semi-direct product $C_3 \rtimes D_{16}$.

2. Prerequisite and notation

We use the following notation. For a number field $K$, we let $h_K$, $d_K$, $\omega_K$, and $\zeta_K$ denote the class number, the absolute value of the discriminant, the number of roots of unity in $K$, and the Dedekind zeta function of $K$, respectively. If $K$
is a CM-field, we let $h_K$, $K^+$ and $Q_K \in \{1, 2\}$ be the relative class number, the maximal real subfield and the Hasse unit index of $K$, respectively. For an abelian extension $F/K$ we denote by $\mathfrak{f}_{F/K}$ the finite part of its conductor. For a positive integer $n$ we let $\zeta_n = e^{2\pi i/n}$. Before starting the proof of Theorem 1 we recall the well-known results which will be used later in this paper.

**Proposition 1.** (1) ([LOO] Lemma 2) If $K$ is a normal CM-field, then the complex conjugation is in the center of its Galois group.

(2) ([CH] Lemma 13.5) Let $K$ be a CM-field. If there is at least one ramified prime ideal in $K/K^+$ which is lying above an odd prime, then $Q_K = 1$.

(3) ([Ho] Theorem 5, [QR], or [LOO] Theorem 5) Let $k \subset K$ be two CM-fields. Then $h_k$ divides $h_{K^+}$. Moreover, if $[K : k]$ is odd, then $h_k$ divides $h_K$ and $Q_k = Q_K$.

(4) ([LO2] and [Lou1] Proposition 6) Let $K$ be a CM-field and let $t$ be the number of prime ideals of $K^+$ that are ramified in the quadratic extension $K/K^+$. Then $2^{t-1}$ divides $h_K$. Moreover, if $Q_K = 2$, then $2^t$ divides $h_K$.

(5) ([Lou1] Proposition 13) and ([LO2] Proposition 2) Let $K = L_1L_2$ be a CM-field which is a compositum of two CM-fields $L_1$ and $L_2$ with the same maximal totally real subfield. Then

$$h_K = \frac{Q_K}{Q_{L_1}Q_{L_2}} \omega_{L_1}^{-1} \omega_{L_2}^{-1} h_{L_1}^{-1} h_{L_2}^{-1},$$

and $h_{L_1}^{-1} h_{L_2}^{-1}$ divides $4h_K$. In particular, if $L_1$ and $L_2$ are isomorphic, then $\omega_{L_1} = \omega_{L_2} = 2$ and $h_K = \frac{(Q_K/2)(Q_{L_1}/Q_{L_2})^2}{2}$.

(6) ([MI] Corollary 2.2 and 2.3) Let $E/F$ be an extension of number fields. Then $h_F$ divides $\left[ E : F \right] h_E$. Moreover, if no nontrivial abelian subextension of $E/F$ is unramified over $E$, then $h_F$ divides $h_E$.

(7) Let $K = L_1L_2$ be a CM-field which is a compositum of two CM-fields $L_1$ and $L_2$ with the same maximal totally real subfield $L_1^+ = L_2^+$. If $h_K = 1$, then $h_{L_1} = 2$ and $h_{L_2} = 2$. If $h_K = 1$ and $h_{L_1} = h_{L_2} = 2$, then $h_{L_1} = h_{L_2} = 2$.

**Proof.** We only need to prove the last statement of (7). If $h_K = 1$ and $h_{L_1} = h_{L_2} = 2$, then $K$ is the Hilbert class field of $L_1$ and is at the same time that of $L_2$. Hence, $K^+$ is the Hilbert class field of $L_1^+ = L_2^+$ (see [P] Lemma 6.2)].

**Proposition 2.** Let $K$ be a CM-field of degree $2n$.

(1) ([MI] $h_K = Q_K \omega_K(2\pi)^n \cdot \sqrt{d_K/d_K^+} \cdot \text{Res}_{s=1}(\zeta_K) / \text{Res}_{s=1}(\zeta_K^+)$

(2) ([LO2] Proposition 9) Let $\beta_K = 1 - \left( 2/\log h_K \right)$ and

$$\varepsilon_K = \max(1 - 2\pi n^{1/n}/d_K^{1/2n}, 2/5 \exp(-2\pi n/d_K^{1/2n})).$$

If $\zeta_K(\beta_K) \leq 0$, then $\text{Res}_{s=1}(\zeta_K) \geq 2\pi / (e \log d_K)$.

(3) ([Lou2]) There exists a computable constant $\mu_k > 0$ such that for any abelian extension $K/k$ of degree $m$ unramified at all the infinite places we have

$$\text{Res}_{s=1}(\zeta_K) \leq (\text{Res}_{s=1}(\zeta_K))^m \left( \frac{1}{2(m-1)} \log(d_K/d_k) + 2\mu_k \right)^{m-1}.$$

Let $C_m$ denote the cyclic group of order $m$, $D_m$ the dihedral group of order $m$, $Q_m$ the quaternion group of order $m$ and set $G_6 = \langle b, c, z | b^4 = c^2 = z^2 = 1, c^{-1}bc = bz, bz = zb, cz = cz \rangle$ and $G_8 = \langle b, c, z | b^2 = c^2 = z^4 = 1, c^{-1}bc = bz^2, bz = zb, cz = cz \rangle$ (in the notation of [JL]). Throughout this paper, $N$ denotes a non-abelian
normal CM-field of degree 48. We assume that the 3-Sylow subgroup of its Galois group \( \text{Gal}(N/Q) \) is normal, and we let \( M \) denote the normal CM-subfield of degree 16 of \( N \). According to Proposition 1, if \( h_N = 1 \), then \( h_M = 1 \) (moreover, either \( N/M \) is ramified at least one finite place and \( h_M = 1 \) or \( N/M \) is unramified at all places, \( h_M = 3 \), and \( N \) is the Hilbert class field of \( M \)). Now, there are 26 normal CM-fields of degree 16 with relative class number one (see [LO2], [Lou3], [CK1], [PK], and Theorem 2 below). If \( \text{Gal}(M/Q) \) is non-abelian, then it is equal to \( Q_8 \times C_2, G_6, D_{16}, G_9, D_8 \times C_2 \). For proving Theorem 1, we first prove that if \( \text{Gal}(M/Q) \neq D_{16}, G_9, D_8 \times C_2 \), then we can use Proposition 1 and the known solutions to various (relative) class number problems for suitable CM-subfields of \( N \) to prove that \( h_N > 1 \). Now, assume that \( \text{Gal}(M/Q) = D_{16}, G_9, \) or \( D_8 \times C_2 \). We will show that we can find a subfield \( L \) of \( M^+ \) such that \( N/L \) is abelian and such that the use of abelian \( L \)-functions to factorize \( \zeta_N/\zeta_L \) readily yields \( (\zeta_N/\zeta_L)(s) \geq 0 \) for \( 0 < s < 1 \). Since \( M \) is known, \( L \) also is known, we will check that \( \zeta_L(s) \leq 0 \) for \( 0 < s < 1 \) and we will therefore deduce that \( \zeta_N(s) \leq 0 \) for \( 0 < s < 1 \). Using Proposition 2, we will obtain explicit lower bounds for \( h_N^{-1} \), according to which we will be able to compute explicit upper bounds on \( d_N \) when \( h_N = 1 \) and to construct a short list of number fields \( N \) containing all such \( N \)'s with \( h_N = 1 \). We will finally explain how one can use the method expounded in [Lou5] and [Lou6] to compute the relative class numbers of these finitely many CM-fields \( N \) that remain, thus completing the proof of Theorem 1.

3. Case 1: \( M \) is abelian

We will show the following.

**Proposition 3.** If \( N \) contains an abelian number field \( M \) of degree 16, then \( h_N > 1 \).

**Proof.** Let \( K_3 \) be any cubic subfield of \( N \). Since \( N \) is non-abelian, \( K_3 \) is not normal, its normal closure \( K_6 \) is a dihedral real sextic field, and we let \( k_2 \) denote the (real) quadratic subfield of \( K_6 \). The Galois group \( \text{Gal}(M/Q) \) is isomorphic to \( C_{16}, C_8 \times C_2, C_4 \times C_4, C_4 \times C_2 \times C_2, \) or \( C_2 \times C_2 \times C_2 \times C_2 \).

(i) If \( \text{Gal}(M/Q) = C_{16} \), then \( \text{Gal}(N/Q) \) is isomorphic to \( C_3 \times C_{16} = \langle a, \mu | a^3 = b^{16} = 1, b^{-1}ab = a^{-1} \rangle \), and \( N \) is a compositum of \( M \) and the real dihedral field of degree 6 that is fixed by \( \langle b^7 \rangle \). According to [Lou4] Theorem 5 we have \( h_N^{-1} > 1 \).

(ii) If \( \text{Gal}(M/Q) = C_8 \times C_2 \) with \( \text{Gal}(M^+/Q) = C_8 \), then \( \text{Gal}(N/Q) = \langle a, b, c \rangle | a^3 = b^8 = c^2 = 1, b^{-1}ab = a^{-1} \rangle \) with \( \text{Gal}(N/N^+) = \langle b^4 \rangle \). The subfield \( K_{12} \) fixed by \( \langle b^2 \rangle \) is a normal CM-field with Galois group isomorphic to \( Q_{12} \). By [LP1] \( h_{K_{12}}^{-1} > 4 \), whence \( h_N^{-1} > 1 \) by Proposition 1(3). If \( \text{Gal}(M/Q) = C_8 \times C_2 \) with \( \text{Gal}(M^+/Q) = C_4 \times C_2 \), then \( h_N^{-1} > 1 \) by [CK1]. Hence \( h_N^{-1} > 1 \) by Proposition 1(3).

(iii) If \( \text{Gal}(M/Q) = C_4 \times C_4 \), then \( \text{Gal}(N/Q) = Q_{12} \times C_4 \) and \( \text{Gal}(N^+/Q) \) is isomorphic to either \( S_3 \times C_4 \) or \( Q_{12} \times C_2 \). Let \( \psi_1 \) and \( \psi_2 \) be two odd primitive characters of order 4 such that \( M \) is associated with the group \( \langle \psi_1, \psi_2 \rangle \). If \( k_2 \) is associated with \( \langle \psi_2 \rangle \) or \( \langle \psi_2 \rangle \), then \( \text{Gal}(N^+/Q) = S_3 \times C_4 \). Assume that \( k_2 \) is associated with \( \langle \psi_2 \rangle \). Let \( M_{12,1} \) be the compositum of \( K_6 \) and the quartic field associated with \( \langle \psi_1 \rangle \), and \( M_{12,2} \) the compositum of \( K_6 \) and the quartic field associated with \( \langle \psi_1 \psi_2 \rangle \). Then \( M_{12,1} \) and \( M_{12,2} \) are quaternion CM-fields.


of degree 12 with the same maximal real subfield $K_6$. According to [LP1] Theorem 1], there is no pair of $(M_{12,1}, M_{12,2})$ such that $h_{M_{12,1}}^+ | 4$, $h_{M_{12,2}}^+ | 4$, and at the same time $M_{12,1}^+ = M_{12,2}^+$, whence $h_{M}^+ > 1$. By symmetry, if $k_2$ is associated with $\langle \psi_2 \rangle$, then $h_{M}^+ > 1$. Assume now that $k_2$ is associated with $\langle \psi_1 \psi_2 \rangle$. Let $M_{24,1}$ be the compositum of $K_6$ and the imaginary cyclic quartic field associated with $\langle \psi_1 \rangle$, and $M_{24,2}$ the compositum of $K_6$ and the imaginary cyclic quartic field associated with $\langle \psi_2 \rangle$. Then $M_{24,1}$ and $M_{24,2}$ are normal CM-fields with Galois group isomorphic to $S_3 \times C_4$ which have the same maximal real subfield. Using Proposition 1(5) we verify that $h_{M}^+ = h_{M_{24,1}}^+ h_{M_{24,2}}^+$.

By [LP1] Theorem 1] there is only on CM-field of relative class number one with Galois group isomorphic to $S_3 \times C_4$, whence $h_{M}^+ > 1$.

(iv) If $\text{Gal}(M/Q) = C_3 \times C_2 \times C_2$ with $\text{Gal}(M^+/Q) = C_4 \times C_2$, then $\text{Gal}(N/Q)$ is isomorphic to either $Q_{12} \times C_2 \times C_2$ or $S_3 \times C_2 \times C_2$. Let $\psi$ be the odd primitive Dirichlet character of order 4, and let $\chi_1$ and $\chi_2$ be two quadratic odd characters such that $M$ is associated with the group $\langle \psi, \chi_1, \chi_2 \rangle$. If $k_2$ is associated with $\langle \psi_2 \rangle$, then the compositum $M_{12,1}$ of $K_6$ and the field associated with $\langle \psi_1 \rangle$, and the compositum $M_{12,2}$ of $K_6$ and the field associated with $\langle \psi_1 \chi_1 \chi_2 \rangle$ are normal CM-fields with Galois group $Q_{12}$ and $M_{12,1}^+ = M_{12,2}^+$. By [LP1] Theorem 1], $h_{M}^+ > 1$. If $k_2$ is associated with $\langle \psi_1 \chi_1 \chi_2 \rangle$ or $\langle \chi_1 \chi_2 \rangle$, then we let $M_{24,1}$ be the compositum of $K_6$ and the field associated with $\langle \psi \rangle$, and $M_{24,2}$ the compositum of $K_6$ and the field associated with $\langle \psi_1^2 \chi_1 \chi_2 \rangle$. Then $\text{Gal}(M_{24,1}/Q) = S_3 \times C_4$, $\text{Gal}(M_{24,2}/Q) = S_3 \times C_2 \times C_2$, $M_{24,1}^+ = M_{24,2}^+$, and $N = M_{24,1} M_{24,2}$. By [LP1] Theorem 1] $h_{M_{24,1}}^+ > 1$ and $h_{M_{24,2}}^+ > 1$, whence according to Proposition 1(7) we have $h_{M}^+ > 1$.

(v) If $\text{Gal}(M/Q) = C_4 \times C_2 \times C_2$ with $\text{Gal}(M^+/Q) = C_2 \times C_2 \times C_2$, then $h_{M}^+ > 1$ by [CK1]. Hence $h_{M}^+ > 1$.

(vi) If $\text{Gal}(M/Q) = C_2 \times C_2 \times C_2 \times C_2$, then $h_{M}^+ > 1$ by [CK1]. Hence $h_{M}^+ > 1$.

4. Case 2: $\text{Gal}(M/Q) \in \{D_{16}, Q_8 \times C_2\}$

In this section we assume that $\text{Gal}(M/Q) \in \{D_{16}, Q_8 \times C_2\}$ and $h_{M}^+ = 1$. We will prove that there is exactly one field $N$ with $h_{N} = 1$. In subsection 4.1 we assume that $G(M/Q) = D_{16}$, and in subsection 4.2 we assume that $G(M/Q) = Q_8 \times C_2$.

4.1. $G(M/Q) = D_{16}$. There are five dihedral CM-fields $M$ of degree 16 with relative class number one [LO2] Theorem 10]: the narrow Hilbert class fields of $Q(\sqrt{pq})$ with $(p, q) \in \{(2, 257), (5, 101), (5, 181), (13, 53), (13, 61)\}$. The narrow Hilbert class field of $Q(\sqrt{2 \cdot 257})$ has class number three and the remaining four $M$'s have class number one. We set $K = Q(\sqrt{p}, \sqrt{q})$ and $k = Q(\sqrt{pq})$. The field $M$ has three quadratic subfields $L_1$, $L_2$, and $k$ with $\text{Gal}(M/L_1) = \text{Gal}(M/L_2) = D_8$, and $\text{Gal}(M/k) = C_8$. Therefore, the Galois group $\text{Gal}(N/Q)$ is isomorphic to $D_{16} \times C_3$ if $N$ contains only one cubic cyclic subfield, $D_{48} = (D_8 \times C_3)^{\times 2}$ $C_2 = C_3 \times D_{16} = \langle a, b, c | a^3 = b^8 = c^2 = 1, c^{-1}bc = b^{-1}, b^{-1}ab = a^{-1}, ac = ca \rangle$. Otherwise, in [LO] it is proved that if $\text{Gal}(N/Q) = D_{48}$, then $h_{N} > 1$. We deal with the fields $N$ with $\text{Gal}(N/Q) = D_{16} \times C_3$ in 4.1.1 and the fields $N$ with $\text{Gal}(N/Q) = (D_8 \times C_3)^{\times 2}$ $C_2$ in 4.1.2, respectively.
Proposition 1. Therefore, \( K \) and sum up the computational results in Table 1. Note that \( N/k \) is cyclic of degree 24.

**Lemma 1.** Let \( \chi \) be any one of the eight characters of order 24 associated with the cyclic extension \( N/k \).

1. We have \( (\zeta_N / \zeta_K)(s) \geq 0 \) in the range \( 0 < s < 1 \).
2. For each given \( M \) with \( h_M = 1 \) we can compute a bound \( N_{k/\mathbb{Q}}(\mathfrak{p}) \leq C \) on the norms of the conductors \( \mathfrak{p} \) of the cyclic cubic extensions \( kK_3/k \) for the \( N \)'s such that \( h_N = 1 \). These bounds are listed in Table 1.
3. Assume that \( h_N = 1 \). Then \( \mathfrak{p} = (l) \) for some prime \( l \) which splits in \( k \), or \( \mathfrak{p} = \mathfrak{P}_1 \) for some prime ideal \( \mathfrak{P}_1 \) of \( k \) above a prime \( l \) ramified in \( k \).
4. \( h_M \) divides \( h_N \), \( L(0, \chi) \in \mathbb{Q}(\sqrt{2}, \sqrt{-3}) \), and \( h_N/h_M = N_{\mathbb{Q}(\sqrt{2}, \sqrt{-3})}(\frac{1}{4}L(0, \chi))^2 \) is a perfect square which can be computed using the techniques developed in [Lou3] and [Lou6].

**Proof.** (1) It follows from \( (\zeta_N / \zeta_K)(s) = \prod_{s=1}^{11} |L(s, \chi)|^2 \).

(2) We have verified that for the above four \( M \)'s, \( \zeta_K(s) \leq 0 \) in the range \( 0 < s < 1 \). Hence, \( \zeta_N(s) \leq 0 \) for \( 0 < s < 1 \). Using [Lou2, Lemma 12] and Proposition 13 we compute explicitly \( \mu_k \text{ Res}_{s=1}(\zeta_k) \) and apply Proposition 2 to get lower bound for \( h_N \). Since \( M/M^+ \) is unramified at all finite places and \( Q_M = \omega_M = 2, N/N^+ \) is unramified at all finite places, \( d_{N^+} = \sqrt{N_{k/\mathbb{Q}}(\mathfrak{p})}^8 \), and \( Q_N = \omega_N = 2 \). From this lower bound for \( h_N \) we obtain the upper bounds \( C \) on \( N_{k/\mathbb{Q}}(\mathfrak{p}) \) such that \( h_N^2 = 1 \) implies \( N_{k/\mathbb{Q}}(\mathfrak{p}) \leq C \).

(3) If the number of ramified primes in \( K_3/k \) is greater than one, then 3 divides \( h_{K_3} \), whence \( 3|h_{N^+} \) by Proposition 1(6). If there is a prime divisor \( l \) of \( N_{k/\mathbb{Q}}(\mathfrak{p}) \) which is inert in \( k \), then \( 3^4 \) divides \( h_N \). Since \( M \) is the narrow Hilbert class field of \( k \), \( (l) \) splits completely in \( M/k \), whence there are at least 4 prime ideals ramified in \( N^+/M^+ \) which split at the same time in \( M/M^+ \). Hence, \( 3^4|h_N \) by [LOO, Proposition 8].

(4) According to [Lou3], the value \( L(0, \chi) \) is an algebraic integer of \( \mathbb{Q}(\zeta_{24}) \) and

\[
h_N^2/h_M^2 = N_{\mathbb{Q}(\zeta_{24})/\mathbb{Q}}(\frac{1}{4}L(0, \chi)).
\]

Let \( \text{Gal}(N/\mathbb{Q}) = \langle a, b, c | a^3 = b^8 = c^2 = 1, b^{-1}ab = a, c^{-1}ac = a, c^{-1}bc = b^{-1} \rangle \), where \( \text{Gal}(N/k) = \langle a, b \rangle \). The restriction of \( c \) to \( k \) generates \( \text{Gal}(k/\mathbb{Q}) \) and using Artin’s reciprocity theorem we obtain that \( \chi \circ c = \chi^7 \) and

\[
\sigma_7(L(0, \chi)) = L(0, \chi^7) = L(0, \chi \circ c) = L(0, \chi).
\]

Therefore, \( L(0, \chi) \in \mathbb{Q}(\sqrt{2}, \sqrt{-3}) \), the subfield of \( \mathbb{Q}(\zeta_{24}) \) fixed by \( \sigma_7 \). It follows that

\[
h_N^2/h_M^2 = (N_{\mathbb{Q}(\sqrt{2}, \sqrt{-3})/\mathbb{Q}}(\frac{1}{4}L(0, \chi)))^2
\]

is a perfect square.

We verify that \( h_N > 1 \) for all fields \( N \) satisfying points (2) and (3) in Lemma 1, and sum up the computational results in Table 1.
Table 1.

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4.1.2. \(\text{Gal}(N/\mathbb{Q}) = (D_8 \times C_3) \rtimes C_2\). The field \(N\) has three non-normal cubic subfields. Let \(K_3\) be any one of them, \(K_6\) its normal closure, and \(k_2\) the quadratic subfield of \(K_6\). Since if \(\text{Gal}(M/k_2) = C_8\), then \(\text{Gal}(N/\mathbb{Q}) = D_{48}\) and \(h_N > 1\). It follows that \(k_2 = \mathbb{Q}(\sqrt{p})\) or \(\mathbb{Q}(\sqrt{q})\), and \(\text{Gal}(M/k_2) = D_8\).

**Lemma 2.** (1) We have \((\zeta_N/\zeta_K)(s) \geq 0\) in the range \(0 < s < 1\).

(2) There exists some positive integer \(f \geq 1\) such that \(\mathfrak{f}_{K_3/k_2} = (f)\). For each given \(M\) with \(h^{-1}_M = 1\) we can compute a bound \(f \leq C\) on the conductors \((f)\) of the cyclic cubic extensions \(K_6/k_2\) for the \(N\)'s such that \(h_N = 1\). These bounds and the possible \(f\)'s are given in Table 2.

(3) Let \(\chi\) be any one of the four characters of order 12 associated with the cyclic extension \(N/K\). Then \(h_M\) divides \(h_N, L(0, \chi) \in \mathbb{Q}\), and \(h_N/h_M = (L(0, \chi)/16)^4\) is a perfect fourth power which can be computed by using the techniques developed in [Lou5] and [Lou6].
Proof.  (1) Since $N/K$ and $M^+/k$ are cyclic of degree 12 and 4, respectively, then as in point (1) of Lemma 1 we obtain $(\zeta_N/\zeta_{M^+})(s) \geq 0$ and $(\zeta_{M^+}/\zeta_K)(s) \geq 0$ for $0 < s < 1$.

(2) The first part follows from [Mar, Theorem III.1] or [LPL, Theorem 4]. For $K = \mathbb{Q}(\sqrt{2}, \sqrt{257})$ we have verified that $\zeta_K(s) \leq 0$ in the range $0 < s < 1$. Hence, $\zeta_N(s) \leq 0$ for $0 < s < 1$ for every $M$ with $h_M^- = 1$. Since $N^+/k_2$ is abelian and $d_{N^+/k_2} = f^{16}$, using Proposition 2 we get upper bound $C$ on $f$ such that $h_N^- = 1$ implies $f \leq C$. To alleviate the list of possible conductors $f$ we use the same argument as in point (3) of Lemma 1: If there is a prime divisor $l$ of $f$ which is inert in $\mathbb{Q}(\sqrt{pq})$, then $3^4$ divides $h_N^-$.  

(3) Let $K_{12}$ be the compositum of $K$ and $K_6$. We have

$$h_N^-/h_M^- = N_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(L(0, \chi)/2^4).$$

Assume $\text{Gal}(N/K) = (a, b^2)$. Let $\chi_-$ be any one of two quartic characters associated with the cyclic extension $M/K$ and $\chi_+$ any one of two cubic characters associated with the cyclic extension $K_{12}/K$ such that $\chi = \chi_- \chi_+$. Using the Artin reciprocity theorem, it can be easily verified that $\chi_- \circ b = \chi_-$, $\chi_- \circ c = \chi_-^{-1}$, $\chi_+ \circ c = \chi_+^{-1}$, and $\chi_+ \circ c = \chi_+$. Whence $\chi \circ b = \chi^3$ and $\chi \circ c = \chi^7$.  

For a positive integer $n$ let $\sigma_n \in \text{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$ with $\sigma_n(\zeta_{12}) = \zeta_{12}$. We have

$$\sigma_5(L(0, \chi)) = L(0, \chi^5) = L(0, \chi \circ b) = L(0, \chi)$$

and $\sigma_7(L(0, \chi)) = L(0, \chi)$. Since $(\sigma_5, \sigma_7) = \text{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$ we have $L(0, \chi) \in \mathbb{Q}$, whence $h_N^-/h_M^-$ is the 4-th power of some integer.

\[ \square \]
Our computational results are given in Table 2. When \( K = \mathbb{Q}(\sqrt{2}, \sqrt{257}) \), if \( h_N = 1 \), then \( N/M, N^+/M^+ \), and \( K_6/k_2 \) are unramified. Otherwise, \( h_N \equiv 0 \mod 3 \). Since \( \mathbb{Q}(\sqrt{2}) \) has class number one, we must have \( k_2 = \mathbb{Q}(\sqrt{257}) \) and \( f = 1 \). Note that when \( K = \mathbb{Q}(\sqrt{5}, \sqrt{101}) \), \( k_2 = \mathbb{Q}(\sqrt{101}) \), and \( f = 2 \), we have \( K_6 = \mathbb{Q}(\sqrt{101}, \theta) \) with \( \theta^3 - \theta^2 - 5\theta - 1 = 0 \). Using KASH (\([K]\)) we verify that the class group of \( \mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta) \) is isomorphic to \( C_2 \) and the narrow class group of this field is isomorphic to \( C_4 \). It follows that \( N^+ \) is the Hilbert class field of \( \mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta) \) and \( N \) is the narrow Hilbert class field of this field. In addition, thanks to KASH we verify that the class number of \( N^+ \) is equal to 1.

4.2. \( \text{Gal}(M/\mathbb{Q}) = Q_8 \times C_2 \). By \([\text{Lou}3]\) Theorem 1, 
\[
M = \mathbb{Q}\left(\sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{-2(2+\sqrt{2})(3+\sqrt{3})}\right)
\]
is the only normal CM-field of relative class number one with Galois group isomorphic to \( Q_8 \times C_2 \). This field has class number one and \( Q_M = 2 \). In this subsection we assume that \( N \) contains this field \( M \) and will prove that \( h_N > 1 \). The Galois group \( \text{Gal}(N/\mathbb{Q}) \) is isomorphic to either \( Q_8 \times C_2 \times C_3 \) or \( Q_{24} \times C_2 \) according to whether \( N \) has a cyclic cubic subfield or not.

4.2.1. \( \text{Gal}(N/\mathbb{Q}) = Q_8 \times C_2 \times C_3 \). The field \( N \) has only one cyclic cubic subfield \( K_3 \). The composita
\[
N_1 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-(2+\sqrt{2})(3+\sqrt{3})}) \quad \text{and} \quad N_2 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-1})
\]
have the same maximal real subfield \( K_3(\sqrt{2}, \sqrt{3}) \). Suppose that \( h_N = 1 \). By Proposition 1(7) we would have \( h_{N_1} = 1 \) or \( h_{N_2} = 1 \). Since every octic quaternion CM-field has an even relative class number, \( h_{N_1} \) is even. Using \([\text{CK}1]\) we verify that there is no imaginary abelian number field with Galois group isomorphic to \( C_2 \times C_2 \times C_2 \times C_3 \) of relative class number one which contains the field \( \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{-1}) \). Hence \( h_N > 1 \).

4.2.2. \( \text{Gal}(N/\mathbb{Q}) = Q_{24} \times C_2 \). The field \( N \) contains a non-normal cubic subfield \( K_3 \). The compositum
\[
N_1 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-2(2+\sqrt{2})(3+\sqrt{3})})
\]
is a normal CM-field with Galois group isomorphic to \( Q_{24} \). The compositum \( N_2 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-1}) \) is a normal CM-field with Galois group isomorphic to \( D_{12} \times C_2 \). We have \( N = N_1N_2 \) with \( N_1^+ = N_2^+ = K_3(\sqrt{2}, \sqrt{3}) \). Note that \( h_{N_1} \) is even. According to \([\text{LO}2]\) Theorem 1, \( h_{N_2} > 1 \). By Proposition 1(7) it follows that \( h_N > 1 \).

5. Case 3: \( \text{Gal}(M/\mathbb{Q}) \in \{G_9, G_6\} \)

In subsection 5.1 we assume that \( \text{Gal}(M/\mathbb{Q}) = G_9 \), and in subsection 5.2 we assume that \( \text{Gal}(M/\mathbb{Q}) = G_6 \).

5.1. \( \text{Gal}(M/\mathbb{Q}) = G_9 \). There is only one normal CM-field \( M \) of relative class number one with Galois group isomorphic to \( G_9 \) (\([\text{LO}2]\) Theorem 20):
\[
M = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{37}, \sqrt{-2(2+\sqrt{2}+3\sqrt{3})(2+\sqrt{5})})
\]
Assume that \( N \) contains \( M \). The aim of this subsection is to prove that \( h_N > 1 \). Note that \( \text{Gal}(M/Q(\sqrt{2}, \sqrt{5}, \sqrt{37})) = Q_8 \), \( \text{Gal}(M/Q(\sqrt{2})) = \text{Gal}(M/Q(\sqrt{5})) = \text{Gal}(M/Q(\sqrt{37})) = D_8 \), and \( \text{Gal}(M/Q(\sqrt{2}, \sqrt{5})) = \text{Gal}(M/Q(\sqrt{2}, \sqrt{37})) = \text{Gal}(M/Q(\sqrt{5}, \sqrt{37})) = C_4 \times C_2 \). Therefore, \( \text{Gal}(N/Q) \) is isomorphic to \( G_0 \times C_3 \)

if \( N \) contains only one cubic cyclic subfield \( K_3 \), \( (Q_8 \times C_3) \times C_2 \), \( (D_8 \times C_3) \times C_2 \), \( (a, b, c \mid a^3 = b^2 = c^2 = z^4 = 1, c^{-1}bc = b^2, b = zb, cz = zc, b^{-1}ab = a, c^{-1}ac = a, z^{-1}az = a^{-1}) \), or \( (C_4 \times C_2 \times C_3) \times C_2 \) otherwise. We divide this subsection into four parts according to \( \text{Gal}(N/Q) \).

5.1.1. \( \text{Gal}(N/Q) = G_9 \times C_3 \). We will show that \( h_N > 1 \). We first get an upper bound \( C \) on the conductor of \( K_3/Q \) such that if \( h_N^- = 1 \), then the conductor is less than or equal to \( C \). Let \( K = Q(\sqrt{2}, \sqrt{5}, \sqrt{37}) \). Since \( N/K \) is cyclic of degree 12, then as in point (1) of Lemma 1 we obtain \( N = (\zeta_N^s \zeta_M^s)^{0 \leq s \leq 0} \) for \( 0 < s < 1 \). We verify that \( \zeta_N^s \leq s \leq 0 \), which implies \( \zeta_N(s) \leq 0 \) for \( s \in [0, 1] \). Let \( F = Q(\sqrt{2}, \sqrt{5}, \sqrt{37}) \). The extension \( N^+/N < F \) is abelian of degree 12 and \( d_N^+/d_F^2 = N_{F/K}(\zeta_{F/K})^8 \). Using \( \mu_F \text{Res}_{s=1}(\zeta_F) \leq 2.27842 \) and Proposition 2, we obtain that if \( h_N^- \leq 1 \), then \( N_{F/K}(\zeta_{F/K}) < 1300 \), whence \( \zeta_{F/K} \in \{7, (3)^2, (13), (19), (31), (37)\} \) with \( (37) = Q_{37}^2 \) in \( F \).

(i) When \( \zeta_{F/K} = (7) \), \( K_3 \) is of conductor 7. There are four non-normal octic CM-fields containing \( Q(\sqrt{2}) \) with relative class number one. Let \( M_{8,1} = Q(\sqrt{2}, \sqrt{5}, \sqrt{3}) \), \( M_{8,2} = (2 + \sqrt{2})/2 \) and \( M_{8,2} \) its conjugate over \( Q \). Set \( N_{24,1} = M_{8,1}K_3 \) and \( N_{24,2} = M_{8,2}K_3 \). By Proposition 1(5) \( h_N^- = (h_N^-)^2 \). The two prime ideals lying above 7 in \( M_{8,1} = Q(\sqrt{2}, \sqrt{5}) \) split in \( M_{8,1} \). Hence, \( 3^2|\overline{h}_{N_{24,1}} \) by \( [LQ] \) Proposition 8] and \( 3^4|h_N^- \).

(ii) When \( \zeta_{F/K} = (3)^2 \), \( K_3 \) is of conductor 9. Let \( M_{8,1}, M_{8,2}, N_{24,1}, \) and \( N_{24,2} \) be as in (i). One prime ideal lying above 37 in \( Q(\sqrt{2}, \sqrt{5}) \) is ramified in \( M_{8,1} \) and \( 37 \) splits in \( K_3 \), whence \( 2^2|h_{N_{24,1}}^- \) and \( 2^4|h_N^- \). By the same argument we prove that if \( \zeta_{F/K} = (19) \), then \( 2^4|h_N^- \).

(iii) When \( \zeta_{F/K} = (13) \), \( K_3 \) is of conductor 13. Let \( M_{8,1} \) be any one of two non-normal octic CM-fields of \( M \) containing \( Q(\sqrt{2}, \sqrt{37}) \) and \( M_{8,2} \) its conjugate over \( Q \). Let \( N_{24,1} = M_{8,1}K_3 \) and \( N_{24,2} = M_{8,2}K_3 \). One prime ideal lying above 5 in \( Q(\sqrt{2}, \sqrt{37}) \) is ramified in \( M_{8,1} \) and \( 5 \) splits in \( K_3 \), which implies \( 2^2|h_{N_{24,1}}^- \) and \( 2^4|h_N^- \).

(iv) When \( \zeta_{F/K} = (31) \), \( K_3 \) is of conductor 31. We let \( M_{8,1} \) be any one of two non-normal octic subfields of \( M \) containing \( Q(\sqrt{5}, \sqrt{37}) \) and \( M_{8,2} \) its conjugate over \( Q \). One prime ideal lying above 2 is ramified in \( M_{8,1} \) and \( 2 \) splits in \( K_3 \), whence \( 2^2|h_{N_{24,1}}^- \) and \( 2^4|h_N^- \).

(v) When \( \zeta_{F/K} = Q_{37}^2 \), \( K_3 \) is of conductor 37. According to \([Ma]\), the cyclic sextic subfield \( K_3^*(\sqrt{2}, \sqrt{5}) \) of \( N \) has class number 6, whence by Proposition 1(6) \( 3|h_N^- \).

5.1.2. \( \text{Gal}(N/Q) = (D_8 \times C_3) \times C_2 \). Let \( K_3 \) be any one of three non-normal cubic subfields of \( N \), \( K_0 \) its normal closure, and \( k_2 \) the quadratic subfield of \( K_0 \). We have \( \text{Gal}(M/k_2) = D_8 \). Let \( K \) be the intermediate field between \( M^+ \) and \( k_2 \) such that \( G(M/K) = C_4 \). Then \( M/K \) is unramified at all finite primes, \( \text{Gal}(M^+/k_2) = C_2 \times C_2 \), \( \text{Gal}(N/K) = C_{12} \), and \( \text{Gal}(N^+/k_2) = C_2 \times C_2 \times C_3 \). As in point (1) of
Table 3.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$f \leq f$</th>
<th>$h_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Q}(\sqrt{2})$</td>
<td>22 None</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{5})$</td>
<td>8 None</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{37})$</td>
<td>44 2 $2^4$</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.

<table>
<thead>
<tr>
<th>$k_2$</th>
<th>$C$</th>
<th>$f$</th>
<th>$h_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Q}(\sqrt{2 \cdot 5})$</td>
<td>140</td>
<td>37</td>
<td>1021$^2$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{2 \cdot 37})$</td>
<td>21</td>
<td>9</td>
<td>5044$^2$</td>
</tr>
<tr>
<td>$\mathbb{Q}(\sqrt{5 \cdot 37})$</td>
<td>35 None</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Lemma 1 we obtain $(\zeta_N/\zeta_{m^+})(s) \geq 0$ and $\zeta_N(s) \leq 0$ for $s \in ]0,1[$. Since $N^+/k_2$ is abelian of degree 12 and $d_{N^+/k_2} = f^{16} N_{K_0/k_2}(\mathfrak{D}_{N^+/K_0})$, where $(f) = \mathfrak{f}_{K_0/k_2}$ and $\mathfrak{D}_{N^+/K_0}$ denotes the discriminant of the extension $N^+/K_0$, using Proposition 2 we obtain upper bound $C$ on $f$ such that if $h_N = 1$, then $f \leq C$. Our computational results are given in Table 3. As in point (3) of Lemma 2 we can easily verify that $h_N/h_M$ is the 4-th power of some rational integer.

5.1.3. Gal($N/\mathbb{Q}$) = $(C_4 \times C_2 \times C_3) \rtimes C_2 = \langle a,b,c,z \mid a^3 = b^2 = c^2 = z^4 = 1, c^{-1}bc = bz^2, bz = zb, cz = ze, b^{-1}ab = a, c^{-1}ac = a^{-1}, z^{-1}az = a \rangle$. In this case $N$ has three non-normal cubic subfields and its associated quadratic subfield $k_2$ has Gal($M/k_2$) = $C_4 \times C_2$. For a fixed field $k_2$ there are two intermediate fields $K$ between $k_2$ and $M^+$ such that Gal($M/K$) = $C_4$. Let $K$ be any one of these two fields. Then $M/K$ is unramified at all finite primes, Gal($N/K$) = $C_{12}$, Gal($M^+/k_2$) = $C_2 \times C_2$, and Gal($N^+/k_2$) = $C_2 \times C_2 \times C_3$. Analogously as in subsection 5.1.2 we get upper bound $C$ on $f$ such that if $h_N = 1$, then $f \leq C$. Since Gal($N/k_2$) = $C_4 \times C_2 \times C_3$, we compute $h_N$ using Hecke $L$-functions over $k_2$ (see Table 4).

5.1.4. Gal($N/\mathbb{Q}$) = $(Q_8 \times C_3) \rtimes C_2$. In this case $N$ has three non-normal cubic subfields such that its associated quadratic subfield $k_2$ is equal to $\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})$. Let $K$ be any one of three quartic fields containing $\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})$. Then Gal($N/K$) = $C_{12}$ and Gal($N^+/k_2$) = $C_2 \times C_2 \times C_3$. As in subsection 5.1.2 we verify that if $h_N = 1$, then $f \leq 36$ with $\mathfrak{f}_{K_0/k_2} = (f)$. There is no sextic field $K_0$ containing $\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})$ with $f \leq 36$. Therefore, $h_N > 1$.

5.2. Gal($M/\mathbb{Q}$) = $G_6$. In [Lou3] it is proved that there are exactly two such fields $M$ with $h_M = 1$: the composita $M = M_1 M_2$ listed in Table 5. In fact, those are the only fields with Galois group isomorphic to $G_6$ of relative class number one. The Galois group Gal($M^+/\mathbb{Q}$) is isomorphic to $D_8$ or $C_4 \times C_2$. When Gal($M^+/\mathbb{Q}$) = $D_8$, $M$ is a compositum of an octic dihedral CM-field $M_{8,1}$ and an imaginary abelian number field $M_{8,2}$ with Gal($M_{8,2}/\mathbb{Q}$) = $C_4 \times C_2$. Using [YK] and [CK1], we verify
that there is only one such field $M$ with $h_M = 1$: the second field in Table 5. When $\text{Gal}(M^+/Q) = C_4 \times C_2$, $M$ is a compositum of two octic dihedral CM-fields $M_{8,1}$ and $M_{8,2}$ with $M_{8,1}^+ = M_{8,2}^+$. According to [Lou3 Theorem 2] there is only one such field $M$ with $h_M = 1$: the first field in Table 5. We will prove that $h_N > 1$.

5.2.1. $\text{Gal}(N/Q) = G_6 \times C_3$. If $\text{Gal}(M^+/Q) = C_4 \times C_2$, then $\text{Gal}(N_1/Q) = D_8 \times C_3 = \text{Gal}(N_2/Q)$. According to [P Theorem 1], $h_{N_1} > 1$ and $h_{N_2} > 1$, whence $h_N > 1$. If $\text{Gal}(M^+/Q) = D_8$, then $\text{Gal}(N_1/Q) = D_8 \times C_3$, and $\text{Gal}(N_2/Q) = C_4 \times C_2 \times C_3$. Using [CK1], we verify that $h_{N_2} > 4$, whence $h_N > 1$.

5.2.2. $\text{Gal}(N/Q) = C_3 \times G_6$.

(a) Let $M$ be the first field in Table 5. The quadratic field $k_2$ associated with $K_3$ is either $Q(\sqrt{34})$, $Q(\sqrt{2})$, or $Q(\sqrt{17})$. If $k_2 = Q(\sqrt{34})$, then $\text{Gal}(N_1/Q) = D_{24}$, and $\text{Gal}(N_2/Q) = C_3 \times D_8$. If $k_2 = Q(\sqrt{2})$, then $\text{Gal}(N_1/Q) = C_3 \times D_8$, and $\text{Gal}(N_2/Q) = D_{24}$. If $k_2 = Q(\sqrt{17})$, then $\text{Gal}(N_1/Q) = C_3 \times D_8 = \text{Gal}(N_2/Q)$. According to [P Theorem 1] and [Lef Theorem 4.1], we have $h_{N_1} > 1$ and $h_{N_2} > 1$, whence $h_N > 1$.

(b) Let $M$ be the second field in Table 5. The quadratic field $k_2$ associated with $K_3$ is either $Q(\sqrt{221})$, $Q(\sqrt{17})$, or $Q(\sqrt{17})$. If $k_2 = Q(\sqrt{221})$, then $\text{Gal}(N_1/Q) = D_{24}$, and $\text{Gal}(N_2/Q) = S_3 \times C_4$. If $k_2 = Q(\sqrt{17})$, then $\text{Gal}(N_1/Q) = C_3 \times D_8$ and $\text{Gal}(N_2/Q) = Q_{12} \times C_2$. If $k_2 = Q(\sqrt{17})$, then $\text{Gal}(N_1/Q) = C_3 \times D_8$, and $\text{Gal}(N_2/Q) = S_3 \times C_4$. Using [Lef Theorem 4.1] and [P Theorem 1], we have $h_{N_1} > 1$ and $h_{N_2} > 1$, whence $h_N > 1$.

6. CASE 4: $\text{Gal}(M/Q) = D_8 \times C_2$

To begin with, we prove the following:

**Theorem 2** (Compare with [Lou3 Theorems 2 and 3]). There are four normal CM-fields $M$ of degree 16 and Galois group $D_8 \times C_2$ with relative class number one, those given in the following Table 6.
Table 6.

<table>
<thead>
<tr>
<th>$M^+$</th>
<th>$\alpha : M = M^+ (\sqrt{-\alpha})$</th>
<th>$Q_M$</th>
<th>$\omega_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q(\sqrt{2}, \sqrt{3}, \sqrt{17})$</td>
<td>$5(5 + \sqrt{17})/2$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$Q(\sqrt{2}, \sqrt{3}, \sqrt{3 + \sqrt{3}})$</td>
<td>1</td>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>$Q(\sqrt[3]{3}, \sqrt[15]{15 + 8\sqrt{3}})$</td>
<td>1</td>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>$Q(\sqrt{2}, \sqrt{17}, \sqrt{3(5 + \sqrt{17})}/2)$</td>
<td>3</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

Proof. The Galois group $\text{Gal}(M^+/\mathbb{Q})$ is isomorphic to either $D_8$ or $C_2 \times C_2 \times C_2$. When $\text{Gal}(M^+/\mathbb{Q}) = D_8$, $M$ is a compositum of an octic dihedral CM-field $M_{8,1}$ and an imaginary abelian number field $M_{8,2}$ with $\text{Gal}(M_{8,2}/\mathbb{Q}) = C_2 \times C_2 \times C_2$. Let $L$ be any one of four non-normal quartic CM-subfields of $M_{8,1}$. According to [Lom3, Proposition 16], $\hat{h}_L = 1$ if and only if $(h_L, h_{M_{8,2}}) \in \{(1, 4), (1, 2), (2, 1)\}$. In the case that $(h_L, h_{M_{8,2}}) = (1, 4)$ or $(1, 2)$, $M_{8,1}^+ = M_{8,2}^+$ is of the form $Q(\sqrt{p}, \sqrt{q})$, where $(p, q)$ is one of the 19 pairs given in [LO1, Theorem 8]. Then $M_{8,2} = Q(\sqrt{p}, \sqrt{q}, \sqrt{-m})$, where $Q(\sqrt{-m})$ is one of 81 (= $9 + 18 + 54$) imaginary quadratic fields of class number one, two or four. For these 1539 (= $19 \times 81$) CM-fields $M_{8,2}$ we compute $\hat{h}_{M_{8,2}}$ and verify that there is only one field $M$ with $\hat{h}_M = 1$: the fourth field in Table 6. In the case that $(h_L, h_{M_{8,2}}) = (2, 1)$, using [YK] and [CK1] we verify that there are exactly two fields $M$ with $\hat{h}_M = 1$: the second and third fields in Table 6. When $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, $M$ is a compositum of two octic dihedral CM-fields $M_{8,1}$ and $M_{8,2}$. According to [Lom3, Theorem 2], there is only one such $M$ with $\hat{h}_M = 1$: the first field in Table 6.

From now on we assume that $M$ is one of these four fields and we will prove that $h_M > 1$. We classify the Galois group $\text{Gal}(N/\mathbb{Q})$. Let $K_3$ be any cubic subfield of $N$. If $K_3$ is not normal, then its normal closure $K_6$ is a dihedral real sextic field and we let $k_3$ denote the (real) quadratic subfield of $K_6$. If $K_3$ is normal, then $\text{Gal}(N/\mathbb{Q}) = D_8 \times C_6$. If $K_3$ is not normal over $\mathbb{Q}$, then $\text{Gal}(N/\mathbb{Q})$ is isomorphic to either $D_8 \times S_3$, $D_2 \times C_2$, or $(C_2 \times D_8) \times C_2$. Let $K_4 = M_{8,1}^{+\times} = M_{8,2}^{+\times}$. If $K_4 \cap k_3 = \mathbb{Q}$, then $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, and $\text{Gal}(N/\mathbb{Q}) = D_8 \times S_3$. If $K_4 \cap k_2 \not\subseteq \mathbb{Q}$, then $M_{8,1}/k_2$ is either cyclic or biquadratic bicyclic. If $M_{8,1}/k_2$ is cyclic quartic, then $\text{Gal}(N/\mathbb{Q}) = D_{24} \times C_2$. If $M_{8,1}/k_2$ is biquadratic bicyclic, then $\text{Gal}(K_3M_{8,1}/\mathbb{Q}) = C_3 \times D_8 = (a, b, c)a^3 = b^4 = c^2 = 1, b^{-1}ab = a^{-1}, c^{-1}ac = a, c^{-1}bc = b^{-1}$, and $\text{Gal}(N/\mathbb{Q}) = (C_3 \times D_8) \times C_2$.

6.1. $\text{Gal}(N/\mathbb{Q}) = D_8 \times C_6$. If $\text{Gal}(M^+/\mathbb{Q}) = D_8$, then the composita $N_1 = M_{8,1}K_3$ and $N_2 = M_{8,2}K_3$ are normal CM-subfields of $N$ with the same maximal real subfields $K_3K_4$. By [P, Theorem 1], the compositum of the dihedral octic CM-field $Q(\sqrt{13}, \sqrt{17}, \sqrt{-9 + \sqrt{13}}/2)$ and of the cyclic cubic field of conductor 13 is the only normal CM-field of relative class number one with Galois group isomorphic to $D_8 \times C_3$. In Table 6 there is no field $M$ containing $Q(\sqrt{13}, \sqrt{17})$. It follows then that $h_{N_1} > 1$. According to [CK1], there are exactly two imaginary abelian number fields with Galois group isomorphic to $C_2 \times C_2 \times C_2 \times C_2$ of relative class number one: $F_7(\sqrt{-1}, \sqrt{-3}, \sqrt{-7})$ and $F_7(\sqrt{-3}, \sqrt{-7}, \sqrt{-13})$, where $F_7$ denotes the cyclic...
cubic field of conductor 7. In Table 6 there is no field \( N \) containing \( \mathbb{Q}(\sqrt{3}, \sqrt{7}) \), or \( \mathbb{Q}(\sqrt{5}, \sqrt{21}) \), whence \( h_N^{-1} > 1 \). It follows that if \( \text{Gal}(M^+/N) = D_8 \), then \( h_N > 1 \). If \( \text{Gal}(M^+/N) = C_2 \times C_2 \times C_2 \), then \( N_1 = M_{8,1}K_3 \) and \( N_2 = M_{8,2}K_3 \) are normal CM-fields with Galois group isomorphic to \( D_8 \times C_3 \). Note that \( M_{8,1}^+ = M_{8,2}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{17}) \). According to [P, Theorem 1], \( h_{N_1}^- > 1 \) and \( h_{N_2}^- > 1 \), which implies \( h_N > 1 \).

6.2. \( \text{Gal}(N/\mathbb{Q}) = D_8 \times S_3 \). In this case \( N \) has three non-normal real cubic subfields. Let \( K_3, K_6, k_2 \) and \( K_4 \) be as above. We have that \( K_4 \cap k_2 = \mathbb{Q} \), \( \text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2 \), \( M \) is the first field in Table 6. In addition, we have \( K_4 = \mathbb{Q}(\sqrt{2}, \sqrt{17}) \), and \( k_2 = \mathbb{Q}(\sqrt{m}) \) with \( m \in \{5, 2 \cdot 5, 5 \cdot 17, 2 \cdot 5 \cdot 17\} \). Let \( (f) \) be the conductor of the extension \( K_5/k_2 \) with \( f \) a positive integer.

**Lemma 3.**

1. We have \( \zeta_N(s) \leq 0 \) for \( 0 < s < 1 \).

2. For each given \( k_2 \) in the above we can compute a bound of \( f \leq C \) on the conductor \( (f) \) for \( N \)'s such that \( h_N = 1 \). These bounds and the possible \( f \)'s are compiled in Table 7.

3. The quotient \( h_N/h_M \) is the perfect fourth power of some rational integer.

**Proof.**

1. Let \( \chi_{N/M^+} \) be any one of two characters associated with the cyclic sextic extension \( N/M^+ \). We have

\[
\frac{\zeta_N(s)}{\zeta_{M^+}(s)} = \frac{\zeta_M(s)}{\zeta_{M^+}(s)} |L(s, \chi_{N/M^+})L(s, \chi_{N/M^+}^2)|^2
\]

and

\[
\frac{\zeta_M(s)}{\zeta_{M^+}(s)} = \frac{\zeta_{M_{8,1}}(s)}{\zeta_{K_4}(s)} \frac{\zeta_{M_{8,2}}(s)}{\zeta_{K_4}(s)} = L(s, \psi_1)^2 L(s, \psi_2)^2,
\]

where \( \psi_i \) is the unique irreducible character of degree 2 of \( \text{Gal}(M_{8,i}/\mathbb{Q}) \) the dihedral group of order 8, and \( L(s, \psi_i) \) denotes the Artin L-function associated with \( \psi_i \) for \( i = 1, 2 \). Since \( \psi_i \) is real valued, \( L(s, \psi_i) \) is on the real axis and \( L(s, \psi_i)^2 \geq 0 \). For \( M^+ = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{17}) \) we have verified that \( \zeta_{M^+}(s) \leq 0 \) for \( s \in \{0, 1\} \), whence \( \zeta_N(s) \leq 0 \).

2. Since \( M/M^+ \) is unramified at all finite primes, \( N/N^+ \) is unramified at all finite primes and \( d_N/d_{N^+} = d_{N^+} = d_{K_6}^2 f_{16}^2 N_{K_6/\mathbb{Q}}(\mathcal{O}_{N^+/K_6}) \). Using Proposition 2, we get an upper bound on \( f \). Since the prime ideals lying above 2 and those above 17 split in \( M/M^+ \), if \( (f, 2) > 1 \) or \( (f, 17) > 1 \), then \( 3^2 \) divides \( h_N \) by [LOO, Proposition 8]. Note that the prime ideals lying above 13 split in \( M/M^+ \), whence the relative class number of the fourth field \( N \) in Table 7 is divisible by \( 3^4 \).

<table>
<thead>
<tr>
<th>( k_2 )</th>
<th>( \mu_{k_2} )</th>
<th>( \text{Res}<em>{s=1}(\zeta</em>{k_2}) )</th>
<th>( f \leq )</th>
<th>( f )</th>
<th>( h_N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Q}(\sqrt{5}) )</td>
<td>0.0436324</td>
<td>10</td>
<td>NONE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mathbb{Q}(\sqrt{2 \cdot 5}) )</td>
<td>0.4276490</td>
<td>41</td>
<td>37</td>
<td>( 920^4 )</td>
<td></td>
</tr>
<tr>
<td>( \mathbb{Q}(\sqrt{5 \cdot 17}) )</td>
<td>0.5861712</td>
<td>31</td>
<td>9</td>
<td>( 44^4 )</td>
<td></td>
</tr>
<tr>
<td>( \mathbb{Q}(\sqrt{2 \cdot 5 \cdot 17}) )</td>
<td>1.4062136</td>
<td>38</td>
<td>13</td>
<td>( 3^4 h_N )</td>
<td></td>
</tr>
</tbody>
</table>
(3) Note that $M_{8,1}$ and $M_{8,2}$ are cyclic over $\mathbb{Q}(\sqrt{34})$. Let $K$ be the compositum of $\mathbb{Q}((\sqrt{34}))$ and $k_2$. Then $\text{Gal}(N/K) = C_{12}$. Let $\chi$ be any one of the four characters of order 12 associated with the cyclic extension $N/K$. Similarly to point (3) of Lemma 2, we verify that $L(0, \chi) \in \mathbb{Q}$ and $h_N^+ / h_M^- = (L(0, \chi)/2)^4$.

In conclusion, we have proved that every normal CM-field with Galois group isomorphic to $D_8 \times S_3$ has class number greater than one. Our computational results are given in Table 7.

6.3. $\text{Gal}(N/\mathbb{Q}) = D_{24} \times C_2$. In this case $N$ has three non-normal cubic fields and $M_{8,1}/k_2$ is cyclic. Let $N_1 = M_{8,1}K_1$ and $N_2 = M_{8,2}K_3$. Then we have $N = N_1N_2$ with $N_1^2 = N_2^2 = K_1K_3$. If $\text{Gal}(M^+/\mathbb{Q}) = D_8$, then $\text{Gal}(N_1/\mathbb{Q}) = D_{24}$, and $\text{Gal}(N_2/\mathbb{Q}) = S_3 \times C_2 \times C_2$. If $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, then $\text{Gal}(N_1/\mathbb{Q}) = D_{24} = \text{Gal}(N_2/\mathbb{Q})$. Using [Lef, Theorem 4.1] and [P] Theorem 1, we verify that in both cases $h_{N_1} > 1$ and $h_{N_2} > 1$. It follows that the class number of a normal CM-field with Galois group isomorphic to $D_{24} \times C_2$ is greater than one.

6.4. $\text{Gal}(N/\mathbb{Q}) = (C_3 \times D_8) \times C_2$. In this case $N$ has three non-normal cubic fields and $M_{8,1}/k_2$ is biquadratic bicyclic. Then the Galois group of the compositum $N_1 = M_{8,1}K_3$ over $\mathbb{Q}$ is isomorphic to $C_3 \times D_8$, whence $h_{N_1} > 1$ ([P, Theorem 13]). If $\text{Gal}(M^+/\mathbb{Q}) = D_8$, then $\text{Gal}(N_2/\mathbb{Q}) = S_3 \times C_2 \times C_2$. If $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$, then $\text{Gal}(N_1/\mathbb{Q}) = C_3 \times D_8$. By [P] Theorems 1 and 13 $h_{N_2} > 1$. Consequently, if $\text{Gal}(N/\mathbb{Q}) = (C_3 \times D_8) \times C_2$, then $h_N > 1$.

To conclude, Theorem 1 is now proved with completion.

All computations were carried out using Pari-Gp ([Pa]) and KASH ([K]).

Acknowledgments

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