

## THE CLASS NUMBER ONE PROBLEM FOR SOME NON-ABELIAN NORMAL CM-FIELDS OF DEGREE 48

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ABSTRACT. We prove that there is precisely one normal CM-field of degree 48 with class number one which has a normal CM-subfield of degree 16: the narrow Hilbert class field of  $\mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$  with  $\theta^3 - \theta^2 - 5\theta - 1 = 0$ .

### 1. INTRODUCTION

According to [O] and [H], there exist only finitely many normal CM-fields with class number one, and their degrees are less than or equal to 436. All imaginary abelian number fields with class number one are known in [Y]: their degrees are less than or equal to 24. All normal CM-fields of degree less than 48 with class number one are known by many authors ([LO1], [LO2], [Lef], [LLO], [LP1], [LP2], [LOO], [Lou3], [P], [YPK], [PsK], [CK2], and [CK3]). In the following table we sum up the numbers of the non-abelian normal CM-fields  $N$  with class number one according to their degrees.

$[N : \mathbb{Q}]$	nb	$[N : \mathbb{Q}]$	nb	$[N : \mathbb{Q}]$	nb	$[N : \mathbb{Q}]$	nb
8	17	20	1	32	6	42	0
12	9	24	7	36	3	44	0
16	12	28	0	40	1		

In this paper we study the non-abelian normal CM-fields that contain a normal CM-subfield of degree 16, and will prove the following:

**Theorem 1.** *There exists one and only one normal CM-field  $N$  of degree 48 with class number one which has a normal CM-subfield of degree 16: the narrow Hilbert class field of the real dihedral number field  $K_{12} = \mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$  of degree 12 with  $\theta^3 - \theta^2 - 5\theta - 1 = 0$ , narrow class number 4 and class number 2. The extension  $N/K_{12}$  is cyclic quartic,  $d_N = d_{K_{12}}^4 = 2^{32} \cdot 5^{24} \cdot 101^{24}$ , the maximal totally real subfield of  $N$  is the Hilbert class field of  $K_{12}$ , and the Galois group  $\text{Gal}(N/\mathbb{Q})$  is isomorphic to the semi-direct product  $C_3 \rtimes D_{16}$ .*

### 2. PREREQUISITE AND NOTATION

We use the following notation. For a number field  $K$ , we let  $h_K$ ,  $d_K$ ,  $\omega_K$ , and  $\zeta_K$  denote the class number, the absolute value of the discriminant, the number of roots of unity in  $K$ , and the Dedekind zeta function of  $K$ , respectively. If  $K$

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is a CM-field, we let  $h_K^-, K^+$  and  $Q_K \in \{1, 2\}$  be the relative class number, the maximal real subfield and the Hasse unit index of  $K$ , respectively. For an abelian extension  $F/K$  we denote by  $\mathfrak{F}_{F/K}$  the finite part of its conductor. For a positive integer  $n$  we let  $\zeta_n = e^{2i\pi/n}$ . Before starting the proof of Theorem 1 we recall the well-known results which will be used later in this paper.

- Proposition 1.** (1) ([LOO, Lemma 2]) *If  $K$  is a normal CM-field, then the complex conjugation is in the center of its Galois group.*  
 (2) ([CH, Lemma 13.5]) *Let  $K$  be a CM-field. If there is at least one ramified prime ideal in  $K/K^+$  which is lying above an odd prime, then  $Q_K = 1$ .*  
 (3) ([Ho, Theorem 5], [Ok], or [LOO, Theorem 5]) *Let  $k \subset K$  be two CM-fields. Then  $h_k^-$  divides  $4h_K^-$ . Moreover, if  $[K : k]$  is odd, then  $h_k^-$  divides  $h_K^-$  and  $Q_k = Q_K$ .*  
 (4) ([LO2] and [Lou1, Proposition 6]) *Let  $K$  be a CM-field and let  $t$  be the number of prime ideals of  $K^+$  that are ramified in the quadratic extension  $K/K^+$ . Then  $2^{t-1}$  divides  $h_K^-$ . Moreover, if  $Q_K = 2$ , then  $2^t$  divides  $h_K^-$ .*  
 (5) ([Lou1, Proposition 13] and [LO2, Proposition 2]) *Let  $K = L_1L_2$  be a CM-field which is a compositum of two CM-fields  $L_1$  and  $L_2$  with the same maximal totally real subfield. Then*

$$h_K^- = \frac{Q_K}{Q_{L_1}Q_{L_2}} \frac{\omega_K}{\omega_{L_1}\omega_{L_2}} h_{L_1}^- h_{L_2}^-$$

and  $h_{L_1}^- h_{L_2}^-$  divides  $4h_K^-$ . In particular, if  $L_1$  and  $L_2$  are isomorphic, then  $\omega_K = \omega_{L_1} = \omega_{L_2} = 2$  and  $h_K^- = (Q_K/2)(h_{L_1}^-/Q_{L_1})^2$ .

- (6) ([M, Corollary 2.2 and 2.3]) *Let  $E/F$  be an extension of number fields. Then  $h_F$  divides  $[E : F]h_E$ . Moreover, if no nontrivial abelian subextension of  $E/F$  is unramified over  $F$ , then  $h_F$  divides  $h_E$ .*  
 (7) *Let  $K = L_1L_2$  be a CM-field which is a compositum of two CM-fields  $L_1$  and  $L_2$  with the same maximal totally real subfield  $L_1^+ = L_2^+$ . If  $h_K = 1$ , then  $h_{L_1}$  and  $h_{L_2}$  are 1 or 2. If  $h_K = 1$  and  $h_{L_1} = h_{L_2} = 2$ , then  $h_{L_1^+} = h_{L_2^+} = 2$ .*

*Proof.* We only need to prove the last statement of (7). If  $h_K = 1$  and  $h_{L_1} = h_{L_2} = 2$ , then  $K$  is the Hilbert class field of  $L_1$  and is at the same time that of  $L_2$ . Hence,  $K^+$  is the Hilbert class field of  $L_1^+ = L_2^+$  (see [P, Lemma 6.2]). □

**Proposition 2.** *Let  $K$  be a CM-field of degree  $2n$ .*

- (1) ([W])  $h_K^- = Q_K \omega_K / (2\pi)^n \cdot \sqrt{d_K/d_{K^+}} \cdot \text{Res}_{s=1}(\zeta_K) / \text{Res}_{s=1}(\zeta_{K^+})$   
 (2) ([LO2, Proposition 9]) *Let  $\beta_K = 1 - (2/\log d_K)$  and*

$$\varepsilon_K = \max(1 - 2\pi n e^{1/n} / d_K^{1/2n}, 2/5 \exp(-2\pi n / d_K^{1/2n})).$$

*If  $\zeta_K(\beta_K) \leq 0$ , then  $\text{Res}_{s=1}(\zeta_K) \geq 2\varepsilon_K / (e \log d_K)$ .*

- (3) ([Lou2]) *There exists a computable constant  $\mu_k > 0$  such that for any abelian extension  $K/k$  of degree  $m$  unramified at all the infinite places we have*

$$\text{Res}_{s=1}(\zeta_K) \leq (\text{Res}_{s=1}(\zeta_k))^m \left( \frac{1}{2(m-1)} \log(d_K/d_k^m) + 2\mu_k \right)^{m-1}.$$

Let  $C_m$  denote the cyclic group of order  $m$ ,  $D_m$  the dihedral group of order  $m$ ,  $Q_m$  the quaternion group of order  $m$  and set  $G_6 = \langle b, c, z | b^4 = c^2 = z^2 = 1, c^{-1}bc = bz, bz = zb, cz = zc \rangle$  and  $G_9 = \langle b, c, z | b^2 = c^2 = z^4 = 1, c^{-1}bc = bz^2, bz = zb, cz = zc \rangle$  (in the notation of [JL]). Throughout this paper,  $N$  denotes a non-abelian

normal CM-field of degree 48. We assume that the 3-Sylow subgroup of its Galois group  $\text{Gal}(N/Q)$  is normal, and we let  $M$  denote the normal CM-subfield of degree 16 of  $N$ . According to Proposition 1, if  $h_N = 1$ , then  $h_M^- = 1$  (moreover, either  $N/M$  is ramified at least one finite place and  $h_M = 1$  or  $N/M$  is unramified at all places,  $h_M = 3$ , and  $N$  is the Hilbert class field of  $M$ ). Now, there are 26 normal CM-fields of degree 16 with relative class number one (see [LO2], [Lou3], [CK1], [PK], and Theorem 2 below). If  $\text{Gal}(M/Q)$  is non-abelian, then it is equal to  $Q_8 \times C_2, G_6, D_{16}, G_9, D_8 \times C_2$ . For proving Theorem 1, we first prove that if  $\text{Gal}(M/Q) \neq D_{16}, G_9, D_8 \times C_2$ , then we can use Proposition 1 and the known solutions to various (relative) class number problems for suitable CM-subfields of  $N$  to prove that  $h_N > 1$ . Now, assume that  $\text{Gal}(M/Q) = D_{16}, G_9$ , or  $D_8 \times C_2$ . We will show that we can find a subfield  $L$  of  $M^+$  such that  $N/L$  is abelian and such that the use of abelian  $L$ -functions to factorize  $\zeta_N/\zeta_L$  readily yields  $(\zeta_N/\zeta_L)(s) \geq 0$  for  $0 < s < 1$ . Since  $M$  is known,  $L$  also is known, we will check that  $\zeta_L(s) \leq 0$  for  $0 < s < 1$  and we will therefore deduce that  $\zeta_N(s) \leq 0$  for  $0 < s < 1$ . Using Proposition 2, we will obtain explicit lower bounds for  $h_N^-$ , according to which we will be able to compute explicit upper bounds on  $d_N$  when  $h_N = 1$  and to construct a short list of number fields  $N$  containing all such  $N$ 's with  $h_N = 1$ . We will finally explain how one can use the method expounded in [Lou5] and [Lou6] to compute the relative class numbers of these finitely many CM-fields  $N$  that remain, thus completing the proof of Theorem 1.

3. CASE 1:  $M$  IS ABELIAN

We will show the following.

**Proposition 3.** *If  $N$  contains an abelian number field  $M$  of degree 16, then  $h_N > 1$ .*

*Proof.* Let  $K_3$  be any cubic subfield of  $N$ . Since  $N$  is non-abelian,  $K_3$  is not normal, its normal closure  $K_6$  is a dihedral real sextic field, and we let  $k_2$  denote the (real) quadratic subfield of  $K_6$ . The Galois group  $\text{Gal}(M/Q)$  is isomorphic to  $C_{16}, C_8 \times C_2, C_4 \times C_4, C_4 \times C_2 \times C_2$ , or  $C_2 \times C_2 \times C_2 \times C_2$ .

- (i) If  $\text{Gal}(M/Q) = C_{16}$ , then  $\text{Gal}(N/Q)$  is isomorphic to  $C_3 \times C_{16} = \langle a, b | a^3 = b^{16} = 1, b^{-1}ab = a^{-1} \rangle$ , and  $N$  is a compositum of  $M$  and the real dihedral field of degree 6 that is fixed by  $\langle b^2 \rangle$ . According to [Lou4, Theorem 5] we have  $h_N^- > 1$ .
- (ii) If  $\text{Gal}(M/Q) = C_8 \times C_2$  with  $\text{Gal}(M^+/Q) = C_8$ , then  $\text{Gal}(N/Q) = \langle a, b, c | a^3 = b^8 = c^2 = 1, b^{-1}ab = a^{-1}, ac = ca, bc = cb \rangle$  with  $\text{Gal}(N/N^+) = \langle b^4c \rangle$ . The subfield  $K_{12}$  fixed by  $\langle b^2c \rangle$  is a normal CM-field with Galois group isomorphic to  $Q_{12}$ . By [LP1]  $h_{K_{12}}^- > 4$ , whence  $h_N^- > 1$  by Proposition 1(3). If  $\text{Gal}(M/Q) = C_8 \times C_2$  with  $\text{Gal}(M^+/Q) = C_4 \times C_2$ , then  $h_N^- > 1$  by [CK1]. Hence  $h_N^- > 1$  by Proposition 1(3).
- (iii) If  $\text{Gal}(M/Q) = C_4 \times C_4$ , then  $\text{Gal}(N/Q) = Q_{12} \times C_4$  and  $\text{Gal}(N^+/Q)$  is isomorphic to either  $S_3 \times C_4$  or  $Q_{12} \times C_2$ . Let  $\psi_1$  and  $\psi_2$  be two odd primitive characters of order 4 such that  $M$  is associated with the group  $\langle \psi_1, \psi_2 \rangle$ . If  $k_2$  is associated with  $\langle \psi_1^2 \rangle$  or  $\langle \psi_2^2 \rangle$ , then  $\text{Gal}(N^+/Q) = S_3 \times C_4$ . Assume that  $k_2$  is associated with  $\langle \psi_1^2 \rangle$ . Let  $M_{12,1}$  be the compositum of  $K_6$  and the quartic field associated with  $\langle \psi_1 \rangle$ , and  $M_{12,2}$  the compositum of  $K_6$  and the quartic field associated with  $\langle \psi_1 \psi_2^2 \rangle$ . Then  $M_{12,1}$  and  $M_{12,2}$  are quaternion CM-fields

of degree 12 with the same maximal real subfield  $K_6$ . According to [LP1, Theorem 1], there is no pair of  $(M_{12,1}, M_{12,2})$  such that  $h_{M_{12,1}}^- | 4$ ,  $h_{M_{12,2}}^- | 4$ , and at the same time  $M_{12,1}^+ = M_{12,2}^+$ , whence  $h_N^- > 1$ . By symmetry, if  $k_2$  is associated with  $\langle \psi_2^2 \rangle$ , then  $h_N^- > 1$ . Assume now that  $k_2$  is associated with  $\langle \psi_1^2 \psi_2^2 \rangle$ . Let  $M_{24,1}$  be the compositum of  $K_6$  and the imaginary cyclic quartic field associated with  $\langle \psi_1 \rangle$ , and  $M_{24,2}$  the compositum of  $K_6$  and the imaginary cyclic quartic field associated with  $\langle \psi_2 \rangle$ . Then  $M_{24,1}$  and  $M_{24,2}$  are normal CM-fields with Galois group isomorphic to  $S_3 \times C_4$  which have the same maximal real subfield. Using Proposition 1(5) we verify that  $h_N^- = h_{M_{24,1}}^- h_{M_{24,2}}^-$ . By [P, Theorem 1] there is only one CM-field of relative class number one with Galois group isomorphic to  $S_3 \times C_4$ , whence  $h_N^- > 1$ .

- (iv) If  $\text{Gal}(M/\mathbb{Q}) = C_4 \times C_2 \times C_2$  with  $\text{Gal}(M^+/\mathbb{Q}) = C_4 \times C_2$ , then  $\text{Gal}(N/\mathbb{Q})$  is isomorphic to either  $Q_{12} \times C_2 \times C_2$  or  $S_3 \times C_2 \times C_4$ . Let  $\psi$  be the odd primitive Dirichlet character of order 4, and let  $\chi_1$  and  $\chi_2$  be two quadratic odd characters such that  $M$  is associated with the group  $\langle \psi, \chi_1, \chi_2 \rangle$ . If  $k_2$  is associated with  $\langle \psi^2 \rangle$ , then the compositum  $M_{12,1}$  of  $K_6$  and the field associated with  $\langle \psi \rangle$ , and the compositum  $M_{12,2}$  of  $K_6$  and the field associated with  $\langle \psi \chi_1 \chi_2 \rangle$  are normal CM-fields with Galois group  $Q_{12}$  and  $M_{12,1}^+ = M_{12,2}^+$ . By [LP1, Theorem 1],  $h_N^- > 1$ . If  $k_2$  is associated with  $\langle \psi^2 \chi_1 \chi_2 \rangle$  or  $\langle \chi_1 \chi_2 \rangle$ , then we let  $M_{24,1}$  be the compositum of  $K_6$  and the field associated with  $\langle \psi \rangle$ , and  $M_{24,2}$  the compositum of  $K_6$  and the field associated with  $\langle \psi^2, \chi_1, \chi_2 \rangle$ . Then  $\text{Gal}(M_{24,1}/\mathbb{Q}) = S_3 \times C_4$ ,  $\text{Gal}(M_{24,2}/\mathbb{Q}) = S_3 \times C_2 \times C_2$ ,  $M_{24,1}^+ = M_{24,2}^+$ , and  $N = M_{24,1} M_{24,2}$ . By [P, Theorem 1]  $h_{M_{24,1}}^- > 1$  and  $h_{M_{24,2}}^- > 1$ , whence according to Proposition 1(7) we have  $h_N^- > 1$ .
- (v) If  $\text{Gal}(M/\mathbb{Q}) = C_4 \times C_2 \times C_2$  with  $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$ , then  $h_M^- > 1$  by [CK1]. Hence  $h_N^- > 1$ .
- (vi) If  $\text{Gal}(M/\mathbb{Q}) = C_2 \times C_2 \times C_2 \times C_2$ , then  $h_M^- > 1$  by [CK1]. Hence  $h_N^- > 1$ . □

#### 4. CASE 2: $\text{Gal}(M/\mathbb{Q}) \in \{D_{16}, Q_8 \times C_2\}$

In this section we assume that  $\text{Gal}(M/\mathbb{Q}) \in \{D_{16}, Q_8 \times C_2\}$  and  $h_M^- = 1$ . We will prove that there is exactly one field  $N$  with  $h_N = 1$ . In subsection 4.1 we assume that  $G(M/\mathbb{Q}) = D_{16}$ , and in subsection 4.2 we assume that  $G(M/\mathbb{Q}) = Q_8 \times C_2$ .

4.1.  $G(M/\mathbb{Q}) = D_{16}$ . There are five dihedral CM-fields  $M$  of degree 16 with relative class number one [LO2, Theorem 10]: the narrow Hilbert class fields of  $\mathbb{Q}(\sqrt{pq})$  with  $(p, q) \in \{(2, 257), (5, 101), (5, 181), (13, 53), (13, 61)\}$ . The narrow Hilbert class field of  $\mathbb{Q}(\sqrt{2 \cdot 257})$  has class number three and the remaining four  $M$ 's have class number one. We set  $K = \mathbb{Q}(\sqrt{p}, \sqrt{q})$  and  $k = \mathbb{Q}(\sqrt{pq})$ . The field  $M$  has three quadratic subfields  $L_1, L_2$ , and  $k$  with  $\text{Gal}(M/L_1) = \text{Gal}(M/L_2) = D_8$ , and  $\text{Gal}(M/k) = C_8$ . Therefore, the Galois group  $\text{Gal}(N/\mathbb{Q})$  is isomorphic to  $D_{16} \times C_3$  if  $N$  contains only one cubic cyclic subfield,  $D_{48}$  or  $(D_8 \times C_3) \rtimes^1 C_2 = C_3 \rtimes D_{16} = \langle a, b, c \mid a^3 = b^8 = c^2 = 1, c^{-1}bc = b^{-1}, b^{-1}ab = a^{-1}, ac = ca \rangle$ , otherwise. In [Lef] it is proved that if  $\text{Gal}(N/\mathbb{Q}) = D_{48}$ , then  $h_N^- > 1$ . We deal with the fields  $N$  with  $\text{Gal}(N/\mathbb{Q}) = D_{16} \times C_3$  in 4.1.1 and the fields  $N$  with  $\text{Gal}(N/\mathbb{Q}) = (D_8 \times C_3) \rtimes^1 C_2$  in 4.1.2, respectively.

4.1.1.  $\text{Gal}(N/\mathbb{Q}) = D_{16} \times C_3$ . Let  $K_3$  denote the cyclic cubic subfield of  $N$ . Since  $K_3/\mathbb{Q}$  is ramified,  $N/M$  is ramified and if  $h_N = 1$ , then  $h_M = 1$  by point (6) of Proposition 1. Therefore,  $M$  cannot be equal to the narrow Hilbert class field of  $\mathbb{Q}(\sqrt{2 \cdot 257})$ . Note that  $N/k$  is cyclic of degree 24.

**Lemma 1.** *Let  $\chi$  be any one of the eight characters of order 24 associated with the cyclic extension  $N/k$ .*

- (1) *We have  $(\zeta_N/\zeta_K)(s) \geq 0$  in the range  $0 < s < 1$ .*
- (2) *For each given  $M$  with  $h_M = 1$  we can compute a bound  $N_{k/\mathbb{Q}}(\mathfrak{F}) \leq C$  on the norms of the conductors  $\mathfrak{F}$  of the cyclic cubic extensions  $kK_3/k$  for the  $N$ 's such that  $h_N^- = 1$ . These bounds are listed in Table 1.*
- (3) *Assume that  $h_N = 1$ . Then  $\mathfrak{F} = (l)$  for some prime  $l$  which splits in  $k$ , or  $\mathfrak{F} = \mathfrak{B}_l$  for some prime ideal  $\mathfrak{B}_l$  of  $k$  above a prime  $l$  ramified in  $k$ .*
- (4)  *$h_M^-$  divides  $h_N^-$ ,  $L(0, \chi) \in \mathbb{Q}(\sqrt{2}, \sqrt{-3})$ , and  $h_N^-/h_M^- = N_{\mathbb{Q}(\sqrt{2}, \sqrt{-3})}(\frac{1}{4}L(0, \chi))^2$  is a perfect square which can be computed using the techniques developed in [Lou5] and [Lou6].*

*Proof.* (1) It follows from  $(\zeta_N/\zeta_K)(s) = \prod_{i=1}^{11} |L(s, \chi^i)|^2$ .

- (2) We have verified that for the above four  $M$ 's,  $\zeta_K(s) \leq 0$  in the range  $0 < s < 1$ . Hence,  $\zeta_N(s) \leq 0$  for  $0 < s < 1$ . Using [Lou2, Lemma 12 and Proposition 13] we compute explicitly  $\mu_k \text{Res}_{s=1}(\zeta_k)$  and apply Proposition 2 to get lower bound for  $h_N^-$ . Since  $M/M^+$  is unramified at all finite places and  $Q_M = \omega_M = 2$ ,  $N/N^+$  is unramified at all finite places,  $d_{N^+} = d_k^{12} N_{k/\mathbb{Q}}(\mathfrak{F})^8$ , and  $Q_N = \omega_N = 2$ . From this lower bound for  $h_N^-$  we obtain the upper bounds  $C$  on  $N_{k/\mathbb{Q}}(\mathfrak{F})$  such that  $h_N^- = 1$  implies  $N_{k/\mathbb{Q}}(\mathfrak{F}) \leq C$ .
- (3) If the number of ramified primes in  $K_3/\mathbb{Q}$  is greater than one, then 3 divides  $h_{K_3}$ , whence  $3|h_{N^+}$  by Proposition 1(6). If there is a prime divisor  $l$  of  $N_{k/\mathbb{Q}}(\mathfrak{F})$  which is inert in  $k$ , then  $3^4$  divides  $h_N^-$ . Since  $M$  is the narrow Hilbert class field of  $k$ ,  $(l)$  splits completely in  $M/k$ , whence there are at least 4 prime ideals ramified in  $N^+/M^+$  which split at the same time in  $M/M^+$ . Hence,  $3^4|h_N^-$  by [LOO, Proposition 8].
- (4) According to [Lou6], the value  $L(0, \chi)$  is an algebraic integer of  $\mathbb{Q}(\zeta_{24})$  and

$$h_N^-/h_M^- = N_{\mathbb{Q}(\zeta_{24})/\mathbb{Q}}(\frac{1}{4}L(0, \chi)).$$

Let  $\text{Gal}(N/\mathbb{Q}) = \langle a, b, c | a^3 = b^8 = c^2 = 1, b^{-1}ab = a, c^{-1}ac = a, c^{-1}bc = b^{-1} \rangle$ , where  $\text{Gal}(N/k) = \langle a, b \rangle$ . The restriction of  $c$  to  $k$  generates  $\text{Gal}(k/\mathbb{Q})$  and using Artin's reciprocity theorem we obtain that  $\chi \circ c = \chi^7$  and

$$\sigma_7(L(0, \chi)) = L(0, \chi^7) = L(0, \chi \circ c) = L(0, \chi).$$

Therefore,  $L(0, \chi) \in \mathbb{Q}(\sqrt{2}, \sqrt{-3})$ , the subfield of  $\mathbb{Q}(\zeta_{24})$  fixed by  $\sigma_7$ . It follows that

$$h_N^-/h_M^- = (N_{\mathbb{Q}(\sqrt{2}, \sqrt{-3})/\mathbb{Q}}(\frac{1}{4}L(0, \chi)))^2$$

is a perfect square. □

We verify that  $h_N^- > 1$  for all fields  $N$  satisfying points (2) and (3) in Lemma 1, and sum up the computational results in Table 1.

TABLE 1.

$k$	$\mu_k \operatorname{Res}_{s=1}(\zeta_k) \leq$	$N_{k/\mathbb{Q}}(\mathfrak{F}) \leq$	$\mathfrak{F}$	$h_N^-$
$\mathbb{Q}(\sqrt{5 \cdot 101})$	2.160235	36000	(7)	$193^2$
			(9)	$463^2$
			(19)	$9337^2$
			(31)	$49252^2$
			(67)	$4540144^2$
			(73)	$1987279^2$
			(79)	$3111696^2$
			(103)	$8589721^2$
			(127)	$51439012^2$
			(163)	$180487972^2$
(181)	$330579292^2$			
$\mathbb{Q}(\sqrt{5 \cdot 181})$	2.154802	4700	(7)	$964^2$
			$\mathfrak{P}_{181}$	$4032^2$
$\mathbb{Q}(\sqrt{13 \cdot 53})$	1.976174	6000	$\mathfrak{P}_{13}$	$28^2$
			(19)	$23863^2$
			(31)	$130729^2$
			(43)	$538657^2$
			(67)	$8539888^2$
(73)	$11762791^2$			
$\mathbb{Q}(\sqrt{13 \cdot 61})$	2.531191	18000	$\mathfrak{P}_{13}$	$63^2$
			(7)	$964^2$
			(9)	$1519^2$
			(31)	$189724^2$
			(37)	$540127^2$
			$\mathfrak{P}_{61}$	$487^2$
			(67)	$4314289^2$
			(103)	$22194844^2$
(127)	$73351108^2$			

4.1.2.  $\operatorname{Gal}(N/\mathbb{Q}) = (D_8 \times C_3) \rtimes^1 C_2$ . The field  $N$  has three non-normal cubic subfields. Let  $K_3$  be any one of them,  $K_6$  its normal closure, and  $k_2$  the quadratic subfield of  $K_6$ . Since if  $\operatorname{Gal}(M/k_2) = C_8$ , then  $\operatorname{Gal}(N/\mathbb{Q}) = D_{48}$  and  $h_N^- > 1$ . It follows that  $k_2 = \mathbb{Q}(\sqrt{p})$  or  $\mathbb{Q}(\sqrt{q})$ , and  $\operatorname{Gal}(M/k_2) = D_8$ .

- Lemma 2.** (1) We have  $(\zeta_N/\zeta_K)(s) \geq 0$  in the range  $0 < s < 1$ .  
 (2) There exists some positive integer  $f \geq 1$  such that  $\mathfrak{F}_{K_6/k_2} = (f)$ . For each given  $M$  with  $h_M^- = 1$  we can compute a bound  $f \leq C$  on the conductors  $(f)$  of the cyclic cubic extensions  $K_6/k_2$  for the  $N$ 's such that  $h_N^- = 1$ . These bounds and the possible  $f$ 's are given in Table 2.  
 (3) Let  $\chi$  be any one of the four characters of order 12 associated with the cyclic extension  $N/K$ . Then  $h_M^-$  divides  $h_N^-$ ,  $L(0, \chi) \in \mathbb{Q}$ , and  $h_N^-/h_M^- = (L(0, \chi)/16)^4$  is a perfect fourth power which can be computed by using the techniques developed in [Lou5] and [Lou6].

TABLE 2.

$K$	$k_2$	$\mu_{k_2} \operatorname{Res}_{s=1}(\zeta_{k_2}) <$	$f <$	$f$	$h_N^-$
$\mathbb{Q}(\sqrt{2}, \sqrt{257})$	$\mathbb{Q}(\sqrt{257})$			1	$2^4$
$\mathbb{Q}(\sqrt{5}, \sqrt{101})$	$\mathbb{Q}(\sqrt{5})$	0.043633	13	None	
	$\mathbb{Q}(\sqrt{101})$	0.562340	22	2	1
$\mathbb{Q}(\sqrt{5}, \sqrt{181})$	$\mathbb{Q}(\sqrt{5})$		105	None	
	$\mathbb{Q}(\sqrt{181})$	0.855096	30	17	$166^4$
$\mathbb{Q}(\sqrt{13}, \sqrt{53})$	$\mathbb{Q}(\sqrt{13})$	0.146745	19	2·5	$35^4$
	$\mathbb{Q}(\sqrt{53})$	0.407728	52	2·5	$50^4$
				2·13	$31^4$
$\mathbb{Q}(\sqrt{13}, \sqrt{61})$	$\mathbb{Q}(\sqrt{13})$		130	11	$85^4$
				2·61	$175^4$
	$\mathbb{Q}(\sqrt{61})$	0.487910	95	13	$6^4$
				2·9	$142^4$
				2·11	$233^4$

- Proof.* (1) Since  $N/K$  and  $M^+/k$  are cyclic of degree 12 and 4, respectively, then as in point (1) of Lemma 1 we obtain  $(\zeta_N/\zeta_{M^+})(s) \geq 0$  and  $(\zeta_{M^+}/\zeta_K)(s) \geq 0$  for  $0 < s < 1$ .
- (2) The first part follows from [Mar, Theorem III.1] or [LPL, Theorem 4]. For  $K = \mathbb{Q}(\sqrt{2}, \sqrt{257})$  we have verified that  $\zeta_K(s) \leq 0$  in the range  $0 < s < 1$ . Hence,  $\zeta_N(s) \leq 0$  for  $0 < s < 1$  for every  $M$  with  $h_M^- = 1$ . Since  $N^+/k_2$  is abelian and  $d_{N^+}/d_{k_2}^{12} = f^{16}$ , using Proposition 2 we get upper bound  $C$  on  $f$  such that  $h_N^- = 1$  implies  $f \leq C$ . To alleviate the list of possible conductors  $f$  we use the same argument as in point (3) of Lemma 1: If there is a prime divisor  $l$  of  $f$  which is inert in  $\mathbb{Q}(\sqrt{pq})$ , then  $3^4$  divides  $h_N^-$ .
- (3) Let  $K_{12}$  be the compositum of  $K$  and  $K_6$ . We have

$$h_N^-/h_M^- = N_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}(L(0, \chi)/2^4).$$

Assume  $\operatorname{Gal}(N/K) = \langle a, b^2 \rangle$ . Let  $\chi_-$  be any one of two quartic characters associated with the cyclic extension  $M/K$  and  $\chi_+$  any one of two cubic characters associated with the cyclic extension  $K_{12}/K$  such that  $\chi = \chi_- \chi_+$ . Using the Artin reciprocity theorem, it can be easily verified that  $\chi_- \circ b = \chi_-$ ,  $\chi_- \circ c = \chi_-^{-1}$ ,  $\chi_+ \circ b = \chi_+^{-1}$ , and  $\chi_+ \circ c = \chi_+$ , whence  $\chi \circ b = \chi^5$  and  $\chi \circ c = \chi^7$ . For a positive integer  $n$  let  $\sigma_n \in \operatorname{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$  with  $\sigma_n(\zeta_{12}) = \zeta_{12}^n$ . We have

$$\sigma_5(L(0, \chi)) = L(0, \chi^5) = L(0, \chi \circ b) = L(0, \chi)$$

and  $\sigma_7(L(0, \chi)) = L(0, \chi)$ . Since  $\langle \sigma_5, \sigma_7 \rangle = \operatorname{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q})$  we have  $L(0, \chi) \in \mathbb{Q}$ , whence  $h_N^-/h_M^-$  is the 4-th power of some integer. □

Our computational results are given in Table 2. When  $K = \mathbb{Q}(\sqrt{2}, \sqrt{257})$ , if  $h_N = 1$ , then  $N/M$ ,  $N^+/M^+$ , and  $K_6/k_2$  are unramified. Otherwise,  $h_N \equiv 0 \pmod{3}$ . Since  $\mathbb{Q}(\sqrt{2})$  has class number one, we must have  $k_2 = \mathbb{Q}(\sqrt{257})$  and  $(f) = 1$ . Note that when  $K = \mathbb{Q}(\sqrt{5}, \sqrt{101})$ ,  $k_2 = \mathbb{Q}(\sqrt{101})$ , and  $f = 2$ , we have  $K_6 = \mathbb{Q}(\sqrt{101}, \theta)$  with  $\theta^3 - \theta^2 - 5\theta - 1 = 0$ . Using KASH ([K]) we verify that the class group of  $\mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$  is isomorphic to  $C_2$  and the narrow class group of this field is isomorphic to  $C_4$ . It follows that  $N^+$  is the Hilbert class field of  $\mathbb{Q}(\sqrt{5}, \sqrt{101}, \theta)$  and  $N$  is the narrow Hilbert class field of this field. In addition, thanks to KASH we verify that the class number of  $N^+$  is equal to 1.

4.2.  $\text{Gal}(M/\mathbb{Q}) = Q_8 \times C_2$ . By [Lou3, Theorem 1],

$$M = \mathbb{Q} \left( \sqrt{-1}, \sqrt{2}, \sqrt{3}, \sqrt{-(2 + \sqrt{2})(3 + \sqrt{3})} \right)$$

is the only normal CM-field of relative class number one with Galois group isomorphic to  $Q_8 \times C_2$ . This field has class number one and  $Q_M = 2$ . In this subsection we assume that  $N$  contains this field  $M$  and will prove that  $h_N > 1$ . The Galois group  $\text{Gal}(N/\mathbb{Q})$  is isomorphic to either  $Q_8 \times C_2 \times C_3$  or  $Q_{24} \times C_2$  according to whether  $N$  has a cyclic cubic subfield or not.

4.2.1.  $\text{Gal}(N/\mathbb{Q}) = Q_8 \times C_2 \times C_3$ . The field  $N$  has only one cyclic cubic subfield  $K_3$ . The composita

$$N_1 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-(2 + \sqrt{2})(3 + \sqrt{3})}) \quad \text{and} \quad N_2 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-1})$$

have the same maximal real subfield  $K_3(\sqrt{2}, \sqrt{3})$ . Suppose that  $h_N = 1$ . By Proposition 1(7) we would have  $h_{N_1}^- = 1$  or  $h_{N_2}^- = 1$ . Since every octic quaternion CM-field has an even relative class number,  $h_{N_1}^-$  is even. Using [CK1] we verify that there is no imaginary abelian number field with Galois group isomorphic to  $C_2 \times C_2 \times C_2 \times C_3$  of relative class number one which contains the field  $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{-1})$ . Hence  $h_N > 1$ .

4.2.2.  $\text{Gal}(N/\mathbb{Q}) = Q_{24} \times C_2$ . The field  $N$  contains a non-normal cubic subfield  $K_3$ . The compositum

$$N_1 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-(2 + \sqrt{2})(3 + \sqrt{3})})$$

is a normal CM-field with Galois group isomorphic to  $Q_{24}$ . The compositum  $N_2 = K_3(\sqrt{2}, \sqrt{3}, \sqrt{-1})$  is a normal CM-field with Galois group isomorphic to  $D_{12} \times C_2$ . We have  $N = N_1N_2$  with  $N_1^+ = N_2^+ = K_3(\sqrt{2}, \sqrt{3})$ . Note that  $h_{N_1}^-$  is even. According to [P, Theorem 1],  $h_{N_2} > 1$ . By Proposition 1(7) it follows that  $h_N > 1$ .

### 5. CASE 3: $\text{Gal}(M/\mathbb{Q}) \in \{G_9, G_6\}$

In subsection 5.1 we assume that  $G(M/\mathbb{Q}) = G_9$ , and in subsection 5.2 we assume that  $G(M/\mathbb{Q}) = G_6$ .

5.1.  $\text{Gal}(M/\mathbb{Q}) = G_9$ . There is only one normal CM-field  $M$  of relative class number one with Galois group isomorphic to  $G_9$  ([LO2, Theorem 20]):

$$M = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{37}, \sqrt{-(2\sqrt{2} + 3\sqrt{5})(2 + \sqrt{5})}).$$

Assume that  $N$  contains  $M$ . The aim of this subsection is to prove that  $h_N > 1$ . Note that  $\text{Gal}(M/\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})) = Q_8$ ,  $\text{Gal}(M/\mathbb{Q}(\sqrt{2})) = \text{Gal}(M/\mathbb{Q}(\sqrt{5})) = \text{Gal}(M/\mathbb{Q}(\sqrt{37})) = D_8$ , and  $\text{Gal}(M/\mathbb{Q}(\sqrt{2 \cdot 5})) = \text{Gal}(M/\mathbb{Q}(\sqrt{2 \cdot 37})) = \text{Gal}(M/\mathbb{Q}(\sqrt{5 \cdot 37})) = C_4 \times C_2$ . Therefore,  $\text{Gal}(N/\mathbb{Q})$  is isomorphic to  $G_9 \times C_3$  if  $N$  contains only one cubic cyclic subfield  $K_3$ ,  $(Q_8 \times C_3) \rtimes C_2$ ,  $(D_8 \times C_3) \rtimes^2 C_2 = \langle a, b, c \mid a^3 = b^2 = c^2 = z^4 = 1, c^{-1}bc = bz^2, bz = zb, cz = zc, b^{-1}ab = a, c^{-1}ac = a, z^{-1}az = a^{-1} \rangle$ , or  $(C_4 \times C_2 \times C_3) \rtimes C_2$  otherwise. We divide this subsection into four parts according to  $\text{Gal}(N/\mathbb{Q})$ .

5.1.1.  $\text{Gal}(N/\mathbb{Q}) = G_9 \times C_3$ . We will show that  $h_N > 1$ . We first get an upper bound  $C$  on the conductor of  $K_3/\mathbb{Q}$  such that if  $h_N^- = 1$ , then the conductor is less than or equal to  $C$ . Let  $K = \mathbb{Q}(\sqrt{2}, \sqrt{5 \cdot 37})$ . Since  $N/K$  is cyclic of degree 12, then as in point (1) of Lemma 1 we obtain  $(\zeta_N/\zeta_{M^+})(s) \geq 0$  for  $0 < s < 1$ . We verify that  $\zeta_{M^+}(s) \leq 0$  for  $s \in ]0, 1[$ , which implies  $\zeta_N(s) \leq 0$  for  $s \in ]0, 1[$ . Let  $F = \mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})$ . The extension  $N^+/F$  is abelian of degree 12 and  $d_{N^+}/d_F^{12} = N_{F/\mathbb{Q}}(\mathfrak{F}_{FK_3/F})^8$ . Using  $\mu_F \text{Res}_{s=1}(\zeta_F) \leq 2.227842$  and Proposition 2, we obtain that if  $h_N^- \leq 1$ , then  $N_{F/\mathbb{Q}}(\mathfrak{F}_{FK_3/F}) \leq 1300$ , whence  $\mathfrak{F}_{FK_3/F} \in \{(7), (3)^2, (13), (19), (31), \mathfrak{P}_{37}\}$  with  $(37) = \mathfrak{P}_{37}^2$  in  $F$ .

- (i) When  $\mathfrak{F}_{FK_3/F} = (7)$ ,  $K_3$  is of conductor 7. There are four non-normal octic CM-fields containing  $\mathbb{Q}(\sqrt{2})$  with relative class number one. Let  $M_{8,1} = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{-(2\sqrt{2} + 3\sqrt{5})(2 + \sqrt{5})})$  and  $M_{8,2}$  its conjugate over  $\mathbb{Q}$ . Set  $N_{24,1} = M_{8,1}K_3$  and  $N_{24,2} = M_{8,2}K_3$ . By Proposition 1(5)  $h_N^- = (h_{N_{24,1}}^-)^2$ . The two prime ideals lying above 7 in  $M_{8,1}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{5})$  split in  $M_{8,1}$ . Hence,  $3^2|h_{N_{24,1}}^-$  by [LOO, Proposition 8] and  $3^4|h_N^-$ .
- (ii) When  $\mathfrak{F}_{FK_3/F} = (3)^2$ ,  $K_3$  is of conductor 9. Let  $M_{8,1}, M_{8,2}, N_{24,1}$ , and  $N_{24,2}$  be as in (i). One prime ideal lying above 37 in  $\mathbb{Q}(\sqrt{2}, \sqrt{5})$  is ramified in  $M_{8,1}$  and 37 splits in  $K_3$ , whence  $2^2|h_{N_{24,1}}^-$  and  $2^4|h_N^-$ . By the same argument we prove that if  $\mathfrak{F}_{FK_3/F} = (19)$ , then  $2^4|h_N^-$ .
- (iii) When  $\mathfrak{F}_{FK_3/F} = (13)$ ,  $K_3$  is of conductor 13. Let  $M_{8,1}$  be any one of two non-normal octic CM-fields of  $M$  containing  $\mathbb{Q}(\sqrt{2}, \sqrt{37})$  and  $M_{8,2}$  its conjugate over  $\mathbb{Q}$ . Let  $N_{24,1} = M_{8,1}K_3$  and  $N_{24,2} = M_{8,2}K_3$ . One prime ideal lying above 5 in  $\mathbb{Q}(\sqrt{2}, \sqrt{37})$  is ramified in  $M_{8,1}$  and 5 splits in  $K_3$ , which implies  $2^2|h_{N_{24,1}}^-$  and  $2^4|h_N^-$ .
- (iv) When  $\mathfrak{F}_{FK_3/F} = (31)$ ,  $K_3$  is of conductor 31. We let  $M_{8,1}$  be any one of two non-normal octic subfields of  $M$  containing  $\mathbb{Q}(\sqrt{5}, \sqrt{37})$  and  $M_{8,2}$  its conjugate over  $\mathbb{Q}$ . One prime ideal lying above 2 is ramified in  $M_{8,1}$  and 2 splits in  $K_3$ , whence  $2^2|h_{N_{24,1}}^-$  and  $2^4|h_N^-$ .
- (v) When  $\mathfrak{F}_{FK_3/F} = \mathfrak{P}_{37}$ ,  $K_3$  is of conductor 37. According to [Ma], the cyclic sextic subfield  $K_3(\sqrt{2 \cdot 5})$  of  $N$  has class number 6, whence by Proposition 1(6)  $3|h_N$ .

5.1.2.  $\text{Gal}(N/\mathbb{Q}) = (D_8 \times C_3) \rtimes^2 C_2$ . Let  $K_3$  be any one of three non-normal cubic subfields of  $N$ ,  $K_6$  its normal closure, and  $k_2$  the quadratic subfield of  $K_6$ . We have  $\text{Gal}(M/k_2) = D_8$ . Let  $K$  be the intermediate field between  $M^+$  and  $k_2$  such that  $G(M/K) = C_4$ . Then  $M/K$  is unramified at all finite primes,  $\text{Gal}(M^+/k_2) = C_2 \times C_2$ ,  $\text{Gal}(N/K) = C_{12}$ , and  $\text{Gal}(N^+/k_2) = C_2 \times C_2 \times C_3$ . As in point (1) of

TABLE 3.

$k$	$f \leq$	$f$	$h_N^-$
$\mathbb{Q}(\sqrt{2})$	22	None	
$\mathbb{Q}(\sqrt{5})$	8	None	
$\mathbb{Q}(\sqrt{37})$	44	2	$2^4$

TABLE 4.

$k_2$	$C$	$f$	$h_N^-$
$\mathbb{Q}(\sqrt{2 \cdot 5})$	140	37	$1021^2$
$\mathbb{Q}(\sqrt{2 \cdot 37})$	21	9	$5044^2$
$\mathbb{Q}(\sqrt{5 \cdot 37})$	35	None	

Lemma 1 we obtain  $(\zeta_N/\zeta_{M^+})(s) \geq 0$  and  $\zeta_N(s) \leq 0$  for  $s \in ]0, 1[$ . Since  $N^+/k_2$  is abelian of degree 12 and  $d_{N^+}/d_{k_2}^{12} = f^{16} N_{K_6/\mathbb{Q}}(\mathfrak{D}_{N^+/K_6})$ , where  $(f) = \mathfrak{F}_{K_6/k_2}$  and  $\mathfrak{D}_{N^+/K_6}$  denotes the discriminant of the extension  $N^+/K_6$ , using Proposition 2 we obtain upper bound  $C$  on  $f$  such that if  $h_N^- = 1$ , then  $f \leq C$ . Our computational results are given in Table 3. As in point (3) of Lemma 2 we can easily verify that  $h_N^-/h_M^-$  is the 4-th power of some rational integer.

5.1.3.  $\text{Gal}(N/\mathbb{Q}) = (C_4 \times C_2 \times C_3) \rtimes C_2 = \langle a, b, c, z \mid a^3 = b^2 = c^2 = z^4 = 1, c^{-1}bc = bz^2, bz = zb, cz = zc, b^{-1}ab = a, c^{-1}ac = a^{-1}, z^{-1}az = a \rangle$ . In this case  $N$  has three non-normal cubic subfields and its associated quadratic subfield  $k_2$  has  $\text{Gal}(M/k_2) = C_4 \times C_2$ . For a fixed field  $k_2$  there are two intermediate fields  $K$  between  $k_2$  and  $M^+$  such that  $\text{Gal}(M/K) = C_4$ . Let  $K$  be any one of these two fields. Then  $M/K$  is unramified at all finite primes,  $\text{Gal}(N/K) = C_{12}$ ,  $\text{Gal}(M^+/k_2) = C_2 \times C_2$ , and  $\text{Gal}(N^+/k_2) = C_2 \times C_2 \times C_3$ . Analogously as in subsection 5.1.2 we get upper bound  $C$  on  $f$  such that if  $h_N^- = 1$ , then  $f \leq C$ . Since  $\text{Gal}(N/k_2) = C_4 \times C_2 \times C_3$ , we compute  $h_N^-$  using Hecke  $L$ -functions over  $k_2$  (see Table 4).

5.1.4.  $\text{Gal}(N/\mathbb{Q}) = (Q_8 \times C_3) \rtimes C_2$ . In this case  $N$  has three non-normal cubic subfields such that its associated quadratic subfield  $k_2$  is equal to  $\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})$ . Let  $K$  be any one of three quartic fields containing  $\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})$ . Then  $\text{Gal}(N/K) = C_{12}$  and  $\text{Gal}(N^+/k_2) = C_2 \times C_2 \times C_3$ . As in subsection 5.1.2 we verify that if  $h_N^- = 1$ , then  $f \leq 36$  with  $\mathfrak{F}_{K_6/k_2} = (f)$ . There is no sextic field  $K_6$  containing  $\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 37})$  with  $f \leq 36$ . Therefore,  $h_N^- > 1$ .

5.2.  $\text{Gal}(M/\mathbb{Q}) = G_6$ . In [Lou3] it is proved that there are exactly two such fields  $M$  with  $h_M = 1$ : the composita  $M = M_1 M_2$  listed in Table 5. In fact, those are the only fields with Galois group isomorphic to  $G_6$  of relative class number one. The Galois group  $\text{Gal}(M^+/\mathbb{Q})$  is isomorphic to  $D_8$  or  $C_4 \times C_2$ . When  $\text{Gal}(M^+/\mathbb{Q}) = D_8$ ,  $M$  is a compositum of an octic dihedral CM-field  $M_{8,1}$  and an imaginary abelian number field  $M_{8,2}$  with  $\text{Gal}(M_{8,2}/\mathbb{Q}) = C_4 \times C_2$ . Using [YK] and [CK1], we verify

TABLE 5.

$M_{8,1}$	$M_{8,2}$	$h_{M_{8,1}}^-$	$h_{M_{8,2}}^-$
$\mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(5 + \sqrt{17})/2})$ $\text{Gal}(M_{8,1}/\mathbb{Q}) = D_8$ $\text{Gal}(M_{8,1}/\mathbb{Q}(\sqrt{34})) = C_4$	$\mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{-(17 + 3\sqrt{17})/2})$ $\text{Gal}(M_{8,2}/\mathbb{Q}) = D_8$ $\text{Gal}(M_{8,2}/\mathbb{Q}(\sqrt{2})) = C_4$	1	2
$\mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{-(9 + \sqrt{13})/2})$ $\text{Gal}(M_{8,1}/\mathbb{Q}) = D_8$ $\text{Gal}(M_{8,1}/\mathbb{Q}(\sqrt{221})) = C_4$	$\mathbb{Q}(\sqrt{17}, \sqrt{-(13 + 2\sqrt{13})})$ $\text{Gal}(M_{8,2}/\mathbb{Q}) = C_4 \times C_2$ $\text{Gal}(\mathbb{Q}(\sqrt{-(13 + 2\sqrt{13})})/\mathbb{Q}) = C_4$	1	2

that there is only one such field  $M$  with  $h_M^- = 1$ : the second field in Table 5. When  $\text{Gal}(M^+/\mathbb{Q}) = C_4 \times C_2$ ,  $M$  is a compositum of two octic dihedral CM-fields  $M_{8,1}$  and  $M_{8,2}$  with  $M_{8,1}^+ = M_{8,2}^+$ . According to [Lou3, Theorem 2] there is only one such field  $M$  with  $h_M^- = 1$ : the first field in Table 5. We will prove that  $h_N > 1$ . If  $K_3/\mathbb{Q}$  is normal, then  $\text{Gal}(N/\mathbb{Q}) = G_6 \times C_3$ . Otherwise,  $\text{Gal}(N/\mathbb{Q}) = C_3 \rtimes G_6$ . Let  $N_1 = M_{8,1}K_3$  and  $N_2 = M_{8,2}K_3$ . Then  $N = N_1N_2$  with  $N_1^+ = N_2^+ = K_3K_4$ .

5.2.1.  $\text{Gal}(N/\mathbb{Q}) = G_6 \times C_3$ . If  $\text{Gal}(M^+/\mathbb{Q}) = C_4 \times C_2$ , then  $\text{Gal}(N_1/\mathbb{Q}) = D_8 \times C_3 = \text{Gal}(N_2/\mathbb{Q})$ . According to [P, Theorem 1],  $h_{N_1}^- > 1$  and  $h_{N_2}^- > 1$ , whence  $h_N > 1$ . If  $\text{Gal}(M^+/\mathbb{Q}) = D_8$ , then  $\text{Gal}(N_1/\mathbb{Q}) = D_8 \times C_3$ , and  $\text{Gal}(N_2/\mathbb{Q}) = C_4 \times C_2 \times C_3$ . Using [CK1], we verify that  $h_{N_2}^- > 4$ , whence  $h_N^- > 1$ .

5.2.2.  $\text{Gal}(N/\mathbb{Q}) = C_3 \rtimes G_6$ .

- (a) Let  $M$  be the first field in Table 5. The quadratic field  $k_2$  associated with  $K_3$  is either  $\mathbb{Q}(\sqrt{34})$ ,  $\mathbb{Q}(\sqrt{2})$ , or  $\mathbb{Q}(\sqrt{17})$ . If  $k_2 = \mathbb{Q}(\sqrt{34})$ , then  $\text{Gal}(N_1/\mathbb{Q}) = D_{24}$ , and  $\text{Gal}(N_2/\mathbb{Q}) = C_3 \rtimes D_8$ . If  $k_2 = \mathbb{Q}(\sqrt{2})$ , then  $\text{Gal}(N_1/\mathbb{Q}) = C_3 \rtimes D_8$ , and  $\text{Gal}(N_2/\mathbb{Q}) = D_{24}$ . If  $k_2 = \mathbb{Q}(\sqrt{17})$ , then  $\text{Gal}(N_1/\mathbb{Q}) = C_3 \rtimes D_8 = \text{Gal}(N_2/\mathbb{Q})$ . According to [P, Theorem 1] and [Lef, Theorem 4.1], we have  $h_{N_1}^- > 1$  and  $h_{N_2}^- > 1$ , whence  $h_N > 1$ .
- (b) Let  $M$  be the second field in Table 5. The quadratic field  $k_2$  associated with  $K_3$  is either  $\mathbb{Q}(\sqrt{221})$ ,  $\mathbb{Q}(\sqrt{13})$ , or  $\mathbb{Q}(\sqrt{17})$ . If  $k_2 = \mathbb{Q}(\sqrt{221})$ , then  $\text{Gal}(N_1/\mathbb{Q}) = D_{24}$ , and  $\text{Gal}(N_2/\mathbb{Q}) = S_3 \times C_4$ . If  $k_2 = \mathbb{Q}(\sqrt{13})$ , then  $\text{Gal}(N_1/\mathbb{Q}) = C_3 \rtimes D_8$  and  $\text{Gal}(N_2/\mathbb{Q}) = Q_{12} \times C_2$ . If  $k_2 = \mathbb{Q}(\sqrt{17})$ , then  $\text{Gal}(N_1/\mathbb{Q}) = C_3 \rtimes D_8$ , and  $\text{Gal}(N_2/\mathbb{Q}) = S_3 \times C_4$ . Using [Lef, Theorem 4.1] and [P, Theorem 1], we have  $h_{N_1}^- > 1$  and  $h_{N_2}^- > 1$ , whence  $h_N > 1$ .

6. CASE 4:  $\text{Gal}(M/\mathbb{Q}) = D_8 \times C_2$

To begin with, we prove the following:

**Theorem 2** (Compare with [Lou3, Theorems 2 and 3]). *There are four normal CM-fields  $M$  of degree 16 and Galois group  $D_8 \times C_2$  with relative class number one, those given in the following Table 6.*

TABLE 6.

$M^+$	$\alpha : M = M^+(\sqrt{-\alpha})$	$Q_M$	$\omega_M$
$\mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{17})$	$5(5 + \sqrt{17})/2$	2	2
$\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{3 + \sqrt{3}})$	1	2	24
$\mathbb{Q}(\sqrt{3}, \sqrt{11}, \sqrt{15 + 8\sqrt{3}})$	1	2	12
$\mathbb{Q}(\sqrt{2}, \sqrt{17}, \sqrt{3(5 + \sqrt{17})/2})$	3	2	6

*Proof.* The Galois group  $\text{Gal}(M^+/\mathbb{Q})$  is isomorphic to either  $D_8$  or  $C_2 \times C_2 \times C_2$ . When  $\text{Gal}(M^+/\mathbb{Q}) = D_8$ ,  $M$  is a compositum of an octic dihedral CM-field  $M_{8,1}$  and an imaginary abelian number field  $M_{8,2}$  with  $\text{Gal}(M_{8,2}/\mathbb{Q}) = C_2 \times C_2 \times C_2$ . Let  $L$  be any one of four non-normal quartic CM-subfields of  $M_{8,1}$ . According to [Lou3, Proposition 16],  $h_M^- = 1$  if and only if  $(h_L^-, h_{M_{8,2}}^-) \in \{(1, 4), (1, 2), (2, 1)\}$ . In the case that  $(h_L^-, h_{M_{8,2}}^-) = (1, 4)$  or  $(1, 2)$ ,  $M_{8,1}^+ = M_{8,2}^+$  is of the form  $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ , where  $(p, q)$  is one of the 19 pairs given in [LO1, Theorem 8]. Then  $M_{8,2} = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{-m})$ , where  $\mathbb{Q}(\sqrt{-m})$  is one of 81 (= 9 + 18 + 54) imaginary quadratic fields of class number one, two or four. For these 1539 (= 19 × 81) CM-fields  $M_{8,2}$  we compute  $h_{M_{8,2}}^-$  and verify that there is only one field  $M$  with  $h_M^- = 1$ : the fourth field in Table 6. In the case that  $(h_L^-, h_{M_{8,2}}^-) = (2, 1)$ , using [YK] and [CK1] we verify that there are exactly two fields  $M$  with  $h_M^- = 1$ : the second and third fields in Table 6. When  $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$ ,  $M$  is a compositum of two octic dihedral CM-fields  $M_{8,1}$  and  $M_{8,2}$ . According to [Lou3, Theorem 2], there is only one such  $M$  with  $h_M^- = 1$ : the first field in Table 6.  $\square$

From now on we assume that  $M$  is one of these four fields and we will prove that  $h_N > 1$ . We classify the Galois group  $\text{Gal}(N/\mathbb{Q})$ . Let  $K_3$  be any cubic subfield of  $N$ . If  $K_3$  is not normal, then its normal closure  $K_6$  is a dihedral real sextic field and we let  $k_2$  denote the (real) quadratic subfield of  $K_6$ . If  $K_3$  is normal, then  $\text{Gal}(N/\mathbb{Q}) = D_8 \times C_6$ . If  $K_3$  is not normal over  $\mathbb{Q}$ , then  $\text{Gal}(N/\mathbb{Q})$  is isomorphic to either  $D_8 \times S_3$ ,  $D_{24} \times C_2$ , or  $(C_3 \rtimes D_8) \times C_2$ . Let  $K_4 = M_{8,1}^+ = M_{8,2}^+$ . If  $K_4 \cap k_2 = \mathbb{Q}$ , then  $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$ , and  $\text{Gal}(N/\mathbb{Q}) = D_8 \times S_3$ . If  $K_4 \cap k_2 \supsetneq \mathbb{Q}$ , then  $M_{8,1}/k_2$  is either cyclic or biquadratic bicyclic. If  $M_{8,1}/k_2$  is cyclic quartic, then  $\text{Gal}(N/\mathbb{Q}) = D_{24} \times C_2$ . If  $M_{8,1}/k_2$  is biquadratic bicyclic, then  $\text{Gal}(K_3 M_{8,1}/\mathbb{Q}) = C_3 \rtimes D_8 = \langle a, b, c \mid a^3 = b^4 = c^2 = 1, b^{-1}ab = a^{-1}, c^{-1}ac = a, c^{-1}bc = b^{-1} \rangle$ , and  $\text{Gal}(N/\mathbb{Q}) = (C_3 \rtimes D_8) \times C_2$ .

6.1.  $\text{Gal}(N/\mathbb{Q}) = D_8 \times C_6$ . If  $\text{Gal}(M^+/\mathbb{Q}) = D_8$ , then the composita  $N_1 = M_{8,1}K_3$  and  $N_2 = M_{8,2}K_3$  are normal CM-subfields of  $N$  with the same maximal real subfields  $K_3K_4$ . By [P, Theorem 1], the compositum of the dihedral octic CM-field  $\mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{-(9 + \sqrt{13})/2})$  and of the cyclic cubic field of conductor 13 is the only normal CM-field of relative class number one with Galois group isomorphic to  $D_8 \times C_3$ . In Table 6 there is no field  $M$  containing  $\mathbb{Q}(\sqrt{13}, \sqrt{17})$ . It follows then that  $h_{N_1}^- > 1$ . According to [CK1], there are exactly two imaginary abelian number fields with Galois group isomorphic to  $C_2 \times C_2 \times C_2 \times C_3$  of relative class number one:  $F_7(\sqrt{-1}, \sqrt{-3}, \sqrt{-7})$  and  $F_7(\sqrt{-3}, \sqrt{-7}, \sqrt{-15})$ , where  $F_7$  denotes the cyclic

cubic field of conductor 7. In Table 6 there is no field  $M$  containing  $\mathbb{Q}(\sqrt{3}, \sqrt{7})$ , or  $\mathbb{Q}(\sqrt{5}, \sqrt{21})$ , whence  $h_{N_2}^- > 1$ . It follows that if  $\text{Gal}(M^+/\mathbb{Q}) = D_8$ , then  $h_N > 1$ . If  $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$ , then  $N_1 = M_{8,1}K_3$  and  $N_2 = M_{8,2}K_3$  are normal CM-fields with Galois group isomorphic to  $D_8 \times C_3$ . Note that  $M_{8,1}^+ = M_{8,2}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{17})$ . According to [P, Theorem 1],  $h_{N_1}^- > 1$  and  $h_{N_2}^- > 1$ , which implies  $h_N > 1$ .

6.2.  $\text{Gal}(N/\mathbb{Q}) = D_8 \times S_3$ . In this case  $N$  has three non-normal real cubic subfields. Let  $K_3, K_6, k_2$  and  $K_4$  be as above. We have that  $K_4 \cap k_2 = \mathbb{Q}$ ,  $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$ ,  $M$  is the first field in Table 6. In addition, we have  $K_4 = \mathbb{Q}(\sqrt{2}, \sqrt{17})$ , and  $k_2 = \mathbb{Q}(\sqrt{m})$  with  $m \in \{5, 2 \cdot 5, 5 \cdot 17, 2 \cdot 5 \cdot 17\}$ . Let  $(f)$  be the conductor of the extension  $K_6/k_2$  with  $f$  a positive integer.

- Lemma 3.**
1. We have  $\zeta_N(s) \leq 0$  for  $0 < s < 1$ .
  2. For each given  $k_2$  in the above we can compute a bound of  $f \leq C$  on the conductor  $(f)$  for  $N$ 's such that  $h_N^- = 1$ . These bounds and the possible  $f$ 's are compiled in Table 7.
  3. The quotient  $h_N^-/h_M^-$  is the perfect fourth power of some rational integer.

*Proof.* (1) Let  $\chi_{N/M^+}$  be any one of two characters associated with the cyclic sextic extension  $N/M^+$ . We have

$$\frac{\zeta_N(s)}{\zeta_{M^+}(s)} = \frac{\zeta_M(s)}{\zeta_{M^+}(s)} |L(s, \chi_{N/M^+})L(s, \chi_{N/M^+}^2)|^2$$

and

$$\frac{\zeta_M(s)}{\zeta_{M^+}(s)} = \frac{\zeta_{M_{8,1}}(s)}{\zeta_{K_4}(s)} \frac{\zeta_{M_{8,2}}(s)}{\zeta_{K_4}(s)} = L(s, \psi_1)^2 L(s, \psi_2)^2,$$

where  $\psi_i$  is the unique irreducible character of degree 2 of  $\text{Gal}(M_{8,i}/\mathbb{Q})$  the dihedral group of order 8, and  $L(s, \psi_i)$  denotes the Artin  $L$ -function associated with  $\psi_i$  for  $i = 1, 2$ . Since  $\psi_i$  is real valued,  $L(s, \psi_i)$  is on the real axis and  $L(s, \psi_i)^2 \geq 0$ . For  $M^+ = \mathbb{Q}(\sqrt{2}, \sqrt{5}, \sqrt{17})$  we have verified that  $\zeta_{M^+}(s) \leq 0$  for  $s \in ]0, 1[$ , whence  $\zeta_N(s) \leq 0$ .

- (2) Since  $M/M^+$  is unramified at all finite primes,  $N/N^+$  is unramified at all finite primes and  $d_N/d_{N^+} = d_{N^+} = d_{k_2}^{12} f^{16} N_{K_6/\mathbb{Q}}(\mathfrak{D}_{N^+/K_6})$ . Using Proposition 2, we get an upper bound on  $f$ . Since the prime ideals lying above 2 and those above 17 split in  $M/M^+$ , if  $(f, 2) > 1$  or  $(f, 17) > 1$ , then  $3^2$  divides  $h_N^-$  by [LOO, Proposition 8]. Note that the prime ideals lying above 13 split in  $M/M^+$ , whence the relative class number of the fourth field  $N$  in Table 7 is divisible by  $3^4$ .

TABLE 7.

$k_2$	$\mu_{k_2} \text{Res}_{s=1}(\zeta_{k_2}) \leq$	$f \leq$	$f$	$h_N^-$
$\mathbb{Q}(\sqrt{5})$	0.0436324	10	NONE	
$\mathbb{Q}(\sqrt{2 \cdot 5})$	0.4276490	41	37	$(920)^4$
$\mathbb{Q}(\sqrt{5 \cdot 17})$	0.5861712	31	9	$(44)^4$
$\mathbb{Q}(\sqrt{2 \cdot 5 \cdot 17})$	1.4062136	38	13	$3^4   h_N^-$

- (3) Note that  $M_{8,1}$  and  $M_{8,2}$  are cyclic over  $\mathbb{Q}(\sqrt{34})$ . Let  $K$  be the compositum of  $\mathbb{Q}(\sqrt{34})$  and  $k_2$ . Then  $\text{Gal}(N/K) = C_{12}$ . Let  $\chi$  be any one of the four characters of order 12 associated with the cyclic extension  $N/K$ . Similarly to point (3) of Lemma 2, we verify that  $L(0, \chi) \in \mathbb{Q}$  and  $h_N^-/h_M^- = (L(0, \chi)/2^4)^4$ .  $\square$

In conclusion, we have proved that every normal CM-field with Galois group isomorphic to  $D_8 \times S_3$  has class number greater than one. Our computational results are given in Table 7.

6.3.  $\text{Gal}(N/\mathbb{Q}) = D_{24} \times C_2$ . In this case  $N$  has three non-normal cubic fields and  $M_{8,1}/k_2$  is cyclic. Let  $N_1 = M_{8,1}K_3$  and  $N_2 = M_{8,2}K_3$ . Then we have  $N = N_1N_2$  with  $N_1^+ = N_2^+ = K_3K_4$ . If  $\text{Gal}(M^+/\mathbb{Q}) = D_8$ , then  $\text{Gal}(N_1/\mathbb{Q}) = D_{24}$ , and  $\text{Gal}(N_2/\mathbb{Q}) = S_3 \times C_2 \times C_2$ . If  $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$ , then  $\text{Gal}(N_1/\mathbb{Q}) = D_{24} = \text{Gal}(N_2/\mathbb{Q})$ . Using [Lef, Theorem 4.1] and [P, Theorem 1], we verify that in both cases  $h_{N_1}^- > 1$  and  $h_{N_2}^- > 1$ . It follows that the class number of a normal CM-field with Galois group isomorphic to  $D_{24} \times C_2$  is greater than one.

6.4.  $\text{Gal}(N/\mathbb{Q}) = (C_3 \times D_8) \times C_2$ . In this case  $N$  has three non-normal cubic fields and  $M_{8,1}/k_2$  is biquadratic bicyclic. Then the Galois group of the compositum  $N_1 = M_{8,1}K_3$  over  $\mathbb{Q}$  is isomorphic to  $C_3 \times D_8$ , whence  $h_{N_1}^- > 1$  ([P, Theorem 13]). If  $\text{Gal}(M^+/\mathbb{Q}) = D_8$ , then  $\text{Gal}(N_2/\mathbb{Q}) = S_3 \times C_2 \times C_2$ . If  $\text{Gal}(M^+/\mathbb{Q}) = C_2 \times C_2 \times C_2$ , then  $\text{Gal}(N_2/\mathbb{Q}) = C_3 \times D_8$ . By [P, Theorems 1 and 13]  $h_{N_2}^- > 1$ . Consequently, if  $\text{Gal}(N/\mathbb{Q}) = (C_3 \times D_8) \times C_2$ , then  $h_N > 1$ .

To conclude, Theorem 1 is now proved with completion.

All computations were carried out using Pari-Gp ([Pa]) and KASH ([K]).

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