MATHEMATICS OF COMPUTATION
Volume 72, Number 242, Pages 1019–1033
S 0025-5718(02)01461-8
Article electronically published on August 14, 2002

LINEAR QUINTUPLE-PRODUCT IDENTITIES

RICHARD BLECKSMITH AND JOHN BRILLHART

Dedicated to our longtime friend John Selfridge

ABSTRACT. In the first part of this paper, series and product representations of four single-variable triple products T_0, T_1, T_2, T_3 and four single-variable quintuple products Q_0, Q_1, Q_2, Q_3 are defined. Reduced forms and reduction formulas for these eight functions are given, along with formulas which connect them. The second part of the paper contains a systematic computer search for linear trinomial Q identities. The complete set of such families is found to consist of two 2-parameter families, which are proved using the formulas in the first part of the paper.

1. INTRODUCTION

It is our purpose in this paper to find all identities of the form

$$Q(k,l) = Q(a,b) \pm x^f Q(c,d), \tag{1}$$

where Q is the series that arises in the quintuple-product identity,

$$Q(k,l) = \sum_{n=-\infty}^{\infty} x^{kn(3n+1)/2}\left(x^{-3nl} - x^{(3n+1)l}\right),$$

and a, b, c, d, f are linear functions of the independent parameters k and l. That such identities exist became obvious one day when we examined a large collection of quadratic trinomial Q identities we had found by a computer search. Most of these turned out to have terms with a nonzero Q factor that could be divided through the equation, giving a linear Q identity. (See [6] for three examples of quadratic Q identities that do not reduce to a linear identity in this way.) The set of these linear identities was readily recognizable as a two-parameter family. A further investigation uncovered a second two-parameter family. Each of these was easy to prove.

At this point in our investigation we became aware of the fact that most of the output of our search program consisted of these bogus Q^2 identities, whose unwelcome presence had to be eliminated so the true quadratic identities could emerge. It therefore became important to us to find all such parametrized linear Q identities.

The results of the search for these trinomial identities are given here, following a discussion and development of the properties of the family of eight functions we

Received by the editor August 29, 2001.
2000 *Mathematics Subject Classification.* Primary 11F11.
Key words and phrases. Triple product, quintuple product, linear identity, search algorithm.
Research was supported in part by Northern Illinois University Research and Artistry grant.

are involved with in this and other studies. In Section 2 we give an introductory account of the four T and four Q functions. We then derive some of the basic properties of these functions. In Section 3 we describe the nature of the search for the linear identities and give the two mod 2 families that were found. In Section 4 (Theorems 9 and 10) we present the corresponding eight solutions over the integers that come from the two mod 2 solutions and give their proofs using the properties developed in Section 2.

2. Definitions and basic relations

The well-known Jacobi triple-product expansion [8, p. 282 (19.8.1)] is

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1})=\sum_{-\infty}^{\infty}x^{n^2}z^n. \tag{2}$$

This expansion can be specialized by replacing the variables (x,z) by the pair $(i^\delta x^k, i^{\delta+2\epsilon}x^l)$, respectively, to give the four two-parameter, single-variable functions $T(k,l)=T_0(k,l)$, $T_1(k,l)$, $T_2(k,l)$, and $T_3(k,l)$, written generally as $T_{2\delta+\epsilon}(k,l)$. Here, as elsewhere in this paper, we assume that $\delta,\epsilon\in\{0,\,1\}$.

Definition 1. For $(k,l)\in\{(\frac{i}{2},\frac{j}{2}):(i,j)\in Z^+\times Z,\ i\equiv j\ (\operatorname{mod}2)\}$,

(3)
$$\begin{aligned}T_{2\delta+\epsilon}(k,l;x)&=T_{2\delta+\epsilon}(k,l)\\&\stackrel{\text{def}}{=}\prod_{n=1}^{\infty}[1-(-1)^{\delta n}x^{2kn}][1+(-1)^{\delta n+\epsilon}x^{2kn-k+l}][1+(-1)^{\delta n+\delta+\epsilon}x^{2kn-k-l}]\\&=\sum_{-\infty}^{\infty}(-1)^{\delta\frac{n(n+1)}{2}+\epsilon n}x^{kn^2+ln}.\end{aligned}$$

The series in (3) is a Laurent series with a finite number of negative-degree terms, which is a power series iff $l\in[-k,k]$.

It is worth mentioning that the expansions for $T_0(k,l)$ and $T_1(k,l)$ appear in [8, p. 283 (19.9.1), (19.9.2)]. The four T expansions are also directly related to Ramanujan's function

$$f(a,b)\stackrel{\text{def}}{=}\sum_{-\infty}^{\infty}a^{\frac{n(n+1)}{2}}b^{\frac{n(n-1)}{2}}$$

by the equation

$$T_{2\delta+\epsilon}(k,l)=f\Big((-1)^{\delta+\epsilon}x^{k+l},(-1)^{\epsilon}x^{k-l}\Big)$$

(see [1, p. 29]). Note that T_0, T_1, T_2, and T_3 correspond to the functions $f(a,b)$, $f(-a,-b)$, $f(-a,b)$, and $f(a,-b)$, respectively.

Definition 2. We say that $T_{2\delta+\epsilon}(k,l)$ is in *reduced form* whenever $0\le l\le k$.

The next two formulas allow us to put any $T_{2\delta+\epsilon}(k,l)$ into reduced form (cf. [4, p. 780]).

Theorem 1. *We have that*

$$T_{2\delta+\epsilon}(k,-l)=T_{2\delta+\epsilon+(-1)^{\epsilon}\delta}(k,l) \tag{4}$$

and

$$T_{2\delta+\epsilon}(k,l)=(-1)^{\epsilon}x^{k-l}T_{2\delta+\epsilon}(k,2k-l). \tag{5}$$

Proof. (4): From (3), it follows that

$$\begin{aligned}
T_{2\delta+\epsilon}(k,-l) &= \prod_{n=1}^{\infty}[1-(-1)^{\delta n}x^{2kn}] \\
&\quad \cdot [1+(-1)^{\delta n+\epsilon}x^{2kn-k-l}][1+(-1)^{\delta n+\delta+\epsilon}x^{2kn-k+l}] \\
&= \prod_{n=1}^{\infty}[1-(-1)^{\delta n}x^{2kn}] \\
&\quad \cdot [1+(-1)^{\delta n+\delta+\{\epsilon+(-1)^{\epsilon}\delta\}}x^{2kn-k-l}][1+(-1)^{\delta n+\{\epsilon+(-1)^{\epsilon}\delta\}}x^{2kn-k+l}] \\
&= T_{2\delta+\epsilon+(-1)^{\epsilon}\delta}(k,l).
\end{aligned}$$

(5): By re-indexing the sum in (3) by $n \to -n-1$, we find that

$$\begin{aligned}
T_{2\delta+\epsilon}(k,l) &= \sum_{-\infty}^{\infty}(-1)^{\delta\frac{n(n+1)}{2}-\epsilon n-\epsilon}x^{kn^2+(2k-l)n+k-l} \\
&= (-1)^{\epsilon}x^{k-l}\sum_{-\infty}^{\infty}(-1)^{\delta\frac{n(n+1)}{2}+\epsilon n}x^{kn^2+(2k-l)n} \\
&= (-1)^{\epsilon}x^{k-l}T_{2\delta+\epsilon}(k,2k-l). \quad \square
\end{aligned}$$

An example of reduction using (5) is

$$x^5T_1(18,21) = -x^5x^{-3}T_1(18,15) = -x^2T_1(18,15). \tag{6}$$

The final T_1 is in reduced form.

Note that the use of $(-1)^{\epsilon}\delta$ in the subscript in (4) interchanges T_2 and T_3 while leaving T_0 and T_1 unchanged.

The next identity is a linear relation that expresses any T in terms of T_0 or T_1.

Theorem 2 (cf. [3, p. 781, Lemma 1])**.** *We have that*

$$T_{2\delta+\epsilon}(k,l) = T_{\delta}(4k,2l) + (-1)^{\epsilon}x^{k-l}T_{\delta}(4k,4k-2l). \tag{7}$$

Proof. Replacing n by $2n$ and $2n-1$ in the summation in (3) gives

$$\begin{aligned}
T_{2\delta+\epsilon}(k,l) &= \sum_{-\infty}^{\infty}(-1)^{\delta n}x^{4kn^2+2ln} + (-1)^{\epsilon}x^{k-l}\sum_{-\infty}^{\infty}(-1)^{\delta n}x^{4kn^2-(4k-2l)n} \\
&= T_{\delta}(4k,2l) + (-1)^{\epsilon}x^{k-l}T_{\delta}(4k,4k-2l),
\end{aligned}$$

where we have used (4) in the last term. $\square$

Remark. Equation (7) is an example of a two-parameter trinomial identity like (1), which, however, involves T functions.

The next theorem is a generalization of (7) in the case when $\delta = 0$.

Theorem 3. *If $m \in Z^+$, then*

$$T_{\epsilon}(k,l) = \sum_{r\in R}(-1)^{\epsilon r}x^{kr^2-lr}T_{\sigma}(km^2, 2kmr-lm), \tag{8}$$

where R is a complete residue system (mod m) *and $\sigma = 0$ or 1 exactly as ϵm is even or odd.*

Proof. From (3) (with $\delta = 0$), we take $T_\epsilon(k,l) = \sum_{-\infty}^{\infty}(-1)^{\epsilon n}x^{kn^2-ln}$. Setting $n = ms + r$, we obtain

$$\begin{aligned} T_\epsilon(k,l) &= \sum_{r\in R}\sum_{s=-\infty}^{\infty}(-1)^{\epsilon(ms+r)}x^{k(ms+r)^2-l(ms+r)} \\ &= \sum_{r\in R}(-1)^{\epsilon r}x^{kr^2-lr}\sum_{s=-\infty}^{\infty}(-1)^{\epsilon ms}x^{km^2s^2+(2kmr-lm)s}. \end{aligned}$$

The result follows. □

It is evident by examining the three factors in (3) for $n = 1$ that $T_{2\delta+\epsilon}(k,l)$, with $l \in [-k,\, k]$, can vanish identically only in the following four cases:

$$T_1(k,k) = T_1(k,-k) = T_2(k,-k) = T_3(k,k) = 0. \tag{9}$$

The familiar quintuple-product expansion [7, p. 306] is

$$\begin{aligned} \prod_{n=1}^{\infty}(1-x^n)(1-x^nz)(1-x^{n-1}z^{-1})(1-x^{2n-1}z^2)(1-x^{2n-1}z^{-2}) \\ = \sum_{-\infty}^{\infty}x^{\frac{n(3n+1)}{2}}(z^{3n}-z^{-(3n+1)}). \end{aligned}\tag{10}$$

This expansion can also be specialized by replacing the variables (x,z) by the pair $((-1)^{\delta}x^k, (-1)^{\delta+\epsilon}x^{-l})$, respectively, to give four two-parameter, single-variable functions $Q(k,l) = Q_0(k,l)$, $Q_1(k,l)$, $Q_2(k,l)$, and $Q_3(k,l)$,[1] expressed generally as $Q_{2\delta+\epsilon}(k,l)$.

Definition 3. Let $k \in Z^+$ and $l \in Z$. Then

$$\begin{aligned} &Q_{2\delta+\epsilon}(k,l;x) = Q_{2\delta+\epsilon}(k,l) \\ &\stackrel{\text{def}}{=} \prod_{n=1}^{\infty}[1-(-1)^{\delta n}x^{kn}][1-(-1)^{\delta n+\delta+\epsilon}x^{kn-l}][1-(-1)^{\delta n+\epsilon}x^{kn-k+l}] \\ &\qquad \cdot [1-(-1)^{\delta}x^{2kn-k-2l}][1-(-1)^{\delta}x^{2kn-k+2l}] \\ &= \sum_{-\infty}^{\infty}(-1)^{\delta\frac{n(n+1)}{2}+\epsilon n}x^{\frac{n(3n+1)}{2}k}\left(x^{-3nl}-(-1)^{\delta+\epsilon}x^{(3n+1)l}\right). \end{aligned}\tag{11}$$

The series for $Q_{2\delta+\epsilon}(k,l)$ in (11) will be a Laurent series with a finite number of negative-degree terms, which is a power series iff $0 \le 2l \le k$.

The following useful formula expresses the simple relationship that holds between the respective T's and Q's.

Theorem 4. *We have that*

$$Q_{2\delta+\epsilon}(k,l) = T_{2\delta+\epsilon}(\tfrac{3k}{2}, \tfrac{k}{2}-3l) - (-1)^{\delta+\epsilon}x^l\, T_{2\delta+\epsilon}(\tfrac{3k}{2}, \tfrac{k}{2}+3l). \tag{12}$$

[1] We have renamed the Q_1, Q_2, and Q_3 as defined in [2, p. 304] to be Q_3, Q_1, and Q_2, respectively, so that here the subscripts of the T's and Q's in Theorem 4 agree.

Proof. Using (11) and (3), it follows that

$$\begin{aligned} Q_{2\delta+\epsilon}(k,l) &= \sum_{-\infty}^{\infty}(-1)^{\delta\frac{n(n+1)}{2}+\epsilon n}x^{\frac{n(3n+1)}{2}k}\Big(x^{-3ln}-(-1)^{\delta+\epsilon}x^{(3n+1)l}\Big)\\ &= \sum_{-\infty}^{\infty}(-1)^{\delta\frac{n(n+1)}{2}+\epsilon n}x^{\frac{3k}{2}n^2+(\frac{k}{2}-3l)n}\\ &\qquad -(-1)^{\delta+\epsilon}x^l\sum_{-\infty}^{\infty}(-1)^{\delta\frac{n(n+1)}{2}+\epsilon n}x^{\frac{3k}{2}n^2+(\frac{k}{2}+3l)n}\\ &= T_{2\delta+\epsilon}(\tfrac{3k}{2},\tfrac{k}{2}-3l)-(-1)^{\delta+\epsilon}x^l\,T_{2\delta+\epsilon}(\tfrac{3k}{2},\tfrac{k}{2}+3l). \qquad \square \end{aligned}$$

In the next theorem, we give reduction formulas for $Q_{2\delta+\epsilon}(k,l)$, analogous to the reduction formulas (4) and (5) for $T_{2\delta+\epsilon}(k,l)$ (cf. [7, (8) and (9)]).

Theorem 5. *We have that*

$$Q_{2\delta+\epsilon}(k,-l)=(-1)^{\delta+\epsilon+1}x^{-l}Q_{2\delta+\epsilon}(k,l) \tag{13}$$

and

$$Q_{2\delta+\epsilon}(k,l)=(-1)^{\delta+\epsilon}x^{2k-3l}Q_{2\delta+\epsilon+(-1)^{\epsilon}\delta}(k,l-k). \tag{14}$$

Proof. (13): From (12) it follows that

$$\begin{aligned} x^lQ_{2\delta+\epsilon}(k,-l) &= x^lT_{2\delta+\epsilon}\Big(\frac{3k}{2},\frac{k}{2}+3l\Big)-(-1)^{\delta+\epsilon}T_{2\delta+\epsilon}\Big(\frac{3k}{2},\frac{k}{2}-3l\Big)\\ &= (-1)^{\delta+\epsilon+1}Q_{2\delta+\epsilon}(k,l). \end{aligned}$$

(14): Put $\mu=2\delta+\epsilon$ and $\nu=2\delta+\epsilon'$, where $\epsilon'=\epsilon+(-1)^{\epsilon}\delta$. Note that $\epsilon'=0$ or 1 and, furthermore, that $(-1)^{\epsilon'}=(-1)^{\delta+\epsilon}$. Using (12), (4), (5), (4), and (12) in turn, we have

$$\begin{aligned} Q_\mu(k,k+m) &= T_\mu\Big(\frac{3k}{2},-\frac{5k}{2}-3m\Big)-(-1)^{\delta+\epsilon}x^{k+m}T_\mu\Big(\frac{3k}{2},\frac{7k}{2}+3m\Big)\\ &= T_\nu\Big(\frac{3k}{2},\frac{5k}{2}+3m\Big)-(-1)^{\delta+\epsilon}x^{k+m}T_\mu\Big(\frac{3k}{2},\frac{7k}{2}+3m\Big)\\ &= (-1)^{\epsilon'}x^{-k-3m}T_\nu\Big(\frac{3k}{2},\frac{k}{2}-3m\Big)-(-1)^{\delta}x^{-k-2m}T_\mu\Big(\frac{3k}{2},-\frac{k}{2}-3m\Big)\\ &= (-1)^{\epsilon'}x^{-k-3m}\Big[T_\nu\Big(\frac{3k}{2},\frac{k}{2}-3m\Big)-(-1)^{\delta+\epsilon'}x^mT_\nu\Big(\frac{3k}{2},\frac{k}{2}+3m\Big)\Big]\\ &= (-1)^{\delta+\epsilon}x^{-k-3m}Q_{2\delta+\epsilon+(-1)^{\epsilon}\delta}(k,m). \end{aligned}$$

Putting $l=k+m$ into this result gives (14). $\square$

Definition 4. We say that $Q_{2\delta+\epsilon}(k,l)$ is in *reduced form* whenever $0\le l\le \frac{k}{2}$.

By applying Theorem 5, we can write any Q as $\pm x^r$ times a reduced Q. For example, $Q(10,27)=x^{20-81}Q(10,17)=x^{-61}x^{20-51}Q(10,7)=x^{-92}x^{20-21}Q(10,-3)=-x^{-96}Q(10,3)$.

The following theorem gives the cases where $Q_{2\delta+\epsilon}(k,l;x)$ vanishes.

Theorem 6. *Suppose that $k\in Z^+$ and that $0\le l\le\frac{k}{2}$, where $l\in Z$. Then*

$$Q_{2\delta+\epsilon}(k,l;x)=0 \iff l=0,\ \delta=\epsilon \ \text{ or } \ l=\frac{k}{2},\ \delta=0. \tag{15}$$

Also,

$$Q_{2\delta+\epsilon}(k,l;x) \equiv 0 \pmod 2 \Longleftrightarrow l=0 \text{ or } l=\frac{k}{2}. \tag{16}$$

Proof. (15): It is clear from (11) that $Q_{2\delta+\epsilon}(k,l)$ vanishes identically iff one of the factors in the quintuple product is zero. Since $n \geq 1$ and $l \leq \frac{k}{2}$, it follows that the first, second, and fifth factors in this product are not zero. The third factor clearly vanishes iff $n=1$, $l=0$, and $\delta=\epsilon$. The fourth factor vanishes iff $n=1$, $l=\frac{k}{2}$, and $\delta=0$.

(16): This is the same as the proof in (15), except that no condition on δ or ϵ enters. □

The fact that the four quintuple products $Q_1(k,0)$, $Q_2(k,0)$, $Q_2(k,\frac{k}{2})$, and $Q_3(k,\frac{k}{2})$ are congruent to 0 (mod 2), but are not identically 0, implies that they are nonzero multiples of 2. A formula for each of these functions, with its factor 2 in evidence, is given in the next corollary.

Corollary 1. *For $\epsilon = 0$ or 1, we have that*

$$Q_{\epsilon+1}(k,0) = 2\,T_{\epsilon+1}\Big(\frac{3k}{2},\,\frac{k}{2}\Big) \tag{17}$$

and

$$Q_{2+\epsilon}(2k,k) = 2\,T_1(12k,4k). \tag{18}$$

Proof. (17): The result follows directly from (12).

(18): Using (12), (4), (5), and (7), we find that

$$\begin{aligned} Q_{2+\epsilon}(2k,k) &= T_{2+\epsilon}(3k,-2k) - (-1)^{\epsilon+1}x^k T_{2+\epsilon}(3k,\,4k) \\ &= T_{2+\epsilon+(-1)^\epsilon}(3k,2k) + T_{2+\epsilon}(3k,2k) = T_3(3k,\,2k) + T_2(3k,\,2k) \\ &= [T_1(12k,\,4k) - x^k T_1(12k,\,8k)] + [T_1(12k,\,4k) + x^k T_1(12k,\,8k)] \\ &= 2\,T_1(12k,\,4k). \quad \square \end{aligned}$$

Corollary 2. *For $k \in Z^+$ and $m \in Z$, we have that*

$$Q_0(k,\,mk) = Q_3(k,\,2mk) = 0 \tag{19}$$

and

$$Q_0(2k,(2m+1)k) = Q_1(2k,(2m+1)k) = 0. \tag{20}$$

Proof. Repeatedly transforming the two Q's in (19) by (14) changes them into a power of x times $Q_0(k,0)$ and $Q_3(k,0)$, respectively, which are zero by (15). A similar transformation in (20) produces a power of x times $Q_\epsilon(2k,k)$, which is also zero by the other case in (15). □

We conclude this section by discussing some occurrences of the formulas in (7) and (12) that have previously appeared in the literature.

In [9], L. Slater published 130 Rogers-Ramanujan type identities. Among these were 30 that contained a binomial whose terms involved infinite products. (These are identities #47, 48, 62–65, 67, 80–84, 86, 87, 94, 96, 98, 99, 105–108, 113, 117–119, 121, 123–125.) Using a method of Bailey, she replaced each of these binomials with a single infinite product. Each of these 30 simplifications can readily be shown to be an instance of (7) or (12), with possible reductions using (4) or (5).

For example, formula #67 contains the identity

(21)
$$\prod_{n=1}^{\infty}(1-x^{16n})(1-x^{16n-6})(1-x^{16n-10})-x\prod_{n=1}^{\infty}(1-x^{16n})(1-x^{16n-2})(1-x^{16n-14})$$
$$=\prod_{n=1}^{\infty}[1-(-1)^n x^{4n}][1-(-1)^n x^{4n-1}][1+(-1)^n x^{4n-3}].$$

Using (3), this can be expressed as the identity $T_1(8,2)-xT_1(8,6)=T_3(2,1)$, which is an instance of (7) with $\delta=\epsilon=1$, $k=2$, and $l=1$.

Another example is formula #108 which contains the identity

(22)
$$\prod_{n=1}^{\infty}(1-x^{36n})(1-x^{36n-9})(1-x^{36n-27})-x^2\prod_{n=1}^{\infty}(1-x^{36n})(1-x^{36n-3})(1-x^{36n-33})$$
$$=\prod_{n=1}^{\infty}(1-x^{12n})(1+x^{12n-5})(1+x^{12n-7})(1-x^{24n-2})(1-x^{24n-22}).$$

Using the product forms in (3) and (11), we can identity these three terms as
$$T_1(18,9)-x^2T_1(18,15)=Q_1(12,5).$$
But we obtain from (12) that $Q_1(12,5)=T_1(18,9)+x^5T_1(18,21)$, which reduces to the previous left-hand side as shown in (6).

A final example is formula #47. This has a product with four factors on its right-hand side which can be expressed as a product of three factors (as in the right-hand side of #48), viz.,

(23)
$$\prod_{n=1}^{\infty}(1-x^{12n})(1-x^{12n-5})(1-x^{12n-7})+x\prod_{n=1}^{\infty}(1-x^{12n})(1-x^{12n-1})(1-x^{12n-11})$$
$$=\prod_{n=1}^{\infty}(1+x^{2n-1})(1-x^{8n-6})(1-x^{8n-2})(1-x^{4n})=\prod_{n=1}^{\infty}(1+x^{2n-1})(1-x^{2n})$$
$$=\prod_{n=1}^{\infty}[1-(-1)^n x^{3n}][1+(-1)^n x^{3n-1}][1-(-1)^n x^{3n-2}].$$

Using (3), this translates into $T_1(6,1)+xT_1(6,5)=T_2(\frac{3}{2},\,\frac{1}{2})$, an instance of (7).

3. The search for linear Q families

In Theorem 4, two T's are linearly combined, using powers of x, to give a Q, while in Theorem 2, two T's are linearly combined to give a T. It is reasonable then to raise the question whether there is an identity of this type where two Q's linearly combine to give a Q, even without knowing that such a relationship exists. In answering this question, we will simplify the search by working modulo 2, so that among other things, Q_1, Q_2, and Q_3 will coincide with Q itself.

In particular, we will find all two-parameter identities of the form

$$Q(k,l)\equiv Q(a,b)+x^fQ(c,d)\pmod{2}, \tag{24}$$

where a,b,c,d, and f are linear functions of the independent parameters k and l.

We assume that no Q in (24) is zero (mod 2). We may also assume, without loss of generality, that each Q is in reduced form. If not, we could use Theorem 5 to put each Q into reduced form, resulting in a mod 2 congruence where each Q is multiplied by a power of x. Dividing through by the smallest power of x gives a term of the form $x^0 Q$. Since the constant term in the expansion of a Q is 1, provided Q is not zero (mod 2), it follows that a second Q in the congruence must also have a coefficient x^0. Switching pairs of letters, if necessary, results in the congruence (24) with each Q in reduced form.

Before we discuss the method of finding these identities, we will deal with a question that relates to the search. Recall from (11) that

$$Q(k,l) = \sum_{-\infty}^{\infty} \left(x^{\frac{n(3n+1)}{2}k - 3nl} - x^{\frac{n(3n+1)}{2}k + (3n+1)l} \right). \tag{25}$$

Assume we are given some function $Q(a_1k+a_2l,\ b_1k+b_2l)$ and we wish to determine if some term $x^{c_1k+c_2l}$ is in the series expansion of this Q. The next theorem gives a necessary condition for this to be the case.

Theorem 7. *Let $c_1, c_2 \in \mathbb{Z}$ and $a_1, a_2, b_1, b_2 \in \mathbb{Q}$ such that $a_1b_2 - a_2b_1 \neq 0$. Then $x^{c_1k+c_2l}$ is a term in the series expansion of $Q(a_1k+a_2l,\ b_1k+b_2l)$ only when $t \in \mathbb{Z}$ or $\frac{1}{3}(t-1) \in \mathbb{Z}$, where*

$$t = \frac{a_2c_1 - a_1c_2}{3(a_1b_2 - a_2b_1)}.$$

Proof. If $x^{c_1k+c_2l}$ is in the given expansion, then by (25) there is an $n \in \mathbb{Z}$ such that either

$$c_1k + c_2l = \frac{n(3n+1)}{2}(a_1k + a_2l) - 3n(b_1k + b_2l)$$

or

$$c_1k + c_2l = \frac{n(3n+1)}{2}(a_1k + a_2l) + (3n+1)(b_1k + b_2l).$$

If we equate the corresponding coefficients of k and l in the first equation, we find that

$$\frac{n(3n+1)}{2}a_1 - 3nb_1 = c_1 \quad \text{and} \quad \frac{n(3n+1)}{2}a_2 - 3nb_2 = c_2.$$

Eliminating the leading term between these two equations and solving for n gives the first result. The second result is obtained in a similar way. □

The approach to finding all two-parameter solutions of (24) is to substitute the series expansion of each Q into congruence (24) and require that all the powers of x match and thereby cancel (mod 2). Fortunately, as it happens, most of the impossible cases are detected by a disagreement of powers by the 6th term on the two sides.

To be able to carry out the matching, we must first know how the terms in the power series expansion of a Q function are ordered by increasing degree. To find this ordering, it is convenient to split the sum in (11) into four sums by separating the positive and negative index values:

$$Q(k,l) = 1 - x^l + \sum_{n=1}^{\infty} \left(x^{A(n)} + x^{B(n)} - x^{C(n)} - x^{D(n)} \right), \tag{26}$$

where

$$
\begin{aligned}
A(n) &= \frac{n(3n+1)}{2}k - 3nl, & B(n) &= \frac{n(3n-1)}{2}k + 3nl,\\
C(n) &= \frac{n(3n+1)}{2}k + (3n+1)l, & D(n) &= \frac{n(3n-1)}{2}k - (3n-1)l.
\end{aligned} \tag{27}
$$

It is a remarkable fact that the exponents in this expansion have only three different orderings and that each of these orderings repeats and continues to infinity—a kind of braid. Which of the three orderings occurs depends only on which of the three intervals $(0, \frac{k}{6})$, $(\frac{k}{6}, \frac{k}{3})$, $(\frac{k}{3}, \frac{k}{2})$ the number l lies in. The cases $l = 0$ and $l = \frac{k}{2}$ are omitted here, because by (16) each of these gives a zero term (mod 2), a trivial case. We also omit the cases where $l = \frac{k}{6}$ and $l = \frac{k}{3}$, since k and l are supposed to be independent parameters.

The next theorem shows explicitly how the exponents are ordered when l is in each of the three intervals.

Theorem 8. *Given the integers $k \geq 1$ and $l \in (0, \frac{k}{2})$, where $l \neq \frac{k}{6}, \frac{k}{3}$. Then for $n \geq 1$, the exponents in* (27) *are ordered in* (26) *as follows:*

$$
\begin{aligned}
&(i) && D(n) < B(n) < A(n) < C(n) < D(n+1), && \textit{if} \;\; l \in \left(0, \frac{k}{6}\right),\\
&(ii) && D(n) < A(n) < B(n) < C(n) < D(n+1), && \textit{if} \;\; l \in \left(\frac{k}{6}, \frac{k}{3}\right),\\
&(iii) && A(n) < B(n) < D(n+1) < C(n) < A(n+1), && \textit{if} \;\; l \in \left(\frac{k}{3}, \frac{k}{2}\right).
\end{aligned}
$$

Proof. It is clear by comparing the exponents in (27) that for each $n \geq 1$ and $l \in (0, \frac{k}{2})$, we have $D(n) < B(n)$, $A(n) < C(n)$, and $B(n) < C(n)$. Also, $A(n) - D(n) = kn - l > k - l > 0$, since $l < \frac{k}{2}$. Next, $A(n) - B(n) = n(k - 6l) > 0 \iff 0 < l < \frac{k}{6}$. Further, $D(n+1) - C(n) = (2n+1)(k-3l) > 0 \iff 0 < l < \frac{k}{3}$. Finally, $D(n+1) - B(n) = (3n+1)(k-2l) > 0$ and $A(n+1) - C(n) = (3n+2)(k-2l) > 0$, since $l < \frac{k}{2}$. Combining these results proves the theorem. ☐

Remarks. 1. Note in each of the three cases in Theorem 8 that the first and the fifth letter in each exponent sequence is the same and the arguments are n and $n+1$, respectively. This shows inductively that these exponent sequences repeat to infinity.

2. Using Theorem 8, we can write down the first six terms in the expansion of a $Q(r,s)$ (mod 2) for the three intervals, viz.,

$$
\begin{aligned}
&\textit{Case } 1. && 0 < s < \frac{r}{6} && Q(r,s) = 1 + x^{s} + x^{r-2s} + x^{r+3s} + x^{2r-3s} + x^{2r+4s} + \cdots,\\
&\textit{Case } 2. && \frac{r}{6} < s < \frac{r}{3} && Q(r,s) = 1 + x^{s} + x^{r-2s} + x^{2r-3s} + x^{r+3s} + x^{2r+4s} + \cdots,\\
&\textit{Case } 3. && \frac{r}{3} < s < \frac{r}{2} && Q(r,s) = 1 + x^{r-2s} + x^{s} + x^{2r-3s} + x^{r+3s} + x^{5r-5s} + \cdots.
\end{aligned}
$$

Since any one of these three orderings can be chosen for each of the three Q's in (24), there are 27 cases to be considered. For any one of these choices, we must then determine what the possibilities are for the terms in (24) to match. To clarify this question, consider the case in which we take the first, second, and third orderings for the three respective Q's in (24). We would then be considering the following congruence (with the k and l exponents on the right):

(28)
$$\begin{aligned}1 &+ x^b + x^{a-2b} + x^{2a-3b} + x^{a+3b} + x^{2a+4b}\\ &+ \cdots + x^f + x^{c-2d+f} + x^{d+f} + x^{2c-3d+f} + x^{c+3d+f} + x^{5c-3d+f}\\ &+ \cdots \equiv 1 + x^l + x^{k-2l} + x^{k+3l} + x^{2k-3l} + x^{2k+4l} \cdots \pmod 2.\end{aligned}$$

The 1's cancel on the two sides leaving three possibilities as to which of the three terms of smallest degree are equal. These are $b = f$, $b = l$, or $f = l$. Suppose, for example, that we take the third possibility where $f = l$. Then the terms x^f and x^l cancel and we record the condition $f = l$ as the first equation in a linear system we will be building from these conditions. The next candidates for the smallest degree terms in the three series are x^b, x^{c-2d+f}, and x^{k-2l}. Here again there are three choices: $b = c - 2d + f$, $b = k - 2l$, and $c - 2d + f = k - 2l$.

Choosing the first of these, say, then the terms x^b and x^{c-2d+f} will cancel (mod 2). This equation will then be the second equation in our growing linear system. Continuing four more times in this way will produce a linear system with six equations in the seven variables. Counting the total number of alternatives in this process, we see there are $27 \cdot 3^6 = 19683$ linear systems that must be examined to find all solutions.

Next we have to consider the possible results we might obtain from one of these linear systems. To correspond to the original problem, we first write each system so that the variables a, b, c, d, f are on the left and the k and l, which are considered to be given, on the right. We then put this system in reduced echelon form. If the six equations turn out to be linearly independent, we get a linear dependency between k and l. Since we are only interested here in genuine two-parameter identities, we ignore such independent systems, and restrict our attention to systems with rank ≤ 5. To ensure that k and l are independent variables, we also require that the sixth and seventh columns of the augmented matrix do not contain only 0's. With these restrictions, the rank of the system will be 5 and the six equations will therefore have a unique solution in terms of the independent parameters k and l. Such a solution is a candidate for an actual solution of (24).

To illustrate the latter process, we return to the previous example. Assume, for example, that we had continued to match the smallest exponents in (28) until we obtained the linear system

$$\begin{aligned}f &= l,\\ b &= c - 2d + f,\\ a - 2b &= d + f,\\ 2c - 3d + f &= k - 2l,\\ 2a - 3b &= k + 3l,\\ a + 3b &= 2k - 3l.\end{aligned}$$

The augmented matrix for this system, with only k and l terms on the right, now row-reduces as follows:

$$\left[\begin{array}{ccccc|cc} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 & -1 & 0 & 0 \\ 1 & -2 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 1 & 1 & -2 \\ 2 & -3 & 0 & 0 & 0 & 1 & 3 \\ 1 & 3 & 0 & 0 & 0 & 2 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|cc} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Since the rank of this matrix is 5, the system has the unique solution $a = k$, $b = \frac{1}{3}k - l$, $c = k$, $d = \frac{1}{3} + l$, and $f = l$, so we get the potential parametric solution

$$Q(k,l) \equiv Q\left(k, \frac{k}{3} - l\right) + x^l\, Q\left(k, \frac{k}{3} + l\right) \pmod 2. \tag{29}$$

To determine whether (29) is a genuine mod 2 identity, we examine more exponents in the series expansions of these Q's. In a true identity, each of the terms in one of its series must occur in one of the other two series. If this is not true, then the trial solution is not a genuine solution to (24). As it happens, checking the first ten terms of the series for a potential identity is enough to eliminate most bogus solutions.

In the search algorithm below, we use Theorem 7 as a necessary condition to check if a term occurs in a particular Q series. Moreover, when n is an integer, we actually check that the given exponent on the left is the n^{th} exponent on the right.

As it turns out, using Theorem 7, the ten terms of each of the three Q's in our potential identity (29) do match a term in one of the other two Q's. This is very strong evidence that (29) is an actual identity. Indeed (29) corresponds to the parametric identity

$$Q(k,l) = Q\left(k, \frac{k}{3} - l\right) - x^l\, Q\left(k, \frac{k}{3} + l\right), \tag{30}$$

whose proof appears in the next section.

The Search Algorithm

Given two independent parameters k and l, find all choices for a, b, c, d, and f in terms of k and l such that

$$Q(k,l) \equiv Q(a,b) + x^f Q(c,d) \pmod 2,$$

where each Q is reduced and nonzero (mod 2).

STEP 1. Order the exponent sequences.

Choose one of the three possible orderings for each of $Q(k,l)$, $Q(a,b)$ and $Q(c,d)$. Use Theorem 8 to make a list $\{\kappa_m k + \lambda_m l\}$ of the exponents of $Q(k,l)$ in increasing order. Similarly make lists $\{\alpha_i a + \beta_i b\}$ of the exponents of $Q(a,b)$ and $\{\gamma_j c + \delta_j d + f\}$ of the exponents of $x^f Q(c,d)$.

STEP 2. Determine all potential re-arrangement schemes. Construct augmented matrix A for each linear system.

Use a ternary digit t_i to indicate each match, where $t_i = 0$ or 1 indicates that a term of $Q(k,l)$ matches $Q(a,b)$ or $x^f Q(c,d)$, respectively, while $t_i = 2$ indicates that a term of $Q(a,b)$ matches a term of $Q(c,d)$.

For $t = 0$ to $3^6 - 1$

Write $t = (t_6\, t_5\, t_4\, t_3\, t_2\, t_1)_3$ in ternary.

Initialize $m = 1$, $i = 1$, $j = 0$.
For $r = 1$ to 6
Case $t_r = 0$:
Set r^{th} row of $A = [\alpha_i, \beta_i, 0, 0, 0, \kappa_m, \lambda_m]$.
Increment m and i.
Case $t_r = 1$:
Set r^{th} row of $A = [0, 0, \gamma_j, \delta_j, 1, \kappa_m, \lambda_m]$.
Increment m and j.
Case $t_r = 2$:
Set r^{th} row of $A = [\alpha_i, \beta_i, -\gamma_j, -\delta_j, -1, 0, 0]$.
Increment i and j.
Next r.
Row-reduce matrix A to reduced echelon form $A' = [H|B]$, where H is a 6×5 matrix and B is 6×2.
Next t.

STEP 3. Check each system A' found in STEP 2.

Test I. *Inconsistency.*

Does a row of the matrix A' have the form $[0, 0, 0, 0, 0, *, *]$, where at least one of the last two entries is nonzero? This either forces k or l to be zero, or else gives a linear dependency between them. Neither alternative is allowed, so we can ignore this matrix

If the rank of A' is 5, does the 6th or 7th column of A' consist of all 0's? If so, then the variable k or l does not occur on the right hand side of (1), which is impossible. Ignore this matrix.

Test II. *Discard trivial solutions.*

If the first two rows of A' are $[1, 0, 0, 0, 0, 1, 0]$ and $[0, 1, 0, 0, 0, 0, 1]$, then this gives the solution $a = k$ and $b = l$, forcing $Q(c, d) = 0$. Since we are not interested in trivial solutions where one term is 0, ignore this matrix.

If the last three nonzero rows of A' are $[0, 0, 1, 0, 0, 1, 0]$, $[0, 0, 0, 1, 0, 1]$ and $[0, 0, 0, 0, 1, 0, 0]$ or $[0, 0, 1, 0, 0, 1, 0]$, $[0, 0, 0, 1, 0, -1]$ and $[0, 0, 0, 0, 1, 0, 1]$, then this gives the trivial solution where $Q(a, b) = 0$. Ignore this matrix.

Test III. *Does re-arrangement continue?*

Assume the rank of A' is 5, so that H is the 5×5 identity matrix, above a row of 0's. Here we explicitly know the values of a, b, c, d and f as linear combinations of k and l. The coefficients of these linear combinations are found in matrix B. Let $\{u_m(k, l)\}$, $\{v_m(k, l)\}$, and $\{w_m(k, l)\}$, be the respective exponent sequences $\{\kappa_m k + \lambda_m l\}$, $\{\alpha_i a + \beta_i b\}$, and $\{\gamma_j c + \delta_j d + f\}$, when each variable a, b, c, d, and f is written in terms of k and l.

Look at the first ten terms $u_m(k, l)$ for $m = 1$ to 10. If u_m does not occur in the $\{v_i\}$ or $\{w_j\}$ sequence, then discard this matrix.

Look at the first ten terms $v_i(k, l)$ for $i = 1$ to 10. If v_i does not occur in the $\{u_m\}$ or $\{w_j\}$ sequence, then discard this matrix.

Look at the first ten terms $w_j(k, l)$ for $j = 1$ to 10. If w_j does not occur in the $\{u_m\}$ or $\{v_j\}$ sequence, then discard this matrix.

Any matrix which passes all three tests has the property that the first ten terms of the three series in (24) enter successfully into the re-arrangement scheme. This is a good candidate for an identity. Output the corresponding parameters.

If A' passes Tests I and II, but the rank of $A' < 5$, then output A' for special consideration.

Remark. This algorithm can be used to find other types of identities for functions involving power series, as long as the following three conditions are satisfied:

(1) The exponents of the power series are linear combinations of the parameters;
(2) We can determine all possible orderings of the exponents, as in Theorem 8;
(3) We can determine when a term occurs in a given exponent sequence, as in Theorem 7.

We ran this algorithm on an IBM PC-compatible computer. The program created the $3^6 = 729$ possible sets of equations for each of the 27 possible orderings of the three Q series, giving a total of $3^9 = 19683$ A matrices. Each A matrix was put in reduced echelon form A'. Of these A' matrices, 18812 were found to be inconsistent [Step 3, Test I] and 757 gave trivial solutions [Step 3, Test II]. This left 114 matrices, which was reduced to 38 when duplications were removed. (It should be pointed out that different values of t in Step 2 can lead to identical A''s, using different orderings in Step 1. For example, the reduced matrix in (32) below appears six times in the list of 114 matrices.) Each of the remaining 38 matrices had rank 5, so the third test was applied to them. In Step 3, Test III, 34 of these matrices were eliminated because the two sides of (24) did not agree. The 4 matrices that were left turned out to lead to genuine mod 2 identities.

In what follows we examine the details for these four solutions.

Solution 1.

$$Q(k,l) \equiv Q(4k, k-2l) + x^l Q(4k, k+2l) \pmod 2. \tag{31}$$

We obtain this solution from the following exponent orderings of the three series in (24):

$$\begin{aligned}
&Q(k,l): &&0,\ l,\ k-2l,\ k+3l,\ 2k-3l,\ 2k+4l &&\left(l < \frac{k}{6}\right),\\
&Q(a,b): &&0,\ b,\ a-2b &&\left(b < \frac{a}{3}\right),\\
&x^f Q(c,d): &&f,\ d+f,\ c-2d+f &&\left(c < \frac{d}{3}\right).
\end{aligned}$$

(See Cases 1 and 2 in the second remark following the proof of Theorem 8.) Then for the particular assignment vector $t = (01101)_3$, the three series mesh to give the following system of equations:

$$\begin{aligned}
f &= l,\\
b &= k-2l,\\
d+f &= k+3l,\\
c-2d+f &= 2k-3l,\\
a-2b &= 2k+4l.
\end{aligned}$$

(The sixth equation is redundant and so has been omitted.) The augmented matrix of this system row-reduces as

(32)
$$\left[\begin{array}{ccccc|cc}
0 & 0 & 0 & 0 & 1 & 0 & 1\\
0 & 1 & 0 & 0 & 0 & 1 & -2\\
0 & 0 & 0 & 1 & 1 & 1 & 3\\
0 & 0 & 1 & -2 & 1 & 2 & -3\\
1 & -2 & 0 & 0 & 0 & 2 & 4
\end{array}\right]
\longrightarrow
\left[\begin{array}{ccccc|cc}
1 & 0 & 0 & 0 & 0 & 4 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & -2\\
0 & 0 & 1 & 0 & 0 & 4 & 0\\
0 & 0 & 0 & 1 & 0 & 1 & 2\\
0 & 0 & 0 & 0 & 1 & 0 & 1
\end{array}\right].$$

This gives the two-parameter solution $a = 4k$, $b = k - 2l$, $c = 4k$, $d = k + 2l$, and $f = l$, yielding (31).

Solution 2.

$$Q(3k, l) \equiv Q(3k, k - l) + x^l Q(3k, k + l) \pmod 2. \tag{33}$$

This is just (30) with k replaced by $3k$. The details were given in the example and the discussion preceding the Search Algorithm. Note that the first 5 ternary digits of t used in Step 2 of the Search Algorithm are $(01221)_3$ for the ordering in (28).

Solutions 3 and 4. The remaining two reduced matrices are

$$\left[\begin{array}{ccccc|cc} 1 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{8} & -\frac{1}{2} \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccccc|cc} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{3} & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{3} & -1 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} & -1 \end{array}\right].$$

These give the identities

$$Q(k, l) \equiv Q\Big(\frac{k}{4}, \frac{k}{8} - \frac{l}{2}\Big) + x^{\frac{k}{8} - \frac{l}{2}} Q\Big(k, \frac{k}{2} - l\Big) \pmod 2$$

and

$$Q(k, l) \equiv Q\Big(k, \frac{k}{3} - l\Big) + x^{\frac{k}{3} - l} Q\Big(k, \frac{2k}{3} - l\Big) \pmod 2,$$

which are not new identities, since the transformations $k = 4u$, $l = u - 2v$ and $k = 3u$, $l = u - v$ change them into (31) and (33), respectively.

4. Linear Q identities over the integers

To find all the linear Q identities over the integers, we next replace each of the three Q's in (31) and (33) by the four possible choices $Q_{2\delta+\epsilon}$ and see which of these choices yields a true identity. We find by comparing coefficients of like powers on the two sides that each of (31) and (33) yields just four identities, given in Theorems 9 and 10. The proofs of these theorems well illustrate how the properties of the T and Q functions allow for proofs that are simple and direct.

Theorem 9. *For $(k, l) \in Z^+ \times Z$, we have*

$$Q_{2\delta+\epsilon}(k, l) = Q_\delta(4k, k - 2l) - (-1)^{\delta+\epsilon} x^l Q_\delta(4k, k + 2l). \tag{34}$$

Proof. Expanding $Q_{2\delta+\epsilon}(k, l)$ and then using (12), (7), (4), and (12), we find that

$$\begin{aligned} Q_{2\delta+\epsilon}(k, l) &= T_{2\delta+\epsilon}(\tfrac{3k}{2}, \tfrac{k}{2} - 3l) - (-1)^{\delta+\epsilon} x^l\, T_{2\delta+\epsilon}(\tfrac{3k}{2}, \tfrac{k}{2} + 3l) \\ &= [T_\delta(6k, k - 6l) + (-1)^\epsilon x^{k+3l} T_\delta(6k, 5k + 6l)] \\ &\quad - (-1)^{\delta+\epsilon} x^l [T_\delta(6k, k + 6l) + (-1)^\epsilon x^{k-3l} T_\delta(6k, 5k - 6l)] \\ &= [T_\delta(6k, -(k - 6l)) - (-1)^\delta x^{k-2l} T_\delta(6k, 5k - 6l)] \\ &\quad - (-1)^{\delta+\epsilon} x^l [T_\delta(6k, -(k + 6l)) - (-1)^\delta x^{k+2l} T_\delta(6k, 5k + 6l)] \\ &= Q_\delta(4k, k - 2l) - (-1)^{\delta+\epsilon} x^l Q_\delta(4k, k + 2l). \quad \square \end{aligned}$$

Note that equation (35) is balanced.

Theorem 10. *For $(k,l) \in Z^+ \times Z$, we have*

(35)
$$Q_{2\delta+\epsilon}(3k,l) = Q_{2\delta+\epsilon+(-1)^\epsilon\delta}(3k,k-l) - (-1)^{\delta+\epsilon}x^l\, Q_{2\delta+\epsilon+(-1)^\epsilon\delta}(3k,k+l).$$

Proof. Put $\mu = 2\delta+\epsilon$ and $\nu = 2\delta+\epsilon'$, where $\epsilon' = \epsilon + (-1)^\epsilon\delta$. Note that $\epsilon' = 0$ or 1 and furthermore that $(-1)^{\epsilon'} = (-1)^{\delta+\epsilon}$. Then by (12), (4), (5), and (12), we have that

$$\begin{aligned}
&Q_\nu(3k,k-l) - (-1)^{\delta+\epsilon}x^l Q_\nu(3k,k+l)\\
&\quad= [T_\nu(\tfrac{9k}{2}, -\tfrac{3k}{2}+3l) - (-1)^\epsilon x^{k-l}\, T_\nu(\tfrac{9k}{2}, \tfrac{9k}{2}-3l)]\\
&\qquad - (-1)^{\delta+\epsilon}x^l[T_\nu(\tfrac{9k}{2}, -\tfrac{3k}{2}-3l) - (-1)^\epsilon x^{k+l}T_\nu(\tfrac{9k}{2}, \tfrac{9k}{2}+3l)]\\
&\quad= [T_\nu(\tfrac{9k}{2}, -\tfrac{3k}{2}+3l) - (-1)^{\delta+\epsilon}x^l T_\nu(\tfrac{9k}{2}, -\tfrac{3k}{2}-3l)]\\
&\qquad - (-1)^\epsilon x^{k-l}[T_\nu(\tfrac{9k}{2}, \tfrac{9k}{2}-3l) - (-1)^{\delta+\epsilon}x^{3l}T_\nu(\tfrac{9k}{2}, \tfrac{9k}{2}+3l)]\\
&\quad= [T_\mu(\tfrac{9k}{2}, \tfrac{3k}{2}-3l) - (-1)^{\delta+\epsilon}x^l\, T_\mu(\tfrac{9k}{2}, \tfrac{3k}{2}+3l)]\\
&\qquad - (-1)^\epsilon x^{k-l}[T_\nu(\tfrac{9k}{2}, \tfrac{9k}{2}-3l) - T_\nu(\tfrac{9k}{2}, \tfrac{9k}{2}-3l)]\\
&\quad= Q_\mu(3k,l). \quad\square
\end{aligned}$$

We would like to conclude this work by expressing our appreciation to Greg Manning for his programming assistance.

References

1. C. Adiga, B. C. Berndt, S. Bhargava, and G. N. Watson, *Chapter* 16 *of Ramanujan's second notebook: theta-functions and q-series*, Mem. Amer. Math. Soc., **53** (1985), Providence, RI. MR **86e:**33004
2. R. Blecksmith, J. Brillhart, and I. Gerst, Some infinite product identities, *Math. Comp.*, **51** (1988), 301–314. MR **89f:**05017
3. R. Blecksmith, J. Brillhart, and I. Gerst, On a certain (mod 2) identity and a method of proof by expansion, *Math. Comp.*, **56** (1991), 775–794. MR **91j:**11087
4. R. Blecksmith, J. Brillhart, and I. Gerst, New proofs for two infinite product identities, *Rocky Mountain J. Math.*, **22** (1992), 819–823. MR **93h:**11114
5. R. Blecksmith, J. Brillhart, and I. Gerst, A fundamental modular identity and some applications, *Math. Comp.*, **61** (1993), 83–95. MR **94c:**11100
6. R. Blecksmith, J. Brillhart, and I. Gerst, Four identities involving the single-variable quintuple product, *Abstracts AMS*, **14** (1993), 584.
7. J. M. Borwein and P. B. Borwein, *Pi and the AGM*, Wiley & Sons, New York, 1987. MR **89a:**11134
8. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, fourth ed., Clarendon Press, Oxford, 1960.
9. L. Slater, Further identities of the Rogers-Ramanujan type, *Proc. London Math. Soc.* (2), **54** (1952), 147–167. MR **14:**138e

Department of Mathematical Sciences, Northern Illinois University, DeKalb, Illinois 60115

E-mail address: richard@math.niu.edu

Department of Mathematics, University of Arizona, Tucson, Arizona 85721

E-mail address: jdb@math.arizona.edu