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RELATIVE INVARIANTS OF SOME 2-SIMPLE PREHOMOGENEOUS VECTOR SPACES

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ABSTRACT. In this paper, we shall construct explicitly irreducible relative invariants of two 2-simple prehomogeneous vector spaces. Together with a preprint by the same authors, this completes the list of all relative invariants of regular 2-simple prehomogeneous vector spaces of type I.

1. INTRODUCTION

Let G be a connected reductive algebraic group defined over the complex number field $\mathbb{C}$, V a finite dimensional vector space, and $\rho : G \longrightarrow GL(V)$ a rational representation of G. Such a triplet (G, ρ, V) is called a prehomogeneous vector space (abbreviated P.V.) if V has an open G-orbit, and a triplet is called irreducible if ρ is an irreducible representation. Furthermore, such a triplet (G, ρ, V) is called simple (resp. n-simple) if the derived subgroup $[G, G]$ is a simple algebraic group (resp. the product of n-simple algebraic groups). A nonzero rational function $F(x)$ is called a relative invariant corresponding to a character $\chi : G \to GL_1$ if it satisfies the relation $F(\rho(g)x) = \chi(g)F(x)$ as a rational function for all $g \in G$.

A complete list of irreducible prehomogeneous vector spaces is given by M. Sato and T. Kimura in [1]. At the same time, the relative invariants are constructed for almost all of these spaces. However, for some complicated prehomogeneous vector spaces, such as classification numbers (6), (7), (10), (20), (21) and (24) in [1], the construction of relative invariants had not been settled. In 1971, an irreducible relative invariant of (20) was constructed in [6] and, in 1981, that of (6), (7) was constructed in [3]. In 1990, relative invariants for (10), (21), (24) were constructed in [7] by some complicated calculations. In 1995, that of (10) and (21) was constructed in [8] by using the notion of the quotient space. For the case of nonirreducible prehomogeneous vector spaces, in 1983, T. Kimura studied the case of nonirreducible simple prehomogeneous vector spaces. In 1988, T. Kimura, S. Kasai, M. Inuzuka and O. Yasukura [9] completed the classifications of nonirreducible reduced 2-simple prehomogeneous vector spaces of type I. See [9] for the definition of type I and type II.

Recently, the relative invariants are constructed in [10] and [11] for almost all of these spaces of type I except for the following two cases:

(regular 8) $(GL_1^2 \times SL_5 \times SL_8, \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*, \mathrm{Alt}_5^{\oplus 8} \oplus V(8)^*)$,

(regular 40) $(GL_1^2 \times \mathrm{Spin}_{10} \times SL_{14}, (\text{a half-spin rep.}) \otimes \Lambda_1 + 1 \otimes \Lambda_1^*, V(16)^{\oplus 14} \oplus V(14)^*)$.

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These cases are the most complicated and difficult cases in [9]. The purpose of this paper is to construct explicitly irreducible relative invariants of the above 2 cases and to complete the construction of irreducible relative invariants for all 2-simple prehomogeneous vector spaces of type I. In this paper, we have reduced this construction problem to determine some polynomials. Although we used the computer software *Mathematica* [12] to decide the explicit form of polynomials, once we obtain them, it is not necessary to use the computer to check them.

2. Notations and preliminaries

We denote by Alt_n (resp. Sym_n) the totality of $n \times n$ alternating matrices (resp. $n \times n$ symmetric matrices). For $X \in \mathrm{Alt}_{2n}$, let $Pf(X)$ be the Pfaffian of X so that we have $Pf(X)^2 = \det X$ and $Pf(AX{}^tA) = \det A \cdot Pf(X)$ for $A \in GL_{2n}$. We denote Λ' (resp. χ) the even half-spin representation (resp. the vector representation) of Spin_{10}. Note that the dual representation Λ'^* of Λ' is the odd half-spin representation of Spin_{10}. For the infinitesimal representation of Λ', see (5.38) in [1].

For $X \in M_n$ (=the totality of $n \times n$ matrices), let $\Delta(X)$ be the cofactor matrix of X so that we have $X \cdot \Delta(X) = \Delta(X) \cdot X = \det X \cdot I_n$.

When we prove the irreducibility of relative invariants, we often use the following facts.

Lemma 2.1 (cf. [1]). *Let (G, ρ, V) be a P.V. with a generic point v_0.*

(1) *For a character $\chi : G \to GL_1$, there exists a relative invariant corresponding to χ if and only if $\chi|_{G_{v_0}} = 1$, where $G_{v_0} = \{g \in G; \rho(g)v_0 = v_0\}$.*
(2) *Any irreducible component of a relative invariant is also a relative invariant.*

Lemma 2.2 (cf. [1], §4 Proposition 18). *There is a one-to-one correspondence between relative invariants $f(x)$ of $(G \times SL_n, \rho \otimes \Lambda_1, V(m) \otimes V(n))$ $(m > n \geq 1)$ and relative invariants $\tilde{f}(\tilde{x})$ of its castling transform $(G \times SL_{m-n}, \rho^* \otimes \Lambda_1, V(m)^* \otimes V(m-n))$. Moreover, there exists a positive integer d for each $f(x)$ such that $deg f(x) = nd$ and $deg \tilde{f}(\tilde{x}) = (m-n)d$. If f is irreducible, then $\tilde{f}$ is also irreducible.*

Moreover, we prove the following lemma to construct some G-equivariant mapping in the first example.

Lemma 2.3. *For $X = (x_{ij}) = (x_1 \mid \cdots \mid x_n) \in M(n+2, n)$, let $X^{(i,j)} \in M(n)$ be the matrix obtained from X by subtracting i-th and j-th rows. For $i < j$, put $\tilde{x_{ij}} = (-1)^{i+j+1} \det X^{(i,j)}$ and define the alternating matrix $\varphi(X) = (\tilde{x_{ij}}) \in \mathrm{Alt}_{n+2}$. Then the map $\varphi : M(n+2, n) \to \mathrm{Alt}_{n+2}$ satisfies $\varphi(AX) = \det A \cdot {}^tA^{-1}\varphi(X)A^{-1}$ for $A \in GL_{n+2}$.*

Proof. Put $Y = AX = (y_1 \mid \cdots \mid y_n)$. For $i < j$, we have

$$\begin{aligned} &e_i \wedge e_j \wedge x_1 \wedge \cdots \wedge x_n \\ &\quad = \det X^{(i,j)} e_i \wedge e_j \wedge e_1 \wedge \cdots \wedge e_{i-1} \wedge e_{i+1} \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_{n+2} \\ &\quad = \tilde{x_{ij}} e_1 \wedge \cdots \wedge e_{n+2}. \end{aligned}$$

Then, by the action of $A = (a_{ij}) \in GL_{n+2}$, we have

$$\begin{aligned}\tilde{x}_{ij} \det A \cdot e_1 \wedge \cdots \wedge e_{n+2} &= \tilde{x_{ij}}(Ae_1) \wedge \cdots \wedge (Ae_{n+2}) \\ &= \left(\sum_{l=1}^{n+2} e_l a_{li}\right) \wedge \left(\sum_{k=1}^{n+2} e_k a_{ki}\right) \wedge y_1 \wedge \cdots \wedge y_n \\ &= \sum_{l<k} a_{li}a_{kj}e_l \wedge e_k \wedge y_1 \wedge \cdots \wedge y_n - \sum_{k<l} a_{li}a_{kj}e_k \wedge e_l \wedge y_1 \wedge \cdots \wedge y_n \\ &= \sum_{l<k}(a_{li}a_{kj} - a_{ki}a_{lj})e_l \wedge e_k \wedge y_1 \wedge \cdots \wedge y_n \\ &= \sum_{l<k}(a_{li}a_{kj} - a_{ki}a_{lj})\tilde{y_{lk}} \wedge e_1 \wedge \cdots \wedge e_{n+2}.\end{aligned}$$

Hence, we have $\sum_{l<k}(a_{li}a_{kj} - a_{ki}a_{lj})\tilde{y_{lk}} = \det A \cdot \tilde{x_{ij}}$, which is equivalent to ${}^tA\varphi(Y)A = \det A \cdot \varphi(X)$. This implies that $\varphi(AX) = \det A \cdot {}^tA^{-1}\varphi(X)A^{-1}$. □

3. Explicit construction of irreducible relative invariants

In this section, H denotes the generic isotropy subgroup of a P.V. (G, ρ, V), and $H_1 \sim H_2$ implies that H_1 and H_2 are locally isomorphic, namely their Lie algebras are isomorphic. N denotes the number of the basic irreducible relative invariants.

3.1. **Explicit construction of irreducible relative invariants of** $(GL_1^2 \times SL_5 \times SL_8, \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$ **with** $H \sim SO_2,\ N = 2$**.** To investigate relative invariants, we may assume that $G = GL_5 \times GL_8$ acts on $V = \mathrm{Alt}_5^{\oplus 8} \oplus M(8,1)$ by

$$x = ((X_1, X_2, \cdots, X_8), Y) \mapsto ((AX_1{}^tA, \cdots, AX_8{}^tA){}^tB, {}^tB^{-1}Y)$$

for $x = ((X_1, \cdots, X_8), Y) \in V$ and $g = (A, B) \in G$. We shall construct the irreducible relative invariants of this prehomogeneous vector space by the following steps.

Step 1. For $Y = {}^t(y_1, \cdots, y_8) \in M(8,1)$, we put $X \cdot Y = \sum_{i=1}^{8} y_i X_i \in \mathrm{Alt}_5$. Then we obtain that

$$X \cdot Y \mapsto AX \cdot Y{}^tA \tag{3.1}$$

and hence we have

$$\Delta(X \cdot Y) \mapsto (\det A)^{2t}A^{-1}\Delta(X \cdot Y)A^{-1}, \tag{3.2}$$

where $\Delta(X \cdot Y)$ is the cofactor matrix of the odd size alternating matrix the $X \cdot Y \in \mathrm{Alt}_5$. Note that $\Delta(X \cdot Y) \in \mathrm{Sym}_5$.

Step 2. For

$$X_i = \begin{pmatrix} 0 & x_{12}^{(i)} & x_{13}^{(i)} & x_{14}^{(i)} & x_{15}^{(i)} \\ -x_{12}^{(i)} & 0 & x_{23}^{(i)} & x_{24}^{(i)} & x_{25}^{(i)} \\ -x_{13}^{(i)} & -x_{23}^{(i)} & 0 & x_{34}^{(i)} & x_{35}^{(i)} \\ -x_{14}^{(i)} & -x_{24}^{(i)} & -x_{34}^{(i)} & 0 & x_{45}^{(i)} \\ -x_{15}^{(i)} & -x_{25}^{(i)} & -x_{35}^{(i)} & -x_{45}^{(i)} & 0 \end{pmatrix} \in \mathrm{Alt}_5 \ (i = 1, 2, \cdots, 8),$$

we put $\tilde{X}_i := {}^t(x_{12}^{(i)}, x_{13}^{(i)}, x_{14}^{(i)}, x_{15}^{(i)}, x_{23}^{(i)}, x_{24}^{(i)}, x_{25}^{(i)}, x_{34}^{(i)}, x_{35}^{(i)}, x_{45}^{(i)}) \in M(10,1)$.

The action $X_i \mapsto AX_i{}^tA$ induces $\tilde{X}_i \mapsto \Lambda_2(A)\tilde{X}_i$ $(\Lambda_2(A) \in GL_{10})$. Then we put $Z = (\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_8) \in M(10,8)$ and define the 10×10-alternating matrix $\tilde{Z} = (z_{ij})_{1 \le i < j \le 10}$ with $z_{ij} = (-1)^{i+j} \det Z^{(i,j)}$, where $Z^{(i,j)}$ is the 8×8-matrix obtained from Z by substracting the i-th and the j-th rows. Then, by Lemma 2.3, we have $\tilde{Z} \mapsto (\det \Lambda_2(A))(\det B)^t\Lambda_2(A)^{-1}\tilde{Z}\Lambda_2(A)^{-1}$. Note that $\det \Lambda_2(A) = (\det A)^4$.

Step 3. If we put $\Phi(\tilde{Z}) := (\varphi(\tilde{Z})_{i,j})_{1 \le i \le j \le 5} \in \mathrm{Sym}_5$ for $\tilde{Z} \in \mathrm{Alt}_{10}$ with the following entries, then we have Lemma 3.1.

$$
\begin{aligned}
\varphi_{11} &= z_{510}^2 + z_{69}^2 + z_{78}^2 - 2z_{59}z_{610} - 2z_{68}z_{79} + 2z_{58}z_{710} - 2z_{67}z_{89} + 2z_{57}z_{810} - 2z_{56}z_{910},\\
\varphi_{22} &= z_{210}^2 + z_{39}^2 + z_{48}^2 - 2z_{29}z_{310} + 2z_{28}z_{410} - 2z_{38}z_{49} + 2z_{24}z_{810} - 2z_{34}z_{89} - 2z_{23}z_{910},\\
\varphi_{33} &= z_{110}^2 + z_{37}^2 + z_{46}^2 - 2z_{17}z_{310} - 2z_{36}z_{47} + 2z_{16}z_{410} - 2z_{34}z_{67} + 2z_{14}z_{610} - 2z_{13}z_{710},\\
\varphi_{44} &= z_{19}^2 + z_{27}^2 + z_{45}^2 - 2z_{17}z_{29} - 2z_{25}z_{47} + 2z_{15}z_{49} - 2z_{24}z_{57} + 2z_{14}z_{59} - 2z_{12}z_{79},\\
\varphi_{55} &= z_{18}^2 + z_{26}^2 + z_{35}^2 - 2z_{16}z_{28} - 2z_{25}z_{36} + 2z_{15}z_{38} - 2z_{23}z_{56} + 2z_{13}z_{58} - 2z_{12}z_{68},\\
\varphi_{12} &= -z_{210}z_{510} - z_{410}z_{58} + z_{310}z_{59} + z_{29}z_{610} + z_{49}z_{68} - z_{39}z_{69}\\
&\quad - z_{28}z_{710} - z_{48}z_{78} + z_{38}z_{79} - z_{27}z_{810} + z_{45}z_{810} + z_{37}z_{89} - z_{46}z_{89} + z_{26}z_{910} - z_{35}z_{910},\\
\varphi_{13} &= z_{110}z_{510} + z_{410}z_{56} - z_{310}z_{57} - z_{19}z_{610} - z_{45}z_{610} + z_{39}z_{67}\\
&\quad - z_{48}z_{67} - z_{47}z_{68} + z_{46}z_{69} + z_{18}z_{710} + z_{35}z_{710} + z_{37}z_{78} - z_{36}z_{79} + z_{17}z_{810} - z_{16}z_{910},\\
\varphi_{14} &= z_{45}z_{510} - z_{49}z_{56} + z_{210}z_{57} + z_{48}z_{57} + z_{47}z_{58} - z_{110}z_{59}\\
&\quad - z_{46}z_{59} - z_{29}z_{67} + z_{19}z_{69} - z_{25}z_{710} - z_{27}z_{78} - z_{18}z_{79} + z_{26}z_{79} - z_{17}z_{89} + z_{15}z_{910},\\
\varphi_{15} &= -z_{35}z_{510} - z_{210}z_{56} + z_{39}z_{56} - z_{38}z_{57} + z_{110}z_{58} - z_{37}z_{58}\\
&\quad + z_{36}z_{59} + z_{25}z_{610} + z_{28}z_{67} - z_{19}z_{68} + z_{27}z_{68} - z_{26}z_{69} + z_{18}z_{78} - z_{15}z_{810} + z_{16}z_{89},\\
\varphi_{23} &= -z_{110}z_{210} + z_{19}z_{310} + z_{27}z_{310} - z_{37}z_{39} - z_{18}z_{410} - z_{26}z_{410}\\
&\quad + z_{38}z_{47} - z_{46}z_{48} + z_{36}z_{49} - z_{24}z_{610} + z_{34}z_{69} + z_{23}z_{710} - z_{34}z_{78} - z_{14}z_{810} + z_{13}z_{910},\\
\varphi_{24} &= -z_{210}z_{27} + z_{110}z_{29} + z_{29}z_{37} - z_{19}z_{39} + z_{25}z_{410} - z_{28}z_{47}\\
&\quad + z_{45}z_{48} + z_{18}z_{49} - z_{35}z_{49} + z_{24}z_{510} - z_{34}z_{59} + z_{24}z_{78} - z_{23}z_{79} + z_{14}z_{89} - z_{12}z_{910},\\
\varphi_{25} &= z_{210}z_{26} - z_{110}z_{28} - z_{25}z_{310} - z_{29}z_{36} + z_{19}z_{38} + z_{35}z_{39}\\
&\quad - z_{38}z_{45} + z_{28}z_{46} - z_{18}z_{48} - z_{23}z_{510} + z_{34}z_{58} - z_{24}z_{68} + z_{23}z_{69} + z_{12}z_{810} - z_{13}z_{89},\\
\varphi_{34} &= -z_{110}z_{19} + z_{17}z_{210} - z_{27}z_{37} + z_{17}z_{39} - z_{15}z_{410} - z_{45}z_{46}\\
&\quad + z_{26}z_{47} + z_{35}z_{47} - z_{16}z_{49} - z_{14}z_{510} + z_{34}z_{57} + z_{24}z_{67} - z_{14}z_{69} + z_{12}z_{710} + z_{13}z_{79},\\
\varphi_{35} &= z_{110}z_{18} - z_{16}z_{210} + z_{15}z_{310} + z_{27}z_{36} - z_{35}z_{37} - z_{17}z_{38}\\
&\quad + z_{36}z_{45} - z_{26}z_{46} + z_{16}z_{48} + z_{13}z_{510} - z_{34}z_{56} - z_{12}z_{610} - z_{23}z_{67} + z_{14}z_{68} - z_{13}z_{78},\\
\varphi_{45} &= -z_{18}z_{19} - z_{26}z_{27} + z_{17}z_{28} + z_{16}z_{29} + z_{25}z_{37} - z_{15}z_{39}\\
&\quad - z_{35}z_{45} + z_{25}z_{46} - z_{15}z_{48} + z_{24}z_{56} + z_{23}z_{57} - z_{14}z_{58} - z_{13}z_{59} + z_{12}z_{69} + z_{12}z_{78}.
\end{aligned}
$$

Lemma 3.1. *For every $A \in GL_5, B \in GL_8$, $\tilde{Z} \in \mathrm{Alt}_{10}$, we have*

$$\tilde{Z} \mapsto (\det A)^4(\det B)^t\Lambda_2(A)^{-1}\tilde{Z}\Lambda_2(A)^{-1}$$

and

$$\Phi((\det A)^4(\det B)\ {}^t\Lambda_2(A)^{-1}\tilde{Z}\Lambda_2(A)^{-1}) = (\det A)^6(\det B)^2 \cdot A\Phi(\tilde{Z})^tA. \tag{3.3}$$

Proof. It is enough to prove the equivariance (3.3) in the case when A is one of the fundamental matrices

$$A_u = \begin{pmatrix} 1 & \varepsilon & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad A_d = \mathrm{diag}(a,1,1,1,1),$$

or permutation matrices.

Checking (3.3) for diagonal or permutation matrices is easy. Note that $\det A_d = a$ and $\Phi(a^{4t}\Lambda_2(A_d)^{-1}\tilde{Z}\Lambda_2(A_d)^{-1}) = a^6 A_d \Phi(\tilde{Z})^t A_d$. For A_u, we consider the action of A_u. Since $\det A_u = 1$,

$$\Phi(\tilde{Z}) \mapsto \Phi({}^t\Lambda_2(A_u)^{-1}\tilde{Z}\Lambda_2(A_u)^{-1}).$$

Then we have $\varphi_{11} \mapsto \varphi_{11} + 2\varepsilon\varphi_{12} + \varepsilon^2\varphi_{22}$, $\varphi_{1j} \mapsto \varphi_{1j} + \varepsilon\varphi_{2j}$ $(2 \le j \le 5)$, $\varphi_{lk} \mapsto \varphi_{lk}$ $(2 \le l \le k \le 5)$. Hence, we have $\Phi({}^t\Lambda_2(A_u)^{-1}\tilde{Z}\Lambda_2(A_u)^{-1}) = A_u\Phi(\tilde{Z})^t A_u$. □

Remark A. By using the computer software *Mathematica* [12], we calculate the above polynomials φ_{ij} $(1 \le i \le j \le 5)$ along the following program.

(i) First, we construct the polynomial φ_{11}. For the action of the diagonal matrix $\mathrm{diag}(a_1,a_2,a_3,a_4,a_5)$, denote by T_1 the polynomial corresponding to the weight $a_1^2(\det A)^{10}$:

$$\begin{aligned} T_1 = {} & k_1 z_{56}z_{910} + k_2 z_{57}z_{810} + k_3 z_{58}z_{710} + k_4 z_{59}z_{610} + k_5 z_{510}^2 + k_6 z_{510}z_{69} \\ & + k_7 z_{510}z_{78} + k_8 z_{69}^2 + k_9 z_{69}z_{78} + k_{10} z_{78}^2 + k_{11} z_{67}z_{89} + k_{12} z_{68}z_{79}. \end{aligned}$$

(ii) Next, we calculate the invariant polynomial TT_1 under the action of permutation matrices (23), (24), (25), (35), (45) on T_1:

$$\begin{aligned} TT_1 = {} & l_1(z_{56}z_{910} + z_{67}z_{89} - z_{57}z_{810} + z_{68}z_{79} - z_{58}z_{710} + z_{59}z_{610}) \\ & + l_2(z_{510}^2 + z_{69}^2 + z_{78}^2). \end{aligned}$$

(iii) Next, we calculate the invariant polynomial TTT_1 under the action of unipotent matrices:

$$\begin{pmatrix} 1 & & & & \\ & 1 & \varepsilon & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & \varepsilon & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \varepsilon \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & \varepsilon & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \varepsilon \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \varepsilon \\ & & & & 1 \end{pmatrix},$$

and we can construct the polynomial φ_{11} uniquely up to the constant

$$\begin{aligned} \varphi_{11} = {} & z_{510}^2 + z_{69}^2 + z_{78}^2 - 2z_{59}z_{610} - 2z_{68}z_{79} \\ & - 2z_{58}z_{710} - 2z_{67}z_{89} - 2z_{57}z_{810} - 2z_{56}z_{910}. \end{aligned}$$

(iv) From the explicit form of φ_{11}, we can construct the other φ_{ij} by the action of the generators of GL_5 on φ_{11}.

Step 4. If we put $\Psi(\tilde{Z}) := (\psi(\tilde{z})_{i,j})_{1\le i<j\le 5} \in \mathrm{Alt}_5$ for $\tilde{Z} \in \mathrm{Alt}_{10}$ with the entries below, then we have Lemma 3.2.

$$\begin{aligned}
\psi_{12} = &-z_{110}z_{15}z_{210} + z_{110}^2 z_{25} - z_{110}z_{19}z_{26} + 2z_{17}z_{210}z_{26} + z_{110}z_{18}z_{27}\\
&- 2z_{16}z_{210}z_{27} - z_{110}z_{17}z_{28} + z_{110}z_{16}z_{29} + z_{15}z_{19}z_{310} - 3z_{17}z_{25}z_{310}\\
&+ 2z_{15}z_{27}z_{310} - z_{110}z_{19}z_{35} + z_{17}z_{210}z_{35} + z_{19}^2 z_{36} + 2z_{27}^2 z_{36}\\
&- 3z_{17}z_{29}z_{36} - z_{18}z_{19}z_{37} - 2z_{26}z_{27}z_{37} + z_{17}z_{28}z_{37} + 2z_{16}z_{29}z_{37}\\
&- 2z_{27}z_{35}z_{37} + 2z_{25}z_{37}^2 + z_{17}z_{19}z_{38} - z_{17}z_{27}z_{38} - z_{16}z_{19}z_{39} + z_{17}z_{26}z_{39}\\
&+ 2z_{17}z_{35}z_{39} - 2z_{15}z_{37}z_{39} - z_{15}z_{18}z_{410} + 3z_{16}z_{25}z_{410}\\
&- 2z_{15}z_{26}z_{410} - z_{15}z_{35}z_{410} + z_{110}z_{18}z_{45} - z_{16}z_{210}z_{45} + z_{15}z_{310}z_{45}\\
&+ 2z_{27}z_{36}z_{45} - 2z_{35}z_{37}z_{45} - 2z_{17}z_{38}z_{45} + 2z_{36}z_{45}^2 - z_{18}z_{19}z_{46} - 2z_{26}z_{27}z_{46}\\
&+ 2z_{17}z_{28}z_{46} + z_{16}z_{29}z_{46} + 2z_{25}z_{37}z_{46} - z_{15}z_{39}z_{46} - 2z_{26}z_{45}z_{46} - 2z_{35}z_{45}z_{46}\\
&+ 2z_{25}z_{46}^2 + z_{18}^2 z_{47} + 2z_{26}^2 z_{47} - 3z_{16}z_{28}z_{47} + 2z_{26}z_{35}z_{47} + 2z_{35}^2 z_{47} - 6z_{25}z_{36}z_{47}\\
&+ 3z_{15}z_{38}z_{47} - z_{17}z_{18}z_{48} + z_{16}z_{27}z_{48} - z_{15}z_{37}z_{48} + 2z_{16}z_{45}z_{48}\\
&- 2z_{15}z_{46}z_{48} + z_{16}z_{18}z_{49} - z_{16}z_{26}z_{49} - 2z_{16}z_{35}z_{49}\\
&+ 3z_{15}z_{36}z_{49} + z_{110}z_{12}z_{510} - z_{17}z_{23}z_{510} + z_{16}z_{24}z_{510} - z_{14}z_{26}z_{510}\\
&+ z_{13}z_{27}z_{510} - 2z_{14}z_{35}z_{510} + 2z_{13}z_{45}z_{510} - z_{14}z_{210}z_{56} - z_{27}z_{34}z_{56}\\
&+ z_{24}z_{37}z_{56} + z_{14}z_{39}z_{56} + z_{12}z_{410}z_{56} - z_{34}z_{45}z_{56} + z_{24}z_{46}z_{56}\\
&- 2z_{23}z_{47}z_{56} - z_{13}z_{49}z_{56} + z_{13}z_{210}z_{57} - z_{12}z_{310}z_{57} + z_{26}z_{34}z_{57} + z_{34}z_{35}z_{57}\\
&- 2z_{24}z_{36}z_{57} + z_{23}z_{37}z_{57} - z_{14}z_{38}z_{57} + z_{23}z_{46}z_{57} + z_{13}z_{48}z_{57} + z_{110}z_{14}z_{58}\\
&+ z_{17}z_{34}z_{58} - 2z_{14}z_{37}z_{58} - z_{14}z_{46}z_{58} + 3z_{13}z_{47}z_{58} - z_{110}z_{13}z_{59} - z_{16}z_{34}z_{59}\\
&+ 3z_{14}z_{36}z_{59} - z_{13}z_{37}z_{59} - 2z_{13}z_{46}z_{59} - z_{12}z_{19}z_{610} - z_{15}z_{24}z_{610}\\
&+ 3z_{14}z_{25}z_{610} - z_{12}z_{27}z_{610} - 2z_{12}z_{45}z_{610} + z_{24}z_{26}z_{67} - z_{23}z_{27}z_{67} + z_{14}z_{28}z_{67}\\
&- z_{13}z_{29}z_{67} - 2z_{25}z_{34}z_{67} + z_{24}z_{35}z_{67} + z_{12}z_{39}z_{67} - z_{23}z_{45}z_{67} - z_{12}z_{48}z_{67}\\
&- z_{14}z_{19}z_{68} - z_{17}z_{24}z_{68} + 2z_{14}z_{27}z_{68} + z_{14}z_{45}z_{68} - 3z_{12}z_{47}z_{68} + z_{13}z_{19}z_{69}\\
&+ z_{17}z_{23}z_{69} - 2z_{14}z_{26}z_{69} + z_{15}z_{34}z_{69} - z_{14}z_{35}z_{69} + z_{12}z_{37}z_{69}\\
&+ 2z_{12}z_{46}z_{69} + z_{12}z_{18}z_{710} + z_{15}z_{23}z_{710} - 3z_{13}z_{25}z_{710} + z_{12}z_{26}z_{710}\\
&+ 2z_{12}z_{35}z_{710} + z_{14}z_{18}z_{78} + z_{16}z_{24}z_{78} - 2z_{13}z_{27}z_{78} - z_{15}z_{34}z_{78}\\
&+ 2z_{12}z_{37}z_{78} - z_{13}z_{45}z_{78} + z_{12}z_{46}z_{78} - z_{13}z_{18}z_{79} - z_{16}z_{23}z_{79}\\
&+ 2z_{13}z_{26}z_{79} + z_{13}z_{35}z_{79} - 3z_{12}z_{36}z_{79} - 2z_{14}z_{15}z_{810} + 2z_{12}z_{17}z_{810}\\
&+ 2z_{14}z_{16}z_{89} - 2z_{13}z_{17}z_{89} + 2z_{13}z_{15}z_{910} - 2z_{12}z_{16}z_{910},
\end{aligned}$$

$$\begin{aligned}
\psi_{13} = &-z_{15}z_{210}^2 + z_{110}z_{210}z_{25} + z_{19}z_{210}z_{26} - z_{18}z_{210}z_{27} - 2z_{110}z_{19}z_{28} + z_{17}z_{210}z_{28}\\
&+ 2z_{110}z_{18}z_{29} - z_{16}z_{210}z_{29} - 2z_{19}z_{25}z_{310} - z_{25}z_{27}z_{310}\\
&+ 3z_{15}z_{29}z_{310} + z_{210}z_{27}z_{35} - z_{110}z_{29}z_{35} - z_{19}z_{29}z_{36} + z_{27}z_{29}z_{36} - z_{27}z_{28}z_{37}\\
&+ z_{18}z_{29}z_{37} - 2z_{29}z_{35}z_{37} + 2z_{19}^2 z_{38} + z_{27}^2 z_{38} - 3z_{17}z_{29}z_{38} - 2z_{18}z_{19}z_{39}\\
&- z_{26}z_{27}z_{39} + 2z_{17}z_{28}z_{39} + z_{16}z_{29}z_{39} + 2z_{19}z_{35}z_{39} + 2z_{25}z_{37}z_{39}\\
&- 2z_{15}z_{39}^2 + 2z_{18}z_{25}z_{410} + z_{25}z_{26}z_{410} - 3z_{15}z_{28}z_{410} - z_{25}z_{35}z_{410} - z_{210}z_{26}z_{45}\\
&+ z_{110}z_{28}z_{45} + z_{25}z_{310}z_{45} + 2z_{29}z_{36}z_{45} - 2z_{19}z_{38}z_{45} - 2z_{35}z_{39}z_{45}\\
&+ 2z_{38}z_{45}^2 + z_{19}z_{28}z_{46} - z_{26}z_{29}z_{46} + z_{25}z_{39}z_{46} - 2z_{28}z_{45}z_{46} - z_{18}z_{28}z_{47}
\end{aligned}$$

$$
\begin{aligned}
&+ z_{26}z_{28}z_{47} + 2z_{28}z_{35}z_{47} - 3z_{25}z_{38}z_{47} - 2z_{18}z_{19}z_{48} - z_{26}z_{27}z_{48} + z_{17}z_{28}z_{48} \\
&+ 2z_{16}z_{29}z_{48} + z_{25}z_{37}z_{48} - 2z_{15}z_{39}z_{48} + 2z_{18}z_{45}z_{48} - 2z_{35}z_{45}z_{48} \\
&+ 2z_{25}z_{46}z_{48} - 2z_{15}z_{48}^2 + 2z_{18}^2z_{49} + z_{26}^2z_{49} - 3z_{16}z_{28}z_{49} - 2z_{18}z_{35}z_{49} + 2z_{35}^2z_{49} \\
&- 3z_{25}z_{36}z_{49} + 6z_{15}z_{38}z_{49} + z_{12}z_{210}z_{510} - z_{19}z_{23}z_{510} + z_{18}z_{24}z_{510} - z_{14}z_{28}z_{510} \\
&+ z_{13}z_{29}z_{510} - 2z_{24}z_{35}z_{510} + 2z_{23}z_{45}z_{510} - z_{210}z_{24}z_{56} - z_{29}z_{34}z_{56} + 2z_{24}z_{39}z_{56} \\
&+ z_{24}z_{48}z_{56} - 3z_{23}z_{49}z_{56} + z_{210}z_{23}z_{57} + z_{28}z_{34}z_{57} - 3z_{24}z_{38}z_{57} + z_{23}z_{39}z_{57} \\
&+ 2z_{23}z_{48}z_{57} + z_{110}z_{24}z_{58} + z_{19}z_{34}z_{58} - z_{24}z_{37}z_{58} - z_{14}z_{39}z_{58} + z_{12}z_{410}z_{58} \\
&- z_{34}z_{45}z_{58} + z_{23}z_{47}z_{58} - z_{14}z_{48}z_{58} + 2z_{13}z_{49}z_{58} - z_{110}z_{23}z_{59} - z_{12}z_{310}z_{59} \\
&- z_{18}z_{34}z_{59} + z_{34}z_{35}z_{59} + z_{24}z_{36}z_{59} + 2z_{14}z_{38}z_{59} - z_{13}z_{39}z_{59} - z_{23}z_{46}z_{59} \\
&- z_{13}z_{48}z_{59} + 2z_{24}z_{25}z_{610} - 2z_{12}z_{29}z_{610} + 2z_{24}z_{28}z_{67} - 2z_{23}z_{29}z_{67} \\
&- 2z_{19}z_{24}z_{68} + z_{24}z_{27}z_{68} + z_{14}z_{29}z_{68} + z_{24}z_{45}z_{68} - 3z_{12}z_{49}z_{68} + 2z_{19}z_{23}z_{69} \\
&- z_{24}z_{26}z_{69} - z_{14}z_{28}z_{69} - z_{25}z_{34}z_{69} + 2z_{12}z_{39}z_{69} - z_{23}z_{45}z_{69} + z_{12}z_{48}z_{69} \\
&- 2z_{23}z_{25}z_{710} + 2z_{12}z_{28}z_{710} + 2z_{18}z_{24}z_{78} - z_{23}z_{27}z_{78} - z_{13}z_{29}z_{78} + z_{25}z_{34}z_{78} \\
&- z_{24}z_{35}z_{78} + z_{12}z_{39}z_{78} + 2z_{12}z_{48}z_{78} - 2z_{18}z_{23}z_{79} + z_{23}z_{26}z_{79} \\
&+ z_{13}z_{28}z_{79} + z_{23}z_{35}z_{79} - 3z_{12}z_{38}z_{79} + z_{12}z_{19}z_{810} \\
&- 3z_{15}z_{24}z_{810} + z_{14}z_{25}z_{810} + z_{12}z_{27}z_{810} - 2z_{12}z_{45}z_{810} + z_{14}z_{18}z_{89} \\
&- z_{13}z_{19}z_{89} - z_{17}z_{23}z_{89} + z_{16}z_{24}z_{89} + 2z_{15}z_{34}z_{89} \\
&- z_{14}z_{35}z_{89} - z_{12}z_{37}z_{89} + z_{13}z_{45}z_{89} + z_{12}z_{46}z_{89} - z_{12}z_{18}z_{910} \\
&+ 3z_{15}z_{23}z_{910} - z_{13}z_{25}z_{910} - z_{12}z_{26}z_{910} + 2z_{12}z_{35}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{14} = {} & 2z_{110}z_{18}z_{210} - 2z_{16}z_{210}^2 + 2z_{110}z_{210}z_{26} - 2z_{110}^2z_{28} - 2z_{18}z_{19}z_{310} \\
&+ z_{15}z_{210}z_{310} - z_{110}z_{25}z_{310} - z_{19}z_{26}z_{310} - z_{18}z_{27}z_{310} - 2z_{26}z_{27}z_{310} \\
&+ 3z_{17}z_{28}z_{310} + 3z_{16}z_{29}z_{310} + 2z_{210}z_{27}z_{36} \\
&- 2z_{110}z_{29}z_{36} + z_{25}z_{310}z_{37} - z_{210}z_{35}z_{37} - z_{29}z_{36}z_{37} - z_{28}z_{37}^2 \\
&+ 2z_{110}z_{19}z_{38} - 2z_{17}z_{210}z_{38} + z_{27}z_{37}z_{38} - z_{15}z_{310}z_{39} \\
&+ z_{110}z_{35}z_{39} + z_{19}z_{36}z_{39} + z_{18}z_{37}z_{39} + z_{26}z_{37}z_{39} \\
&- z_{17}z_{38}z_{39} - z_{16}z_{39}^2 + 2z_{18}^2z_{410} + 2z_{18}z_{26}z_{410} + 2z_{26}^2z_{410} \\
&- 6z_{16}z_{28}z_{410} + z_{35}^2z_{410} - 3z_{25}z_{36}z_{410} + 3z_{15}z_{38}z_{410} - z_{310}z_{35}z_{45} \\
&+ z_{210}z_{36}z_{45} - z_{110}z_{38}z_{45} - 2z_{210}z_{26}z_{46} + 2z_{110}z_{28}z_{46} \\
&+ 2z_{25}z_{310}z_{46} + z_{29}z_{36}z_{46} - z_{19}z_{38}z_{46} - z_{35}z_{39}z_{46} \\
&+ 2z_{38}z_{45}z_{46} - 2z_{28}z_{46}^2 + 3z_{28}z_{36}z_{47} - z_{1}z_{38}z_{47} \\
&- 2z_{26}z_{38}z_{47} - z_{35}z_{38}z_{47} - 2z_{110}z_{18}z_{48} + 2z_{16}z_{210}z_{48} \\
&- 2z_{15}z_{310}z_{48} - z_{27}z_{36}z_{48} + z_{35}z_{37}z_{48} + z_{17}z_{38}z_{48} - 2z_{36}z_{45}z_{48} + 2z_{18}z_{46}z_{48} \\
&+ 2z_{26}z_{46}z_{48} - 2z_{16}z_{48}^2 - 2z_{18}z_{36}z_{49} - z_{26}z_{36}z_{49} + z_{35}z_{36}z_{49} \\
&+ 3z_{16}z_{38}z_{49} + 2z_{13}z_{210}z_{510} - 2z_{110}z_{23}z_{510} - z_{34}z_{35}z_{510} - z_{24}z_{36}z_{510} \\
&+ z_{14}z_{38}z_{510} + z_{23}z_{46}z_{510} - z_{13}z_{48}z_{510} + z_{24}z_{310}z_{56} - 2z_{210}z_{34}z_{56} + z_{34}z_{39}z_{56} \\
&- 3z_{23}z_{410}z_{56} + z_{34}z_{48}z_{56} + 2z_{23}z_{310}z_{57} - 2z_{34}z_{38}z_{57} - z_{14}z_{310}z_{58} \\
&+ 2z_{110}z_{34}z_{58} - z_{34}z_{37}z_{58} + 3z_{13}z_{410}z_{58} - z_{34}z_{46}z_{58} \\
&- 2z_{13}z_{310}z_{59} + 2z_{34}z_{36}z_{59} - z_{12}z_{210}z_{610} + z_{19}z_{23}z_{610} + z_{18}z_{24}z_{610}
\end{aligned}
$$

$$
\begin{aligned}
&+ z_{24}z_{26}z_{610} - 2z_{14}z_{28}z_{610} - z_{13}z_{29}z_{610} + z_{25}z_{34}z_{610}\\
&+ z_{23}z_{45}z_{610} + z_{12}z_{48}z_{610} - z_{210}z_{23}z_{67}\\
&+ 3z_{28}z_{34}z_{67} - z_{24}z_{38}z_{67} - z_{23}z_{39}z_{67}\\
&+ 2z_{23}z_{48}z_{67} + z_{14}z_{210}z_{68} - z_{110}z_{24}z_{68} - z_{19}z_{34}z_{68} + z_{27}z_{34}z_{68}\\
&- 2z_{12}z_{410}z_{68} + z_{24}z_{46}z_{68} + z_{23}z_{47}z_{68} - z_{14}z_{48}z_{68} - z_{13}z_{49}z_{68}\\
&+ z_{110}z_{23}z_{69} + z_{12}z_{310}z_{69} - z_{18}z_{34}z_{69} - 2z_{26}z_{34}z_{69} + z_{14}z_{38}z_{69}\\
&+ z_{13}z_{39}z_{69} - 2z_{23}z_{46}z_{69} - 2z_{18}z_{23}z_{710} - z_{23}z_{26}z_{710}\\
&+ 3z_{13}z_{28}z_{710} - z_{23}z_{35}z_{710} - z_{12}z_{38}z_{710} - z_{13}z_{210}z_{78} + z_{12}z_{310}z_{78}\\
&+ 2z_{18}z_{34}z_{78} + z_{26}z_{34}z_{78} - z_{24}z_{36}z_{78} - z_{23}z_{37}z_{78} + 2z_{13}z_{48}z_{78}\\
&+ 2z_{23}z_{36}z_{79} - 2z_{13}z_{38}z_{79} + z_{110}z_{12}z_{810} + z_{14}z_{18}z_{810}\\
&- z_{17}z_{23}z_{810} - 2z_{16}z_{24}z_{810} + z_{14}z_{26}z_{810} + z_{13}z_{27}z_{810} - z_{15}z_{34}z_{810}\\
&- z_{13}z_{45}z_{810} - z_{12}z_{46}z_{810} - z_{110}z_{13}z_{89}\\
&+ 3z_{16}z_{34}z_{89} - z_{14}z_{36}z_{89} - z_{13}z_{37}z_{89} + 2z_{13}z_{46}z_{89} - z_{13}z_{18}z_{910}\\
&+ 3z_{16}z_{23}z_{910} - 2z_{13}z_{26}z_{910} + z_{13}z_{35}z_{910} + z_{12}z_{36}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{15} = {}& 2z_{110}z_{19}z_{210} - 2z_{17}z_{210}^2 + 2z_{110}z_{210}z_{27} - 2z_{110}^2z_{29} - 2z_{19}^2z_{310}\\
&- 2z_{19}z_{27}z_{310} - 2z_{27}^2z_{310} + 6z_{17}z_{29}z_{310} + 2z_{210}z_{27}z_{37} - 2z_{110}z_{29}z_{37} - 2z_{29}z_{37}^2\\
&+ 2z_{110}z_{19}z_{39} - 2z_{17}z_{210}z_{39} + 2z_{19}z_{37}z_{39} + 2z_{27}z_{37}z_{39} - 2z_{17}z_{39}^2\\
&+ 2z_{18}z_{19}z_{410} + z_{15}z_{210}z_{410} - z_{110}z_{25}z_{410} + z_{19}z_{26}z_{410} + z_{18}z_{27}z_{410}\\
&+ 2z_{26}z_{27}z_{410} - 3z_{17}z_{28}z_{410} - 3z_{16}z_{29}z_{410} - 2z_{25}z_{37}z_{410}\\
&+ 2z_{15}z_{39}z_{410} + z_{35}z_{410}z_{45} - z_{310}z_{45}^2\\
&- z_{25}z_{410}z_{46} + z_{210}z_{45}z_{46} + z_{39}z_{45}z_{46} - z_{29}z_{46}^2 - 2z_{210}z_{26}z_{47} + 2z_{110}z_{28}z_{47}\\
&+ 3z_{25}z_{310}z_{47} - z_{210}z_{35}z_{47} + 3z_{29}z_{36}z_{47} + z_{28}z_{37}z_{47}\\
&- 2z_{19}z_{38}z_{47} - z_{27}z_{38}z_{47} - z_{26}z_{39}z_{47}\\
&- 2z_{35}z_{39}z_{47} + z_{38}z_{45}z_{47} - z_{28}z_{46}z_{47} + z_{15}z_{410}z_{48} - z_{110}z_{45}z_{48} - z_{37}z_{45}z_{48}\\
&+ z_{19}z_{46}z_{48} + z_{27}z_{46}z_{48} + z_{18}z_{47}z_{48} - z_{17}z_{48}^2 - 2z_{110}z_{18}z_{49} + 2z_{16}z_{210}z_{49}\\
&- 3z_{15}z_{310}z_{49} + z_{110}z_{35}z_{49} - z_{19}z_{36}z_{49} - 2z_{27}z_{36}z_{49} - z_{18}z_{37}z_{49} + 2z_{35}z_{37}z_{49}\\
&+ 3z_{17}z_{38}z_{49} + z_{16}z_{39}z_{49} - z_{36}z_{45}z_{49} + z_{26}z_{46}z_{49} - z_{16}z_{48}z_{49} + 2z_{14}z_{210}z_{510}\\
&- 2z_{110}z_{24}z_{510} - z_{24}z_{37}z_{510} + z_{14}z_{39}z_{510} - z_{34}z_{45}z_{510} + z_{23}z_{47}z_{510}\\
&- z_{13}z_{49}z_{510} - 2z_{24}z_{410}z_{56} + 2z_{34}z_{49}z_{56} + 3z_{24}z_{310}z_{57} - 2z_{210}z_{34}z_{57}\\
&- z_{34}z_{39}z_{57} - z_{23}z_{410}z_{57} - z_{34}z_{48}z_{57}\\
&+ 2z_{14}z_{410}z_{58} - 2z_{34}z_{47}z_{58} - 3z_{14}z_{310}z_{59} + 2z_{110}z_{34}z_{59}\\
&+ z_{34}z_{37}z_{59} + z_{13}z_{410}z_{59} + z_{34}z_{46}z_{59}\\
&+ 2z_{19}z_{24}z_{610} + z_{24}z_{27}z_{610} - 3z_{14}z_{29}z_{610} + z_{24}z_{45}z_{610}\\
&+ z_{12}z_{49}z_{610} - z_{210}z_{24}z_{67} + 3z_{29}z_{34}z_{67}\\
&- 2z_{24}z_{39}z_{67} + z_{24}z_{48}z_{67} + z_{23}z_{49}z_{67} + 2z_{24}z_{47}z_{68} - 2z_{14}z_{49}z_{68} + z_{14}z_{210}z_{69}\\
&- 2z_{19}z_{34}z_{69} - z_{27}z_{34}z_{69} + 2z_{14}z_{39}z_{69} - z_{12}z_{410}z_{69}\\
&- z_{24}z_{46}z_{69} - z_{23}z_{47}z_{69} - z_{12}z_{210}z_{710}\\
&- z_{19}z_{23}z_{710} - z_{18}z_{24}z_{710} - z_{23}z_{27}z_{710} + z_{14}z_{28}z_{710}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2z_{13}z_{29}z_{710} + z_{25}z_{34}z_{710} - z_{24}z_{35}z_{710}\\
&- z_{12}z_{39}z_{710} - z_{110}z_{24}z_{78} + z_{19}z_{34}z_{78}\\
&+ 2z_{27}z_{34}z_{78} - 2z_{24}z_{37}z_{78} - z_{12}z_{410}z_{78}\\
&+ z_{14}z_{48}z_{78} + z_{13}z_{49}z_{78} - z_{13}z_{210}z_{79} + z_{110}z_{23}z_{79}\\
&+ 2z_{12}z_{310}z_{79} + z_{18}z_{34}z_{79} - z_{26}z_{34}z_{79}\\
&+ z_{24}z_{36}z_{79} + z_{23}z_{37}z_{79} - z_{14}z_{38}z_{79} - z_{13}z_{39}z_{79} + z_{14}z_{19}z_{810} - 3z_{17}z_{24}z_{810}\\
&+ 2z_{14}z_{27}z_{810} - z_{14}z_{45}z_{810} - z_{12}z_{47}z_{810}\\
&- z_{110}z_{14}z_{89} + 3z_{17}z_{34}z_{89} - 2z_{14}z_{37}z_{89}\\
&+ z_{14}z_{46}z_{89} + z_{13}z_{47}z_{89} + z_{110}z_{12}z_{910}\\
&- z_{13}z_{19}z_{910} + 2z_{17}z_{23}z_{910} + z_{16}z_{24}z_{910}\\
&- z_{14}z_{26}z_{910} - z_{13}z_{27}z_{910} - z_{15}z_{34}z_{910} + z_{14}z_{35}z_{910} + z_{12}z_{37}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{23} = &-z_{15}z_{210}z_{510} + z_{110}z_{25}z_{510} + z_{19}z_{35}z_{510} - z_{27}z_{35}z_{510}\\
&- z_{18}z_{45}z_{510} + z_{26}z_{45}z_{510} + z_{12}z_{510}^2 + z_{19}z_{210}z_{56}\\
&- 2z_{210}z_{27}z_{56} + z_{110}z_{29}z_{56} + z_{29}z_{37}z_{56} - 2z_{19}z_{39}z_{56}\\
&+ z_{27}z_{39}z_{56} + 2z_{25}z_{410}z_{56} - z_{28}z_{47}z_{56} + z_{45}z_{48}z_{56}\\
&+ 2z_{18}z_{49}z_{56} - z_{26}z_{49}z_{56} - z_{35}z_{49}z_{56} + z_{24}z_{510}z_{56}\\
&- z_{18}z_{210}z_{57} + 2z_{210}z_{26}z_{57} - z_{110}z_{28}z_{57} - 2z_{25}z_{310}z_{57}\\
&- z_{29}z_{36}z_{57} + 2z_{19}z_{38}z_{57} - z_{27}z_{38}z_{57} + z_{35}z_{39}z_{57}\\
&- z_{38}z_{45}z_{57} + z_{28}z_{46}z_{57} - 2z_{18}z_{48}z_{57} + z_{26}z_{48}z_{57}\\
&- z_{23}z_{510}z_{57} - 2z_{110}z_{19}z_{58} + z_{17}z_{210}z_{58} + z_{110}z_{27}z_{58}\\
&+ z_{19}z_{37}z_{58} - 2z_{27}z_{37}z_{58} + z_{17}z_{39}z_{58} - 2z_{15}z_{410}z_{58}\\
&- z_{45}z_{46}z_{58} - z_{18}z_{47}z_{58} + 2z_{26}z_{47}z_{58} + z_{35}z_{47}z_{58}\\
&- z_{16}z_{49}z_{58} - z_{14}z_{510}z_{58} + 2z_{34}z_{57}z_{58} + 2z_{110}z_{18}z_{59}\\
&- z_{16}z_{210}z_{59} - z_{110}z_{26}z_{59} + 2z_{15}z_{310}z_{59} - z_{19}z_{36}z_{59}\\
&+ 2z_{27}z_{36}z_{59} - z_{35}z_{37}z_{59} - z_{17}z_{38}z_{59} + z_{36}z_{45}z_{59}\\
&+ z_{18}z_{46}z_{59} - 2z_{26}z_{46}z_{59} + z_{16}z_{48}z_{59} + z_{13}z_{510}z_{59}\\
&- 2z_{34}z_{56}z_{59} - 2z_{19}z_{25}z_{610} + z_{25}z_{27}z_{610} + z_{15}z_{29}z_{610}\\
&- z_{25}z_{45}z_{610} - 3z_{12}z_{59}z_{610} - z_{19}z_{28}z_{67} + z_{27}z_{28}z_{67}\\
&+ z_{18}z_{29}z_{67} - z_{26}z_{29}z_{67} + z_{25}z_{39}z_{67} - z_{25}z_{48}z_{67} + z_{24}z_{58}z_{67}\\
&- z_{23}z_{59}z_{67} + 2z_{19}^2z_{68} - 2z_{19}z_{27}z_{68} + 2z_{27}^2z_{68} - 2z_{17}z_{29}z_{68}\\
&+ z_{45}^2z_{68} - 3z_{25}z_{47}z_{68} + 3z_{15}z_{49}z_{68} - 3z_{24}z_{57}z_{68}\\
&+ 3z_{14}z_{59}z_{68} - 2z_{18}z_{19}z_{69} + 2z_{19}z_{26}z_{69} - 2z_{26}z_{27}z_{69}\\
&+ z_{17}z_{28}z_{69} + z_{16}z_{29}z_{69} + z_{25}z_{37}z_{69} - 2z_{15}z_{39}z_{69}\\
&- z_{35}z_{45}z_{69} + 2z_{25}z_{46}z_{69} - z_{15}z_{48}z_{69} + z_{24}z_{56}z_{69}\\
&+ 2z_{23}z_{57}z_{69} - 2z_{14}z_{58}z_{69} - z_{13}z_{59}z_{69} + 2z_{12}z_{69}^2\\
&+ 2z_{18}z_{25}z_{710} - z_{25}z_{26}z_{710} - z_{15}z_{28}z_{710} + z_{25}z_{35}z_{710}\\
&+ 3z_{12}z_{58}z_{710} - 2z_{18}z_{19}z_{78} + 2z_{18}z_{27}z_{78} - 2z_{26}z_{27}z_{78}\\
&+ z_{17}z_{28}z_{78} + z_{16}z_{29}z_{78} + 2z_{25}z_{37}z_{78} - z_{15}z_{39}z_{78}
\end{aligned}
$$

$$
\begin{aligned}
&- z_{35}z_{45}z_{78} + z_{25}z_{46}z_{78} - 2z_{15}z_{48}z_{78} + 2z_{24}z_{56}z_{78} \\
&+ z_{23}z_{57}z_{78} - z_{14}z_{58}z_{78} - 2z_{13}z_{59}z_{78} + 2z_{12}z_{69}z_{78} \\
&+ 2z_{12}z_{78}^2 + 2z_{18}^2z_{79} - 2z_{18}z_{26}z_{79} + 2z_{26}^2z_{79} - 2z_{16}z_{28}z_{79} \\
&+ z_{35}^2z_{79} - 3z_{25}z_{36}z_{79} + 3z_{15}z_{38}z_{79} - 3z_{23}z_{56}z_{79} \\
&+ 3z_{13}z_{58}z_{79} - 6z_{12}z_{68}z_{79} + z_{15}z_{19}z_{810} + z_{17}z_{25}z_{810} \\
&- 2z_{15}z_{27}z_{810} + z_{15}z_{45}z_{810} + 3z_{12}z_{57}z_{810} + z_{17}z_{18}z_{89} \\
&- z_{16}z_{19}z_{89} - z_{17}z_{26}z_{89} + z_{16}z_{27}z_{89} + z_{15}z_{37}z_{89} \\
&- z_{15}z_{46}z_{89} + z_{14}z_{56}z_{89} - z_{13}z_{57}z_{89} - 2z_{12}z_{67}z_{89} \\
&- z_{15}z_{18}z_{910} - z_{16}z_{25}z_{910} + 2z_{15}z_{26}z_{910} - z_{15}z_{35}z_{910} - 3z_{12}z_{56}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{24} = {} & 2z_{110}z_{18}z_{510} - 2z_{16}z_{210}z_{510} + z_{15}z_{310}z_{510} + 2z_{110}z_{35}z_{510} \\
&+ z_{27}z_{36}z_{510} - 2z_{35}z_{37}z_{510} - z_{17}z_{38}z_{510} + 2z_{36}z_{45}z_{510} \\
&- z_{26}z_{46}z_{510} + z_{16}z_{48}z_{510} + 2z_{13}z_{510}^2 + 2z_{110}z_{210}z_{56} \\
&- z_{19}z_{310}z_{56} - z_{27}z_{310}z_{56} - z_{210}z_{37}z_{56} - z_{110}z_{39}z_{56} \\
&+ 2z_{37}z_{39}z_{56} + 2z_{18}z_{410}z_{56} + z_{26}z_{410}z_{56} + z_{35}z_{410}z_{56} \\
&- z_{38}z_{47}z_{56} + z_{46}z_{48}z_{56} - 2z_{36}z_{49}z_{56} - z_{34}z_{510}z_{56} - z_{18}z_{310}z_{57} \\
&- z_{310}z_{35}z_{57} + z_{210}z_{36}z_{57} + z_{110}z_{38}z_{57} - z_{37}z_{38}z_{57} \\
&+ z_{36}z_{48}z_{57} - 2z_{110}^2z_{58} + 2z_{17}z_{310}z_{58} + 2z_{110}z_{37}z_{58} - 2z_{37}^2z_{58} \\
&- 3z_{16}z_{410}z_{58} - z_{46}^2z_{58} + 3z_{36}z_{47}z_{58} + z_{16}z_{310}z_{59} \\
&- 2z_{110}z_{36}z_{59} + z_{36}z_{37}z_{59} - z_{36}z_{46}z_{59} - 2z_{18}z_{19}z_{610} \\
&- z_{110}z_{25}z_{610} - z_{26}z_{27}z_{610} + z_{17}z_{28}z_{610} + 2z_{16}z_{29}z_{610} - z_{19}z_{35}z_{610} \\
&+ 2z_{25}z_{37}z_{610} - z_{15}z_{39}z_{610} - z_{18}z_{45}z_{610} - 2z_{35}z_{45}z_{610} \\
&+ z_{25}z_{46}z_{610} - z_{15}z_{48}z_{610} - z_{12}z_{510}z_{610} + 2z_{24}z_{56}z_{610} + z_{23}z_{57}z_{610} \\
&- 3z_{14}z_{58}z_{610} - 3z_{13}z_{59}z_{610} + z_{210}z_{26}z_{67} - 2z_{110}z_{28}z_{67} - z_{25}z_{310}z_{67} \\
&- 2z_{29}z_{36}z_{67} + z_{28}z_{37}z_{67} + z_{19}z_{38}z_{67} + z_{18}z_{39}z_{67} \\
&+ 2z_{35}z_{39}z_{67} - z_{38}z_{45}z_{67} + z_{28}z_{46}z_{67} - 2z_{18}z_{48}z_{67} - z_{35}z_{48}z_{67} \\
&- 2z_{23}z_{510}z_{67} + 3z_{34}z_{58}z_{67} + 2z_{110}z_{19}z_{68} - z_{17}z_{210}z_{68} \\
&- z_{110}z_{27}z_{68} - z_{19}z_{37}z_{68} + 2z_{27}z_{37}z_{68} - z_{17}z_{39}z_{68} + z_{15}z_{410}z_{68} \\
&+ z_{45}z_{46}z_{68} - z_{18}z_{47}z_{68} - z_{26}z_{47}z_{68} - 2z_{35}z_{47}z_{68} + 2z_{16}z_{49}z_{68} \\
&+ 2z_{14}z_{510}z_{68} - z_{34}z_{57}z_{68} - 2z_{24}z_{67}z_{68} + z_{110}z_{26}z_{69} + z_{19}z_{36}z_{69} \\
&- z_{26}z_{37}z_{69} - z_{16}z_{39}z_{69} + z_{18}z_{46}z_{69} + z_{35}z_{46}z_{69} - z_{34}z_{56}z_{69} \\
&+ z_{12}z_{610}z_{69} + z_{23}z_{67}z_{69} + z_{14}z_{68}z_{69} + z_{13}z_{69}^2 \\
&+ 2z_{18}^2z_{710} + z_{26}^2z_{710} - 3z_{16}z_{28}z_{710} + 2z_{18}z_{35}z_{710} + 2z_{35}^2z_{710} \\
&- 3z_{25}z_{36}z_{710} + 2z_{15}z_{38}z_{710} - 3z_{23}z_{56}z_{710} + 6z_{13}z_{58}z_{710} \\
&- 3z_{12}z_{68}z_{710} - 2z_{110}z_{18}z_{78} + z_{16}z_{210}z_{78} - z_{15}z_{310}z_{78} \\
&- 2z_{27}z_{36}z_{78} + 2z_{18}z_{37}z_{78} + 2z_{35}z_{37}z_{78} + z_{17}z_{38}z_{78} \\
&- z_{36}z_{45}z_{78} + z_{26}z_{46}z_{78} - 2z_{16}z_{48}z_{78} - 2z_{13}z_{510}z_{78} \\
&+ 2z_{34}z_{56}z_{78} + 2z_{12}z_{610}z_{78} + z_{23}z_{67}z_{78} - z_{14}z_{68}z_{78} \\
&+ 2z_{13}z_{78}^2 - 2z_{18}z_{36}z_{79} + z_{26}z_{36}z_{79} - z_{35}z_{36}z_{79} + z_{16}z_{38}z_{79}
\end{aligned}
$$

$$
\begin{aligned}
&- 3z_{13}z_{68}z_{79} + z_{110}z_{15}z_{810} + z_{17}z_{18}z_{810} - z_{16}z_{27}z_{810}\\
&+ z_{17}z_{35}z_{810} - z_{15}z_{37}z_{810} + z_{16}z_{45}z_{810} + z_{14}z_{56}z_{810}\\
&+ 2z_{13}z_{57}z_{810} + z_{12}z_{67}z_{810} - z_{110}z_{16}z_{89} - z_{17}z_{36}z_{89}\\
&+ 2z_{16}z_{37}z_{89} - z_{16}z_{46}z_{89} - 3z_{13}z_{67}z_{89} - z_{16}z_{18}z_{910}\\
&+ z_{16}z_{26}z_{910} - 2z_{16}z_{35}z_{910} + z_{15}z_{36}z_{910} - 3z_{13}z_{56}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{25} = {}& 2z_{110}z_{19}z_{510} - 2z_{17}z_{210}z_{510} + z_{27}z_{37}z_{510} - z_{17}z_{39}z_{510}\\
&+ z_{15}z_{410}z_{510} + 2z_{110}z_{45}z_{510} + 2z_{45}z_{46}z_{510} - z_{26}z_{47}z_{510}\\
&- 2z_{35}z_{47}z_{510} + z_{16}z_{49}z_{510} + 2z_{14}z_{510}^2 + z_{19}z_{410}z_{56}\\
&+ z_{410}z_{45}z_{56} - z_{210}z_{47}z_{56} + z_{39}z_{47}z_{56} - z_{110}z_{49}z_{56} - z_{46}z_{49}z_{56}\\
&+ 2z_{110}z_{210}z_{57} - 2z_{19}z_{310}z_{57} - z_{27}z_{310}z_{57} + z_{37}z_{39}z_{57}\\
&+ z_{18}z_{410}z_{57} + z_{26}z_{410}z_{57} - z_{310}z_{45}z_{57} + z_{210}z_{46}z_{57}\\
&- 2z_{38}z_{47}z_{57} + z_{110}z_{48}z_{57} + 2z_{46}z_{48}z_{57} - z_{36}z_{49}z_{57} - z_{34}z_{510}z_{57}\\
&- z_{17}z_{410}z_{58} + 2z_{110}z_{47}z_{58} - z_{37}z_{47}z_{58} + z_{46}z_{47}z_{58} - 2z_{110}^2z_{59}\\
&+ 3z_{17}z_{310}z_{59} - z_{37}^2z_{59} - 2z_{16}z_{410}z_{59} - 2z_{110}z_{46}z_{59} - 2z_{46}^2z_{59}\\
&+ 3z_{36}z_{47}z_{59} - 2z_{19}^2z_{610} - z_{27}^2z_{610} + 3z_{17}z_{29}z_{610} - 2z_{19}z_{45}z_{610}\\
&- 2z_{45}^2z_{610} + 3z_{25}z_{47}z_{610} - 2z_{15}z_{49}z_{610} + 3z_{24}z_{57}z_{610}\\
&- 6z_{14}z_{59}z_{610} + z_{210}z_{27}z_{67} - 2z_{110}z_{29}z_{67} - z_{29}z_{37}z_{67}\\
&+ 2z_{19}z_{39}z_{67} - z_{25}z_{410}z_{67} + z_{39}z_{45}z_{67} - z_{29}z_{46}z_{67}\\
&+ 2z_{28}z_{47}z_{67} - z_{19}z_{48}z_{67} - 2z_{45}z_{48}z_{67} - z_{18}z_{49}z_{67} + z_{35}z_{49}z_{67}\\
&- 2z_{24}z_{510}z_{67} + 3z_{34}z_{59}z_{67} - 2z_{19}z_{47}z_{68} + z_{27}z_{47}z_{68}\\
&- z_{45}z_{47}z_{68} + z_{17}z_{49}z_{68} + 2z_{110}z_{19}z_{69} - z_{17}z_{210}z_{69}\\
&+ z_{27}z_{37}z_{69} - 2z_{17}z_{39}z_{69} + z_{15}z_{410}z_{69} + 2z_{19}z_{46}z_{69}\\
&+ 2z_{45}z_{46}z_{69} - 2z_{26}z_{47}z_{69} - z_{35}z_{47}z_{69} + z_{16}z_{49}z_{69}\\
&+ 2z_{14}z_{510}z_{69} - 2z_{34}z_{57}z_{69} - z_{24}z_{67}z_{69} + 2z_{14}z_{69}^2\\
&+ 2z_{18}z_{19}z_{710} - z_{110}z_{25}z_{710} + z_{26}z_{27}z_{710} - 2z_{17}z_{28}z_{710}\\
&- z_{16}z_{29}z_{710} + z_{19}z_{35}z_{710} - z_{25}z_{37}z_{710} + z_{15}z_{39}z_{710} + z_{18}z_{45}z_{710}\\
&+ 2z_{35}z_{45}z_{710} - 2z_{25}z_{46}z_{710} + z_{15}z_{48}z_{710} - z_{12}z_{510}z_{710} - z_{24}z_{56}z_{710}\\
&- 2z_{23}z_{57}z_{710} + 3z_{14}z_{58}z_{710} + 3z_{13}z_{59}z_{710} - 2z_{12}z_{69}z_{710}\\
&- z_{110}z_{27}z_{78} + z_{19}z_{37}z_{78} + z_{37}z_{45}z_{78} - z_{27}z_{46}z_{78} + z_{18}z_{47}z_{78}\\
&- z_{17}z_{48}z_{78} + z_{34}z_{57}z_{78} - z_{24}z_{67}z_{78} - z_{12}z_{710}z_{78} + z_{14}z_{78}^2\\
&- 2z_{110}z_{18}z_{79} + z_{16}z_{210}z_{79} + z_{110}z_{26}z_{79} - z_{15}z_{310}z_{79}\\
&- z_{19}z_{36}z_{79} - z_{27}z_{36}z_{79} + z_{35}z_{37}z_{79} + 2z_{17}z_{38}z_{79} - 2z_{36}z_{45}z_{79}\\
&- z_{18}z_{46}z_{79} + 2z_{26}z_{46}z_{79} - z_{16}z_{48}z_{79} - 2z_{13}z_{510}z_{79}\\
&+ z_{34}z_{56}z_{79} + 3z_{12}z_{610}z_{79} + 2z_{23}z_{67}z_{79} - 3z_{14}z_{68}z_{79}\\
&- z_{13}z_{69}z_{79} + z_{13}z_{78}z_{79} + z_{17}z_{19}z_{810} - z_{17}z_{27}z_{810}\\
&+ 2z_{17}z_{45}z_{810} - z_{15}z_{47}z_{810} + 3z_{14}z_{57}z_{810} - z_{110}z_{17}z_{89}\\
&+ z_{17}z_{37}z_{89} - 2z_{17}z_{46}z_{89} + z_{16}z_{47}z_{89} - 3z_{14}z_{67}z_{89} + z_{110}z_{15}z_{910}\\
&- z_{16}z_{19}z_{910} + z_{17}z_{26}z_{910} - z_{17}z_{35}z_{910} - z_{16}z_{45}z_{910}
\end{aligned}
$$

$$+ z_{15}z_{46}z_{910} - 2z_{14}z_{56}z_{910} - z_{13}z_{57}z_{910} + z_{12}z_{67}z_{910},$$

$$\begin{aligned}
\psi_{34} = &-2z_{210}z_{26}z_{510} + 2z_{110}z_{28}z_{510} + z_{25}z_{310}z_{510} + 2z_{210}z_{35}z_{510}\\
&+ z_{29}z_{36}z_{510} - z_{19}z_{38}z_{510} - 2z_{35}z_{39}z_{510} + 2z_{38}z_{45}z_{510} - z_{28}z_{46}z_{510}\\
&+ z_{18}z_{48}z_{510} + 2z_{23}z_{510}^2 + 2z_{210}^2z_{56} - 2z_{29}z_{310}z_{56} - 2z_{210}z_{39}z_{56}\\
&+ 2z_{39}^2z_{56} + 3z_{28}z_{410}z_{56} + z_{48}^2z_{56} - 3z_{38}z_{49}z_{56} - z_{28}z_{310}z_{57}\\
&+ 2z_{210}z_{38}z_{57} - z_{38}z_{39}z_{57} + z_{38}z_{48}z_{57} - 2z_{110}z_{210}z_{58} + z_{19}z_{310}z_{58}\\
&+ z_{27}z_{310}z_{58} + z_{210}z_{37}z_{58} + z_{110}z_{39}z_{58} - 2z_{37}z_{39}z_{58} - z_{18}z_{410}z_{58}\\
&- 2z_{26}z_{410}z_{58} + z_{35}z_{410}z_{58} + 2z_{38}z_{47}z_{58} - z_{46}z_{48}z_{58} + z_{36}z_{49}z_{58}\\
&- z_{34}z_{510}z_{58} + z_{26}z_{310}z_{59} - z_{310}z_{35}z_{59} - z_{210}z_{36}z_{59} - z_{110}z_{38}z_{59}\\
&+ z_{36}z_{39}z_{59} - z_{38}z_{46}z_{59} - z_{210}z_{25}z_{610} - z_{19}z_{28}z_{610} + z_{26}z_{29}z_{610}\\
&- z_{29}z_{35}z_{610} + z_{25}z_{39}z_{610} - z_{28}z_{45}z_{610} - z_{24}z_{58}z_{610} - 2z_{23}z_{59}z_{610}\\
&- z_{210}z_{28}z_{67} - z_{29}z_{38}z_{67} + 2z_{28}z_{39}z_{67} - z_{28}z_{48}z_{67} + z_{19}z_{210}z_{68}\\
&- 2z_{210}z_{27}z_{68} + z_{110}z_{29}z_{68} + z_{29}z_{37}z_{68} - 2z_{19}z_{39}z_{68} + z_{27}z_{39}z_{68}\\
&+ z_{25}z_{410}z_{68} - 2z_{28}z_{47}z_{68} + z_{45}z_{48}z_{68} + z_{18}z_{49}z_{68} + z_{26}z_{49}z_{68}\\
&- 2z_{35}z_{49}z_{68} + 2z_{24}z_{510}z_{68} - z_{34}z_{59}z_{68} + 2z_{210}z_{26}z_{69} - z_{110}z_{28}z_{69}\\
&- z_{25}z_{310}z_{69} - z_{29}z_{36}z_{69} + 2z_{19}z_{38}z_{69} - 2z_{26}z_{39}z_{69} + 2z_{35}z_{39}z_{69}\\
&- z_{38}z_{45}z_{69} + 2z_{28}z_{46}z_{69} - z_{18}z_{48}z_{69} - 2z_{23}z_{510}z_{69}\\
&+ 2z_{34}z_{58}z_{69} - z_{24}z_{68}z_{69} + 2z_{23}z_{69}^2 + z_{18}z_{28}z_{710} - z_{26}z_{28}z_{710}\\
&+ 2z_{28}z_{35}z_{710} - z_{25}z_{38}z_{710} + 3z_{23}z_{58}z_{710} - z_{18}z_{210}z_{78} + z_{28}z_{37}z_{78}\\
&- z_{27}z_{38}z_{78} + z_{18}z_{39}z_{78} - z_{26}z_{48}z_{78} + z_{35}z_{48}z_{78} - z_{34}z_{58}z_{78}\\
&+ z_{24}z_{68}z_{78} + z_{23}z_{78}^2 - z_{28}z_{36}z_{79} - z_{18}z_{38}z_{79} + 2z_{26}z_{38}z_{79}\\
&- z_{35}z_{38}z_{79} - 3z_{23}z_{68}z_{79} - z_{18}z_{19}z_{810} + z_{15}z_{210}z_{810} - 2z_{26}z_{27}z_{810}\\
&+ 2z_{17}z_{28}z_{810} + z_{16}z_{29}z_{810} + z_{27}z_{35}z_{810} + z_{25}z_{37}z_{810}\\
&- 2z_{15}z_{39}z_{810} + z_{26}z_{45}z_{810} - 2z_{35}z_{45}z_{810} + z_{25}z_{46}z_{810} - z_{15}z_{48}z_{810}\\
&- z_{12}z_{510}z_{810} + 3z_{24}z_{56}z_{810} + 3z_{23}z_{57}z_{810} - 2z_{14}z_{58}z_{810} - z_{13}z_{59}z_{810}\\
&+ 2z_{12}z_{69}z_{810} + z_{12}z_{78}z_{810} + z_{110}z_{18}z_{89} - 2z_{16}z_{210}z_{89} + z_{15}z_{310}z_{89}\\
&+ z_{27}z_{36}z_{89} + z_{26}z_{37}z_{89} - 2z_{35}z_{37}z_{89} - 2z_{17}z_{38}z_{89} + z_{16}z_{39}z_{89}\\
&+ z_{36}z_{45}z_{89} - 2z_{26}z_{46}z_{89} + z_{35}z_{46}z_{89} + z_{16}z_{48}z_{89} + 2z_{13}z_{510}z_{89}\\
&- 3z_{34}z_{56}z_{89} - z_{12}z_{610}z_{89} - 3z_{23}z_{67}z_{89} + 2z_{14}z_{68}z_{89} - z_{13}z_{69}z_{89}\\
&- z_{13}z_{78}z_{89} + z_{18}^2z_{910} + 2z_{26}^2z_{910} - 3z_{16}z_{28}z_{910} - 2z_{26}z_{35}z_{910} + 2z_{35}^2z_{910}\\
&- 2z_{25}z_{36}z_{910} + 3z_{15}z_{38}z_{910} - 6z_{23}z_{56}z_{910} + 3z_{13}z_{58}z_{910} - 3z_{12}z_{68}z_{910},
\end{aligned}$$

$$\begin{aligned}
\psi_{35} = &-2z_{210}z_{27}z_{510} + 2z_{110}z_{29}z_{510} + z_{29}z_{37}z_{510} - z_{19}z_{39}z_{510} + z_{25}z_{410}z_{510}\\
&+ 2z_{210}z_{45}z_{510} - z_{28}z_{47}z_{510} + 2z_{45}z_{48}z_{510} + z_{18}z_{49}z_{510}\\
&- 2z_{35}z_{49}z_{510} + 2z_{24}z_{510}^2 + z_{29}z_{410}z_{56} - 2z_{210}z_{49}z_{56} + z_{39}z_{49}z_{56} - z_{48}z_{49}z_{56}\\
&+ 2z_{210}^2z_{57} - 3z_{29}z_{310}z_{57} + z_{39}^2z_{57} + 2z_{28}z_{410}z_{57} + 2z_{210}z_{48}z_{57}\\
&+ 2z_{48}^2z_{57} - 3z_{38}z_{49}z_{57} - z_{27}z_{410}z_{58} + z_{410}z_{45}z_{58} + z_{210}z_{47}z_{58}\\
&+ z_{47}z_{48}z_{58} + z_{110}z_{49}z_{58} - z_{37}z_{49}z_{58} - 2z_{110}z_{210}z_{59} + z_{19}z_{310}z_{59}
\end{aligned}$$

$$
\begin{aligned}
&+ 2z_{27}z_{310}z_{59} - z_{37}z_{39}z_{59} - z_{18}z_{410}z_{59} - z_{26}z_{410}z_{59} - z_{310}z_{45}z_{59} \\
&- z_{210}z_{46}z_{59} + z_{38}z_{47}z_{59} - z_{110}z_{48}z_{59} - 2z_{46}z_{48}z_{59} + 2z_{36}z_{49}z_{59} \\
&- z_{34}z_{510}z_{59} - z_{19}z_{29}z_{610} + z_{27}z_{29}z_{610} - 2z_{29}z_{45}z_{610} + z_{25}z_{49}z_{610} \\
&- 3z_{24}z_{59}z_{610} - z_{210}z_{29}z_{67} + z_{29}z_{39}z_{67} - 2z_{29}z_{48}z_{67} + z_{28}z_{49}z_{67} \\
&- z_{29}z_{47}z_{68} - z_{19}z_{49}z_{68} + 2z_{27}z_{49}z_{68} - z_{45}z_{49}z_{68} + z_{19}z_{210}z_{69} \\
&- z_{27}z_{39}z_{69} + z_{39}z_{45}z_{69} + z_{29}z_{46}z_{69} + z_{19}z_{48}z_{69} - z_{26}z_{49}z_{69} \\
&+ z_{34}z_{59}z_{69} + z_{24}z_{69}^2 - z_{210}z_{25}z_{710} - z_{27}z_{28}z_{710} + z_{18}z_{29}z_{710} \\
&+ z_{29}z_{35}z_{710} + z_{28}z_{45}z_{710} - z_{25}z_{48}z_{710} + 2z_{24}z_{58}z_{710} + z_{23}z_{59}z_{710} \\
&- 2z_{210}z_{27}z_{78} + z_{110}z_{29}z_{78} + 2z_{29}z_{37}z_{78} - z_{19}z_{39}z_{78} + z_{25}z_{410}z_{78} \\
&- z_{28}z_{47}z_{78} - 2z_{27}z_{48}z_{78} + 2z_{45}z_{48}z_{78} + 2z_{18}z_{49}z_{78} - z_{35}z_{49}z_{78} \\
&+ 2z_{24}z_{510}z_{78} - 2z_{34}z_{59}z_{78} + 2z_{24}z_{78}^2 - z_{18}z_{210}z_{79} + 2z_{210}z_{26}z_{79} \\
&- z_{110}z_{28}z_{79} - z_{25}z_{310}z_{79} - 2z_{29}z_{36}z_{79} + z_{19}z_{38}z_{79} + z_{27}z_{38}z_{79} + z_{35}z_{39}z_{79} \\
&- 2z_{38}z_{45}z_{79} + z_{28}z_{46}z_{79} - 2z_{18}z_{48}z_{79} + z_{26}z_{48}z_{79} - 2z_{23}z_{510}z_{79} \\
&+ z_{34}z_{58}z_{79} - 3z_{24}z_{68}z_{79} + z_{23}z_{69}z_{79} - z_{23}z_{78}z_{79} - z_{19}^2z_{810} - 2z_{27}^2z_{810} \\
&+ 3z_{17}z_{29}z_{810} + 2z_{27}z_{45}z_{810} - 2z_{45}^2z_{810} + 2z_{25}z_{47}z_{810} - 3z_{15}z_{49}z_{810} \\
&+ 6z_{24}z_{57}z_{810} - 3z_{14}z_{59}z_{810} + 3z_{12}z_{79}z_{810} + z_{110}z_{19}z_{89} - 2z_{17}z_{210}z_{89} \\
&+ 2z_{27}z_{37}z_{89} - z_{17}z_{39}z_{89} + z_{15}z_{410}z_{89} - z_{37}z_{45}z_{89} - z_{27}z_{46}z_{89} \\
&+ 2z_{45}z_{46}z_{89} - z_{26}z_{47}z_{89} - z_{35}z_{47}z_{89} - z_{17}z_{48}z_{89} + 2z_{16}z_{49}z_{89} \\
&+ 2z_{14}z_{510}z_{89} - 3z_{34}z_{57}z_{89} - 3z_{24}z_{67}z_{89} + z_{14}z_{69}z_{89} - z_{12}z_{710}z_{89} \\
&+ z_{14}z_{78}z_{89} - 2z_{13}z_{79}z_{89} + z_{18}z_{19}z_{910} + z_{15}z_{210}z_{910} + 2z_{26}z_{27}z_{910} \\
&- z_{17}z_{28}z_{910} - 2z_{16}z_{29}z_{910} - z_{27}z_{35}z_{910} - z_{25}z_{37}z_{910} + z_{15}z_{39}z_{910} \\
&- z_{26}z_{45}z_{910} + 2z_{35}z_{45}z_{910} - z_{25}z_{46}z_{910} + 2z_{15}z_{48}z_{910} \\
&- z_{12}z_{510}z_{910} - 3z_{24}z_{56}z_{910} - 3z_{23}z_{57}z_{910} + z_{14}z_{58}z_{910} \\
&+ 2z_{13}z_{59}z_{910} - z_{12}z_{69}z_{910} - 2z_{12}z_{78}z_{910},
\end{aligned}
$$

$$
\begin{aligned}
\psi_{45} = &-z_{210}z_{37}z_{510} + z_{110}z_{39}z_{510} - z_{35}z_{410}z_{510} + z_{310}z_{45}z_{510} \\
&+ z_{210}z_{46}z_{510} - z_{110}z_{48}z_{510} + z_{34}z_{510}^2 - z_{210}z_{410}z_{56} + 2z_{39}z_{410}z_{56} \\
&- z_{410}z_{48}z_{56} - z_{310}z_{49}z_{56} + z_{210}z_{310}z_{57} - z_{310}z_{39}z_{57} - z_{38}z_{410}z_{57} \\
&+ 2z_{310}z_{48}z_{57} + z_{110}z_{410}z_{58} - 2z_{37}z_{410}z_{58} + z_{410}z_{46}z_{58} + z_{310}z_{47}z_{58} \\
&- z_{110}z_{310}z_{59} + z_{310}z_{37}z_{59} + z_{36}z_{410}z_{59} - 2z_{310}z_{46}z_{59} - z_{210}z_{27}z_{610} \\
&+ z_{110}z_{29}z_{610} + 2z_{29}z_{37}z_{610} - 2z_{19}z_{39}z_{610} + 2z_{25}z_{410}z_{610} - z_{39}z_{45}z_{610} \\
&- z_{29}z_{46}z_{610} - z_{28}z_{47}z_{610} + z_{19}z_{48}z_{610} + 2z_{45}z_{48}z_{610} + z_{18}z_{49}z_{610} \\
&- z_{35}z_{49}z_{610} + z_{24}z_{510}z_{610} - 3z_{34}z_{59}z_{610} + z_{210}^2z_{67} - 3z_{29}z_{310}z_{67} \\
&+ 2z_{39}^2z_{67} + 3z_{28}z_{410}z_{67} - 2z_{39}z_{48}z_{67} + 2z_{48}^2z_{67} - 2z_{38}z_{49}z_{67} \\
&- z_{19}z_{410}z_{68} + z_{27}z_{410}z_{68} - z_{39}z_{47}z_{68} + z_{47}z_{48}z_{68} + z_{37}z_{49}z_{68} \\
&- z_{46}z_{49}z_{68} - z_{110}z_{210}z_{69} + 2z_{19}z_{310}z_{69} + z_{27}z_{310}z_{69} - 2z_{37}z_{39}z_{69} \\
&- z_{18}z_{410}z_{69} - 2z_{26}z_{410}z_{69} + 2z_{39}z_{46}z_{69} + z_{38}z_{47}z_{69} - 2z_{46}z_{48}z_{69} \\
&+ z_{36}z_{49}z_{69} - z_{24}z_{610}z_{69} + 2z_{34}z_{69}^2 + z_{210}z_{26}z_{710} - z_{110}z_{28}z_{710} \\
&- 2z_{25}z_{310}z_{710} - z_{29}z_{36}z_{710} - z_{28}z_{37}z_{710} + z_{19}z_{38}z_{710} + z_{18}z_{39}z_{710}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2z_{35}z_{39}z_{710} - z_{38}z_{45}z_{710} + 2z_{28}z_{46}z_{710} - 2z_{18}z_{48}z_{710} - z_{35}z_{48}z_{710} \\
&- z_{23}z_{510}z_{710} + 3z_{34}z_{58}z_{710} - z_{24}z_{68}z_{710} + 2z_{23}z_{69}z_{71} + z_{110}z_{210}z_{78} \\
&- z_{19}z_{310}z_{78} - 2z_{27}z_{310}z_{78} + 2z_{37}z_{39}z_{78} + 2z_{18}z_{410}z_{78} + z_{26}z_{410}z_{78} \\
&- z_{38}z_{47}z_{78} - 2z_{37}z_{48}z_{78} + 2z_{46}z_{48}z_{78} - z_{36}z_{49}z_{78} + 2z_{24}z_{610}z_{78} \\
&- 2z_{34}z_{69}z_{78} - z_{23}z_{710}z_{78} + 2z_{34}z_{78}^2 - z_{18}z_{310}z_{79} + z_{26}z_{310}z_{79} \\
&+ z_{37}z_{38}z_{79} - z_{36}z_{39}z_{79} - z_{38}z_{46}z_{79} + z_{36}z_{48}z_{79} - z_{23}z_{610}z_{79} \\
&- 2z_{34}z_{68}z_{79} - z_{110}z_{19}z_{810} + z_{17}z_{210}z_{810} - 2z_{27}z_{37}z_{810} + 2z_{17}z_{39}z_{810} \\
&- 2z_{15}z_{410}z_{810} + z_{37}z_{45}z_{810} + z_{27}z_{46}z_{810} - 2z_{45}z_{46}z_{810} + z_{26}z_{47}z_{810} \\
&+ z_{35}z_{47}z_{810} - z_{17}z_{48}z_{810} - z_{16}z_{49}z_{810} - z_{14}z_{510}z_{810} + 3z_{34}z_{57}z_{810} \\
&+ 3z_{24}z_{67}z_{810} - 2z_{14}z_{69}z_{810} + 2z_{12}z_{710}z_{810} + z_{14}z_{78}z_{810} + z_{13}z_{79}z_{810} \\
&+ z_{110}^2z_{89} - 3z_{17}z_{310}z_{89} + 2z_{37}^2z_{89} + 3z_{16}z_{410}z_{89} - 2z_{37}z_{46}z_{89} \\
&+ 2z_{46}^2z_{89} - 2z_{36}z_{47}z_{89} + 3z_{14}z_{610}z_{89} - 6z_{34}z_{67}z_{89} - 3z_{13}z_{710}z_{89} \\
&+ z_{110}z_{18}z_{910} - z_{16}z_{210}z_{910} + 2z_{15}z_{310}z_{910} + z_{27}z_{36}z_{910} + z_{26}z_{37}z_{910} \\
&- 2z_{35}z_{37}z_{910} - z_{17}z_{38}z_{910} - z_{16}z_{39}z_{910} + z_{36}z_{45}z_{910} - 2z_{26}z_{46}z_{910} \\
&+ z_{35}z_{46}z_{910} + 2z_{16}z_{48}z_{910} + z_{13}z_{510}z_{910} - 3z_{34}z_{56}z_{910} - 2z_{12}z_{610}z_{910} \\
&- 3z_{23}z_{67}z_{910} + z_{14}z_{68}z_{910} + z_{13}z_{69}z_{910} - 2z_{13}z_{78}z_{910}.
\end{aligned}
$$

Lemma 3.2. *For every* $A \in GL_5,\ B \in GL_8,\ \tilde{Z} \in \mathrm{Alt}_{10}$, *we have*

$$\tilde{Z} \mapsto (\det A)^4(\det B)\ {}^t\Lambda_2(A)^{-1}\tilde{Z}\Lambda_2(A)^{-1}$$

and

$$\Psi((\det A)^4(\det B){}^t\Lambda_2(A)^{-1}\tilde{Z}\Lambda_2(A)^{-1}) = (\det A)^{10}(\det B)^3\ {}^tA^{-1}\Psi(\tilde{Z})A^{-1}. \tag{3.4}$$

Proof. It is enough to prove the equivariance (3.4) in the case when A is one of the fundamental matrices,

$$A_u = \begin{pmatrix} 1 & \varepsilon & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \quad A_d = \mathrm{diag}(a, 1, 1, 1, 1),$$

or permutation matrices.

For the diagonal or permutation matrices, verifying (3.4) is easy. Note that, for $A_d = \mathrm{diag}(a_1, a_2, a_3, a_4, a_5)$, we have

$$\det A_d = a_1a_2a_3a_4a_5 \text{ and } z_{12} \mapsto (\det A_d)^3 a_1^{-1}a_4a_5,$$

$$z_{13} \mapsto (\det A_d)^3 a^{-1}a_3a_5, \cdots, z_{910} \mapsto (\det A_d)^3 a_1a_2a_5^{-1}.$$

Hence, we have $\psi_{ij} \mapsto (\det A_d)^{10}a_i^{-1}a_j^{-1}$, $(1 \le i < j \le 5)$ and then

$$\Psi((\det A_d){}^t\Lambda_2(A_d)\tilde{Z}\Lambda_2(A_d)^{-1}) = (\det A_d)^{10\,t}A_d^{-1}\Psi(\tilde{Z})A_d^{-1}.$$

For A_u, we consider the action of A_u. Since $\det A_u = 1$, and

$$\Psi(\tilde{Z}) \mapsto \Psi({}^t\Lambda_2(A_u)\tilde{Z}\Lambda_2(A_u)^{-1}),$$

then we have $\psi_{2j} \mapsto \psi_{2j} - \varepsilon\psi_{1j}$ $(3 \le j \le 5)$, $\psi_{1j} \mapsto \psi_{1j}$ and $\psi_{lk} \mapsto \psi_{lk}$ $(3 \le l < k \le 5)$. Hence, we have $\Psi({}^t\Lambda_2(A_u)^{-1}\tilde{Z}\Lambda_2(A_u)^{-1}) = {}^tA_u^{-1}\Psi(\tilde{Z})A_u^{-1}$. □

Remark B. We can construct the above polynomials ψ_{ij} $(1 \leq i < j \leq 5)$ in the same program as in Remark A. This case is much more complicated than the case of $\varphi'_{ij}s$. Here we note that we consider constructing the polynomial ψ_{45}. First, we consider the polynomial corresponding to the weight $(\det A)^{10} a_4^{-1} a_5^{-1}$. This polynomial is constructed of 205-term monomials. After the action of the generators of GL_5, we have uniquely the polynomial ψ_{45} constructed of 148-term monomials. Hence from the explicit form of ψ_{45}, we can construct the other polynomials ψ_{ij} by the action of GL_5.

Step 5. From (3.2) and (3.3), $\Phi(\tilde{Z})\Delta(X \cdot Y) \mapsto (\det A)^8 (\det B)^2 A \Phi(\tilde{Z})\Delta(X \cdot Y) A^{-1}$ and hence $F_1(x) = \operatorname{tr} \Phi(\tilde{Z})\Delta(X \cdot Y)$ is a relative invariant of degree 24 corresponding to the character $(\det A)^8 (\det B)^2$.

From (3.1) and (3.4), $(X \cdot Y)\Psi(\tilde{Z}) \mapsto (\det A)^{10} (\det B)^3 A (X \cdot Y)\Psi(\tilde{Z}) A^{-1}$ and hence $F_2(x) = \operatorname{tr}((X \cdot Y)\Psi(\tilde{Z}))$ is a relative invariant of degree 26 corresponding to the character $(\det A)^{10} (\det B)^3$.

For a generic point $x_0 = (e_1 \wedge e_2\ ,\ 2e_1 \wedge e_3\ ,\ 2e_2 \wedge e_3\ ,\ e_1 \wedge e_5\ ,\ e_1 \wedge e_4 - e_2 \wedge e_5\ ,\ e_2 \wedge e_4 - e_3 \wedge e_5\ ,\ e_3 \wedge e_4\ ,\ e_4 \wedge e_5\ ,\ e'_2 + e'_8)$, we have

$$X_0 \cdot Y_0 = \begin{pmatrix} 0 & & 2 & & \\ & 0 & & & \\ -2 & & 0 & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \text{ and } \Delta(X_0 \cdot Y_0) = \begin{pmatrix} 0 & & & & \\ & 4 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

$$Z_0 = \begin{pmatrix} 1 & & & & & & & \\ & 2 & & & & & & \\ & & & & 1 & & & \\ & & & 1 & & & & \\ & & 2 & & & & & \\ & & & & & 1 & & \\ & & & & -1 & & & \\ & & & & & & 1 & \\ & & & & & -1 & & \\ & & & & & & & 1 \end{pmatrix},\ \tilde{Z}_0 = \begin{pmatrix} 0 & & & & & & & & & \\ & 0 & & & & & & & & \\ & & 0 & & & -4 & & & -4 & \\ & & & 0 & & & & & & \\ & & & & 0 & & & & & \\ & & 4 & & & 0 & 4 & & & \\ & & & & & -4 & 0 & & -4 & \\ & & & & & & & 0 & & \\ & & 4 & & & & 4 & & 0 & \\ & & & & & & & & & 0 \end{pmatrix},$$

$$\Phi(\tilde{Z}_0) = \begin{pmatrix} 0 & & & & \\ & 16 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}, \Psi(\tilde{Z}_0) = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 192 \\ & & & -192 & 0 \end{pmatrix},$$

and hence

$$F_1(X_1^0, \cdots, X_8^0, Y_0) = \operatorname{tr}(\Phi(\tilde{Z}_0)\ \Delta(X_0 \cdot Y_0))$$

$$= \operatorname{tr}\left(\begin{pmatrix} 0 & & & & \\ & 16 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix} \begin{pmatrix} 0 & & & & \\ & 4 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}\right) = 64 \neq 0, \tag{3.5}$$

(3.6)

$$F_2(X_1^0,\cdots,X_8^0,Y_0)=\mathrm{tr}((X_0\cdot Y_0)\Psi(\tilde{Z}_0))$$
$$=\mathrm{tr}\left(\begin{pmatrix} 0 & & 2 & & \\ & 0 & & & \\ -2 & & 0 & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & 192 \\ & & & -192 & 0 \end{pmatrix}\right)=-384\neq 0.$$

Thus they are not identically zero.

Since $\deg F_1 = 24 < \deg F_2 = 26$ and $\deg_Y F_1 = 4 > \deg_Y F_2 = 1$, we have $F_1 \nmid F_2$ and $F_2 \nmid F_1$. Hence, if F_1 or F_2 is not irreducible, it is a power of an irreducible invariant. Since the character corresponding to F_2 is $(\det A)^{10}(\det B)^3$, we have the irreducibility of F_2. If F_1 is not irreducible, we have $F_1 = G^2$ for some relative invariant corresponding to the character $(\det A)^4(\det B)$. For

$$\tilde{x}=((z_1^{(1)}e_1\wedge e_2, 2z_2^{(2)}e_1\wedge e_3, 2z_5^{(3)}e_2\wedge e_3, z_4^{(4)}e_1\wedge e_5, z_3^{(5)}e_1\wedge e_4 - z_7^{(5)}e_2\wedge e_5, z_6^{(6)}e_2\wedge e_4 - z_9^{(6)}e_3\wedge e_5, z_8^{(7)}e_3\wedge e_4, z_{10}^{(8)}e_4\wedge e_5), e_2'+e_8')\in V,$$

we have

$$F_1(\tilde{x}) = 64(z_1^{(1)}z_2^{(2)}z_5^{(3)}z_4^{(4)}z_5^{(7)}z_6^{(6)}z_8^{(7)}z_{10}^{(8)})^2 z_2^{(2)}z_{10}^{(8)}$$

and hence $F_1(x)$ is not of the form $G(x)^2$ for some polynomial $G(x)$. This shows the irreducibility of $F_1(x)$.

Theorem 3.3. *The prehomogeneous vector space* $(GL_5\times GL_8, \Lambda_2\otimes\Lambda_1 + 1\otimes\Lambda_1^*, \mathrm{Alt}_5^{\oplus 8}\oplus V(8)^*)$ *has* 2 *basic relative invariants:*

(1) $F_1(x)=\mathrm{tr}\,\Phi(\tilde{Z})\Delta(X\cdot Y)\longleftrightarrow(\det A)^8(\det B)^2$, $\deg .F_1 = 24$,
(2) $F_2(x)=\mathrm{tr}(X\cdot Y)\Psi(\tilde{Z})\longleftrightarrow(\det A)^{10}(\det B)^3$, $\deg .F_2 = 26$.

3.2. **Explicit construction of the irreducible relative invariants of** $(GL_1^2\times \mathrm{Spin}_{10}\times SL_{14}, \Lambda'\otimes\Lambda_1 + 1\otimes\Lambda_1^*)$ **with** $H\sim SL_2\times SO_2$**,** $N=2$**.** We may assume that $GL_1\times\mathrm{Spin}_{10}\times GL_{14}$ acts on $V=V(16)^{\oplus 14}\oplus M(14,1)$ by $x=((X_1,X_2,\cdots,X_{14}),Y)\mapsto((\Lambda'(A)X_1,\Lambda'(A)X_2,\cdots,\Lambda'(A)X_{14})^tB, \alpha^tB^{-1}Y)$ for $x=((X_1,X_2,\cdots,X_{14}),Y)\in V$ and $g=(\alpha,A,B)\in G$.

For $Y={}^t(y_1,\cdots,y_{14})$, $X\cdot Y := X_1\ y_1+\cdots+X_{14}\ y_{14}\in V(16)$. Then we have $X\cdot Y\longmapsto \alpha\ \Lambda'(A)\ X\cdot Y$. Now we define the mapping

$$\eta: V(16)\longrightarrow V(10),\ X=(x_0,x_{12},\cdots,x_{2345})\longmapsto \eta(X)=(\eta_1(x),\ldots,\eta_{10}(x))$$

as follows:

$$\begin{array}{lcllll}
\eta_1(x) &=& -x_{12}x_{1345} & +x_{13}x_{1245} & -x_{14}x_{1235} & +x_{15}x_{1234},\\
\eta_2(x) &=& x_{23}x_{1245} & -x_{24}x_{1235} & +x_{25}x_{1234} & -x_{12}x_{2345},\\
\eta_3(x) &=& -x_{34}x_{1235} & +x_{35}x_{1234} & -x_{13}x_{2345} & +x_{23}x_{1345}\\
\eta_4(x) &=& x_{45}x_{1234} & -x_{14}x_{2345} & +x_{24}x_{1345} & -x_{34}x_{1245},\\
\eta_5(x) &=& -x_{15}x_{2345} & +x_{25}x_{1345} & -x_{35}x_{1245} & +x_{45}x_{1235},\\
\eta_6(x) &=& x_0x_{2345} & -x_{23}x_{45} & +x_{24}x_{35} & -x_{25}x_{34},\\
\eta_7(x) &=& -x_0x_{1345} & +x_{34}x_{15} & -x_{35}x_{14} & +x_{13}x_{45},\\
\eta_8(x) &=& x_0x_{1245} & -x_{45}x_{12} & +x_{14}x_{25} & -x_{24}x_{15},\\
\eta_9(x) &=& -x_0x_{1235} & +x_{15}x_{23} & -x_{25}x_{13} & +x_{35}x_{12},\\
\eta_{10}(x) &=& x_0x_{1234} & -x_{12}x_{34} & +x_{13}x_{24} & -x_{14}x_{23}
\end{array}$$

(cf. [7], [8]). Then we have

$$\eta(\alpha\ \Lambda'(A)(X\cdot Y)) = \alpha^2\ \chi(A)\ \eta(X\cdot Y), \tag{3.7}$$

where χ is the vector representation of Spin_{10} . Since the infinitesimal representation of χ (resp. Λ') is given by (5.28) (resp. (5.38)) in [1], we have

$$\chi(\mathrm{Spin}_{10}) = SO(10,K) = \{A \in SL_{10}\ ;\ {}^tAKA = K\} \text{ for } K = \left(\begin{array}{c|c} O & I_5 \\ \hline I_5 & O \end{array}\right).$$

On the other hand, we put

$$\begin{aligned}\tilde{X}_i = {}^t(&x_0^{(i)}, x_{12}^{(i)}, x_{13}^{(i)}, x_{14}^{(i)}, x_{15}^{(i)}, x_{23}^{(i)}, x_{24}^{(i)}, x_{25}^{(i)},\\ &x_{34}^{(i)}, x_{35}^{(i)}, x_{45}^{(i)}, x_{1234}^{(i)}, x_{1235}^{(i)}, x_{1245}^{(i)}, x_{1345}^{(i)}, x_{2345}^{(i)})\\ \in{}& V(16)\end{aligned}$$

and consider that $Z = \left[\tilde{X}_1, \cdots, \tilde{X_{14}}\right] \in M(16,14)$.

Let $Z^{(i,j)}$ be the 14×14-matrix obtained from $\tilde{Z}$ by substracting the i-th and the j-th rows and $\tilde{Z} := (z_{ij}) \in \mathrm{Alt}_{16}$ with $z_{ij} := (-1)^{i+j}\det Z^{(i,j)}$. Then, by Lemma 2.3, we have

$$\tilde{Z} \longmapsto (\det\Lambda'(A))\ (\det B)\ {}^t\Lambda'(A)^{-1}\ \tilde{Z}\ \Lambda'(A)^{-1}. \tag{3.8}$$

Note that $\det\Lambda'(A) = 1$. Now we shall construct $\Phi(\tilde{Z}) \in \mathrm{Sym}_{10}$ such that

$$\Phi(\tilde{Z}) \longmapsto (\det B)^2\ {}^t\chi(A)^{-1}\ \Phi(\tilde{Z})\ \chi(A)^{-1}. \tag{3.9}$$

If we put

$$\begin{aligned}\Phi(\tilde{Z}) &= (\varphi(\tilde{Z})_{i,j})_{1\le i<j\le 10}\\ &= \Phi(\tilde{Z}) = \left(\begin{array}{ccc|ccc} \varphi_{11} & \cdots & \varphi_{15} & \varphi_{16} & \cdots & \varphi_{110}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \varphi_{15} & \cdots & \varphi_{55} & \varphi_{56} & \cdots & \varphi_{510}\\ \hline \varphi_{16} & \cdots & \varphi_{56} & \varphi_{66} & \cdots & \varphi_{610}\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ \varphi_{110} & \cdots & \varphi_{510} & \varphi_{610} & \cdots & \varphi_{1010} \end{array}\right) \in \mathrm{Sym}_{10}\end{aligned}$$

with the following entries, then we have Lemma 3.4.

$\varphi_{11} := -z_{215}^2 + 2z_{1415}z_{23} - 2z_{1315}z_{24} + 2z_{1215}z_{25} - z_{314}^2 + 2z_{214}z_{315} + 2z_{1314}z_{34} - 2z_{1214}z_{35} - z_{413}^2 + 2z_{313}z_{414} - 2z_{213}z_{415} + 2z_{1213}z_{45} - z_{512}^2 + 2z_{412}z_{513} - 2z_{312}z_{514} + 2z_{212}z_{515}$,

$\varphi_{22} := -z_{216}^2 + 2z_{1416}z_{26} - 2z_{1316}z_{27} + 2z_{1216}z_{28} - z_{614}^2 + 2z_{214}z_{616} + 2z_{1314}z_{67} - 2z_{1214}z_{68} - z_{713}^2 + 2z_{613}z_{714} - 2z_{213}z_{716} + 2z_{1213}z_{78} - z_{812}^2 + 2z_{712}z_{813} - 2z_{612}z_{814} + 2z_{212}z_{816}$,

$\varphi_{33} := -z_{1012}^2 + 2z_{1216}z_{310} + 2z_{1016}z_{312} - z_{316}^2 + 2z_{1516}z_{36} - 2z_{1316}z_{39} - 2z_{1215}z_{610} - 2z_{1015}z_{612} - z_{615}^2 + 2z_{315}z_{616} + 2z_{1315}z_{69} + 2z_{1213}z_{910} + 2z_{1013}z_{912} - z_{913}^2 + 2z_{613}z_{915} - 2z_{313}z_{916}$,

$\varphi_{44} := -z_{1112}^2 + 2z_{1216}z_{411} + 2z_{1116}z_{412} - z_{416}^2 + 2z_{1516}z_{47} - 2z_{1416}z_{49} - 2z_{1215}z_{711} - 2z_{1115}z_{712} - z_{715}^2 + 2z_{415}z_{716} + 2z_{1415}z_{79} + 2z_{1214}z_{911} + 2z_{1114}z_{912} - z_{914}^2 + 2z_{714}z_{915} - 2z_{414}z_{916}$,

$\varphi_{55} := -z_{1014}^2 - z_{1113}^2 + 2z_{1013}z_{1114} + 2z_{1011}z_{1314} - 2z_{1416}z_{510} + 2z_{1316}z_{511} + 2z_{1116}z_{513} - 2z_{1016}z_{514} - z_{516}^2 + 2z_{1516}z_{58} + 2z_{1415}z_{810} - 2z_{1315}z_{811} - 2z_{1115}z_{813} + 2z_{1015}z_{814} - z_{815}^2 + 2z_{515}z_{816}$,

$\varphi_{66} := -z_{116}^2 + 2z_{1116}z_{16} - 2z_{1016}z_{17} - z_{611}^2 + 2z_{111}z_{616} + 2z_{1011}z_{67} - z_{710}^2 + 2z_{610}z_{711} - 2z_{110}z_{716} + 2z_{79}z_{810} - 2z_{69}z_{811} + 2z_{19}z_{816} - z_{89}^2 + 2z_{78}z_{910} - 2z_{68}z_{911} + 2z_{18}z_{916},$

$\varphi_{77} := -z_{115}^2 + 2z_{1115}z_{13} - 2z_{1015}z_{14} - z_{311}^2 + 2z_{111}z_{315} + 2z_{1011}z_{34} - z_{410}^2 + 2z_{310}z_{411} - 2z_{110}z_{415} + 2z_{49}z_{510} - 2z_{39}z_{511} + 2z_{19}z_{515} - z_{59}^2 + 2z_{45}z_{910} - 2z_{35}z_{911} + 2z_{15}z_{915},$

$\varphi_{88} := -z_{114}^2 + 2z_{1114}z_{12} - z_{211}^2 + 2z_{111}z_{214} + 2z_{28}z_{411} - 2z_{18}z_{414} - z_{48}^2 - 2z_{27}z_{511} + 2z_{17}z_{514} - z_{57}^2 + 2z_{47}z_{58} - 2z_{25}z_{711} + 2z_{15}z_{714} + 2z_{45}z_{78} + 2z_{24}z_{811} - 2z_{14}z_{814}$

$\varphi_{99} := -z_{113}^2 + 2z_{1013}z_{12} - z_{210}^2 + 2z_{110}z_{213} + 2z_{28}z_{310} - 2z_{18}z_{313} - z_{38}^2 - 2z_{26}z_{510} + 2z_{16}z_{513} - z_{56}^2 + 2z_{36}z_{58} - 2z_{25}z_{610} + 2z_{15}z_{613} + 2z_{35}z_{68} + 2z_{23}z_{810} - 2z_{13}z_{813},$

$\varphi_{1010} := -z_{112}^2 + 2z_{19}z_{212} - z_{29}^2 - 2z_{17}z_{312} - z_{37}^2 + 2z_{27}z_{39} + 2z_{16}z_{412} - z_{46}^2 + 2z_{36}z_{47} - 2z_{26}z_{49} + 2z_{14}z_{612} + 2z_{34}z_{67} - 2z_{24}z_{69} - 2z_{13}z_{712} + 2z_{23}z_{79} + 2z_{12}z_{912},$

$\varphi_{12} := -z_{215}z_{216} + z_{1416}z_{23} - z_{1316}z_{24} + z_{1216}z_{25} + z_{1415}z_{26} - z_{1315}z_{27} + z_{1215}z_{28} + z_{214}z_{316} + z_{1314}z_{37} - z_{1214}z_{38} - z_{213}z_{416} - z_{1314}z_{46} + z_{1213}z_{48} + z_{212}z_{516} + z_{1214}z_{56} - z_{1213}z_{57} - z_{514}z_{612} + z_{414}z_{613} - z_{314}z_{614} + z_{214}z_{615} + z_{513}z_{712} - z_{413}z_{713} + z_{313}z_{714} - z_{213}z_{715} - z_{512}z_{812} + z_{412}z_{813} - z_{312}z_{814} + z_{212}z_{815},$

$\varphi_{13} := z_{1215}z_{210} + z_{1015}z_{212} + z_{1516}z_{23} - z_{1315}z_{29} - z_{1214}z_{310} - z_{1014}z_{312} - z_{216}z_{315} + z_{314}z_{316} - z_{1316}z_{34} + z_{1216}z_{35} + z_{1415}z_{36} + z_{1314}z_{39} + z_{1213}z_{410} + z_{1013}z_{412} - z_{313}z_{416} - z_{1315}z_{46} - z_{1012}z_{512} + z_{312}z_{516} + z_{1215}z_{56} - z_{1213}z_{59} - z_{515}z_{612} + z_{415}z_{613} - z_{315}z_{614} + z_{215}z_{615} + z_{513}z_{912} - z_{413}z_{913} + z_{313}z_{914} - z_{213}z_{915},$

$\varphi_{14} := z_{1215}z_{211} + z_{1115}z_{212} + z_{1516}z_{24} - z_{1415}z_{29} - z_{1214}z_{311} - z_{1114}z_{312} - z_{1416}z_{34} + z_{1415}z_{37} + z_{1213}z_{411} + z_{1113}z_{412} + z_{316}z_{414} - z_{216}z_{415} - z_{413}z_{416} + z_{1216}z_{45} - z_{1315}z_{47} + z_{1314}z_{49} - z_{1112}z_{512} + z_{412}z_{516} + z_{1215}z_{57} - z_{1214}z_{59} - z_{515}z_{712} + z_{415}z_{713} - z_{315}z_{714} + z_{215}z_{715} + z_{514}z_{912} - z_{414}z_{913} + z_{314}z_{914} - z_{214}z_{915},$

$\varphi_{15} := -z_{1415}z_{210} + z_{1315}z_{211} + z_{1115}z_{213} - z_{1015}z_{214} + z_{1516}z_{25} - z_{1314}z_{311} - z_{1114}z_{313} + z_{1014}z_{314} - z_{1416}z_{35} + z_{1415}z_{38} + z_{1314}z_{410} + z_{1113}z_{413} - z_{1013}z_{414} + z_{1316}z_{45} - z_{1315}z_{48} - z_{1214}z_{510} + z_{1213}z_{511} - z_{1112}z_{513} - z_{416}z_{513} + z_{1012}z_{514} + z_{316}z_{514} - z_{216}z_{515} + z_{512}z_{516} + z_{1215}z_{58} - z_{515}z_{812} + z_{415}z_{813} - z_{315}z_{814} + z_{215}z_{815},$

$\varphi_{16} := z_{1316}z_{14} - z_{1216}z_{15} - z_{1415}z_{16} + z_{115}z_{216} - z_{1116}z_{23} + z_{1015}z_{27} - z_{114}z_{316} - z_{1014}z_{37} + z_{113}z_{416} - z_{1113}z_{46} - z_{112}z_{516} + z_{1112}z_{56} - z_{1012}z_{57} + z_{511}z_{612} - z_{411}z_{613} + z_{311}z_{614} - z_{211}z_{615} + z_{410}z_{713} - z_{310}z_{714} + z_{210}z_{715} - z_{513}z_{79} - z_{412}z_{810} + z_{59}z_{812} + z_{39}z_{814} - z_{29}z_{815} - z_{48}z_{913} + z_{38}z_{914} - z_{28}z_{915},$

$\varphi_{17} := -z_{1315}z_{14} + z_{13}z_{1415} + z_{1215}z_{15} - z_{115}z_{215} + z_{1115}z_{23} - z_{1015}z_{24} - z_{311}z_{314} + z_{114}z_{315} + z_{211}z_{315} + z_{1014}z_{34} - z_{1113}z_{34} + z_{1112}z_{35} + z_{313}z_{411} - z_{410}z_{413} + z_{310}z_{414} - z_{113}z_{415} - z_{210}z_{415} - z_{1012}z_{45} + z_{412}z_{510} - z_{312}z_{511} + z_{49}z_{513} - z_{39}z_{514} + z_{112}z_{515} + z_{29}z_{515} - z_{512}z_{59} + z_{45}z_{913} - z_{35}z_{914} + z_{25}z_{915},$

$\varphi_{18} := z_{1314}z_{14} - z_{12}z_{1415} - z_{1214}z_{15} + z_{115}z_{214} - z_{211}z_{215} - z_{1114}z_{23} + z_{1113}z_{24} - z_{1112}z_{25} + z_{214}z_{311} - z_{114}z_{314} - z_{213}z_{411} + z_{113}z_{414} - z_{38}z_{414} + z_{28}z_{415} + z_{413}z_{48} + z_{212}z_{511} - z_{47}z_{513} - z_{112}z_{514} + z_{37}z_{514} - z_{27}z_{515} + z_{512}z_{57} - z_{412}z_{58} - z_{45}z_{713} + z_{35}z_{714} - z_{25}z_{715} + z_{45}z_{812} - z_{34}z_{814} + z_{24}z_{815},$

$\varphi_{19} := -z_{13}z_{1314} + z_{12}z_{1315} + z_{1213}z_{15} - z_{115}z_{213} + z_{210}z_{215} + z_{1014}z_{23} - z_{1013}z_{24} + z_{1012}z_{25} - z_{214}z_{310} + z_{114}z_{313} - z_{28}z_{315} + z_{314}z_{38} + z_{213}z_{410} - z_{113}z_{413} - z_{313}z_{48} - z_{212}z_{510} + z_{112}z_{513} + z_{46}z_{513} - z_{36}z_{514} + z_{26}z_{515} - z_{512}z_{56} + z_{312}z_{58} + z_{45}z_{613} - z_{35}z_{614} + z_{25}z_{615} - z_{35}z_{812} + z_{34}z_{813} - z_{23}z_{815},$

$\varphi_{110} := -z_{12}z_{1215} + z_{1214}z_{13} - z_{1213}z_{14} + z_{115}z_{212} - z_{215}z_{29} - z_{114}z_{312} + z_{27}z_{315} - z_{314}z_{37} + z_{214}z_{39} + z_{113}z_{412} + z_{36}z_{414} - z_{26}z_{415} - z_{413}z_{46} + z_{313}z_{47} - z_{213}z_{49} - z_{112}z_{512} + z_{412}z_{56} - z_{312}z_{57} + z_{212}z_{59} - z_{45}z_{612} + z_{34}z_{614} - z_{24}z_{615} + z_{35}z_{712} - z_{34}z_{713} + z_{23}z_{715} - z_{25}z_{912} + z_{24}z_{913} - z_{23}z_{914},$

$\varphi_{23} := z_{1216}z_{210} + z_{1016}z_{212} + z_{1516}z_{26} - z_{1316}z_{29} - z_{216}z_{316} + z_{1416}z_{36} - z_{1316}z_{37} + z_{1216}z_{38} - z_{1214}z_{610} - z_{1014}z_{612} - z_{614}z_{615} + z_{215}z_{616} + z_{314}z_{616} + z_{1315}z_{67} - z_{1215}z_{68} + z_{1314}z_{69} + z_{1213}z_{710} + z_{1013}z_{712} + z_{613}z_{715} - z_{313}z_{716} - z_{1012}z_{812} - z_{612}z_{815} + z_{312}z_{816} -$

$z_{1213}z_{89} + z_{813}z_{912} - z_{713}z_{913} + z_{613}z_{914} - z_{213}z_{916},$
$\varphi_{24} := z_{1216}z_{211} + z_{1116}z_{212} + z_{1516}z_{27} - z_{1416}z_{29} - z_{216}z_{416} + z_{1416}z_{46} - z_{1316}z_{47} +$
$z_{1216}z_{48} - z_{1214}z_{611} - z_{1114}z_{612} + z_{414}z_{616} + z_{1415}z_{67} + z_{1213}z_{711} + z_{1113}z_{712} - z_{615}z_{714} +$
$z_{713}z_{715} + z_{215}z_{716} - z_{413}z_{716} - z_{1215}z_{78} + z_{1314}z_{79} - z_{1112}z_{812} - z_{712}z_{815} + z_{412}z_{816} -$
$z_{1214}z_{89} + z_{814}z_{912} - z_{714}z_{913} + z_{614}z_{914} - z_{214}z_{916},$
$\varphi_{25} := -z_{1416}z_{210} + z_{1316}z_{211} + z_{1116}z_{213} - z_{1016}z_{214} + z_{1516}z_{28} - z_{216}z_{516} + z_{1416}z_{56} -$
$z_{1316}z_{57} + z_{1216}z_{58} - z_{1314}z_{611} - z_{1114}z_{613} + z_{1014}z_{614} + z_{514}z_{616} + z_{1415}z_{68} + z_{1314}z_{710} +$
$z_{1113}z_{713} - z_{1013}z_{714} - z_{513}z_{716} - z_{1315}z_{78} - z_{1214}z_{810} + z_{1213}z_{811} - z_{1112}z_{813} + z_{715}z_{813} +$
$z_{1012}z_{814} - z_{615}z_{814} - z_{812}z_{815} + z_{215}z_{816} + z_{512}z_{816},$
$\varphi_{26} := -z_{1416}z_{16} + z_{1316}z_{17} - z_{1216}z_{18} + z_{116}z_{216} - z_{1116}z_{26} + z_{1016}z_{27} + z_{611}z_{614} -$
$z_{114}z_{616} - z_{211}z_{616} - z_{1014}z_{67} + z_{1113}z_{67} - z_{1112}z_{68} - z_{613}z_{711} + z_{710}z_{713} - z_{610}z_{714} +$
$z_{113}z_{716} + z_{210}z_{716} + z_{1012}z_{78} - z_{712}z_{810} + z_{612}z_{811} - z_{79}z_{813} + z_{69}z_{814} - z_{112}z_{816} -$
$z_{29}z_{816} + z_{812}z_{89} - z_{78}z_{913} + z_{68}z_{914} - z_{28}z_{916},$
$\varphi_{27} := z_{13}z_{1416} - z_{1315}z_{17} + z_{1215}z_{18} - z_{116}z_{215} - z_{1016}z_{24} + z_{1115}z_{26} + z_{211}z_{316} -$
$z_{1113}z_{37} + z_{1112}z_{38} - z_{210}z_{416} - z_{1014}z_{46} - z_{1012}z_{48} + z_{29}z_{516} + z_{414}z_{610} - z_{314}z_{611} +$
$z_{114}z_{615} - z_{514}z_{69} - z_{413}z_{710} + z_{313}z_{711} + z_{510}z_{712} - z_{113}z_{715} - z_{312}z_{811} + z_{49}z_{813} +$
$z_{112}z_{815} - z_{512}z_{89} - z_{57}z_{913} + z_{56}z_{914} + z_{25}z_{916},$
$\varphi_{28} := -z_{12}z_{1416} + z_{1314}z_{17} - z_{1214}z_{18} + z_{116}z_{214} - z_{211}z_{216} - z_{1114}z_{26} + z_{1113}z_{27} -$
$z_{1112}z_{28} + z_{28}z_{416} - z_{27}z_{516} + z_{214}z_{611} - z_{114}z_{614} + z_{514}z_{67} - z_{414}z_{68} - z_{213}z_{711} -$
$z_{58}z_{712} + z_{57}z_{713} + z_{113}z_{714} - z_{56}z_{714} - z_{25}z_{716} + z_{413}z_{78} - z_{512}z_{78} + z_{212}z_{811} + z_{48}z_{812} -$
$z_{47}z_{813} - z_{112}z_{814} + z_{46}z_{814} + z_{24}z_{816},$
$\varphi_{29} := z_{12}z_{1316} - z_{1314}z_{16} + z_{1213}z_{18} - z_{116}z_{213} + z_{210}z_{216} + z_{1014}z_{26} - z_{1013}z_{27} +$
$z_{1012}z_{28} - z_{28}z_{316} + z_{26}z_{516} - z_{214}z_{610} + z_{58}z_{612} + z_{114}z_{613} - z_{57}z_{613} + z_{56}z_{614} +$
$z_{25}z_{616} - z_{513}z_{67} + z_{314}z_{68} + z_{512}z_{68} + z_{213}z_{710} - z_{113}z_{713} - z_{313}z_{78} - z_{212}z_{810} -$
$z_{38}z_{812} + z_{112}z_{813} + z_{37}z_{813} - z_{36}z_{814} - z_{23}z_{816},$
$\varphi_{210} := -z_{12}z_{1216} + z_{1214}z_{16} - z_{1213}z_{17} + z_{116}z_{212} - z_{216}z_{29} + z_{27}z_{316} - z_{26}z_{416} -$
$z_{114}z_{612} - z_{48}z_{612} + z_{47}z_{613} - z_{46}z_{614} - z_{24}z_{616} - z_{314}z_{67} + z_{413}z_{67} - z_{412}z_{68} + z_{214}z_{69} +$
$z_{113}z_{712} + z_{38}z_{712} - z_{37}z_{713} + z_{36}z_{714} + z_{23}z_{716} + z_{312}z_{78} - z_{213}z_{79} - z_{112}z_{812} + z_{212}z_{89} -$
$z_{28}z_{912} + z_{27}z_{913} - z_{26}z_{914},$
$\varphi_{34} := -z_{1012}z_{1112} + z_{1216}z_{311} + z_{1116}z_{312} + z_{1516}z_{37} - z_{1416}z_{39} + z_{1216}z_{410} + z_{1016}z_{412} -$
$z_{316}z_{416} + z_{1516}z_{46} - z_{1316}z_{49} - z_{1215}z_{611} - z_{1115}z_{612} + z_{415}z_{616} + z_{1415}z_{69} - z_{1215}z_{710} -$
$z_{1015}z_{712} - z_{615}z_{715} + z_{315}z_{716} + z_{1315}z_{79} + z_{1214}z_{910} + z_{1213}z_{911} + z_{1014}z_{912} + z_{1113}z_{912} -$
$z_{913}z_{914} + z_{614}z_{915} + z_{713}z_{915} - z_{314}z_{916} - z_{413}z_{916},$
$\varphi_{35} := z_{1012}z_{1014} - z_{1013}z_{1112} + z_{1011}z_{1213} - z_{1416}z_{310} + z_{1316}z_{311} + z_{1116}z_{313} -$
$z_{1016}z_{314} + z_{1516}z_{38} + z_{1216}z_{510} + z_{1016}z_{512} - z_{316}z_{516} + z_{1516}z_{56} - z_{1316}z_{59} + z_{1415}z_{610} -$
$z_{1315}z_{611} - z_{1115}z_{613} + z_{1015}z_{614} + z_{515}z_{616} - z_{1215}z_{810} - z_{1015}z_{812} - z_{615}z_{815} + z_{315}z_{816} +$
$z_{1315}z_{89} + z_{1314}z_{910} + z_{1113}z_{913} - z_{1013}z_{914} + z_{813}z_{915} - z_{513}z_{916},$
$\varphi_{36} := -z_{1016}z_{112} - z_{110}z_{1216} - z_{1516}z_{16} + z_{1316}z_{19} + z_{116}z_{316} - z_{1116}z_{36} + z_{1016}z_{37} -$
$z_{1112}z_{610} + z_{1011}z_{612} + z_{611}z_{615} - z_{115}z_{616} - z_{311}z_{616} - z_{1015}z_{67} + z_{1113}z_{69} + z_{1012}z_{710} -$
$z_{610}z_{715} + z_{310}z_{716} - z_{1013}z_{79} + z_{69}z_{815} - z_{39}z_{816} + z_{713}z_{910} - z_{812}z_{910} - z_{613}z_{911} -$
$z_{810}z_{912} + z_{89}z_{913} + z_{68}z_{915} + z_{113}z_{916} - z_{38}z_{916},$
$\varphi_{37} := z_{1015}z_{112} + z_{110}z_{1215} + z_{13}z_{1516} - z_{1315}z_{19} + z_{1112}z_{310} - z_{1011}z_{312} - z_{116}z_{315} +$
$z_{311}z_{316} - z_{1016}z_{34} + z_{1115}z_{36} - z_{1113}z_{39} - z_{1012}z_{410} - z_{310}z_{416} - z_{1015}z_{46} + z_{1013}z_{49} +$
$z_{39}z_{516} + z_{415}z_{610} - z_{315}z_{611} + z_{115}z_{615} - z_{515}z_{69} - z_{413}z_{910} + z_{512}z_{910} + z_{313}z_{911} +$
$z_{510}z_{912} - z_{59}z_{913} - z_{113}z_{915} + z_{56}z_{915} + z_{35}z_{916},$
$\varphi_{38} := -z_{1014}z_{112} - z_{110}z_{1214} - z_{12}z_{1516} + z_{1314}z_{19} - z_{1112}z_{210} + z_{1011}z_{212} + z_{1113}z_{29} -$
$z_{216}z_{311} + z_{116}z_{314} - z_{1114}z_{36} + z_{38}z_{416} - z_{1013}z_{47} - z_{37}z_{516} + z_{215}z_{611} - z_{115}z_{614} +$
$z_{515}z_{67} - z_{415}z_{68} - z_{512}z_{710} + z_{59}z_{713} - z_{56}z_{715} - z_{35}z_{716} + z_{410}z_{812} + z_{46}z_{815} + z_{34}z_{816} -$
$z_{413}z_{89} - z_{213}z_{911} - z_{58}z_{912} + z_{113}z_{914},$

$\varphi_{39} := z_{1013}z_{112} + z_{110}z_{1213} + z_{13}z_{1316} - z_{1315}z_{16} + z_{1012}z_{210} - z_{1016}z_{23} + z_{1015}z_{26} - z_{1013}z_{29} + z_{216}z_{310} - z_{116}z_{313} - z_{316}z_{38} + z_{36}z_{516} - z_{215}z_{610} + z_{512}z_{610} + z_{510}z_{612} + z_{115}z_{613} - z_{59}z_{613} + z_{56}z_{615} + z_{35}z_{616} + z_{315}z_{68} - z_{513}z_{69} - z_{312}z_{810} - z_{310}z_{812} + z_{39}z_{813} - z_{36}z_{815} + z_{313}z_{89} + z_{213}z_{910} - z_{113}z_{913}$,

$\varphi_{310} := -z_{1012}z_{112} - z_{1216}z_{13} + z_{1215}z_{16} - z_{1213}z_{19} + z_{116}z_{312} + z_{316}z_{37} - z_{216}z_{39} - z_{36}z_{416} - z_{412}z_{610} - z_{115}z_{612} - z_{410}z_{612} + z_{49}z_{613} - z_{46}z_{615} - z_{34}z_{616} - z_{315}z_{67} + z_{215}z_{69} + z_{413}z_{69} + z_{312}z_{710} + z_{310}z_{712} - z_{39}z_{713} + z_{36}z_{715} - z_{313}z_{79} - z_{212}z_{910} + z_{113}z_{912} - z_{210}z_{912} + z_{29}z_{913} - z_{26}z_{915} + z_{23}z_{916}$,

$\varphi_{45} := -z_{1112}z_{1113} + z_{1012}z_{1114} + z_{1011}z_{1214} - z_{1416}z_{410} + z_{1316}z_{411} + z_{1116}z_{413} - z_{1016}z_{414} + z_{1516}z_{48} + z_{1216}z_{511} + z_{1116}z_{512} - z_{416}z_{516} + z_{1516}z_{57} - z_{1416}z_{59} + z_{1415}z_{710} - z_{1315}z_{711} - z_{1115}z_{713} + z_{1015}z_{714} + z_{515}z_{716} - z_{1215}z_{811} - z_{1115}z_{812} - z_{715}z_{815} + z_{415}z_{816} + z_{1415}z_{89} + z_{1314}z_{911} + z_{1114}z_{913} - z_{1014}z_{914} + z_{814}z_{915} - z_{514}z_{916}$,

$\varphi_{46} := -z_{1116}z_{112} - z_{111}z_{1216} - z_{1516}z_{17} + z_{1416}z_{19} + z_{116}z_{416} - z_{1116}z_{46} + z_{1016}z_{47} - z_{1112}z_{611} - z_{411}z_{616} - z_{1115}z_{67} + z_{1114}z_{69} + z_{1012}z_{711} + z_{615}z_{711} + z_{1011}z_{712} - z_{710}z_{715} - z_{115}z_{716} + z_{410}z_{716} - z_{1014}z_{79} + z_{79}z_{815} - z_{49}z_{816} + z_{714}z_{910} - z_{614}z_{911} - z_{812}z_{911} - z_{811}z_{912} + z_{89}z_{914} + z_{78}z_{915} + z_{114}z_{916} - z_{48}z_{916}$,

$\varphi_{47} := z_{1115}z_{112} + z_{111}z_{1215} + z_{14}z_{1516} - z_{1415}z_{19} + z_{1112}z_{311} - z_{1116}z_{34} + z_{1115}z_{37} - z_{1114}z_{39} - z_{1012}z_{411} + z_{316}z_{411} - z_{1011}z_{412} - z_{116}z_{415} - z_{410}z_{416} - z_{1015}z_{47} + z_{1014}z_{49} + z_{49}z_{516} + z_{415}z_{710} - z_{315}z_{711} + z_{115}z_{715} - z_{515}z_{79} - z_{414}z_{910} + z_{314}z_{911} + z_{512}z_{911} + z_{511}z_{912} - z_{59}z_{914} - z_{114}z_{915} + z_{57}z_{915} + z_{45}z_{916}$,

$\varphi_{48} := -z_{1114}z_{112} - z_{111}z_{1214} - z_{14}z_{1416} + z_{1415}z_{17} - z_{1112}z_{211} + z_{1116}z_{24} - z_{1115}z_{27} + z_{1114}z_{29} - z_{216}z_{411} + z_{116}z_{414} + z_{416}z_{48} - z_{47}z_{516} + z_{215}z_{711} - z_{512}z_{711} - z_{511}z_{712} - z_{115}z_{714} + z_{59}z_{714} - z_{57}z_{715} - z_{45}z_{716} - z_{415}z_{78} + z_{514}z_{79} + z_{412}z_{811} + z_{411}z_{812} - z_{49}z_{814} + z_{47}z_{815} - z_{414}z_{89} - z_{214}z_{911} + z_{114}z_{914}$,

$\varphi_{49} := z_{1113}z_{112} + z_{111}z_{1213} - z_{12}z_{1516} + z_{1314}z_{19} + z_{1012}z_{211} + z_{1011}z_{212} - z_{1014}z_{29} - z_{1114}z_{36} + z_{216}z_{410} - z_{116}z_{413} - z_{1013}z_{47} - z_{316}z_{48} + z_{46}z_{516} + z_{512}z_{611} - z_{59}z_{614} + z_{57}z_{615} + z_{45}z_{616} + z_{515}z_{67} - z_{215}z_{710} + z_{115}z_{713} + z_{315}z_{78} - z_{311}z_{812} - z_{37}z_{815} + z_{34}z_{816} + z_{314}z_{89} + z_{214}z_{910} - z_{58}z_{912} - z_{114}z_{913}$,

$\varphi_{410} := -z_{1112}z_{112} - z_{1216}z_{14} + z_{1215}z_{17} - z_{1214}z_{19} + z_{116}z_{412} - z_{416}z_{46} + z_{316}z_{47} - z_{216}z_{49} - z_{412}z_{611} - z_{411}z_{612} + z_{49}z_{614} - z_{47}z_{615} - z_{415}z_{67} + z_{414}z_{69} + z_{312}z_{711} - z_{115}z_{712} + z_{311}z_{712} - z_{39}z_{714} + z_{37}z_{715} - z_{34}z_{716} + z_{215}z_{79} - z_{314}z_{79} - z_{212}z_{911} + z_{114}z_{912} - z_{211}z_{912} + z_{29}z_{914} - z_{27}z_{915} + z_{24}z_{916}$,

$\varphi_{56} := -z_{1116}z_{113} + z_{1016}z_{114} - z_{111}z_{1316} + z_{110}z_{1416} - z_{1516}z_{18} + z_{116}z_{516} - z_{1116}z_{56} + z_{1016}z_{57} + z_{1114}z_{610} - z_{1113}z_{611} - z_{1011}z_{614} - z_{511}z_{616} - z_{1115}z_{68} - z_{1014}z_{710} + z_{1013}z_{711} + z_{1011}z_{713} + z_{510}z_{716} + z_{1015}z_{78} - z_{715}z_{810} + z_{615}z_{811} - z_{115}z_{816} - z_{59}z_{816} + z_{815}z_{89} + z_{814}z_{910} - z_{813}z_{911} - z_{811}z_{913} + z_{810}z_{914} - z_{58}z_{916}$,

$\varphi_{57} := z_{1115}z_{113} - z_{1015}z_{114} + z_{111}z_{1315} - z_{110}z_{1415} + z_{15}z_{1516} - z_{1114}z_{310} + z_{1113}z_{311} + z_{1011}z_{314} - z_{1116}z_{35} + z_{1115}z_{38} + z_{1014}z_{410} - z_{1013}z_{411} - z_{1011}z_{413} + z_{1016}z_{45} - z_{1015}z_{48} - z_{416}z_{510} + z_{316}z_{511} - z_{116}z_{515} + z_{516}z_{59} + z_{415}z_{810} - z_{315}z_{811} + z_{115}z_{815} - z_{515}z_{89} - z_{514}z_{910} + z_{513}z_{911} + z_{511}z_{913} - z_{510}z_{914} + z_{58}z_{915}$,

$\varphi_{58} := -z_{1114}z_{113} + z_{1014}z_{114} - z_{111}z_{1314} - z_{1416}z_{15} + z_{1415}z_{18} + z_{1114}z_{210} - z_{1113}z_{211} - z_{1011}z_{214} + z_{1116}z_{25} - z_{1115}z_{28} - z_{216}z_{511} + z_{116}z_{514} - z_{516}z_{57} + z_{416}z_{58} + z_{514}z_{710} - z_{513}z_{711} - z_{511}z_{713} + z_{510}z_{714} - z_{58}z_{715} - z_{515}z_{78} - z_{414}z_{810} + z_{215}z_{811} + z_{413}z_{811} + z_{411}z_{813} - z_{115}z_{814} - z_{410}z_{814} + z_{48}z_{815} - z_{45}z_{816}$,

$\varphi_{59} := z_{1113}z_{113} - z_{1013}z_{114} + z_{110}z_{1314} + z_{1316}z_{15} - z_{1315}z_{18} - z_{1014}z_{210} + z_{1013}z_{211} + z_{1011}z_{213} - z_{1016}z_{25} + z_{1015}z_{28} + z_{216}z_{510} - z_{116}z_{513} + z_{516}z_{56} - z_{316}z_{58} - z_{514}z_{610} + z_{513}z_{611} + z_{511}z_{613} - z_{510}z_{614} + z_{58}z_{615} + z_{515}z_{68} - z_{215}z_{810} + z_{314}z_{810} - z_{313}z_{811} + z_{115}z_{813} - z_{311}z_{813} + z_{310}z_{814} - z_{38}z_{815} + z_{35}z_{816}$,

$\varphi_{510} := -z_{1112}z_{113} + z_{1012}z_{114} + z_{111}z_{1213} - z_{110}z_{1214} - z_{12}z_{1516} - z_{1114}z_{36} - z_{1013}z_{47} +$

$$z_{116}z_{512}-z_{416}z_{56}+z_{316}z_{57}-z_{216}z_{59}-z_{413}z_{611}+z_{410}z_{614}-z_{48}z_{615}+z_{45}z_{616}-z_{415}z_{68}-z_{314}z_{710}+z_{311}z_{713}+z_{38}z_{715}-z_{35}z_{716}+z_{315}z_{78}-z_{115}z_{812}+z_{215}z_{89}+z_{214}z_{910}-z_{213}z_{911}-z_{58}z_{912}-z_{211}z_{913}+z_{210}z_{914},$$

$$\varphi_{67} := z_{115}z_{116}-z_{1116}z_{13}+z_{1016}z_{14}-z_{1115}z_{16}+z_{1015}z_{17}-z_{111}z_{316}-z_{1011}z_{37}+z_{110}z_{416}+z_{1011}z_{46}-z_{19}z_{516}-z_{411}z_{610}+z_{311}z_{611}-z_{111}z_{615}+z_{511}z_{69}+z_{410}z_{710}-z_{310}z_{711}+z_{110}z_{715}-z_{510}z_{79}-z_{49}z_{810}+z_{39}z_{811}-z_{19}z_{815}+z_{59}z_{89}-z_{48}z_{910}+z_{57}z_{910}+z_{38}z_{911}-z_{56}z_{911}-z_{18}z_{915}-z_{15}z_{916},$$

$$\varphi_{68} := -z_{114}z_{116}+z_{1116}z_{12}+z_{1114}z_{16}-z_{1014}z_{17}+z_{111}z_{216}+z_{1011}z_{27}-z_{18}z_{416}+z_{17}z_{516}-z_{211}z_{611}+z_{111}z_{614}-z_{511}z_{67}+z_{411}z_{68}-z_{57}z_{710}+z_{210}z_{711}+z_{56}z_{711}-z_{110}z_{714}+z_{15}z_{716}-z_{410}z_{78}+z_{59}z_{78}+z_{58}z_{79}+z_{47}z_{810}-z_{29}z_{811}-z_{46}z_{811}+z_{19}z_{814}-z_{14}z_{816}-z_{48}z_{89}-z_{28}z_{911}+z_{18}z_{914},$$

$$\varphi_{69} := z_{113}z_{116}-z_{1016}z_{12}-z_{1113}z_{16}+z_{1013}z_{17}-z_{110}z_{216}-z_{1011}z_{26}+z_{18}z_{316}-z_{16}z_{516}+z_{211}z_{610}+z_{57}z_{610}-z_{56}z_{611}-z_{111}z_{613}-z_{15}z_{616}+z_{510}z_{67}-z_{311}z_{68}-z_{59}z_{68}-z_{58}z_{69}-z_{210}z_{710}+z_{110}z_{713}+z_{310}z_{78}+z_{29}z_{810}-z_{37}z_{810}+z_{36}z_{811}-z_{19}z_{813}+z_{13}z_{816}+z_{38}z_{89}+z_{28}z_{910}-z_{18}z_{913},$$

$$\varphi_{610} := -z_{112}z_{116}+z_{1112}z_{16}-z_{1012}z_{17}+z_{19}z_{216}-z_{17}z_{316}+z_{16}z_{416}-z_{47}z_{610}+z_{46}z_{611}+z_{111}z_{612}+z_{14}z_{616}+z_{311}z_{67}-z_{410}z_{67}+z_{49}z_{68}-z_{211}z_{69}+z_{48}z_{69}+z_{37}z_{710}-z_{36}z_{711}-z_{110}z_{712}-z_{13}z_{716}-z_{39}z_{78}+z_{210}z_{79}-z_{38}z_{79}+z_{19}z_{812}-z_{29}z_{89}-z_{27}z_{910}+z_{26}z_{911}+z_{18}z_{912}+z_{12}z_{916},$$

$$\varphi_{78} := z_{114}z_{115}-z_{1115}z_{12}-z_{1114}z_{13}+z_{1014}z_{14}-z_{111}z_{215}-z_{1011}z_{24}+z_{211}z_{311}-z_{111}z_{314}-z_{210}z_{411}-z_{38}z_{411}+z_{110}z_{414}+z_{18}z_{415}+z_{410}z_{48}-z_{47}z_{510}+z_{29}z_{511}+z_{37}z_{511}-z_{19}z_{514}-z_{17}z_{515}-z_{49}z_{58}+z_{57}z_{59}-z_{45}z_{710}+z_{35}z_{711}-z_{15}z_{715}-z_{34}z_{811}+z_{14}z_{815}+z_{45}z_{89}+z_{25}z_{911}-z_{15}z_{914},$$

$$\varphi_{79} := -z_{113}z_{115}+z_{1015}z_{12}+z_{1113}z_{13}-z_{1013}z_{14}+z_{110}z_{215}+z_{1011}z_{23}-z_{211}z_{310}+z_{111}z_{313}-z_{18}z_{315}+z_{311}z_{38}+z_{210}z_{410}-z_{110}z_{413}-z_{310}z_{48}-z_{29}z_{510}+z_{46}z_{510}-z_{36}z_{511}+z_{19}z_{513}+z_{16}z_{515}+z_{39}z_{58}-z_{56}z_{59}+z_{45}z_{610}-z_{35}z_{611}+z_{15}z_{615}+z_{34}z_{810}-z_{13}z_{815}-z_{35}z_{89}-z_{25}z_{910}+z_{15}z_{913},$$

$$\varphi_{710} := z_{112}z_{115}-z_{1112}z_{13}+z_{1012}z_{14}-z_{19}z_{215}-z_{111}z_{312}+z_{17}z_{315}-z_{311}z_{37}+z_{211}z_{39}+z_{36}z_{411}+z_{110}z_{412}-z_{16}z_{415}-z_{410}z_{46}+z_{310}z_{47}-z_{210}z_{49}-z_{19}z_{512}+z_{49}z_{56}-z_{39}z_{57}+z_{29}z_{59}+z_{34}z_{611}-z_{14}z_{615}-z_{45}z_{69}-z_{34}z_{710}+z_{13}z_{715}+z_{35}z_{79}+z_{24}z_{910}-z_{23}z_{911}-z_{15}z_{912}-z_{12}z_{915},$$

$$\varphi_{89} := z_{113}z_{114}-z_{1014}z_{12}-z_{1113}z_{12}+z_{210}z_{211}-z_{111}z_{213}-z_{110}z_{214}-z_{28}z_{311}+z_{18}z_{314}-z_{28}z_{410}+z_{18}z_{413}+z_{38}z_{48}+z_{27}z_{510}+z_{26}z_{511}-z_{17}z_{513}-z_{16}z_{514}+z_{56}z_{57}-z_{37}z_{58}-z_{46}z_{58}+z_{25}z_{611}-z_{15}z_{614}-z_{45}z_{68}+z_{25}z_{710}-z_{15}z_{713}-z_{35}z_{78}-z_{24}z_{810}-z_{23}z_{811}+z_{14}z_{813}+z_{13}z_{814},$$

$$\varphi_{810} := -z_{112}z_{114}+z_{1112}z_{12}+z_{111}z_{212}+z_{19}z_{214}-z_{211}z_{29}+z_{27}z_{311}-z_{17}z_{314}-z_{26}z_{411}-z_{18}z_{412}+z_{16}z_{414}-z_{38}z_{47}+z_{46}z_{48}+z_{28}z_{49}+z_{17}z_{512}-z_{47}z_{56}+z_{37}z_{57}-z_{27}z_{59}-z_{24}z_{611}+z_{14}z_{614}+z_{45}z_{67}+z_{23}z_{711}+z_{15}z_{712}-z_{13}z_{714}+z_{34}z_{78}-z_{25}z_{79}-z_{14}z_{812}+z_{24}z_{89}+z_{12}z_{914},$$

$$\varphi_{910} := z_{112}z_{113}-z_{1012}z_{12}-z_{110}z_{212}-z_{19}z_{213}+z_{210}z_{29}-z_{27}z_{310}+z_{18}z_{312}+z_{17}z_{313}+z_{37}z_{38}-z_{28}z_{39}+z_{26}z_{410}-z_{16}z_{413}-z_{36}z_{48}-z_{16}z_{512}+z_{46}z_{56}-z_{36}z_{57}+z_{26}z_{59}+z_{24}z_{610}-z_{15}z_{612}-z_{14}z_{613}-z_{35}z_{67}-z_{34}z_{68}+z_{25}z_{69}-z_{23}z_{710}+z_{13}z_{713}+z_{13}z_{812}-z_{23}z_{89}-z_{12}z_{913}.$$

Lemma 3.4. *For every* $A \in \mathrm{Spin}_{10}$, $B \in GL_{14}$, $\tilde{Z} \in \mathrm{Alt}_{16}$, *we have* $\tilde{Z} \mapsto (\det B)\tilde{Z}$, $\tilde{Z} \mapsto {}^t\Lambda'(A)^{-1}\tilde{Z}\Lambda'(A)^{-1}$ *and hence*

$$\begin{gathered} \Phi((\det B)\tilde{Z}) \mapsto (\det B)^2\Phi(\tilde{Z}), \\ \Phi({}^t\Lambda'(A)^{-1}\tilde{Z}\Lambda'(A)^{-1}) = {}^t\chi(A)^{-1}\Phi(\tilde{Z})\chi(A)^{-1}, \end{gathered} \tag{3.10}$$

where χ is the vector representation of Spin_{10} *in* (3.9)*. Hence we have, by the action of* $(\alpha, A, B) \in GL_1 \times \mathrm{Spin}_{10} \times GL_{14}$,

$$\Phi(\tilde{Z}) \mapsto (\det B)^t \chi(A)^{-1} \Phi(\tilde{Z}) \chi(A)^{-1}.$$

Proof. We may prove the equivariance (3.10) for $A's$ of one of the following three forms:

$$A_u = \left(\begin{array}{ccccc|ccccc} 1 & \varepsilon & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ \hline & & & & & 1 & & & & \\ & & & & & -\varepsilon & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{array}\right),$$

$$A_{u'} = \left(\begin{array}{ccccc|ccccc} 1 & & & & & & \varepsilon & & & \\ & 1 & & & & -\varepsilon & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ \hline & & & & & 1 & & & & \\ & & & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{array}\right);$$

$$A_{u''} = \left(\begin{array}{ccccc|ccccc} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ \hline & \varepsilon & & & & 1 & & & & \\ -\varepsilon & & & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{array}\right),$$

$$A_d = \mathrm{diag}(a_1, a_2, a_3, a_4, a_5, a_1^{-1}, a_2^{-1}, a_3^{-1}, a_4^{-1}, a_5^{-1});$$

and permutation type matrices such as

$$A_u = \left(\begin{array}{ccccc|ccccc} 0 & 1 & & & & & & & & \\ 1 & 0 & & & & & & & & \\ & & 1 & & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ \hline & & & & & 0 & 1 & & & \\ & & & & & 1 & 0 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{array}\right).$$

Checking (3.10), for diagonal or permutation type matrices is easy. Note that, for A_d, $z_{12} \mapsto a_3a_4a_5z_{12}, z_{13} \mapsto a_2a_4a_5z_{13}, \dots, z_{23} \mapsto a_1^{-1}a_4a_5z_{23}, z_{24} \mapsto a_1^{-1}a_3a_5z_{24}, \dots, z_{1516} \mapsto a_3^{-1}a_4^{-1}a_5^{-1}z_{1516}$. Hence $\varphi_{ij} \mapsto a_ia_j\varphi_{ij}$ for $1 \le i \le j \le 5$, $\varphi_{lk} \mapsto a_l^{-1}a_k^{-1}\varphi_{lk}$ for $6 \le l \le k \le 10$, $\varphi_{lj} \mapsto a_l^{-1}a_j\varphi_{lj}$ for $6 \le l \le 10, 1 \le j \le 5$. Then we have $\Phi({}^t\Lambda'(A_d)^{-1}\tilde{Z}\Lambda'(A_d)^{-1}) = {}^t\chi(A_d)^{-1}\Phi(\tilde{Z})\chi(A_d)^{-1}$. For A_u, we consider the action of A_u. Since $\det A_u = 1$, $\Phi(\tilde{Z}) \mapsto \Phi({}^t\Lambda'(A_u)\tilde{Z}\Lambda'(A_u)^{-1})$. Then we have $\varphi_{11} \mapsto \varphi_{11} + 2\varepsilon\varphi_{12} + \varepsilon^2\varphi_{22}$, $\varphi_{1j} \mapsto \varphi_{1j} + \varepsilon\varphi_{2j}$ $(1 \le j \le 5)$, $\varphi_{77} \mapsto \varphi_{77} - 2\varepsilon\varphi_{67} + \varepsilon^2\varphi_{66}$, $\varphi_{67} \mapsto \varphi_{67} - \varepsilon\varphi_{66}$, $\varphi_{7k} \mapsto \varphi_{7k} - \varepsilon\varphi_{6k}$ $(8 \le k \le 10)$, $\varphi_{17} \mapsto \varphi_{17} + \varepsilon(\varphi_{27} - \varphi_{16}) + \varepsilon^2\varphi_{26}$, $\varphi_{1j} \mapsto \varphi_{1j} + \varepsilon\varphi_{2j}$ $(j \in \{6, 8, 9, 10\})$, $\varphi_{k7} \mapsto \varphi_{k7} - \varepsilon\varphi_{k6}$ $(k \in \{2, 3, 4, 5\})$ and other φ_{ij}'s are invariants. Thus, we have $\Phi({}^t\Lambda'(A_u)^{-1}\tilde{Z}\Lambda'(A_u)^{-1}) = {}^t\chi(A_u)^{-1}\Phi(\tilde{Z}){}^t\chi(A_u)^{-1}$.

In the same way, we have $\Phi({}^t\Lambda'(A_{u'})^{-1}\tilde{Z}\Lambda'(A_{u'})^{-1}) = {}^t\chi(A_{u'})^{-1}\Phi(\tilde{Z})\chi(A_{u'})^{-1}$ and $\Phi({}^t\Lambda'(A_{u''})^{-1}\tilde{Z}\Lambda'(A_{u''})^{-1}) = {}^t\chi(A_{u''})^{-1}\Phi(\tilde{Z})\chi(A_{u''})^{-1}$. □

Remark C. We can construct the above polynomials φ_{ij} $(1 \le i \le j \le 10)$ in the same program as in Remark A in §§3.1. We have

$$K\Phi(\tilde{Z}) \mapsto (\det B)^2K{}^t\chi(A)^{-1}\Phi(\tilde{Z})\chi(A)^{-1} = (\det B)^2\chi(A)K\Phi(\tilde{Z})\chi(A)^{-1}$$

with $K = \left(\begin{array}{c|c} 0 & I_5 \\ \hline I_5 & 0 \end{array}\right)$ and hence $F_1(x) = \operatorname{tr} K\Phi(\tilde{Z})$ is a relative invariant of degree 28 corresponding to the character $(\det B)^2$. Moreover, by (3.7) and Lemma 3.4, we put $F_2(x) = \langle\eta(X\cdot Y)|\Phi(\tilde{Z})|\eta(X\cdot Y)\rangle := {}^t\eta(X\cdot Y)\Phi(\tilde{Z})\eta(X\cdot Y)$ and hence $F_2(x)$ is a relative invariant of degree 36 corresponding to the character $\alpha^4(\det B)^2$.

For a generic point

$$x_0 = (e_1e_5 + e_2e_3e_4e_5\ ,\ e_2e_5\ ,\ e_3e_5\ ,\ e_4e_5\ ,\ -e_1e_3e_4e_5\ , e_1e_2e_4e_5\ , -e_1e_2e_3e_5\ ,$$
$$-1 + e_1e_2e_3e_4\ ,\ e_1e_2\ ,\ e_1e_3\ ,\ e_1e_4\ , -e_3e_4\ ,\ e_2e_4\ ,\ -e_2e_3\ ,\ e_4' + e_{14}'),$$

we have

$$X_0\cdot Y_0 = {}^t(0,\ 0,\ 0,\ 0,\ 0,\ -1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0),$$

$$\eta(X_0\cdot Y_0) = {}^t(0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0),$$

$$\tilde{Z}_0 = (e_1\wedge e_5 - e_1\wedge e_{16}) \in \mathrm{Alt}_{16},$$

$$\Phi(\tilde{Z}_0) = \Phi(\tilde{Z}_0) = \left(\begin{array}{ccccc|ccccc} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \in \mathrm{Sym}_{10}\,.$$

Then we have $F_2(X_1^{(0)}, \cdots, X_{14}^{(0)}, Y_0) = \langle \eta(X_0 \cdot Y_0) \mid \Phi(\tilde{Z}_0) \mid \eta(X_0 \cdot Y_0)\rangle = -1 \neq 0$ and $F_1(X_1^{(0)}, \cdots, X_{14}^{(0)}, Y_0) = \operatorname{tr} K\Phi(\tilde{Z}_0) = -4 \neq 0$ and hence they are not identically zero.

In [6] or [8], the irreducible relative invariant $f(x)$ of degree 4 of a prehomogeneous vector space $(\mathrm{Spin}_{10} \times GL_2, \Lambda' \otimes \Lambda_1, V(16) \oplus V(16))$ is constructed. Through the theory of castling transformation, above polynomial $F_1(x)$ is corresponding to the irreducible relative invariant of a prehomogeneous vector space $(\mathrm{Spin}_{10} \times GL_2, \Lambda' \otimes \Lambda_1, V(16) \oplus V(16))$ studied in [6]. By Lemma 1.2, we can know the irreducibility of $F_1(x)$. Note that we can also check the degree of $F_1(x)$ by compairing the degree of $f(x)$ by Lemma 1.2.

If the polynomial $F_2(x)$ is not irreducible, then $F_2(x) = F_1(x)G(x)$ for some relative invariant $G(x)$ of degree 8 corresponding to the character α^4, or $F_2(x) = H(x)^2$ for some relative invariant $H(x)$ of degree 18 corresponding to the character $\alpha^2(\det B)$. For $x_1 = {}^t(tX_1^{(0)}, X_2^{(0)}, \ldots, X_{14}^{(0)}, Y_1) \in V(16)^{\oplus 14} \oplus V(14)^*$ with $Y_1 = {}^t(y_1, 0, 0, y_4, 0, 0, 0, y_8, 0, 0, y_{11}, 0, 0, y_{14})$, we have $F_1(x_1) = -4t^2$ and $F_2(x_1) = 2t^2y_1^2y_{11}^2y_{14} - y_{14}^2y_4^2 - 2t^2y_1^2y_8^2$. Then we have $F_1(x_1) \nmid F_2(x_1)$ and hence $F_1(x) \nmid F_2(x)$. Moreover, $F_2(x_1)$ cannot be of the form $F_2(x_1) = H(x_1)^2$, and hence $F_2(x)$ cannot be of the form $F_2(x) = H(x)^2$. Thus we have obtained the irreducibility of $F_2(x)$.

Theorem 3.5. *The prehomogeneous vector space* $(GL_1 \times \mathrm{Spin}_{10} \times GL_{14}, \Lambda' \otimes \Lambda_1 + 1 \otimes \Lambda_1^*, V(16)^{\oplus 14} \oplus V(14)^*)$ *has* 2 *basic relative invariants:*

(1) $F_1(x) = \operatorname{tr} K\Phi(\tilde{Z}) \longleftrightarrow (\det B)^2$, $\deg .F_1 = 28$,
(2) $F_2(x) = \langle \eta(X \cdot Y) | \Phi(\tilde{Z}) | \eta(X \cdot Y)\rangle \longleftrightarrow \alpha^4(\det B)^2$, $\deg .F_2 = 36$.

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