THE TAME KERNEL OF IMAGINARY QUADRATIC FIELDS
WITH CLASS NUMBER 2 OR 3
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Abstract. This paper presents improved bounds for the norms of exceptional
finite places of the group $K_2O_F$, where $F$ is an imaginary quadratic field of
class number 2 or 3. As an application we show that $K_2Z[\sqrt{-10}] = 1$.

1. Introduction

Tate [9] has determined the tame kernel of all imaginary quadratic Euclidean
fields $F$ and of $F = Q(\sqrt{-15})$. He and Bass [2] proved that when the norm of
the finite place $v$ of the field $F$ is sufficiently large, then a certain associated map
$\partial_v$ (see section 2 below) is an isomorphism. It follows that in order to compute
the tame kernel we need only investigate the remaining $v$'s (those whose norms
are smaller than the bound of the exceptional $v$'s), and perform the appropriate
computations with Steinberg symbols. Skalba [8] used a generalization of Thue’s
theorem to get a reasonable bound for norms of exceptional $v$'s, and proved that
$K_2O_F = 1$ when $F = Q(\sqrt{-19})$ and $F = Q(\sqrt{-20})$. Modifying the method of
Tate, Qin [6,7] did the same for $F = Q(\sqrt{-24})$ and $F = Q(\sqrt{-35})$. Recently
Browkin [4] improved the estimates of Skalba and Qin for the bounds of norms of
exceptional $v$'s and applied his result to the case $F = Q(\sqrt{-23})$. In the present
paper we present a certain adaptation of Tate, Skalba and Browkin’s method for
computing the tame kernel of imaginary quadratic fields with class number 2 or 3,
and we get much smaller bounds for norms of exceptional $v$'s. We apply this result
in the case $F = Q(\sqrt{-40})$.

2. Notation

Let $F$ be a number field and let $v_1, v_2, v_3, \ldots$ be all finite places of $F$, ordered in
a certain way. For $m \geq 0$ let $S_m = \{v_1, \ldots, v_m\}$, thus $S_0 = \emptyset$. Denote the ring of
$S_m$-integers of $F$ by $O_{S_m}$, the group of $S_m$-units by $U_{S_m}$. Let $P_m$ be the maximal
ideal of $O_{S_m-1}$, corresponding to the place $v_m$. The residue field of the place $v_m$
is denoted by $k_{v_m}$ ($k_{v_m} = O_{S_{m-1}}/P_m$) and the norm of $v_m$ is defined to be $N_{v_m} = \text{card}k_{v_m}$. Thus $O_{S_0}$ is the ring $O_F$ of integers of $F$; we denote it by $O$ below.

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Let $K_2^{S_m}(F)$ be the subgroup of $K_2F$ generated by symbols \{a, b\}, where $a, b \in U_{sm}$. Then $K_2F = \bigcup_{m=1}^{\infty} K_2^{S_m}(F)$.

Let $\partial_{v_m} : K_2F \to k_{v_m}^*$ be the tame symbol corresponding to $v_m$. Since $K_2^{S_{m-1}}(F) \subset \ker(\partial_{v_m})$, there is an induced map (also denoted by $\partial_{v_m}$)

$$\partial_{v_m} : K_2^{S_m}(F)/K_2^{S_{m-1}}(F) \to k_{v_m}^*. $$

By Quillen’s exact sequence

$$(2.1) \quad 1 \to K_2O \to K_2F = \bigcup_{m=1}^{\infty} K_2^{S_m}(F) \xrightarrow{\partial} \prod_{m=1}^{\infty} k_{v_m}^* \to 1,$$

where $\partial = \prod_{m=1}^{\infty} \partial_{v_m}$, we know that if $\partial_{v_j}$ are isomorphisms for all $j > m$, then $K_2O \subset K_2^{S_m}(F)$.

Since in the sequel we shall assume that $m$ is fixed, we simplify the notation as follows:

$$S = S_{m-1}, \quad P = P_m \bigcap O, \quad v = v_m, \quad \partial_v = \partial_{v_m}, \quad U = U_{S_{m-1}}, \quad k^* = k_{v_m}^*. $$

Now let $F = Q(\sqrt{d})$ be the imaginary quadratic field with discriminant $d$ and class number 2 or 3. Under the embedding $\sigma : F \to \mathcal{C} : a + b\sqrt{d} \mapsto a + b\sqrt{d}$, of the field $F$ into $\mathcal{C}$, we can consider $F$ as a subfield of $\mathcal{C}$. For all $x \in O$, $N_x = |x|^2 = \pi^r$.

Suppose that $\lambda = p$ is the least prime number such that $p$ is not inertial in $O$ and all factors in the prime decomposition of $pO$ are nonprincipal. Let $pO = Q_1Q_2$, where $Q_1, Q_2$ are nonprincipal primes in $O$ ($Q_2$ may equal $Q_1$ when the class number of $F$ is 2). Set $A = \{Q_1, Q_2\}$. For any nonprincipal prime $P$ in $O$, there is an element $Q \in A$ such that $QP$ is principal and the norm of $QP$ is $\lambda N(P)$.

**Definition 2.1.** Let $P$ be a prime ideal of $O$. Define the principal norm $M(P)$ of $P$ as follows:

$$M(P) = \begin{cases} N(P), & \text{if } P \text{ is principal in } O, \\ \lambda N(P), & \text{if } P \text{ is nonprincipal in } O. \end{cases}$$

For the finite place $v$ corresponding to a prime ideal $P$ of $O$, we write $M_v = M(P)$ and call $M_v$ the principal norm of $v$. If $P$ is principal in $O$, we say the place $v$ is principal and we have $M_v = N_v$. Otherwise, $v$ is nonprincipal and $M_v = \lambda N_v$.

For finite places $v_i$ and $v_j$, if $i < j$, we say that $v_i$ precedes $v_j$.

Let $M$ be a real number satisfying $\lambda \leq M$. We assume that the ordering on the set of all finite places of $F$ satisfies the property: for all finite places $v$ and $v'$, if $M_v < M_{v'}$ and $M < M_{v'}$, then $v$ precedes $v'$.

When the ordering on the set of finite places satisfies the preceding condition, we say that the ordering is normal for finite places with principal norm greater than $M$.

In the two sections below, we always assume that $M < M_v$. Thus the prime ideal $PO_S$ corresponding to $v$ in $O_S$ is principal. We write $PO_S$ as $\pi O_S$, where $\pi \in O$ and $N\pi = M_v$. Let $\beta$ be the homomorphism from $U$ to $k^*$ given by $\beta(u) = u(\mod \pi)$ (sometimes we write $u(\mod \pi) = u(\mod v)$). Denote by $U_1$ the subgroup of $U$ generated by $(1 + \pi U) \cap U$. It is easy to see that $U_1 \subseteq \ker(\beta)$. Let $a, b \in U$; then $a \sim b$ means $a \in bU_1$. 

3. Preliminary information on the finite place \( v \)

**Lemma 3.1.**
1. If \( x \in O \) and \( 0 < Nx < M_v \), then \( x \in U \).
2. \( W = \{ x \in O \cap U \mid Nx \leq M_v \} \) generates \( U \).

**Proof.** (1) Suppose that \( (x) = (y)(z) \) where \( (y) = P_1P_2 \cdots P_r \), \( (z) = Q_1Q_2 \cdots Q_s \), \( P_i \) \((i = 1, \ldots, r)\) is principal prime in \( O \) and \( Q_j \) \((j = 1, \ldots, s)\) is nonprincipal prime in \( O \) respectively. For \( P_i \), we have \( N(P_i) \leq Nx < M_v \) and for \( Q_j \), we have \( \lambda \leq N(Q_j) \leq Nz/\lambda \leq Nx/\lambda < M_v/\lambda \), i.e., \( M(Q_j) < M_v \). That means the finite places corresponding to \( P_i \) or \( Q_j \) precede \( v \), so \( x \in U \).

(2) In fact \( U \) is generated by the set \( \{ a \in O \cap U \mid Na = 1 \text{ or } Na = M_v, \ v' \text{ precedes } v \} \).

**Lemma 3.2.** Suppose \( v \) is principal. Let \( a, b \in O \cap U \) satisfy \( |a| + |b| < N_v \) and \( a \equiv b(\text{mod } v) \). Then \( a \sim b \).

**Proof.** Assume \( a \neq b \). Since \( a \equiv b(\text{mod } v) \), \( \pi|a−b| \) (note that \( PO_S = \pi O_S \)). By \( |a| + |b| < N_v \), we have \( N(a−b)/\pi < N_v \). Then \( a−b/\pi \in U \) by Lemma 3.1, so \( a−b \in \pi U, a \sim b \).

Now we assume \( v \) is nonprincipal and let \( Q \) be a nonprincipal prime in \( A \) such that \( QP \) is principal in \( O \). Then \( N(QP) = \lambda N_v = M_v \).

**Lemma 3.3.** Suppose \( v \) is nonprincipal and let \( a, b \in O \cap U \).
1. If \( |a| + |b| < N_v \) and \( a \equiv b(\text{mod } v) \), then \( a \sim b \).
2. Further, if \( a, b \in O \) satisfy \( |a| + |b| < M_v \) and \( a \equiv b(\text{mod } v) \), then \( a \sim b \).

**Proof.** Assume \( a \neq b \). As an ideal of \( O \), \( (a−b) = (y)(z) \), where \( (y) = P_1P_2 \cdots P_r \), \( (z) = PQ_1 \cdots Q_s \), \( P_i \) \((i = 1, \ldots, r)\) is principal prime in \( O \) and \( Q_j \) \((j = 1, \ldots, s, s \geq 1)\) is nonprincipal prime in \( O \) respectively and \( PQ_1 \) or \( PQ_1Q_2 \) is a principal ideal denoted by \( (c) \).

(1) First we show that \( PO_S = cO_S \), where \( (c) = PQ_1 \) or \( PQ_1Q_2 \). This follows from the fact that the finite place corresponding to \( Q_j \) \((j = 1, 2, \ldots, s)\) precedes \( v \).

In fact, \( N(Q_j) \leq N(a−b)/N(P) < N_v^2/N_v = N_v \). Then \( a−b/\pi \in U, a−b \in cU, a \sim b \).

(2) Similarly to the proof of (1), suppose \( PQ = (c) \); then \( Nc = M_v, c|a−b \).

We have \( N(a−b)/\pi < M_v \). By Lemma 3.1, \( a−b \in cU, a \sim b \).

4. Conditions for \( \partial_v \) being bijective

Tate [3] has proved the following useful result

**Theorem 4.1 (3 Prop. 1).** Suppose that \( W, C \) and \( G \) are subsets of \( U \) such that

1. \( W \subset CU_1 \) and \( W \) generates \( U \),
2. \( CG \subset CU_1 \) and \( \beta(G) \) generates \( k^* \),
3. \( 1 \subset C \cap \ker \beta \subset U_1 \).

Then \( \partial_v \) is bijective.

In this section we will give some other conditions for \( \partial_v \) being bijective.

**Proposition 4.2.** Suppose that \( W, D, E \) and \( E' \) are subsets of \( U \) such that

1. \( W \subset DD^{-1}U_1, \) where \( D^{-1} = \{ d^{-1} \mid d \in D \} \) and \( W \) generates \( U \),
2. \( 1 \subset E, E' \subset E, \) and the map: \( E' \times E' \times E \rightarrow k^* \times k^* \) given by \((e_1, e_2, e) \rightarrow (\beta(e_2/e_1), \beta(e/e_1))\) is surjective,
3. for any \( x \in \{1\} \cup D, e_1, e_2, e' \in E' \) and \( e \in E \),
Proof. Let \( u \in U \). Write \( u \sim \prod_{i=1}^{n} d_i^s \), where \( d_i \in D \) and \( s_i = 1 \) or \( s_i = -1 \) by condition (1). We want to show, by induction on \( r \), that

\[
(4.1) \quad \text{for any } u \in U, \text{ there are } e' \in E' \text{ and } e \in E \text{ such that } u \sim e'/e.
\]

When \( r = 1 \) we have \( u \sim d^s \), where \( s = 1 \) or \( s = -1 \). If \( s = 1 \), there are \( e_1, e_2 \in E' \) such that \( d/e \equiv e_2/e_1 (\mod v) \) by condition (2) (since \((\beta(d), 1) \) has preimage in \( E' \times E' \times E \)). We have \( d \sim e_2/e_1 \) from condition (3). So (4.1) holds (note that \( E' \subset E \)). When \( s = -1 \), replacing \( u \) by \( u^{-1} \), we have the same proof.

Assume (4.1) is true for \( r - 1 \). Writing \( u \sim d^s \prod_{i=2}^{n} d_i^{s_i} \), we have that there are \( e' \in E' \) and \( e \in E \) such that \( u \sim d^s e'/e \), where \( s = 1 \) or \( s = -1 \), by the inductive assumption.

When \( s = 1 \), by condition (2) there are \( e_1, e_2 \in E' \) and \( e_3 \in E \) such that \( d/e \equiv e_2/e_1 (\mod v) \), \( 1/e' \equiv e_3/e_1 (\mod v) \). Then we have \( d/e \sim e_2/e_1, 1/e' \sim e_3/e_1 \) by condition (3). Hence \( u \sim (e_2/e_1)/(e_3/e_1) = e_2/e_3 \), and (4.1) holds. When \( s = -1 \), replacing \( d/e \equiv e_2/e_1 (\mod v) \) and \( 1/e' \equiv e_3/e_1 (\mod v) \) by \( 1/e \equiv e_2/e_1 (\mod v) \) and \( d/e' \equiv e_3/e_1 (\mod v) \), where \( e_1, e_2 \in E' \) and \( e_3 \in E \), respectively, we can prove it by the same line of argument as the case \( s = 1 \).

Further, we have \( \ker \beta \subset U_1 \) and hence \( \ker \beta = U_1 \), since if \( u \sim e'/e \in \ker \beta \), i.e., \( \beta(e'/e) = 1 \), then \( e'/e \sim 1 \).

Let \( \alpha \) be the homomorphism from \( U \) to \( K_{S_m}^F/K_{S_m-1}^F \) given by \( \alpha(u) = \{u, \pi\} (\mod K_{S_m-1}^F) \), where \( \pi \in \pi \) satisfies \( PO_S = \pi O_S \). Then, as in the proof of Lemma 3.2 in [2], we have \( \beta = \partial \circ \alpha \). To prove \( \partial \) is bijective, it suffices to show that

(i) \( \alpha \) is surjective, and
(ii) \( \beta \) is surjective and \( \ker \beta \subset \ker \alpha \).

From [2] pages 405-406, (i) is true. By condition (2), \( \beta \) is surjective. If \( 1 - u \pi \in U_1 \), then \( \{1 - u \pi, \pi\} = \{1 - u, \pi\}^{-1} \in K_{S_m-1}^F \) and so \( \ker \beta = U_1 \subset \ker \beta \). We are done. \( \blacksquare \)

Let \( h > 0 \) be a real number. Define \( D(h) = \{x \in O \mid 0 < |x| \leq h\} \). For simplicity, let \( K = \frac{2}{\pi} \sqrt{|d|} \).

**Lemma 4.3.** Suppose that \( v \) is nonprincipal and \( N_v > K^2 \). Let \( h^2 = \lambda K N_v^2 \) and \( D = \{x \in O \mid 0 < |x| \leq h\} \). We have \( \text{vol} \{x \in O \mid 0 < |x| \leq h\} = \frac{1}{2} N(Q)^2 N_v \).

**Proof.** We only sketch the line of the proof, since it is similar to that of Lemmas 1-3 in [2].

For any \( a \in O \), define \( K_a : C \times C \to C \) by the formula \( (x, y) \to y - ax \). Let \( K_a = F_{a^{-1}}(P \cap Q) \). Then \( K_a \) is a lattice in \( R^4 \simeq C \times C \), and we have

\[
\text{vol} K_a = \text{vol} K_0 = \text{vol} (Q \times (Q \cap P)) = \left( \frac{\sqrt{|d|}}{2} \right)^2 N(Q)^2 N_v.
\]

Let \( S_h = \{(x, y) \in C \times C \mid |x| \leq h, |y| \leq h\} \). We have \( \text{vol} S_h = (\pi h^2)^2 \) and \( 2^4 \text{vol} K_a = \text{vol} S_h \). Applying Minkowski’s theorem, we can get \( x, y \in S_h \cap K_a \), where \( (x, y) \neq (0, 0) \). Note that \( N_x, N_y \leq h^2 < M_v \). If \( a \in O \setminus P \), then \( x, y \neq 0 \).
By Lemma 3.1, $x, y \in U$. Then $a \equiv y/x (mod v)$ (see Skalba’s GTT in [8] page 306).

**Lemma 4.4.** Suppose that $v$ is nonprincipal and $M_v > K^3$. Let $\tilde{h} > 0$ and $\tilde{h}^2 = KM_{\tilde{h}}^3 (= K\lambda^2 N_{\tilde{h}}^3)$, and let $E = D(\tilde{h}) \cap U, E' = E \cap Q$. For any $a, b \in O \setminus P$, there exist $e_1, e_2 \in E'$ and $e \in E$ such that $a \equiv e_2/e_1 (mod v)$ and $b \equiv e/e_1 (mod v)$. 

**Proof.** For any $a, b \in O$, define $F_{a, b} : C \times C \times C \to C \times C$ by the formula $(x, y, z) \to (y - ax, z - bx)$. Let $K_{a, b} = F_{a, b}^{-1}(P \times P) \cap (Q \times Q \times O)$. The remaining procedure is as in Lemma 4.3 (you also can refer to [5]).

**Theorem 4.5.** Suppose that $v$ is nonprincipal $(M_v = \lambda N_v)$. If the inequalities

1. $N_v > \max\{K^2, K^3/\lambda\}$,
2. $N_v > (1 + M_{\sqrt{v}}^3)^4 \cdot K^2$,
3. $M_v > [(1 + \lambda^2 M_v^{-12})K]^{\frac{1}{2}}$,
4. $N_v > \lambda^2 (4K)^{\frac{1}{2}}$

hold, then $\partial_v$ is bijective.

**Proof.** Let us check that the subsets $W$ as in Lemma 3.1, $D$ as in Lemma 4.3, and $E, E'$ as in Lemma 4.4, of $U$ satisfy the conditions of Proposition 4.2.

(i) Let $w \in W$. Since $N_v > \max\{K^2, K^3/\lambda\}$, by Lemma 4.3 there exist $d_1, d_2 \in D$ such that $w \equiv d_1/d_2 (mod v)$. The condition (2) above provides $|wd_2| + |d_1| < M_v$, so $w \in DD^{-1}U_1$ by Lemma 3.3.

(ii) That $E' \subset E, 1 \in E$ is clear. The condition (1) above allows us to apply Lemma 4.4 to get that the map from $E' \times E' \times E$ to $k^* \times k^*$ is surjective.

(iii) Let $x \in \{1\} \cup D, e_1, e_2, e' \in E'$ and $e \in E$. By conditions (3) and (4) above, the following inequalities hold:

$$|xe_1| + |ee_2| < M_v, \quad |e'| + |e| < N_v.$$ 

Applying Lemma 3.3, we know that condition (3) of Proposition 4.2 holds.

When $v$ is principal, the results of Lemma 3.1 and Lemma 3.2 allow us to apply Theorem 1 in [5] to obtain the following result.

**Theorem 4.6.** Suppose that $v$ is principal and $|d| \geq 3$. If the inequalities

1. $N_v > K^3$,
2. $N_v > (1 + N_v^{-\frac{1}{2}})^4 \cdot K^2$,
3. $N_v > [(1 + N_v^{-12})K]^{\frac{1}{2}}$

hold, then $\partial_v$ is bijective.

**Note.** We take $a = 1$ and $\lambda = 1$ in Theorem 1 of [5].

In Table [11] for every discriminant $d$ such that the class number of $F = Q(\sqrt{d})$ is 2 or 3, we give the estimates of $N_v$ from Theorem 4.5 and Theorem 4.6.

**Note.** 1) From [11], we know that all $d$ such that the class number of $F = Q(\sqrt{d})$ is 3 are listed in the above table.

2) When computing the bound of $N_v$, note that the right sides of the inequalities (2) and (3) in Theorems 4.5 and 4.6 are decreasing functions on $M_v$ and $N_v$ respectively.

**Remark 4.7.** Comparing the estimates of $N_v$ above with Browkin’s in [11], we can see that the bound for norms of exceptional $v$’s is much smaller here.
Table 1.

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5. Computation of $K_2Z[\sqrt{-10}]$

In this section we apply the general method of previous sections to the special case $F = Q(\sqrt{-10})$. Note that the class number of $F$ is 2.

Let $\omega = \sqrt{-10}$. Then $O = Z[\omega]$. Take $Q = (2, \omega)$; we have $Q^2 = (2^2 \omega)$, $N = N(Q) = 2$, $O$ and $Q$ can be considered as a lattice in $C$.

Let $\delta$ and $\delta'$ denote the maximal distance from $C$ to $O$ and $Q$ respectively. Then $\delta^2 = \frac{11}{4}$, $\delta'^2 = \frac{7}{2}$.

**Lemma 5.1.** Let $A$ be a principal ideal of $O$. Then there is a representative element $c$ in every residue class modulo $A$ such that $Nc \leq \frac{1}{4} N(A)$.

**Lemma 5.2.** Let $A$ be a nonprincipal ideal of $O$. Then there is a representative element $c$ in every residue class modulo $A$ such that $Nc \leq \frac{2}{7} N(A)$.

The proofs of the two lemmas are elementary.

Suppose that all finite places of $F$ are ordered in the following way:

1. $v_1, v_2$ and $v_3$ correspond to the prime ideals $(3), (2, \omega)$ and $(5)$, respectively (these ideals are in $O, O_{S_1}$ and $O_{S_2}$, respectively). The remaining finite places with principal norm not greater than 41 correspond to the ideal $(a_m)$, $m = 4, 5, \ldots, 13$, in $O_{S_{m-1}}$, where $a_4 = 2 - \omega, a_6 = 1 + \omega, a_8 = 3 + \omega, a_{10} = 4 + \omega, a_{12} = 1 + 2\omega$, and $a_{2k+1} = \omega_{2k} (k = 2, 3, 4, 5, 6)$.

2. The ordering is normal for finite places with principal norm greater than 41. In fact this ordering is even normal on the set of finite places with principal norm greater than 14.

In the following propositions we set $G = \{x \in O \cap U \mid |x| \leq |g|\}$, where $|g|$ is the least number such that $\beta(G)$ generates $k^*$ and $W$ as in Lemma 3.1, and choose a suitable subset $C$ of $U$ satisfying the conditions of Theorem 4.1 so as to apply Theorem 4.1 to show $\partial_v$ is bijective for some $v$ with small norm.

**Proposition 5.3.** If $v$ is nonprincipal and $N_v > 13$ (i.e., $M_v > 26$), then $\partial_v$ is bijective.

**Proof.** According to the table for the bound of $N_v$ in Section 4, we only need to consider the case $N_v \leq 170$.

Let $C = \{x \in O \mid 0 < Nx \leq \frac{7}{4} N_v\}$. By Lemma 3.1 we know that $C \subset U$. From Lemma 3.3 and Lemma 5.2 we see that if $N_v$ satisfies the inequalities

1. $\sqrt{2N_v} + \sqrt{\frac{7}{4} N_v} < N_v$, i.e., $7.5 \approx (\sqrt{2} + \sqrt{\frac{7}{4}})^2 < N_v$,
(2) $\sqrt{\frac{2}{3}N_v(|g| + 1)} < N_v$, i.e., $\frac{2}{3}(|g| + 1)^2 < N_v$,
then we can apply Theorem 4.1 to obtain that $\partial_v$ is bijective. Thus for the following
cases the consequence holds:

$$N_v = 23, 37, 47, 53, 103, 167, \text{ here } |g| = 2;$$
$$N_v = 127, \text{ here } |g| = 3;$$
$$N_v = 157, \text{ here } |g| = \sqrt{11}. \quad \Box$$

Let $h > 0$. Denote $h' = \max\{|x| \mid x \in D(h)\}$ (note that $D(h) = \{x \in O \mid 0 < |x| \leq h\}$).

**Lemma 5.4** (Lemma 3.2 in [5]). Suppose $N_v > K^2$, $D_1 = D(h_1)$ and $D_2 = D(h_2)$,
where $h_1, h_2 > 0$ satisfy the following conditions:

1. $h_1h_2 = KN_v^\frac{1}{2}$,
2. $\max(h_1^2, h_2^2) < N_v$.

Then for any $a \in O \setminus P$, there exist $x \in D_1$ and $y \in D_2$ such that $a \equiv x/y (\text{mod } v)$.

**Proposition 5.5.** If $v$ is principal and $N_v > 41$, then $\partial_v$ is bijective.

**Proof.** We only need to consider the case $N_v < 281$ by the table in Section 4. Take $h_1, h_2$ and $D_1, D_2$ as in Lemma 5.4 and let $C = D_1D_2^{-1}$. By Lemma 3.1 we know
that $D_1, D_2 \subset U$, so $C \subset U$. From Lemma 3.2 and Lemma 5.4 we see that if the
two inequalities

1. $N_v^{\frac{1}{2}}h_2' + h_1' < N_v$,
2. $(1 + |g|)h_1'h_2' < N_v$

hold, then $\partial_v$ is bijective by Theorem 4.1.

It is easy to show the consequence holds for the following cases. Here we list the
values of $N_v, |g|, h_2'^2, h_1'^2$, where $h_1$ is determined by condition (1) in Lemma
5.4 (in fact the method was used by Skalba [5]).

| $N_v$ | $|g|$ | $h_2'^2$ | $h_1'^2$ | $h_2'^2$ |
|-------|-------|--------|--------|--------|
| 251   | 3     | 18.99  | 211    | 16     |
| 211   | 2     | 58.49  | 56     | 56     |
| 179   | 2     | 53.87  | 49     | 49     |
| 139   | 2     | 18.99  | 115    | 16     |
| 131   | 2     | 18.99  | 110    | 16     |

For the remaining cases we apply Qin’s method [3, 7] to construct $C$.

Let $C' = \{x \in O \mid 0 < Nx \leq \frac{14}{11}N_v\}$, $T = C' \setminus U = \{t_1, t_2, \ldots, t_r\}$. Choose
$S = \{s_1, s_2, \ldots, s_r\} \subset O \cap U$, where $s_i \equiv t_i (\text{mod } v)$, $i = 1, \ldots, r$. Let $C = (C' \setminus T) \cup S$
(the choice of elements in $S$ should make $m(C) = \max\{|x| \mid x \in C\}$ as small as
possible).

By Lemma 3.1 and Lemma 5.1, if the inequalities

1. $m(C) + \sqrt{N_v} < N_v$,
2. $m(C)(1 + |g|) < N_v$

hold, then $\partial_v$ is bijective. So the result is true for the remaining cases:

| $N_v$ | $|g|$ | $\sqrt{14}$ | $3$ | $2$ | $\sqrt{2124}$ | $\sqrt{304}$ | $\Box$ |
|-------|-------|------------|-----|-----|-------------|-------------|-------|
| 241   | 89    | 59         |     |     | 2124        | 304         |       |
We have stated that the finite places with principal norm not greater than 41 correspond to the ideals \((a_m)\), \(m = 4, 5, \ldots, 13\), in \(O_{S_{m-1}}\) (except for \(v_1, v_2, v_3\)). For convenience, we write \(\partial_{m}^a\) as \(\partial_m\), where \(m = 4, 5, \ldots, 13\).

**Proposition 5.6.** \(\partial_m\) is bijective for \(m = 5, 6, \ldots, 13\).

**Proof.** We will construct the subsets \(W, C\) and \(G\) of \(U\) directly. Let \(a_0 = -1, a_1 = 2, a_2 = 3, a_3 = \omega\) (note that the meaning of \(a_m\), \(m = 4, 5, \ldots, 13\), has been given before). Set \(W_m = \{a_j \mid j = 0, 1, \ldots, m - 1\}\), where \(m = 5, 6, \ldots, 13\). We will list the sets \(C\) and \(G\) just for \(\partial_m\), where \(m = \{2r + 1, r = 2, 3, 4, 5, 6\}\). For \(m = 2r, r = 3, 4, 5, 6\), as the sets \(C\) and \(G\) for \(\partial_{2r}\) we take the conjugates of \(C\) and \(G\) for \(\partial_{2r+1}\) respectively. Let \(X\) be a number set. Denote by \(\pm X\) the set \(-X \cup X\), where \(-X = \{-x \mid x \in X\}\).

\[
\begin{align*}
m = 13, & \quad C = \{1, 2, 4, 8, 18, \omega, \omega + 1, 2(\omega + 1), 4(\omega + 1), \omega + 2, 2(\omega + 2), \omega^2, 3(\omega + 2), 3(\omega + 3), 3(\omega + 1), \omega + 2, 2(\omega + 4), \omega + 5, \omega + 6, 9\omega(\omega + 1)/2, 9\omega(\omega + 1)\}, \quad G = \{2, 3\}; \\
m = 11, & \quad C = \{1, 2, \omega, 2\omega, -\omega^5, \omega, -10\}, \quad G = \{2\}; \\
m = 9, & \quad C = \{1, 2, 4, 5, 4/5, \omega/2, \omega/5, \omega/10, 2\omega, 4\omega, 2(\omega + 2)\}, \quad G = \{2\}; \\
m = 7, & \quad C = \{1, 2, 5/2, \omega/2, \omega/3, \omega + 2\}, \quad G = \{2\}; \\
m = 5, & \quad C = \{1, 2, 4\}, \quad G = \{-2\}.
\end{align*}
\]

We can use the method used in [5] and [8] to check that the subsets \(W, C\) and \(G\) of \(U\) satisfy the conditions of Theorem 4.1.

**Theorem 5.7.** \(K_2 Z[\sqrt{-10}] = 1\).

Before proving the assertion, we list some identities on symbols and some results on the elements of \(K_2 F\) which are useful for our proof.

1) Let \(x \in F^*\) and assume that the polynomials of \(x\) make sense of the following symbols. We have

\[
\{x, x\}^2 = \{x, -1\}^2 = 1, \quad \{x, x + 1\}^2 = 1, \quad \{x, x^2 \pm 1\}^2 = 1, \quad \{x, x^2 + x + 1\}^3 = 1.
\]

2) If \(x, y, z \in F^*\) and \(z = x \pm y\), then \(\{z, -y\}^2 = \{x, y\}^2\), and hence \(\{x, y\}^2 \cdot \{y, z\}^2 = \{x, z\}^2\).

3) If \(\left\{\frac{x}{b}, \frac{y}{c}\right\}^n = 1\), where \(a, b, c, d \in U_{S_m}, c' \in F^*\), then \(\{a, c'\}^n \equiv \{b, c'\}^n \pmod{K_2^m}\).

4) From the proof of Proposition 3 in [2], we know that there is no element of order 2 in \(K_2 O\). Note that the class number of \(Q(\sqrt{3 \times 40})\) is prime to 3, so there is no element of order 3 in \(K_2 O\) by the results on \(p\)-rank of \(K_2 O\) in [2]. Thus if \(x \in K_2 O\) and \(x^{2^r} \in A\), where \(A\) is a subgroup of \(K_2 F\) and \(r, k \in N\), we have \(x \in A\).

Now let us complete the proof of Theorem 5.7.

By Quillen’s exact sequence (see (2.1) in Section 2) and the preceding results, we have

\[
(5.1) \quad K_2 O \subseteq K_2^G(F).
\]

a) By (5.1), \(K_2 O\) consists of products of symbols of the form \(\{a, b\}\), where \(a, b \in \{-1, 2, \omega, 3, 2 - \omega\}\). Since \(\omega - (\omega - 2) = 2\), we have

\[
(5.2) \quad \left\{\frac{\omega}{2}, \frac{\omega - 2}{2}\right\}^2 = 1.
\]
Since \((\omega - 2)^2 = \omega^2 - 4\omega + 4 = -(4\omega + 6)\), we have

\[(5.3) \quad \{\frac{-4\omega}{6}, \frac{(\omega - 2)^2}{6}\} = 1.\]

From (5.2), (5.3), we obtain that \(K_2O \subseteq \langle\{2, \omega - 2\}\rangle K_2^{S_3}(F)\). Let \(x = \frac{-2}{4}\). Then \(x^2 + x + 1 = \frac{1}{4}\) and \(\{x, \frac{1}{2}\}\) \(9\) = \(\{x, \frac{1}{2}\}\) \(3\) = 1. So

\[(5.4) \quad \{x, 2\}^9 = 1, \quad \{2, x\}^9 = 1, \quad \text{and} \quad \{2, \omega - 2\}^9 = 1.\]

(Note that \(\{2, 4\} = \{2, 2\}^2 = 1\).) Therefore \(K_2O \subseteq K_2^{S_3}(F)\).

b) Since

\[(5.5) \quad \{2, \omega\}^8 = \{2^4, \omega^2\} = \{16, -10\} = \{4, -10\}^2 = \{4, 5\}^2\{4, -2\}^2 = 1\]

and

\[(5.6) \quad \{3, \omega\}^8 = \{3^2, \omega^2\}^2 = \{9, -10\}^2 = 1,\]

we have \(K_2O \subseteq K_2^{S_3}(F)\).

Since \(\{2, 3\}^2 = 1\), we obtain \(K_2O = 1\). \(\square\)

Remark 5.8. The result agrees with that conjectured in [3], and also that in [4].

Acknowledgments

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