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FINITE ELEMENT APPROXIMATION OF SPECTRAL PROBLEMS WITH NEUMANN BOUNDARY CONDITIONS ON CURVED DOMAINS

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Abstract. This paper deals with the finite element approximation of the spectral problem for the Laplace equation with Neumann boundary conditions on a curved nonconvex domain Ω. Convergence and optimal order error estimates are proved for standard piecewise linear continuous elements on a discrete polygonal domain $\Omega_h \not\subset \Omega$ in the framework of the abstract spectral approximation theory.

1. Introduction

In this paper we analyze an elementary problem: the piecewise linear finite element approximation of eigenvalues and eigenfunctions of the Laplace operator with a Neumann boundary condition on a curved domain Ω.

Since the Neumann condition is a natural condition, in principle, a conforming finite element space could theoretically be defined on a curved triangulation fitting exactly the boundary of Ω (see, for instance, Chapter 8 of [5]). Then, to effectively solve the problem, the integrals for the entries of the stiffness and mass matrices on the curved triangles should be computed by means of a quadrature rule. Thus, the method turns out to be actually nonconforming.

However, for piecewise linear elements, the most common procedure in practice is to consider a mesh $\mathcal{T}_h$ formed by straight triangles. Then, the integrals of the entries of both matrices are actually computed on a discrete domain $\Omega_h := \bigcup\{T \in \mathcal{T}_h\}$, which do not coincide with Ω, thus yielding also a nonconforming method. As we show in this paper, this coarser procedure leads anyway to optimal order approximation results.

When $\Omega_h \subset \Omega$, the standard theory applies almost directly. In fact, for Dirichlet boundary conditions the method turns out to be conforming since the discrete space is a subset of $\mathrm{H}_0^1(\Omega)$ (see, e.g., the book by Raviart and Thomas [16]). For Neumann boundary conditions the method is not conforming, but convergence and optimal error estimates can be proved as rather straightforward consequences of the standard Strang's lemma, estimates like Lemma 5.2–5.3 in Raviart and Thomas's

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book [16], and the classical abstract spectral approximation theory by Osborn [15] (see also the monograph by Babuška and Osborn [1]).

The situation is quite different when $\Omega_h \not\subset \Omega$. The finite element approximation of the linear boundary value problem associated to the Laplace operator has been extensively studied. Different approaches have been considered, for instance by Ciarlet and Raviart [8], Zlámal [23], and Scott [17] (see also [18], [2], and [22]).

However, these results do not lead to a similar analysis for the corresponding eigenvalue problem. Indeed, convergence results for a linear boundary value problem do not necessarily imply similar results for its associated spectral one. This is particularly true for nonconforming and mixed methods (see, for instance, [3, 4] for examples of methods working out for the linear boundary value problem but failing for the associated eigenvalue problem).

In fact, to apply the spectral approximation theory stated in [1], one needs convergence in L^2 norm for the involved discrete operators, as well as optimal order L^2 and H^1 convergence results for the associated linear boundary value problem with homogeneous boundary conditions. For instance, for the Dirichlet problem on a curved domain with a smooth boundary, such results have been only recently proved by Bramble and King in Proposition 1 of [6], where the transfer of nonhomogeneous boundary data is analyzed.

For the eigenvalue problem, the first proof of convergence valid in the case $\Omega_h \not\subset \Omega$ was given by Vanmaele and Ženíšek [19] just a few years ago. They prove it for simple eigenvalues and Dirichlet boundary conditions by using the *min-max* characterization. These results have been further extended by the same authors to include multiple eigenvalues [20] and numerical integration effects [21] (thus allowing us to considering nonconstant coefficients).

Around the same time, Lebaud analyzed in [14] a similar problem also for simple eigenvalues and Dirichlet boundary conditions. She allowed for nonconstant coefficients in the differential operator, but assumed exact integration. She used the spectral approximation theory stated in [1] and optimal convergence results like those in [6]. She proved that, when isoparametric elements of degree k are used and the eigenfunctions are sufficiently smooth, the approximation of the eigenvalues is of order $\mathcal{O}(h^{2k})$ instead of $\mathcal{O}(h^{k+1})$.

In the case of Neumann boundary conditions, to the best of the authors' knowledge, no approximation result has been proved yet for the eigenvalue problem. Moreover, it is not apparent that the theory in [19] nor that in [14] could be extended to provide optimal error estimates in this case. In fact, both theories rely on estimates like (4.8) in [19], which are valid for functions vanishing on $\partial\Omega_h$ or on $\partial\Omega$ like the eigenfunctions of a Dirichlet problem, but not for those of a Neumann one.

For instance, for piecewise linear elements, the techniques in [14] clearly do not apply to the Neumann problem. To see this, notice that these results depend on the estimate (4.4) of that paper:

$$\|(T - T_h)u\|_{0,2,\mathbb{R}^2} = \mathcal{O}(h^2), \tag{1.1}$$

where Tu is the solution of the Laplace equations on Ω with right-hand side u, and $T_h u$ is its piecewise linear finite element approximation on Ω_h, obtained by using as its right-hand side the extension of u by zero outside of Ω (Tu and $T_h u$ are also extended by zero outside of their domains). Estimate (1.1) is proved in [14] as a

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consequence of the following three inequalities:

$$\|(T-T_h)u\|_{m,2,\Omega\cap\Omega_h} \leq Ch^{2-m}\|Tu\|_{2,2,\Omega}, \quad \text{for } m=0,1, \tag{1.2}$$

$$\|Tu\|_{0,2,\Omega\setminus(\Omega\cap\Omega_h)} \leq Ch^2\|Tu\|_{2,2,\Omega}, \tag{1.3}$$

$$\|T_hu\|_{0,2,\Omega_h\setminus(\Omega\cap\Omega_h)} \leq Ch^2\|T_hu\|_{1,2,\Omega\cap\Omega_h}. \tag{1.4}$$

To the best of the authors' knowledge estimate (1.2) has not been proved for $m=0$ and Neumann boundary conditions. Furthermore, estimate (1.4) is clearly not valid in this case.

This shows the necessity of considering a more subtle extension of the data u outside of Ω. One such extension, which preserves the eigenvalues of the Neumann problem, is introduced and analyzed in the next section.

The goal of this paper is to analyze the piecewise linear finite element approximation of the spectral problem for the Laplace equation with Neumann boundary conditions on curved nonconvex domains. We prove optimal order L^2 and H^1 error estimates for the eigenfunctions, and a double order for the eigenvalues. These estimates are proved to be valid for any piecewise smooth Lipschitz domain, for which the eigenfunctions do not necessarily belong to $\mathrm{H}^2(\Omega)$.

The outline of the paper is as follows. In Section 2 we present the model problem, the meshes to be used, the finite element approximation, and some basic results. Then, in Section 3, we prove optimal order error estimates for the approximate eigenfunctions and eigenvalues in the framework of the spectral approximation theory stated in [1].

2. Statement of the problem

Let $\Omega\subset\mathbb{R}^2$ be a bounded open domain, in general nonconvex, with a piecewise smooth (e.g., $\mathcal{C}^2$) Lipschitz boundary $\Gamma := \partial\Omega$. Consider the eigenvalue problem for the shifted Laplace operator in Ω with Neumann boundary conditions below.

Find $\lambda\in\mathbb{R}$ and $u\in\mathrm{H}^1(\Omega)$, $u\neq 0$, such that

$$a(u,v)=\lambda b(u,v) \qquad \forall v\in\mathrm{H}^1(\Omega). \tag{2.1}$$

The bilinear forms $a:\mathrm{H}^1(\Omega)\times\mathrm{H}^1(\Omega)\longrightarrow\mathbb{R}$ and $b:\mathrm{L}^2(\Omega)\times\mathrm{L}^2(\Omega)\longrightarrow\mathbb{R}$ in the equation above are defined by

$$a(v,w) := \int_\Omega (\nabla v\cdot\nabla w + vw)\ dx \qquad \forall v,w\in\mathrm{H}^1(\Omega), \tag{2.2}$$

$$b(f,g) := \int_\Omega fg\ dx \qquad \forall f,g\in\mathrm{L}^2(\Omega). \tag{2.3}$$

It is well known that problem (2.1) attains a sequence of finite multiplicity eigenvalues satisfying

$$1=\lambda_0<\lambda_1\leq\cdots\leq\lambda_k\leq\cdots, \qquad \lambda_k\overset{k\to\infty}{\longrightarrow}+\infty,$$

with corresponding $\mathrm{L}^2(\Omega)$-orthonormal eigenfunctions $\{u_k\}_{k=0}^\infty$ providing a Hilbert basis of $\mathrm{H}^1(\Omega)$.

Remark 2.1. We consider the bilinear form given by (2.2) instead of the standard one for the Laplace equations (i.e., $a(u,v)=\int_\Omega\nabla u\cdot\nabla v\ dx$), because the eigenvalues of both problems only differ by one and the corresponding eigenfunctions coincide. Since the same is true for the discrete problem (2.4) below, we thus avoid dealing with the standard constraints for pure Neumann problems.

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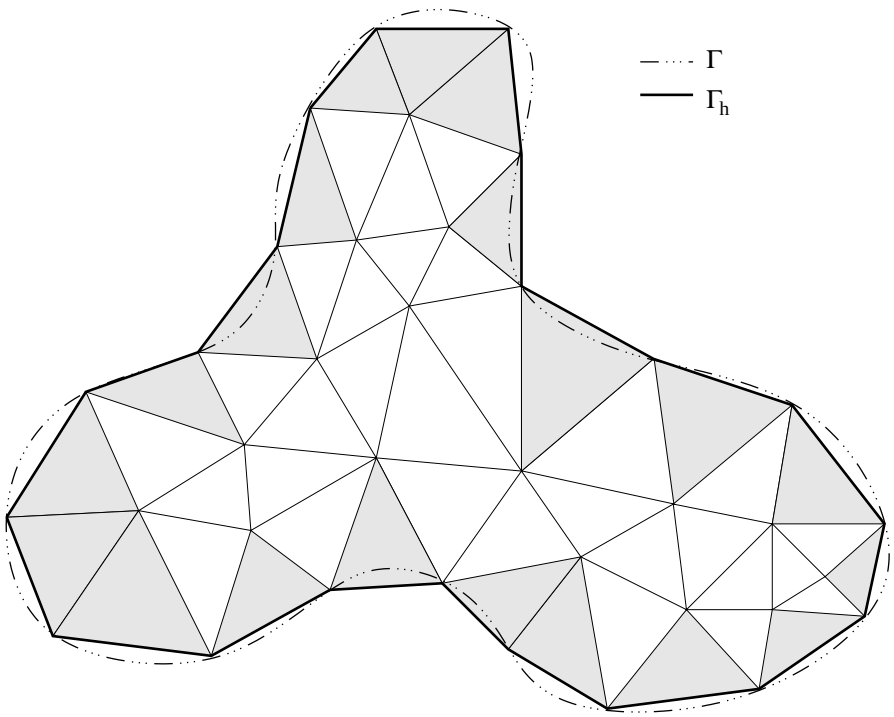

FIGURE 1. Triangulation of $\Omega_h \approx \Omega$. The shaded triangles are the *boundary* ones $T \in \mathcal{T}_h^\partial$.

We consider a family $\{\mathcal{T}_h\}$ of standard finite element triangulations (see Figure 1 and, e.g., [7]) of polygonal domains Ω_h approximating Ω, such that if $\Gamma_h := \partial\Omega_h$ and $\mathcal{N}_h$ is the set of vertices of all the triangles in $\mathcal{T}_h$, then it holds that

- $\mathcal{N}_h \subset \bar{\Omega}_h$ and $\mathcal{N}_h \cap \Gamma_h \subset \Gamma$;
- for all $T \in \mathcal{T}_h$, at most two vertices of T lie on Γ_h,

where the second assumption is only made for the sake of simplicity.

As usual, h stands for the mesh-size, namely, the maximum diameter h_T of all the triangles $T \in \mathcal{T}_h$. We also assume that the family $\{\mathcal{T}_h\}$ is regular in the sense of a minimum angle condition; i.e., there exists $\theta_0 > 0$ such that, if θ_T is the smallest angle of T, then $\theta_T > \theta_0$ for all $T \in \mathcal{T}_h$ and for all the triangulations $\mathcal{T}_h$ in the family.

In what follows, we will use the notation and definition introduced in [11] (see also [22]). For a given triangulation $\mathcal{T}_h$, we denote by $\mathcal{T}_h^\partial$ the subset of the so called *boundary triangles*; namely, those having an edge on Γ_h (see Figure 1). For one such triangle T, let P_1^T, P_2^T, and P_3^T be its three vertices, with $S_T := \overline{P_2^T P_3^T} \subset \Gamma_h$, and let Σ_T be the piece of Γ approximated by S_T (see Figure 2).

We denote by $\widetilde{T}$ the curved triangle of edges $\overline{P_1^T P_2^T}$, Σ_T, and $\overline{P_3^T P_1^T}$, and we call it the *ideal triangle* associated to T. For inner triangles $T \in \mathcal{T}_h \setminus \mathcal{T}_h^\partial$ we simply define $\widetilde{T} \equiv T$ as its associated ideal triangle. We denote by $\widetilde{\mathcal{T}}_h := \left\{\widetilde{T} : \ T \in \mathcal{T}_h\right\}$, the partition of Ω provided by the ideal triangles and we call it the *ideal triangulation*

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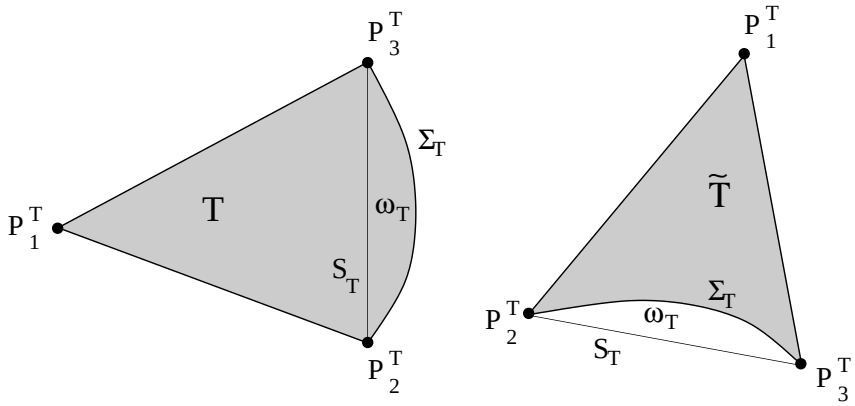

FIGURE 2. Ideal triangles $\widetilde{T}$.

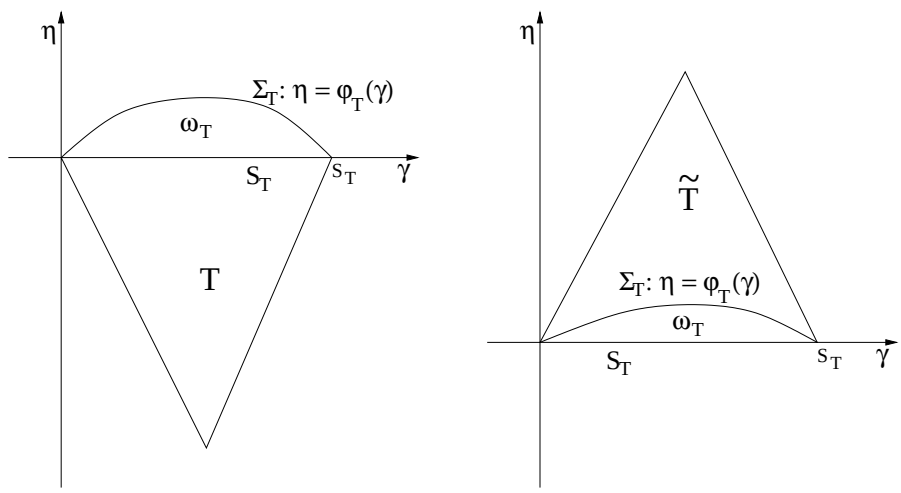

FIGURE 3. Local coordinates (γ, η).

of Ω. We also denote $\widetilde{\mathcal{T}}_h^\partial := \left\{\widetilde{T} : \ T \in \mathcal{T}_h^\partial\right\}$ the family of ideal triangles associated to the boundary ones.

For the sake of simplicity we assume that the triangulations $\mathcal{T}_h$ are such that for each boundary triangles $T \in \mathcal{T}_h^\partial$, either $T \subset \widetilde{T}$ or $T \supset \widetilde{T}$. Let ω_T be the domain bounded by Σ_T and S_T (see Figure 2). Therefore $\bigcup_{T\in\mathcal{T}_h^\partial} \omega_T = (\Omega \setminus \bar{\Omega}_h) \cup (\Omega_h \setminus \bar{\Omega})$.

We introduce in each domain ω_T local coordinates (γ, η), the first one along S_T and the second normal to S_T. Let $\varphi_T(\gamma)$, $0 \le \gamma \le s_T := \text{length}\,(S_T)$, be a parametrization of Σ_T. We choose the orientation of η in such a way that $\varphi_T \ge 0$. Then $\Sigma_T = \{(\gamma, \eta) : \ \eta = \varphi_T(\gamma), \gamma \in [0, s_T]\}$ and $\omega_T \ = \{(\gamma, \eta) : \ 0 < \eta < \varphi_T, \eta \in (0, s_T)\}$ (see Figure 3).

As a consequence of the assumed smoothness of Γ, $\varphi_T \in \mathcal{C}^2([0, s_T])$ and the following estimate holds.

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Lemma 2.2. *There exists a constant $C > 0$, independent of T, such that*

$$0 \le \varphi_T(\gamma) \le C s_T^2 \le C h_T^2 \qquad \forall \gamma \in [0, s_T].$$

Proof. It is a direct consequence of standard interpolation results. □

Here and hereafter, C denotes a positive constant not necessarily the same at each occurrence, but always independent of the mesh-size h.

Consider the discretization of the spectral problem (2.1) obtained by using standard piecewise linear continuous finite elements on $\mathcal{T}_h$ below.

Find $\lambda_h \in \mathbb{R}$ and $u_h \in \mathcal{L}_h(\Omega_h)$, $u_h \neq 0$, such that

$$a_h(u_h, v_h) = \lambda_h b_h(u_h, v_h) \qquad \forall v_h \in \mathcal{L}_h(\Omega_h). \tag{2.4}$$

In the equation above

$$\mathcal{L}_h(\Omega_h) := \{v_h \in \mathrm{H}^1(\Omega_h) : \, v_h|_T \in \mathcal{P}_1(T) \; \forall T \in \mathcal{T}_h\},$$

with $\mathcal{P}_1(T)$ being the space of linear functions defined on T, and the bilinear forms $a_h : \mathrm{H}^1(\Omega_h) \times \mathrm{H}^1(\Omega_h) \longrightarrow \mathbb{R}$ and $b_h : \mathrm{L}^2(\Omega_h) \times \mathrm{L}^2(\Omega_h) \longrightarrow \mathbb{R}$ are defined by

$$a_h(v, w) \;:=\; \int_{\Omega_h} (\nabla v \cdot \nabla w + v w) \; dx \qquad \forall v, w \in \mathrm{H}^1(\Omega_h), \tag{2.5}$$

$$b_h(f, g) \;:=\; \int_{\Omega_h} f g \; dx \qquad \forall f, g \in \mathrm{L}^2(\Omega_h). \tag{2.6}$$

Problem (2.4) reduces to a finite dimensional generalized eigenvalue problem with symmetric positive definite matrices. It attains a finite number of eigenvalues

$$1 = \lambda_{h1} < \lambda_{h2} \le \cdots \le \lambda_{hm},$$

with corresponding $\mathrm{L}^2(\Omega_h)$-orthonormal eigenfunctions $\{u_{hk}\}_{k=0}^m$.

The following three basic lemmas will be used in the rest of this paper.

Lemma 2.3. *There exist positive constants c and C (not depending on T) such that, if $v_h \in \mathcal{P}_1(\mathbb{R}^2)$ and h_T is small enough, then*

$$c\|v_h\|_{0,\widetilde{T}} \le \|v_h\|_{0,T} \le C\|v_h\|_{0,\widetilde{T}},$$
$$c|v_h|_{1,\widetilde{T}} \le |v_h|_{1,T} \le C|v_h|_{1,\widetilde{T}}.$$

Proof. Similar results have been proved in several papers (see, e.g., [11]). For the sake of completeness, we include an elementary proof in the case $\widetilde{T} \subset T$; the case $\widetilde{T} \supset T$ can be dealt with almost identically.

We use the notation in Figure 4. In particular T' is the triangle of vertices P_1^T, Q_2^T, and Q_3^T, with Q_k^T being the midpoint of the edge $\overline{P_1^T P_k^T}$, for $k = 2, 3$.

Because of Lemma 2.2, for h_T small enough, the triangle T' is contained in $\widetilde{T}$ as shown in Figure 4. Then straightforward computations show that $\|v_h\|_{0,T} \le C\,\|v_h\|_{0,T'}$ and $|v_h|_{1,T} = 2\,|v_h|_{1,\widetilde{T}}$. Hence, the lemma is a consequence of the fact that $T' \subset \widetilde{T} \subset T$. □

Lemma 2.4. *There exists a positive constant C such that if h is small enough, then*

$$\|v\|_{0,\Omega\setminus\bar{\Omega}_h} \;\le\; C h^s \|v\|_{s,\Omega} \quad \forall v \in \mathrm{H}^s(\Omega) \quad (0 \le s \le 1),$$
$$\|v\|_{0,\Omega_h\setminus\bar{\Omega}} \;\le\; C h^s \|v\|_{s,\Omega_h} \quad \forall v \in \mathrm{H}^s(\Omega_h) \quad (0 \le s \le 1).$$

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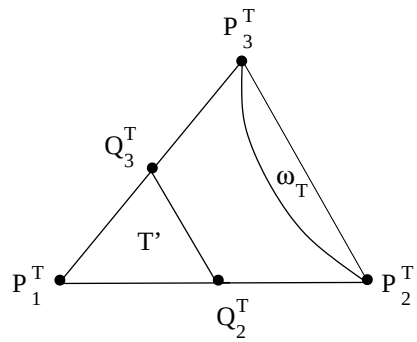

FIGURE 4. Auxiliary triangle T' (case: $\widetilde{T} \subset T$).

Proof. The two inequalities have been essentially proved in the proof of Lemma 5.2–5.3 in [16] for $s = 1$. Since the inequalities are clearly true for $s = 0$, they follow for $0 < s < 1$ from standard results on interpolation in Sobolev spaces (see, for instance, Theorem 1.4 in [12]). □

Lemma 2.5. *There exists a positive constant C such that if h_T is small enough, then, for all $T \in \mathcal{T}_h$,*

$$\|v\|_{0,\widetilde{T}\setminus T} \le C\left(\|v\|_{0,T} + h_T^{2s}\|v\|_{s,\widetilde{T}}\right) \qquad \forall v \in \mathrm{H}^s(\widetilde{T}) \quad (0 \le s \le 1), \qquad \text{if } T \subset \widetilde{T},$$

$$\|v\|_{0,T\setminus\widetilde{T}} \le C\left(\|v\|_{0,\widetilde{T}} + h_T^{2s}\|v\|_{s,T}\right) \qquad \forall v \in \mathrm{H}^s(T) \quad (0 \le s \le 1), \qquad \text{if } \widetilde{T} \subset T.$$

Proof. We only include the proof of the first inequality. The second one can be obtained by applying similar arguments. We use the notation shown in Figure 5 for $T \subset \widetilde{T}$.

Consider polar coordinates centered at the vertex P_1^T. We assume that S_T and Σ_T are parametrized by $r = r_1(\theta)$ and $r = r_2(\theta)$, respectively, with $0 \le \theta \le \theta_1$. Then $\omega_T := \widetilde{T} \setminus T = \{(r,\theta) : \ r_1(\theta) < r < r_2(\theta),\ 0 < \theta < \theta_1\}$. Let d denote the distance of each point on Σ_T to S_T, as shown in Figure 5.

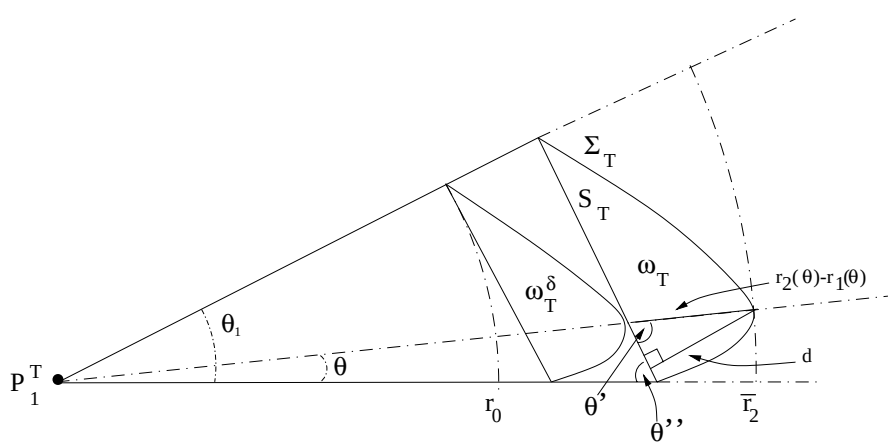

FIGURE 5. Polar coordinates on $\widetilde{T} \supset T$.

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Assume that $\theta' \leq \frac{\pi}{2}$ as in Figure 5 (a similar argument is valid for $\theta' \geq \frac{\pi}{2}$). Since $\theta' = \theta'' + \theta$, then $\sin\theta' \geq \sin\theta'' \geq \sin\theta_T$, with θ_T being the smallest angle of T. Hence,

$$r_2(\theta) - r_1(\theta) = \frac{d}{\sin\theta'} \leq \frac{d}{\sin\theta_T}.$$

Then, because of Lemma 2.2 and the minimum angle condition,

$$\delta := \max_{0\leq\theta\leq\theta_1} [r_2(\theta) - r_1(\theta)] \leq Ch_T^2,$$

and, consequently, for h_T small enough,

$$\begin{aligned} r_0 &:= \min_{0\leq\theta\leq\theta_1} r_1(\theta) - \delta \geq ch_T > 0, \\ \bar{r}_2 &:= \max_{0\leq\theta\leq\theta_1} r_2(\theta) \leq Ch_T. \end{aligned}$$

Hence, for h_T small enough, $\omega_T^\delta := \{(r-\delta,\theta) : \ (r,\theta) \in \omega_T\} \subset T$ (see Figure 5).

Let $v^\delta(r,\theta) := v(r-\delta,\theta)$, for $(r,\theta) \in \omega_T$. We will prove that

$$\|v - v^\delta\|_{0,\omega_T} \leq Ch_T^{2s} \|v\|_{s,\widetilde{T}}. \tag{2.7}$$

Then the lemma follows from this inequality and the fact that, for h_T small enough, $\|v^\delta\|_{0,\omega_T} \leq C\|v\|_{0,\omega_T^\delta} \leq C\|v\|_{0,T}$.

Inequality (2.7) is clear for $s = 0$. We prove it for $s = 1$; then, for $0 < s < 1$, it follows by interpolation between Sobolev spaces.

Let $v \in \mathcal{C}^\infty(\widetilde{T})$ (we prove (2.7) for such smooth v; for any $v \in \mathrm{H}^1(\widetilde{T})$ it follows by the standard density argument). For any $(r,\theta) \in \omega_T$ we may write

$$v(r,\theta) - v^\delta(r,\theta) = \int_{r-\delta}^{r} \frac{\partial v}{\partial r}(t,\theta)\, dt.$$

Hence, applying the Cauchy-Schwarz inequality and integrating on ω_T, we obtain

$$\begin{aligned} \|v - v^\delta\|_{0,\omega_T}^2 &\leq \int_0^{\theta_1} \int_{r_1(\theta)}^{r_2(\theta)} \delta \int_{r-\delta}^{r} \left|\frac{\partial v}{\partial r}(t,\theta)\right|^2 dt\ r\,dr\ d\theta \\ &\leq \delta \int_0^{\theta_1} \left[\int_{r_0}^{r_2(\theta)} \left|\frac{\partial v}{\partial r}(t,\theta)\right|^2 \frac{t}{r_0}\ dt\right] \left[\int_{r_1(\theta)}^{r_2(\theta)} r\,dr\right]\ d\theta \\ &\leq \delta \int_0^{\theta_1} \left[\int_0^{r_2(\theta)} |\nabla v(t,\theta)|^2\, t\,dt\right] \frac{2\bar{r}_2}{r_0} \left[r_2(\theta) - r_1(\theta)\right]\ d\theta \\ &\leq C\delta^2 \|\nabla v\|_{0,\widetilde{T}}^2. \end{aligned}$$

Thus, for $s = 1$, (2.7) is a consequence of the fact that $\delta \leq Ch_T^2$. □

Consider the operator $\mathbf{T}$ defined by

$$\begin{array}{rccl} \mathbf{T}: & \mathrm{L}^2(\Omega) & \longrightarrow & \mathrm{L}^2(\Omega) \\ & f & \longmapsto & u \in \mathrm{H}^1(\Omega) : \ a(u,v) = b(f,v) \quad \forall v \in \mathrm{H}^1(\Omega), \end{array}$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are the bilinear forms defined in (2.2) and (2.3), respectively. By virtue of the Lax-Milgram lemma, it is clear that $\mathbf{T}$ is a well-defined bounded operator and

$$\|u\|_{1,\Omega} \leq C\|f\|_{0,\Omega}. \tag{2.8}$$

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Furthermore, because of the compact inclusion $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$, $\mathbf{T}$ is compact. Clearly, the eigenvalues of $\mathbf{T}$ are given by $\mu_k = 1/\lambda_k$, with λ_k being the eigenvalues of problem (2.1), and the corresponding eigenfunctions u_k coincide.

As a consequence of the classical a priori estimates (see [13]), for any $f \in \mathrm{L}^2(\Omega)$, $u = \mathbf{T}f$ is known to satisfy some further regularity. In fact, $u \in \mathrm{H}^{1+r}(\Omega)$ for some $r > 0$ depending on the geometry of Ω, and there holds

$$\|u\|_{1+r,\Omega} \leq C\|f\|_{0,\Omega}. \tag{2.9}$$

For a pure Neumann problem, if Ω has no reentrant corners, then $r = 1$; otherwise, $r < \frac{2\pi}{\theta}$, with θ being the largest interior angle of Ω. From now on let $r \in (0, 1]$ be fixed such that (2.9) holds. Notice that, as a consequence, the eigenfunctions u_k of $\mathbf{T}$ belong to $\mathrm{H}^{1+r}(\Omega)$ and satisfy

$$\|u_k\|_{1+r,\Omega} \leq C\|u_k\|_{0,\Omega}. \tag{2.10}$$

Let $\overline{\mathbf{T}}_h$ be defined by

$$\begin{array}{rccl} \overline{\mathbf{T}}_h : & \mathrm{L}^2(\Omega_h) & \longrightarrow & \mathrm{L}^2(\Omega_h) \\ & g & \longmapsto & v_h \in \mathcal{L}_h(\Omega_h) : \ a_h(v_h, w_h) = b_h(g, w_h) \quad \forall w_h \in \mathcal{L}_h(\Omega_h), \end{array}$$

with $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ as defined in (2.5) and (2.6), respectively. Clearly

$$\|v_h\|_{1,\Omega_h} \leq C\|g\|_{0,\Omega_h}. \tag{2.11}$$

The positive nonzero eigenvalues of $\overline{\mathbf{T}}_h$ are given by $\mu_{hk} = 1/\lambda_{hk}$, with λ_{hk} being those of the discrete problem (2.4) and coinciding associated eigenfunctions u_{hk}.

The spectral approximation theory stated in [1] cannot be directly applied to the operators $\overline{\mathbf{T}}_h$, since their domains $\mathrm{L}^2(\Omega_h)$ do not coincide with that of $\mathbf{T}$. So, we are going to introduce other discrete operators $\mathbf{T}_h$ defined on $\mathrm{L}^2(\Omega)$ and with spectrum also related to that of problem (2.4).

From now on, we assume that h is sufficiently small in order for Lemmas 2.3, 2.4, and 2.5 to hold. Let

$$\mathcal{L}_h(\Omega) := \{v_h \in \mathrm{H}^1(\Omega) : \ v_h|_{\widetilde{T}} \in \mathcal{P}_1(\widetilde{T}) \ \forall \widetilde{T} \in \widetilde{\mathcal{T}}_h\}.$$

We consider two restriction-extension operators, one from $\mathcal{L}_h(\Omega_h)$ to $\mathcal{L}_h(\Omega)$ and the other from $\mathcal{L}_h(\Omega)$ to $\mathcal{L}_h(\Omega_h)$. Let

$$\begin{array}{rccc} \check{\mathbf{E}} : & \mathcal{L}_h(\Omega_h) & \longrightarrow & \mathcal{L}_h(\Omega) \\ & v_h & \longmapsto & \check{v}_h \end{array}$$

with

$$\check{v}_h|_{\widetilde{T}} := \begin{cases} v_h|_{\widetilde{T}} & \forall \widetilde{T} \in \widetilde{\mathcal{T}}_h : \ \widetilde{T} \subset \Omega_h, \\ (v_h|_T)\check{} & \forall \widetilde{T} \in \widetilde{\mathcal{T}}_h : \ \widetilde{T} \not\subset \Omega_h, \end{cases}$$

where $(v_h|_T)\check{} \in \mathcal{P}_1(\widetilde{T})$ denotes the natural extension of the linear function $v_h|_T \in \mathcal{P}_1(T)$ to the larger set $\widetilde{T}$ (notice that, if $\widetilde{T} \not\subset \Omega_h$, then $T \subset \widetilde{T}$).

Let $\hat{\mathbf{E}} := \check{\mathbf{E}}^{-1}$; namely,

$$\begin{array}{rccc} \hat{\mathbf{E}} : & \mathcal{L}_h(\Omega) & \longrightarrow & \mathcal{L}_h(\Omega_h) \\ & w_h & \longmapsto & \hat{w}_h \end{array}$$

with

$$\hat{w}_h|_T := \begin{cases} w_h|_T & \forall T \in \mathcal{T}_h : \ T \subset \Omega, \\ \left(w_h|_{\widetilde{T}}\right)\hat{} & \forall T \in \mathcal{T}_h : \ T \not\subset \Omega, \end{cases}$$

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where $\left(w_h|_{\widetilde{T}}\right)\hat{} \in \mathcal{P}_1(T)$ denotes the natural extension of the linear function $w_h|_{\widetilde{T}} \in \mathcal{P}_1(\widetilde{T})$ to the larger set T (notice that now if $T \not\subset \Omega$, then $\widetilde{T} \subset T$).

As a consequence of Lemma 2.3, the operators $\check{\mathbf{E}}$ and $\hat{\mathbf{E}}$ are uniformly bounded in L^2 and H^1 norms. More precisely,

$$\|\check{\mathbf{E}}v_h\|_{k,\Omega} \le C\|v_h\|_{k,\Omega_h} \qquad \forall v_h \in \mathcal{L}_h(\Omega_h), \qquad k = 0, 1, \tag{2.12}$$

$$\|\hat{\mathbf{E}}w_h\|_{k,\Omega_h} \le C\|w_h\|_{k,\Omega} \qquad \forall w_h \in \mathcal{L}_h(\Omega), \qquad k = 0, 1. \tag{2.13}$$

We will also use the $\mathrm{L}^2(\Omega)$-projection onto $\mathcal{L}_h(\Omega)$. Let

$$\begin{array}{rccc} \mathbf{P}: & \mathrm{L}^2(\Omega) & \longrightarrow & \mathcal{L}_h(\Omega) \\ & f & \longmapsto & f_h \end{array}$$

with

$$f_h \in \mathcal{L}_h(\Omega): \qquad \int_\Omega (f_h - f)w_h \, dx = 0 \qquad \forall w_h \in \mathcal{L}_h(\Omega). \tag{2.14}$$

Clearly

$$\|f_h\|_{0,\Omega} \le \|f\|_{0,\Omega} \qquad \forall f \in \mathrm{L}^2(\Omega). \tag{2.15}$$

Furthermore, the standard error estimate for this projection yields

$$\|f_h - f\|_{0,\Omega} \le Ch\|f\|_{1,\Omega} \qquad \forall f \in \mathrm{H}^1(\Omega). \tag{2.16}$$

Now we are able to define the discrete operator that we will use in the rest of this paper:

$$\begin{array}{rccc} \mathbf{T}_h: & \mathrm{L}^2(\Omega) & \longrightarrow & \mathrm{L}^2(\Omega) \\ & f & \longmapsto & \check{u}_h = \check{\mathbf{E}}\overline{\mathbf{T}}_h\hat{\mathbf{E}}\mathbf{P}f \end{array}$$

As a consequence of the estimates (2.15), (2.13), (2.11), and (2.12), the operators $\mathbf{T}_h$ are uniformly bounded. Moreover, they satisfy

$$\|\mathbf{T}_h f\|_{1,\Omega} \le C\|f\|_{0,\Omega}. \tag{2.17}$$

The following lemma shows that the eigenvalues of $\mathbf{T}_h$ and $\overline{\mathbf{T}}_h$ coincide and the eigenfunctions of one operator are the restriction-extension of those of the other.

Lemma 2.6. *The nonzero eigenvalues of $\overline{\mathbf{T}}_h$ and $\mathbf{T}_h$ coincide and the respective associated eigenfunctions u_h and $\check{u}_h$ are related by $\check{u}_h = \check{\mathbf{E}}u_h$ and $u_h = \hat{\mathbf{E}}\check{u}_h$.*

Proof. Let $u_h \in \mathcal{L}_h(\Omega_h)$, $u_h \neq 0$, such that $\overline{\mathbf{T}}_h u_h = \lambda_h u_h$. Then

$$\mathbf{T}_h\check{\mathbf{E}}u_h = \check{\mathbf{E}}\overline{\mathbf{T}}_h\hat{\mathbf{E}}\mathbf{P}\check{\mathbf{E}}u_h = \check{\mathbf{E}}\overline{\mathbf{T}}_h\hat{\mathbf{E}}\check{\mathbf{E}}u_h = \check{\mathbf{E}}\overline{\mathbf{T}}_h u_h = \lambda_h\check{\mathbf{E}}u_h.$$

Conversely, let $\check{u}_h \in \mathcal{L}_h(\Omega)$, $\check{u}_h \neq 0$, such that $\mathbf{T}_h\check{u}_h = \lambda_h\check{u}_h$. Then

$$\overline{\mathbf{T}}_h\hat{\mathbf{E}}\check{u}_h = \hat{\mathbf{E}}\check{\mathbf{E}}\overline{\mathbf{T}}_h\hat{\mathbf{E}}\mathbf{P}\check{u}_h = \hat{\mathbf{E}}\mathbf{T}_h\check{u}_h = \lambda_h\hat{\mathbf{E}}\check{u}_h. \qquad \square$$

The aim of this paper is to study how the eigenvalues and eigenfunctions of $\mathbf{T}_h$, approximate those of $\mathbf{T}$, and hence to analyze how the eigenvalue problem (2.4) allows approximating the solutions of the spectral problem (2.1).

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3. Spectral approximation

In order to use the spectral approximation theory stated in [1], we are going to prove that the operators $\mathbf{T}_h$ converge to $\mathbf{T}$ in norm as h goes to zero. From now on and throughout the rest of the section, we use the following notation: given $f \in \mathrm{L}^2(\Omega)$, we denote $u := \mathbf{T}f$, $f_h := \mathbf{P}f$, $\hat{f}_h := \hat{\mathbf{E}}f_h$, $u_h := \overline{\mathbf{T}}_h\hat{f}_h$, and $\check{u}_h := \check{\mathbf{E}}u_h = \mathbf{T}_h f$.

In the proofs, we will also use extensions to $\Omega\cup\Omega_h$ of functions originally defined in Ω. For $v \in \mathrm{H}^s(\Omega)$ $(s > 0)$, let v^e denote an extension of v to $\mathbb{R}^2$ satisfying (see, for instance, Theorem 1.4.3.1 in [13])

$$\|v^e\|_{s,\mathbb{R}^2} \le C\|v\|_{s,\Omega}. \tag{3.1}$$

In particular since, according to the estimate (2.9), $u \in \mathrm{H}^{1+r}(\Omega)$, let u^e be an extension of u satisfying

$$\|u^e\|_{1+r,\mathbb{R}^2} \le C\|u\|_{1+r,\Omega}. \tag{3.2}$$

The following lemma splits $\|(\mathbf{T}-\mathbf{T}_h)f\|_{1,\Omega} = \|u-\check{u}_h\|_{1,\Omega}$ into three terms which can be dealt with separately.

Lemma 3.1. *There exists a positive constant C, not depending on f, such that*

$$\|u-\check{u}_h\|_{1,\Omega} \le C\left[\inf_{v_h\in\mathcal{L}_h(\Omega_h)}\|u^e-v_h\|_{1,\Omega_h} + \sup_{w_h\in\mathcal{L}_h(\Omega_h)}\frac{|b_h(\hat{f}_h,w_h)-a_h(u^e,w_h)|}{\|w_h\|_{1,\Omega_h}}\right] + \|u-\check{u}_h\|_{1,\Omega\setminus\bar{\Omega}_h}. \tag{3.3}$$

Proof. It is enough to notice that

$$\begin{aligned}\|u-\check{u}_h\|_{1,\Omega}^2 &= \|u-u_h\|_{1,\Omega\cap\Omega_h}^2 + \|u-\check{u}_h\|_{1,\Omega\setminus\bar{\Omega}_h}^2\\ &\le \|u^e-u_h\|_{1,\Omega_h}^2 + \|u-\check{u}_h\|_{1,\Omega\setminus\bar{\Omega}_h}^2,\end{aligned}$$

and to use the standard techniques to deal with nonconforming methods to estimate the first term (see Strang's lemma, for instance, in [7]):

$$\|u^e-u_h\|_{1,\Omega_h} \le 2\inf_{v_h\in\mathcal{L}_h(\Omega_h)}\|u^e-v_h\|_{1,\Omega_h} + \sup_{w_h\in\mathcal{L}_h(\Omega_h)}\frac{|b_h(\hat{f}_h,w_h)-a_h(u^e,w_h)|}{\|w_h\|_{1,\Omega_h}}. \tag{3.4}$$

□

In what follows we give estimates of the three terms in the right-hand side of (3.3). For the first one we have

Lemma 3.2. *There exists a positive constant C, not depending on f, and $v_h \in \mathcal{L}_h(\Omega_h)$ such that*

$$\|u^e-v_h\|_{1,\Omega_h} \le Ch^r\|f\|_{0,\Omega}.$$

Proof. Let $v_h \in \mathcal{L}_h(\Omega_h)$ be the Lagrange interpolant of $u^e \in \mathrm{H}^{1+r}(\Omega_h)$. Proceeding as in Example 3 of [9] we have

$$\|u^e-v_h\|_{1,\Omega_h} \le Ch^r\|u^e\|_{1+r,\Omega_h}$$

(see also Theorem 2.27 in [10] for an alternative proof). So, the lemma follows from this inequality and estimates (3.2) and (2.9). □

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Now we estimate the second term in the right-hand side of (3.3).

Lemma 3.3. *There exists a positive constant C, not depending on f, such that*

$$\sup_{w_h \in \mathcal{L}_h(\Omega_h)} \frac{|b_h(\hat{f}_h, w_h) - a_h(u^e, w_h)|}{\|w_h\|_{1,\Omega_h}} \leq Ch^r \|f\|_{0,\Omega}.$$

Proof. Let $w_h \in \mathcal{L}_h(\Omega_h)$ and $\check{w}_h = \check{\mathbf{E}} w_h$. Since $\check{w}_h \in \mathrm{H}^1(\Omega)$, then $a(u, \check{w}_h) = b(f, \check{w}_h)$, and hence

$$b_h(\hat{f}_h, w_h) - a_h(u^e, w_h) = [b_h(\hat{f}_h, w_h) - b(f, \check{w}_h)] + [a(u, \check{w}_h) - a_h(u^e, w_h)].$$

Because of (2.14) and the fact that $\check{w}_h \in \mathcal{L}_h(\Omega)$, we have

$$\begin{aligned} b_h(\hat{f}_h, w_h) - b(f, \check{w}_h) &= \int_{\Omega_h} \hat{f}_h w_h \, dx - \int_{\Omega} f \check{w}_h \, dx \\ &= \int_{\Omega_h \setminus \bar{\Omega}} \hat{f}_h w_h \, dx - \int_{\Omega \setminus \bar{\Omega}_h} f_h \check{w}_h \, dx. \end{aligned}$$

Analogously,

$$\begin{aligned} a(u, \check{w}_h) - a_h(u^e, w_h) = &\int_{\Omega \setminus \bar{\Omega}_h} (\nabla u \cdot \nabla \check{w}_h + u \check{w}_h) \, dx \\ &- \int_{\Omega_h \setminus \bar{\Omega}} (\nabla u^e \cdot \nabla w_h + u^e w_h) \, dx. \end{aligned}$$

Now by using the Cauchy-Schwarz inequality and Lemma 2.4 we obtain

$$\begin{aligned} \left| \int_{\Omega_h \setminus \bar{\Omega}} \hat{f}_h w_h \, dx \right| &\leq \|\hat{f}_h\|_{0,\Omega_h \setminus \bar{\Omega}} Ch \|w_h\|_{1,\Omega_h}, \\ \left| \int_{\Omega \setminus \bar{\Omega}_h} f_h \check{w}_h \, dx \right| &\leq \|f_h\|_{0,\Omega \setminus \bar{\Omega}_h} Ch \|\check{w}_h\|_{1,\Omega}, \\ \left| \int_{\Omega \setminus \bar{\Omega}_h} (\nabla u \cdot \nabla \check{w}_h + u \check{w}_h) \, dx \right| &\leq Ch^r \|u\|_{1+r,\Omega} \|\check{w}_h\|_{1,\Omega \setminus \bar{\Omega}_h}, \\ \left| \int_{\Omega_h \setminus \bar{\Omega}} (\nabla u^e \cdot \nabla w_h + u^e w_h) \, dx \right| &\leq Ch^r \|u^e\|_{1+r,\Omega_h} \|w_h\|_{1,\Omega_h \setminus \bar{\Omega}}. \end{aligned}$$

Finally, we conclude the lemma bounding the four right-hand sides above by $Ch^r \|f\|_{0,\Omega} \|w_h\|_{1,\Omega_h}$. To do this, we use (2.13) and (2.15) for the first one, (2.12) for the second one, (2.9) and (2.12) for the third one, and (3.2) and (2.9) for the last one. □

Remark 3.4. Combining estimate (3.4) in the proof of Lemma 3.1, with Lemmas 3.2 and 3.3, we also have

$$\|u^e - u_h\|_{1,\Omega_h} \leq Ch^r \|f\|_{0,\Omega}.$$

Now we are able to estimate the third term in (3.3).

Lemma 3.5. *There exists a positive constant C, not depending on f, such that*

$$\|u - \check{u}_h\|_{1,\Omega \setminus \bar{\Omega}_h} \leq Ch^r \|f\|_{0,\Omega}.$$

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Proof. For the L^2 norm, we use Lemma 2.4 and estimates (2.8) and (2.17) to obtain

$$\|u - \check{u}_h\|_{0,\Omega\setminus\bar{\Omega}_h} \le Ch\|u\|_{1,\Omega} + Ch\|\check{u}_h\|_{1,\Omega} \le Ch\|f\|_{0,\Omega}.$$

For the H^1 seminorm we use Lemma 2.5:

$$\begin{aligned} |u - \check{u}_h|^2_{1,\Omega\setminus\bar{\Omega}_h} &= \sum_{T\in\mathcal{T}_h^\partial:\, T\subset\widetilde{T}} \|\nabla(u - \check{u}_h)\|^2_{0,\widetilde{T}\setminus T} \\ &\le \sum_{T\in\mathcal{T}_h^\partial:\, T\subset\widetilde{T}} C\left[\|\nabla(u - \check{u}_h)\|_{0,T} + h_T^{2r}\|\nabla(u - \check{u}_h)\|_{r,\widetilde{T}}\right]^2 \\ &\le C\left[\|u^e - u_h\|^2_{1,\Omega_h} + h^{4r}\left(\|u\|^2_{1+r,\Omega} + \|\check{u}_h\|^2_{1,\Omega}\right)\right], \end{aligned}$$

where we have used that $\|\nabla\check{u}_h\|_{r,\widetilde{T}} = \|\nabla\check{u}_h\|_{0,\widetilde{T}}$, since $\check{u}_h|_{\widetilde{T}} \in \mathcal{P}_1(\widetilde{T})$. Therefore, the lemma follows from Remark 3.4 and estimates (2.9) and (2.17). □

Now we may conclude the convergence in norm of $\mathbf{T}_h$ to $\mathbf{T}$.

Lemma 3.6. *There exists a positive constant C such that, for all $f \in \mathrm{L}^2(\Omega)$,*

$$\|(\mathbf{T} - \mathbf{T}_h)f\|_{1,\Omega} \le Ch^r\|f\|_{0,\Omega}.$$

Proof. It is an immediate consequence of the four previous lemmas. □

Therefore, we are able to apply the spectral approximation theory for compact operators (see [1]) to obtain optimal order error estimates for the eigenfunctions. For simplicity we state the result for a simple eigenvalue; see [1] for the general statement.

Theorem 3.7. *Let μ_k be the k-th (simple) eigenvalue of $\mathbf{T}$ and μ_{hk} the k-th eigenvalue of $\mathbf{T}_h$. Then, the corresponding eigenfunctions u_k and $\check{u}_{hk}$ can be chosen such that $\|u_k\|_{0,\Omega} = \|\check{u}_{hk}\|_{0,\Omega} = 1$ and*

$$\|u_k - \check{u}_{hk}\|_{1,\Omega} \le Ch^r,$$

with C a strictly positive constant.

Proof. It is a direct consequence of the convergence in norm of $\mathbf{T}_h$ to $\mathbf{T}$ and the error estimate in the previous lemma, and Lemma 7.1 in [1]. □

In the remainder of the paper we will prove optimal order error estimates for the L^2 norm of the eigenfunctions and for the eigenvalues. To this end, we will use a double order error estimate for $\|u - \check{u}_h\|_{0,\Omega}$. To the best of the authors' knowledge, estimates of this type have not been proved for Neumann boundary conditions on curved domains. The proofs given below for the estimate of $\|u - \check{u}_h\|_{0,\Omega}$ are valid only for smooth right-hand sides f (namely, $f \in \mathrm{H}^1(\Omega)$). However, the results obtained suffice to yield optimal error estimates for the spectral problem without assuming any regularity.

The following two lemmas are valid for $f \in \mathrm{H}^1(\Omega)$. For such functions, we denote by f^e their bounded extensions to $\mathbb{R}^2$ satisfying (3.1) for $s = 1$; namely

$$\|f^e\|_{1,\mathbb{R}^2} \le C\|f\|_{1,\Omega}. \tag{3.5}$$

First we prove the following technical result.

Lemma 3.8. *There exists a positive constant C such that, for all $f \in \mathrm{H}^1(\Omega)$,*

$$\|\hat{f}_h - f^e\|_{0,\Omega_h\setminus\bar{\Omega}} \le Ch\|f\|_{1,\Omega}.$$

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Proof. According to Lemma 2.5, we have $\forall T \in \mathcal{T}_h^\partial$ such that $T \supset \widetilde{T}$,

$$\|\hat{f}_h - f^e\|_{0,T\setminus\widetilde{T}} \le C\left(\|f_h - f\|_{0,\widetilde{T}} + h_T^2\|\hat{f}_h - f^e\|_{1,T}\right).$$

From a standard local inverse inequality (see for instance [7]), we obtain

$$|\hat{f}_h|_{1,T} \le \frac{C}{h_T}\|\hat{f}_h\|_{0,T}.$$

Hence, since $\|\hat{f}_h\|_{0,T} \le C\|f_h\|_{0,\widetilde{T}}$ because of Lemma 2.3, we have

$$h_T^2\|\hat{f}_h - f^e\|_{1,T} \le h_T^2\left(\|\hat{f}_h\|_{1,T} + \|f^e\|_{1,T}\right) \le C\left(h_T\|f_h\|_{0,\widetilde{T}} + h_T^2\|f^e\|_{1,T}\right).$$

Therefore,

$$\|\hat{f}_h - f^e\|_{0,\Omega_h\setminus\bar{\Omega}} \le C\left(\|f_h - f\|_{0,\Omega} + h\|f_h\|_{0,\Omega} + h^2\|f^e\|_{1,\Omega_h}\right),$$

and the lemma follows from (2.16), (2.15), and (3.5). □

Now we can prove an double order $\mathrm{L}^2(\Omega)$-estimate.

Lemma 3.9. *There exists a positive constant C such that, for all $f \in \mathrm{H}^1(\Omega)$,*

$$\|(\mathbf{T} - \mathbf{T}_h)f\|_{0,\Omega} \le Ch^{2r}\|f\|_{1,\Omega}.$$

Proof. Let $f \in \mathrm{H}^1(\Omega)$. Since

$$\|(\mathbf{T} - \mathbf{T}_h)f\|_{0,\Omega} = \sup_{g\in\mathrm{L}^2(\Omega)} \frac{b((\mathbf{T} - \mathbf{T}_h)f, g)}{\|g\|_{0,\Omega}},$$

then it only remains to prove that

$$|b\left((\mathbf{T} - \mathbf{T}_h)f, g\right)| \le Ch^{2r}\|f\|_{1,\Omega}\,\|g\|_{0,\Omega} \quad \forall g \in \mathrm{L}^2(\Omega).$$

Let $g \in \mathrm{L}^2(\Omega)$ and $v = \mathbf{T}g$. We denote $v_h = \overline{\mathbf{T}}_h\hat{\mathbf{E}}\mathbf{P}g$ and $\check{v}_h = \check{\mathbf{E}}v_h = \mathbf{T}_h g$ as above. Since according to (2.9) $v \in \mathrm{H}^{1+r}(\Omega)$, there exists a bounded extension v^e of v satisfying (3.1) for $s = 1 + r$.

Since a and b are symmetric and $u - \check{u}_h = (\mathbf{T} - \mathbf{T}_h)f \in \mathrm{H}^1(\Omega)$, then we have

$$b\left((\mathbf{T} - \mathbf{T}_h)f, g\right) = a(u - \check{u}_h, v) = a(u - \check{u}_h, v - \check{v}_h) + a(u - \check{u}_h, \check{v}_h).$$

Thus, from the continuity of $a(\cdot,\cdot)$ and Lemma 3.6, we only have to estimate the second term in the right-hand side. Since $\check{v}_h \in \mathcal{L}_h(\Omega) \subset \mathrm{H}^1(\Omega)$ and $v_h \in \mathcal{L}_h(\Omega_h)$, then

$$\begin{aligned} a(u - \check{u}_h, \check{v}_h) &= b(f, \check{v}_h) - a(\check{u}_h, \check{v}_h) \\ &= \left[b(f, \check{v}_h) - b_h(\hat{f}_h, v_h)\right] + \left[a_h(u_h, v_h) - a(\check{u}_h, \check{v}_h)\right]. \end{aligned}$$

By using (2.14) and repeating the arguments in the proof of Lemma 3.3, we obtain

$$\begin{aligned} a(u - \check{u}_h, \check{v}_h) = &\int_{\Omega\setminus\bar{\Omega}_h} f_h\check{v}_h\,dx - \int_{\Omega_h\setminus\bar{\Omega}} \hat{f}_h v_h\,dx \\ &+ \int_{\Omega_h\setminus\bar{\Omega}} \left(\nabla u_h\cdot\nabla v_h + u_h v_h\right)\,dx - \int_{\Omega\setminus\bar{\Omega}_h} \left(\nabla\check{u}_h\cdot\nabla\check{v}_h + \check{u}_h\check{v}_h\right)\,dx. \end{aligned}$$

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Thus, to conclude the lemma, we estimate each term in the right-hand side above. From Lemma 2.4 applied to $\check{v}_h$ and f_h, (2.16), and (2.17),

$$\begin{aligned}\left|\int_{\Omega\setminus\bar{\Omega}_h} f_h\check{v}_h\,dx\right| &\leq \|f_h\|_{0,\Omega\setminus\bar{\Omega}_h}\|\check{v}_h\|_{0,\Omega\setminus\bar{\Omega}_h}\\ &\leq \left(\|f_h-f\|_{0,\Omega\setminus\bar{\Omega}_h}+\|f\|_{0,\Omega\setminus\bar{\Omega}_h}\right)Ch\|\check{v}_h\|_{1,\Omega}\\ &\leq Ch^2\|f\|_{1,\Omega}\|g\|_{0,\Omega}.\end{aligned}$$

By using Lemma 2.4 applied to v_h and f^e, Lemma 3.8, (2.12), (3.5), and (2.17),

$$\begin{aligned}\left|\int_{\Omega_h\setminus\bar{\Omega}} \hat{f}_h v_h\,dx\right| &\leq \|\hat{f}_h\|_{0,\Omega_h\setminus\bar{\Omega}}\|v_h\|_{0,\Omega_h\setminus\bar{\Omega}}\\ &\leq \left(\|\hat{f}_h-f^e\|_{0,\Omega_h\setminus\bar{\Omega}}+\|f^e\|_{0,\Omega_h\setminus\bar{\Omega}}\right)Ch\|v_h\|_{1,\Omega_h}\\ &\leq Ch^2\left(\|f\|_{1,\Omega}+\|f^e\|_{1,\Omega_h}\right)\|\check{v}_h\|_{1,\Omega_h}\\ &\leq Ch^2\|f\|_{1,\Omega}\|g\|_{0,\Omega}.\end{aligned}$$

By using Remark 3.4, Lemma 2.4, (3.2), and (2.9),

$$\begin{aligned}\left|\int_{\Omega_h\setminus\bar{\Omega}} (\nabla u_h\cdot\nabla v_h+u_hv_h)\,dx\right| &\leq \|u_h\|_{1,\Omega_h\setminus\bar{\Omega}}\|v_h\|_{1,\Omega_h\setminus\bar{\Omega}}\\ &\leq \left(\|u_h-u^e\|_{1,\Omega_h\setminus\bar{\Omega}}+\|u^e\|_{1,\Omega_h\setminus\bar{\Omega}}\right)\left(\|v_h-v^e\|_{1,\Omega_h\setminus\bar{\Omega}}+\|v^e\|_{1,\Omega_h\setminus\bar{\Omega}}\right)\\ &\leq \left(Ch^r\|f\|_{0,\Omega}+Ch^r\|u^e\|_{1+r,\Omega_h}\right)\left(Ch^r\|g\|_{0,\Omega}+Ch^r\|v^e\|_{1+r,\Omega_h}\right)\\ &\leq Ch^{2r}\|f\|_{0,\Omega}\|g\|_{0,\Omega}.\end{aligned}$$

Finally, from Lemmas 3.6 and 2.4, and (2.9),

$$\begin{aligned}\left|\int_{\Omega\setminus\bar{\Omega}_h} (\nabla\check{u}_h\cdot\nabla\check{v}_h+\check{u}_h\check{v}_h)\,dx\right| &\leq \|\check{u}_h\|_{1,\Omega\setminus\bar{\Omega}_h}\|\check{v}_h\|_{1,\Omega\setminus\bar{\Omega}_h}\\ &\leq \left(\|\check{u}_h-u\|_{1,\Omega\setminus\bar{\Omega}_h}+\|u\|_{1,\Omega\setminus\bar{\Omega}_h}\right)\left(\|\check{v}_h-v\|_{1,\Omega\setminus\bar{\Omega}_h}+\|v\|_{1,\Omega\setminus\bar{\Omega}_h}\right)\\ &\leq \left(Ch^r\|f\|_{0,\Omega}+Ch^r\|u\|_{1+r,\Omega}\right)\left(Ch^r\|g\|_{0,\Omega}+Ch^r\|v\|_{1+r,\Omega}\right)\\ &\leq Ch^{2r}\|f\|_{0,\Omega}\|g\|_{0,\Omega}.\end{aligned}$$

□

As a consequence of this lemma we may prove a double order error estimate for the eigenfunctions in L^2 norm.

Theorem 3.10. *Let μ_k be the k-th (simple) eigenvalue of $\mathbf{T}$ and μ_{hk} the k-th eigenvalue of $\mathbf{T}_h$. Then, the corresponding eigenfunctions u_k and $\check{u}_{hk}$ can be chosen such that $\|u_k\|_{0,\Omega}=\|\check{u}_{hk}\|_{0,\Omega}=1$ and*

$$\|u_k-\check{u}_{hk}\|_{0,\Omega}\leq Ch^{2r},$$

with C a strictly positive constant.

Proof. Because of the previous lemma and the estimate (2.10), the theorem is also a direct consequence of Lemma 7.1 in [1]. □

Finally, a double order of convergence for the eigenvalues can also be proved:

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Theorem 3.11. *Let μ_k be the k-th (simple) eigenvalue of $\mathbf{T}$ and μ_{hk} the k-th eigenvalue of $\mathbf{T}_h$. Then, there exists a strictly positive constant C such that*

$$|\mu_k - \mu_{hk}| \leq Ch^{2r}.$$

Proof. Since $\mathbf{T}$ is self-adjoint with respect to $b(\cdot,\cdot)$, then Theorem 7.3 of [1] applied to our case yields

$$|\mu - \mu_h| \leq C \left[\sup_{f,g\in\mathcal{E}\subset \mathrm{L}^2(\Omega)} \frac{|b\left((\mathbf{T}-\mathbf{T}_h)f,g\right)|}{\|f\|_{0,\Omega}\,\|g\|_{0,\Omega}} + \|(\mathbf{T}-\mathbf{T}_h)|_{\mathcal{E}}\|_{0,\Omega}\|(\mathbf{T}-\mathbf{T}_h^*)|_{\mathcal{E}}\|_{0,\Omega} \right],$$

where $\mathcal{E}$ is the eigenspace associated to μ_k, and $\mathbf{T}_h^*$ is the adjoint operator of $\mathbf{T}_h$ with respect to $b(\cdot,\cdot)$.

Now

$$\sup_{f,g\in\mathcal{E}\subset \mathrm{L}^2(\Omega)} \frac{|b\left((\mathbf{T}-\mathbf{T}_h)f,g\right)|}{\|f\|_{0,\Omega}\,\|g\|_{0,\Omega}} = \sup_{f\in\mathcal{E}} \frac{\|(\mathbf{T}-\mathbf{T}_h)f\|_{0,\Omega}}{\|f\|_{0,\Omega}}$$

and

$$\begin{aligned} \|(\mathbf{T}-\mathbf{T}_h^*)f\|_{0,\Omega} &= \sup_{g\in \mathrm{L}^2(\Omega)} \frac{b((\mathbf{T}-\mathbf{T}_h^*)f,g)}{\|g\|_{0,\Omega}} = \sup_{g\in \mathrm{L}^2(\Omega)} \frac{b(f,(\mathbf{T}-\mathbf{T}_h)g)}{\|g\|_{0,\Omega}} \\ &\leq \|f\|_{0,\Omega} \sup_{g\in \mathrm{L}^2(\Omega)} \frac{\|(\mathbf{T}-\mathbf{T}_h)g\|_{0,\Omega}}{\|g\|_{0,\Omega}} \leq C\|f\|_{0,\Omega}, \end{aligned}$$

the last inequality because of (2.8) and (2.17).

Thus, the theorem is a consequence of Lemma 3.9 and estimate (2.10). □

4. Conclusions

We have analyzed the finite element approximation of eigenvalues and eigenfunctions of the Laplace equations with Neumann boundary conditions on curved (nonconvex) domains Ω. Convergence and optimal order error estimates have been proved for standard piecewise linear elements on a discrete polygonal domain $\Omega_h \not\subset \Omega$, in the framework of the abstract spectral approximation theory as stated in [1]. Similar results had been previously proved by Vanmaele and Ženíšek [19] (based on *min-max* techniques), and Lebaud [14], but in both cases only for Dirichlet boundary conditions.

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