NEW AMICABLE FOUR-CYCLES

KARSTEN BLANKENAGEL, WALTER BORHO, AND AXEL VOM STEIN

Abstract. Fifty new amicable four-cycles are discovered by the constructive
method invented in 1969 by the second author.

1. Let \( \tau(n) \) denote the sum of proper divisors of a natural number \( n \), and let
\( \sigma(n) = n + \tau(n) \). We consider when the sequence \( n, \tau(n), \tau(\tau(n)) = \tau(\tau(n)), \ldots \)
becomes periodic. If \( n = \tau(k)(n) \) with \( k \) minimal, then \( n_1 = n, n_2 = \tau(n), n_3 = \tau(2)(n), \ldots, n_k = \tau(k-1)(n) \) is called an amicable \( k \)-cycle. The study of amicable
1-cycles (perfect numbers) and of amicable 2-cycles (amicable pairs) has a thousand-
year-old history. Here we study amicable four-cycles. The smallest example is
\[
\begin{align*}
n_1 &= 2^2 \cdot 5 \cdot 17 \cdot 3719, & n_3 &= 2^2 \cdot 521 \cdot 829, \\
n_2 &= 2^2 \cdot 5 \cdot 193 \cdot 401, & n_4 &= 2^5 \cdot 40787,
\end{align*}
\]
discovered by H. Cohen in 1970 by an exhaustive trial and error search below
60,000,000.

2. Alternatively, one may try to construct amicable four-cycles of a special form.
This can be done by means of the following theorem, due to the second author.

Theorem 1 \((3)\). Let \( a_1 \) and \( a_2 \) be natural numbers, \( a_1 \neq a_2 \), and let \( D := a_1 a_2 - \tau(a_1) \tau(a_2) \). Let \( d_1 d_2 = a_1 a_2 \) be any factorization into two natural numbers \( d_1, d_2 \). Consider the six numbers \( p_{ij}, r_i \) \((i, j = 1, 2)\)
\[
\begin{align*}
p_{ij} &= \frac{1}{D}(\tau(a_{i+1}) \sigma(a_i) + d_j \sigma(a_{i+1})), & \text{where } a_3 &= a_1, \\
r_i &= \frac{1}{a_i} \tau(a_i p_{11} p_{22}).
\end{align*}
\]
If all six are primes, and \( p_{ij} \nmid a_i, r_i \nmid a_i, p_{11} \neq p_{22} \) \((i, j = 1, 2)\), then the following is an amicable four-cycle:
\[
\begin{align*}
n_1 &= a_1 p_{11} p_{12}, & n_3 &= a_2 p_{21} p_{22}, \\
n_2 &= a_1 r_1, & n_4 &= a_2 r_2.
\end{align*}
\]
The smallest example is
\[
\begin{align*}
n_1 &= 3^3 \cdot 5 \cdot 7 \cdot 83 \cdot 359, & n_3 &= 3^3 \cdot 5 \cdot 11 \cdot 79 \cdot 263, \\
n_2 &= 3^3 \cdot 5 \cdot 7 \cdot 31643, & n_4 &= 3^3 \cdot 5 \cdot 11 \cdot 20183,
\end{align*}
\]
which was found in \((3)\) in 1969 without use of a computer.

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To specify these four numbers, it is enough to give \( a_1, a_2 \) and \( d_1 \), because the rest is then given by formulas (1) and (2) of the theorem. In the example

\[
3^1 \cdot 5 \cdot \begin{cases} 
7 \\
11
\end{cases} = a_1, \\
d_1 = 3^2 \cdot 5 \cdot 7.
\]

3. By a constructive search based on the above theorem, we produced 50 new (amicable) four-cycles specified below. Of the 60 (amicable) four-cycles previously known, 30 were found by this same constructive method, while 30 were found by trial and error (below \( 10^{12} \)). Here is the list of the discoverers:

<table>
<thead>
<tr>
<th>years</th>
<th>Discoverers</th>
<th>number of discoveries</th>
<th>method</th>
<th>references</th>
</tr>
</thead>
<tbody>
<tr>
<td>1969</td>
<td>Borho</td>
<td>1</td>
<td>constructive</td>
<td>[3]</td>
</tr>
<tr>
<td>1970</td>
<td>Cohen</td>
<td>8</td>
<td>trial and error</td>
<td>[5]</td>
</tr>
<tr>
<td>1990</td>
<td>Yuanhua</td>
<td>3</td>
<td>constructive</td>
<td>[10]</td>
</tr>
<tr>
<td>1990</td>
<td>Flammenkamp</td>
<td>7</td>
<td>trial and error</td>
<td>[7]</td>
</tr>
<tr>
<td>1990-98</td>
<td>Moews, Moews</td>
<td>8</td>
<td>constructive</td>
<td>[8]</td>
</tr>
<tr>
<td>1997-00</td>
<td>Pedersen</td>
<td>9</td>
<td>constructive</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>Baader</td>
<td>1</td>
<td>trial and error</td>
<td></td>
</tr>
<tr>
<td>2000-01</td>
<td>Ren Yuanhua</td>
<td>9</td>
<td>constructive</td>
<td></td>
</tr>
<tr>
<td>2000-01</td>
<td>new</td>
<td>50</td>
<td>constructive</td>
<td></td>
</tr>
</tbody>
</table>

4. In the constructive search with the theorem above, the choice of \( a_1 \) and \( a_2 \) is of course essential. It is useful to check the following necessary condition for \( a_1, a_2 \).

**Theorem 2** (Four-cycle condition). In order that the \( p_{ij} \) in Theorem 1 can be integers, it is necessary that

\[
(a_1 - a_2) f g \equiv 0 \mod D,
\]

where \( f \) resp. \( g \) is the greatest common divisor of \( \sigma(a_1), \sigma(a_2) \) resp. of \( a_1, a_2 \).

We call a four-cycle given by Theorem 1 “regular of type \((r,s)\)” if \( a_1, a_2 \) have the form

\[
a_1 = a p_1 p_2 \ldots p_r, \quad a_2 = a q_1 q_2 \ldots q_s,
\]

with some greatest common divisor \( a \), and different primes \( p_1, \ldots, p_r, q_1, \ldots, q_s \) not dividing \( a \). The 50 new amicable four-cycles discovered here are all “regular” in this sense. Nonregular or “exotic” examples were found by Ren Yuanhua and Pedersen.

5. For our constructive search for four-cycles, we must make the denominator

\[
D := D(a_1, a_2) = a_1 a_2 - \tau(a_1) \tau(a_2) = a_1 \sigma(a_2) + a_2 \sigma(a_1) - \sigma(a_1) \sigma(a_2)
\]

small in order to increase the chance that the \( p_{ij} \) become integers. But we can cancel by

\[
g_i := \gcd(a_i, \sigma(a_i)) \text{ for } i = 1, 2, \text{ and in addition by } \gcd(\sigma(a_1)/g_1, \sigma(a_2)/g_2).
\]

Let
\(D(a_1, a_2)\) denote the result. What we really have to make small is this \(D(a_1, a_2)\).

One might take \(d_j\) a multiple of \(g_j\) to make \(p_{ij}\) integer.

Let us describe, for example, our search for regular four-cycles of type (1, 1). So let \(a_1 = a p_1, a_2 = a q_1\) as in section 4. For \(a\) we took all values \(50,000,000\) with \(D(a, a) < 100\). These are 6224 numbers. Next, \(p_1\) must be chosen greater than \(\tau(a) \sigma(a)/D(a, a)\) (to make \(D(a p_1, a)\) positive). All primes \(p_1\) greater than this lower bound were taken in increasing order as long as \(D(a p_1, a) < 100\). Finally, \(q_1\) was chosen in such a way that the four-cycle condition (Theorem 2) was satisfied.

This search produced seven of the thirteen known four-cycles of type (1, 1), plus an addition two new ones, numbers 2), 3) of the list below.

Searches for four-cycles of types (1, 0), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3) were performed in a similar manner. The new results are listed below. Note that no four-cycles of type (1, 0) are known so far. It is an interesting open problem to find a four-cycle of this type.

An interesting phenomenon that occurs so far only in four-cycles of type (1, 1) is that \(p_{11}\) may equal \(p_{21}\). This actually occurs in 9 of the 16 known cases of type (1, 1).

For our search for amicable four-cycles we spent a total of about 3 years of computation on Sun desk calculators of 333 MHz.

The following table specifies all 50 new amicable four-cycles found by the searches described above. In fact, for each of the 50 four-cycles we give the triples \(a_1, a_2\) and \(d_1\) in prime factorization. From these data, the full four-cycle itself can be easily computed by the formulae given in Theorem 1. For instance, for four-cycle number 4 below, one obtains the 14 decimal digit numbers (in prime decomposition):

\[
n_1 = 2 \cdot 7 \cdot 223 \cdot 5 \cdot 11 \cdot 3121 \cdot 73589, \quad n_2 = 2 \cdot 7 \cdot 223 \cdot 23 \cdot 4013 \cdot 215417,
\]

\[
n_3 = 2 \cdot 7 \cdot 223 \cdot 5 \cdot 11 \cdot 288231067, \quad n_4 = 2 \cdot 7 \cdot 223 \cdot 23 \cdot 689238587.
\]

We let each line of the following table begin with an ordinal number of the corresponding four-cycle, followed by the number of decimal digits of the smallest member of the four-cycle, with a “D” for “decimal digits”. Next, \(a_1\) and \(a_2\) are given, with the common divisor given only once. Finally, \(d_1\) is given. The four-cycles are ordered according to their type.

<table>
<thead>
<tr>
<th>Type (1, 1)</th>
<th>1) 23D</th>
<th>2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 101</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>{118169, 236339} d_1 = 2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 101</td>
</tr>
<tr>
<td></td>
<td>2) 26D</td>
<td>3^2 \cdot 5^2 \cdot 19 \cdot 29 \cdot 41</td>
</tr>
<tr>
<td></td>
<td></td>
<td>{491, 294461} d_1 = 3^2 \cdot 5^2 \cdot 19 \cdot 41 \cdot 294461</td>
</tr>
<tr>
<td></td>
<td>3) 27D</td>
<td>3^3 \cdot 7^2 \cdot 13 \cdot 17 \cdot 67</td>
</tr>
<tr>
<td></td>
<td></td>
<td>{179, 49139} d_1 = 3^3 \cdot 7^2 \cdot 13 \cdot 17 \cdot 67</td>
</tr>
</tbody>
</table>
type $(2, 1)$

4) $14D \quad 2 \cdot 7 \cdot 223 \cdot \{ 5 \cdot 11 \quad d_1 = 2 \cdot 7 \cdot 223
13 \quad 23
\}
\{ 7 \cdot 59 \quad d_1 = 3^3 \cdot 5^3
89
\}
5) $15D \quad 3^3 \cdot 5^3 \cdot \{ 7 \cdot 59 \quad d_1 = 3^2 \cdot 5^2 \cdot 31
89
\}
6) $16D \quad 3^2 \cdot 5^2 \cdot 31 \cdot \{ 53 \cdot 379 \quad d_1 = 2 \cdot 3 \cdot 5 \cdot 13 \cdot 19 \cdot 53
71
\}
7) $17D \quad 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot \{ 17 \cdot 109 \quad d_1 = 3 \cdot 5 \cdot 13 \cdot 109
17929
\}
8) $22D \quad 3^5 \cdot 5 \cdot 13 \cdot \{ 11 \cdot 53 \quad d_1 = 2 \cdot 3 \cdot 5 \cdot 13 \cdot 23 \cdot 29 \cdot 137
1871
\}
9) $23D \quad 3^5 \cdot 7^2 \cdot 13 \cdot 83 \cdot \{ 29 \cdot 547 \quad d_1 = 3 \cdot 5 \cdot 7^2 \cdot 13 \cdot 1871
349
\}
10) $24D \quad 3^2 \cdot 5 \cdot 13 \cdot 23 \cdot 137 \cdot \{ 29 \cdot 797 \quad d_1 = 2 \cdot 3 \cdot 5 \cdot 13 \cdot 23 \cdot 29 \cdot 137
113
\}
11) $24D \quad 2^4 \cdot 101 \cdot \{ 29 \cdot 797 \quad d_1 = 2 \cdot 3 \cdot 5 \cdot 13 \cdot 23 \cdot 29 \cdot 137
113
\}
12) $26D \quad 3^7 \cdot 5^2 \cdot 41 \cdot \{ 11 \cdot 79 \quad d_1 = 3 \cdot 5 \cdot 13 \cdot 19 \cdot 53
7529
\}
13) $31D \quad 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot \{ 571 \cdot 162749 \quad d_1 = 3 \cdot 5 \cdot 13 \cdot 19 \cdot 53
29
\}
14) $32D \quad 3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot \{ 379 \cdot 523 \quad d_1 = 3 \cdot 5 \cdot 7^2 \cdot 13 \cdot 1871
5177119
\}
15) $36D \quad 2^8 \cdot 619 \cdot \{ 1487 \cdot 105229 \quad d_1 = 3 \cdot 5 \cdot 7 \cdot 17 \cdot 23^2 \cdot 101
39145599
\}
16) $36D \quad 3^2 \cdot 7 \cdot 11 \cdot 17 \cdot 23 \cdot \{ 101 \cdot 397493 \quad d_1 = 3 \cdot 5 \cdot 7 \cdot 17 \cdot 23 \cdot 29 \cdot 137
2039
\}

type $(2, 2)$

17) $16D \quad 3 \cdot 5 \cdot 13 \cdot \{ 11 \cdot 1279 \quad d_1 = 2 \cdot 3 \cdot 5 \cdot 13 \cdot 1279
23 \quad 127
\}
18) $17D \quad 2^3 \cdot \{ 13 \cdot 3079 \quad d_1 = 2 \cdot 3 \cdot 13
19 \quad 293
\}
19) $19D \quad 3^4 \cdot 11^2 \cdot 17 \cdot \{ 5 \cdot 373 \quad d_1 = 3 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 1049
13 \quad 43
\}
20) $20D \quad 2 \cdot 5 \cdot 13 \cdot \{ 29 \cdot 139 \quad d_1 = 2 \cdot 3 \cdot 5 \cdot 13 \cdot 47 \cdot 139
47 \quad 2099
\}
21) $21D \quad 3^4 \cdot 7^2 \cdot 11^2 \cdot \{ 17 \cdot 97 \quad d_1 = 2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 11^2 \cdot 1049
41 \quad 1049
\}
22) $22D \quad 3^3 \cdot 7^2 \cdot 13 \cdot \{ 11 \cdot 199 \quad d_1 = 2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 199
17 \quad 1559
\}
23) $22D \quad 2^6 \cdot \{ 107 \cdot 2099 \quad d_1 = 2^5 \cdot 179 \cdot 2099
179 \quad 2939
\}
24) $24D \quad 2^5 \cdot 97 \cdot \{ 193 \cdot 2447 \quad d_1 = 2^5 \cdot 97 \cdot 197 \cdot 2447
197 \quad 2909
\}
<table>
<thead>
<tr>
<th>Type</th>
<th>Number</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>25) 25D</td>
<td>$3^2 \cdot 5^3 \cdot 13^2$</td>
<td>${29 \cdot 193, 103 \cdot 14549} \quad d_1 = 3^2 \cdot 5^3 \cdot 13^2 \cdot 103 \cdot 193$</td>
</tr>
<tr>
<td>26) 25D</td>
<td>$3^3 \cdot 5 \cdot 11 \cdot 43$</td>
<td>${101 \cdot 1031, 251 \cdot 1289} \quad d_1 = 3^3 \cdot 5 \cdot 11 \cdot 43 \cdot 101 \cdot 1289$</td>
</tr>
<tr>
<td>27) 25D</td>
<td>$3^2 \cdot 5 \cdot 13 \cdot 41$</td>
<td>${19 \cdot 233, 29 \cdot 2347} \quad d_1 = 3^2 \cdot 5 \cdot 13 \cdot 41^2 \cdot 233$</td>
</tr>
<tr>
<td>28) 26D</td>
<td>$3 \cdot 5 \cdot 7 \cdot 19$</td>
<td>${13 \cdot 509, 577 \cdot 4787} \quad d_1 = 3 \cdot 5 \cdot 7 \cdot 19 \cdot 509$</td>
</tr>
<tr>
<td>29) 26D</td>
<td>$3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 43$</td>
<td>${101 \cdot 1031, 251 \cdot 1289} \quad d_1 = 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 43 \cdot 101 \cdot 1289$</td>
</tr>
<tr>
<td>30) 28D</td>
<td>$3^3 \cdot 5^4 \cdot 19$</td>
<td>${37 \cdot 239, 47 \cdot 9803} \quad d_1 = 3^4 \cdot 5^3 \cdot 19 \cdot 47$</td>
</tr>
<tr>
<td>31) 28D</td>
<td>$2^6$</td>
<td>${79 \cdot 2699, 367 \cdot 23819} \quad d_1 = 2^6 \cdot 367 \cdot 2699$</td>
</tr>
<tr>
<td>32) 30D</td>
<td>$3^4 \cdot 5 \cdot 11$</td>
<td>${23 \cdot 11699, 419 \cdot 3167} \quad d_1 = 3^4 \cdot 5 \cdot 11 \cdot 23$</td>
</tr>
<tr>
<td>33) 30D</td>
<td>$2^2 \cdot 11 \cdot 79$</td>
<td>${17 \cdot 13903, 89 \cdot 112507} \quad d_1 = 2^2 \cdot 11 \cdot 79 \cdot 13903$</td>
</tr>
<tr>
<td>34) 31D</td>
<td>$3^3 \cdot 5^3 \cdot 13$</td>
<td>${199 \cdot 4751, 271 \cdot 336599} \quad d_1 = 3^3 \cdot 5^3 \cdot 13 \cdot 271$</td>
</tr>
<tr>
<td>35) 31D</td>
<td>$3^2 \cdot 5^2 \cdot 13 \cdot 31$</td>
<td>${199 \cdot 4751, 271 \cdot 336599} \quad d_1 = 3^2 \cdot 5^2 \cdot 13 \cdot 31 \cdot 271$</td>
</tr>
<tr>
<td>36) 37D</td>
<td>$3^2 \cdot 5 \cdot 13 \cdot 23$</td>
<td>${19 \cdot 101429, 461 \cdot 22271129} \quad d_1 = 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 23 \cdot 101429$</td>
</tr>
<tr>
<td>37) 41D</td>
<td>$3^2 \cdot 5 \cdot 11 \cdot 19$</td>
<td>${131 \cdot 5651377, 509 \cdot 30557} \quad d_1 = 3^2 \cdot 5 \cdot 11 \cdot 19 \cdot 131$</td>
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**Type (3, 1)**

<table>
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<tr>
<td>38) 32D</td>
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<tr>
<td>39) 48D</td>
<td>$2 \cdot 5 \cdot 11^2 \cdot 241$</td>
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**Type (3, 2)**

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<td>41) 19D</td>
<td>$3^3 \cdot 5^2$</td>
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<tr>
<td>42) 24D</td>
<td>$2^2 \cdot 11$</td>
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<td>43) 25D</td>
<td>$3^2 \cdot 5^3 \cdot 13$</td>
</tr>
<tr>
<td>44) 35D</td>
<td>$3^2 \cdot 7^2 \cdot 11 \cdot 13$</td>
</tr>
<tr>
<td>45) 40D</td>
<td>$3^5 \cdot 5 \cdot 13$</td>
</tr>
</tbody>
</table>
The two four-cycles 1), 13) were found by a different type of search: $a_1, a_2$ were chosen as so-called breeders for amicable number pairs, as studied in [4], [1]. These two new cycles are due to all three authors; the other ones are due to the first two authors. A complete list of all 110 known amicable four-cycles is provided by J. O. M. Pedersen on his internet amicable.adsl.dk pages


**REFERENCES**


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