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NEW AMICABLE FOUR-CYCLES

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Abstract. Fifty new amicable four-cycles are discovered by the constructive method invented in 1969 by the second author.

1. Let $\tau(n)$ denote the sum of proper divisors of a natural number n, and let $\sigma(n) = n + \tau(n)$. We consider when the sequence n, $\tau(n)$, $\tau^{(2)}(n) := \tau(\tau(n))$, ... becomes periodic. If $n = \tau^{(k)}(n)$ with k minimal, then $n_1 = n$, $n_2 = \tau(n)$, $n_3 = \tau^{(2)}(n)$, ..., $n_k = \tau^{(k-1)}(n)$ is called an *amicable k-cycle*. The study of amicable 1-cycles (*perfect numbers*) and of amicable 2-cycles (*amicable pairs*) has a thousand-year-old history. Here we study amicable four-cycles. The smallest example is

$$n_1 = 2^2 \cdot 5 \cdot 17 \cdot 3719, \qquad n_3 = 2^2 \cdot 521 \cdot 829,$$
$$n_2 = 2^2 \cdot 5 \cdot 193 \cdot 401, \qquad n_4 = 2^5 \cdot 40787,$$

discovered by H. Cohen [5] in 1970 by an exhaustive trial and error search below $60,000,000$.

2. Alternatively, one may try to *construct* amicable four-cycles of a special form. This can be done by means of the following theorem, due to the second author.

Theorem 1 ([3]). *Let a_1 and a_2 be natural numbers, $a_1 \neq a_2$, and let $D := a_1 a_2 - \tau(a_1)\,\tau(a_2)$. Let $d_1 d_2 = a_1 a_2$ be any factorization into two natural numbers d_1, d_2. Consider the six numbers p_{ij}, r_i $(i, j = 1, 2)$*

$$(1) \qquad p_{ij} := \frac{1}{D}\left(\tau(a_{i+1})\,\sigma(a_i) + d_j\,\sigma(a_{i+1})\right), \qquad \textit{where } a_3 := a_1,$$

$$(2) \qquad r_i := \frac{1}{a_i}\,\tau(a_i\,p_{i1}\,p_{i2}).$$

If all six are primes, and $p_{ij} \nmid a_i$, $r_i \nmid a_i$, $p_{i1} \neq p_{i2}$ $(i, j = 1, 2)$, then the following is an amicable four-cycle:

$$n_1 = a_1\,p_{11}\,p_{12}, \qquad n_3 = a_2\,p_{21}\,p_{22},$$
$$n_2 = a_1\,r_1, \qquad n_4 = a_2\,r_2\,.$$

The smallest example is

$$n_1 = 3^3 \cdot 5 \cdot 7 \cdot 83 \cdot 359, \qquad n_3 = 3^3 \cdot 5 \cdot 11 \cdot 79 \cdot 263,$$
$$n_2 = 3^3 \cdot 5 \cdot 7 \cdot 31643, \qquad n_4 = 3^3 \cdot 5 \cdot 11 \cdot 20183,$$

which was found in [3] in 1969 without use of a computer.

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To specify these four numbers, it is enough to give a_1, a_2 and d_1, because the rest is then given by formulas (1) and (2) of the theorem. In the example

$$3^3 \cdot 5 \cdot \begin{cases} 7 & = a_1, \\ 11 & = a_2, \end{cases} \qquad d_1 = 3^2 \cdot 5 \cdot 7.$$

3. By a constructive search based on the above theorem, we produced 50 new (amicable) four-cycles specified below. Of the 60 (amicable) four-cycles previously known, 30 were found by this same constructive method, while 30 were found by trial and error (below 10^{12}). Here is the list of the discoverers:

years	Discoverers	number of discoveries	method	references
1969	Borho	1	constructive	[3]
1970	Cohen	8	trial and error	[5]
1972	David / Root	4 / 5	trial and error	[6] / [2]
1990	Yuanhua	3	constructive	[10]
1990	Flammenkamp	7	trial and error	[7]
1990-95	Moews, Moews	9	trial and error	[8] / [9]
1990-98	Moews, Moews	8	constructive	[8]
1997-00	Pedersen	9	constructive	
2000	Baader	1	trial and error	
2000-01	Ren Yuanhua	9	constructive	
2000-01	new	50	constructive	

4. In the constructive search with the theorem above, the choice of a_1 and a_2 is of course essential. It is useful to check the following necessary condition for a_1, a_2.

Theorem 2 (Four-cycle condition)**.** *In order that the p_{ij} in Theorem 1 can be integers, it is necessary that*

$$(a_1 - a_2)\, f\, g \equiv 0 \mod D,$$

where f resp. g is the greatest common divisor of $\sigma(a_1), \sigma(a_2)$ resp. of a_1, a_2.

We call a four-cycle given by Theorem 1 "regular of type (r, s)" if a_1, a_2 have the form

$$a_1 = a\, p_1\, p_2 \dots p_r, \qquad a_2 = a\, q_1\, q_2 \dots q_s,$$

with some greatest common divisor a, and different primes $p_1, \dots, p_r$, $q_1, \dots, q_s$ not dividing a. The 50 new amicable four-cycles discovered here are all "regular" in this sense. Nonregular or "exotic" examples were found by Ren Yuanhua and Pedersen.

5. For our constructive search for four-cycles, we must make the denominator $D := D(a_1, a_2) = a_1\, a_2 - \tau(a_1)\, \tau(a_2) = a_1\, \sigma(a_2) + a_2\, \sigma(a_1) - \sigma(a_1)\, \sigma(a_2)$ small in order to increase the chance that the p_{ij} become integers. But we can cancel by $g_i := \gcd(a_i, \sigma(a_i))$ for $i = 1, 2$, and in addition by $\gcd(\sigma(a_1)/g_1, \sigma(a_2)/g_2)$. Let

$\bar{D}(a_1, a_2)$ denote the result. What we really have to make small is this $\bar{D}(a_1, a_2)$. (One might take d_j a multiple of g_j to make p_{ij} integer.)

Let us describe, for example, our search for regular four-cycles of type $(1, 1)$. So let $a_1 = a\,p_1$, $a_2 = a\,q_1$ as in section 4. For a we took all values $\leq 50,000,000$ with $\bar{D}(a, a) < 100$. These are 6224 numbers. Next, p_1 must be chosen greater than $\tau(a)\,\sigma(a)/D(a, a)$ (to make $D(a\,p_1, a)$ positive). All primes p_1 greater than this lower bound were taken in increasing order as long as $\bar{D}(a\,p_1, a) < 100$. Finally, q_1 was chosen in such a way that the four-cycle condition (Theorem 2) was satisfied. To all pairs $a_1 = a\,p_1$, $a_2 = a\,q_1$ so chosen, Theorem 1 was applied; that is, all factorizations $d_1\,d_2 = a_1\,a_2$ were checked to see whether they give an amicable four-cycle.

This search produced seven of the thirteen known four-cycles of type $(1, 1)$, plus in addition two new ones, numbers 2), 3) of the list below.

Searches for four-cycles of types $(1, 0)$, $(2, 1)$, $(2, 2)$, $(3, 1)$, $(3, 2)$, $(3, 3)$ were performed in a similar manner. The new results are listed below. Note that no four-cycles of type $(1, 0)$ are known so far. It is an interesting open problem to find a four-cycle of this type.

An interesting phenomenon that occurs so far only in four-cycles of type $(1, 1)$ is that p_{11} may equal p_{21}. This actually occurs in 9 of the 16 known cases of type $(1, 1)$.

For our search for amicable four-cycles we spent a total of about 3 years of computation on Sun desk calculators of 333 MHz.

6. The following table specifies all 50 new amicable four-cycles found by the searches described above. In fact, for each of the 50 four-cycles we give the triples a_1, a_2 and d_1 in prime factorization. From these data, the full four-cycle itself can be easily computed by the formulae given in Theorem 1. For instance, for four-cycle number 4 below, one obtains the 14 decimal digit numbers (in prime decomposition):

$$n_1 = 2 \cdot 7 \cdot 223 \cdot 5 \cdot 11 \cdot 3121 \cdot 73589, \quad n_3 = 2 \cdot 7 \cdot 223 \cdot 23 \cdot 4013 \cdot 215417,$$
$$n_2 = 2 \cdot 7 \cdot 223 \cdot 5 \cdot 11 \cdot 288231067, \qquad n_4 = 2 \cdot 7 \cdot 223 \cdot 23 \cdot 689238587.$$

We let each line of the following table begin with an ordinal number of the corresponding four-cycle, followed by the number of decimal digits of the smallest member of the four-cycle, with a "D" for "decimal digits". Next, a_1 and a_2 are given, with the common divisor given only once. Finally, d_1 is given. The four-cycles are ordered according to their type.

type (1, 1)

1)	$23D$	$2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 101 \cdot \begin{cases} 118169 \\ 236339 \end{cases}$	$d_1 = 2 \cdot 7^2 \cdot 13 \cdot 19 \cdot 101$
2)	$26D$	$3^2 \cdot 5^2 \cdot 19 \cdot 29 \cdot 41 \cdot \begin{cases} 491 \\ 294461 \end{cases}$	$d_1 = 3^2 \cdot 5^2 \cdot 19 \cdot 41 \cdot 294461$
3)	$27D$	$3^3 \cdot 7^2 \cdot 13 \cdot 17 \cdot 67 \cdot \begin{cases} 179 \\ 49139 \end{cases}$	$d_1 = 3^3 \cdot 7^2 \cdot 13 \cdot 17 \cdot 67$

type (2, 1)

4)	$14D$	$2 \cdot 7 \cdot 223 \cdot \begin{cases} 5 \cdot 11 \\ 23 \end{cases}$	$d_1 = 2 \cdot 7 \cdot 223$
5)	$15D$	$3^3 \cdot 5^3 \cdot \begin{cases} 7 \cdot 59 \\ 89 \end{cases}$	$d_1 = 3^3 \cdot 5^3$
6)	$16D$	$3^2 \cdot 5^2 \cdot 31 \cdot \begin{cases} 7 \cdot 59 \\ 89 \end{cases}$	$d_1 = 3^2 \cdot 5^2 \cdot 31$
7)	$17D$	$3^2 \cdot 5 \cdot 13 \cdot 19 \cdot \begin{cases} 53 \cdot 379 \\ 71 \end{cases}$	$d_1 = 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 53$
8)	$22D$	$3^5 \cdot 5 \cdot 13 \cdot \begin{cases} 17 \cdot 109 \\ 17929 \end{cases}$	$d_1 = 3^7 \cdot 5 \cdot 13 \cdot 109$
9)	$23D$	$3^5 \cdot 7^2 \cdot 13 \cdot 83 \cdot \begin{cases} 11 \cdot 53 \\ 1871 \end{cases}$	$d_1 = 3^5 \cdot 7^2 \cdot 13 \cdot 1871$
10)	$24D$	$3^2 \cdot 5 \cdot 13 \cdot 23 \cdot 137 \cdot \begin{cases} 29 \cdot 547 \\ 349 \end{cases}$	$d_1 = 3^2 \cdot 5 \cdot 13 \cdot 23 \cdot 29 \cdot 137$
11)	$24D$	$2^4 \cdot 101 \cdot \begin{cases} 29 \cdot 797 \\ 113 \end{cases}$	$d_1 = 2^4 \cdot 29$
12)	$26D$	$3^7 \cdot 5^2 \cdot 41 \cdot \begin{cases} 11 \cdot 79 \\ 7529 \end{cases}$	$d_1 = 3^{11} \cdot 5^2 \cdot 41$
13)	$31D$	$3^2 \cdot 5 \cdot 13 \cdot 19 \cdot \begin{cases} 571 \cdot 162749 \\ 29 \end{cases}$	$d_1 = 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 571$
14)	$32D$	$3^4 \cdot 7^2 \cdot 11 \cdot 19 \cdot \begin{cases} 379 \cdot 523 \\ 5177119 \end{cases}$	$d_1 = 3^4 \cdot 11 \cdot 19 \cdot 5177119$
15)	$36D$	$2^8 \cdot 619 \cdot \begin{cases} 1487 \cdot 105229 \\ 391455599 \end{cases}$	$d_1 = 2^8 \cdot 619 \cdot 105229$
16)	$36D$	$3^2 \cdot 7 \cdot 11 \cdot 17 \cdot 23 \cdot \begin{cases} 101 \cdot 397493 \\ 2039 \end{cases}$	$d_1 = 3^2 \cdot 7 \cdot 17 \cdot 23^2 \cdot 101$

type (2, 2)

17)	$16D$	$3^2 \cdot 5 \cdot 13 \cdot \begin{cases} 11 \cdot 1279 \\ 23 \cdot 127 \end{cases}$	$d_1 = 3^2 \cdot 5 \cdot 13 \cdot 1279$
18)	$17D$	$2^3 \cdot \begin{cases} 13 \cdot 3079 \\ 19 \cdot 293 \end{cases}$	$d_1 = 2^3 \cdot 13$
19)	$19D$	$3^4 \cdot 11^2 \cdot 17 \cdot \begin{cases} 5 \cdot 373 \\ 13 \cdot 43 \end{cases}$	$d_1 = 3^4 \cdot 11^2 \cdot 13 \cdot 17 \cdot 43$
20)	$20D$	$2 \cdot 5 \cdot 13 \cdot \begin{cases} 29 \cdot 139 \\ 47 \cdot 2099 \end{cases}$	$d_1 = 2 \cdot 5 \cdot 13 \cdot 47 \cdot 139$
21)	$21D$	$3^4 \cdot 7^2 \cdot 11^2 \cdot \begin{cases} 17 \cdot 97 \\ 41 \cdot 1049 \end{cases}$	$d_1 = 3^5 \cdot 7^2 \cdot 11^2 \cdot 1049$
22)	$22D$	$3^3 \cdot 7^2 \cdot 13 \cdot \begin{cases} 11 \cdot 199 \\ 17 \cdot 1559 \end{cases}$	$d_1 = 3^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 199$
23)	$22D$	$2^6 \cdot \begin{cases} 107 \cdot 2099 \\ 179 \cdot 2939 \end{cases}$	$d_1 = 2^5 \cdot 179 \cdot 2099$
24)	$24D$	$2^5 \cdot 97 \cdot \begin{cases} 193 \cdot 2447 \\ 197 \cdot 2909 \end{cases}$	$d_1 = 2^5 \cdot 97 \cdot 197 \cdot 2447$

25) $25D$	$3^2 \cdot 5^3 \cdot 13^2 \cdot$	$\begin{cases} 29 \cdot 193 \\ 103 \cdot 14549 \end{cases}$	$d_1 = 3^2 \cdot 5^3 \cdot 13^2 \cdot 103 \cdot 193$
26) $25D$	$3^3 \cdot 5 \cdot 11 \cdot 43 \cdot$	$\begin{cases} 101 \cdot 1031 \\ 251 \cdot 1289 \end{cases}$	$d_1 = 3^3 \cdot 5 \cdot 11 \cdot 43 \cdot 101 \cdot 1289$
27) $25D$	$3^2 \cdot 5 \cdot 13 \cdot 41 \cdot$	$\begin{cases} 19 \cdot 233 \\ 29 \cdot 2347 \end{cases}$	$d_1 = 3^2 \cdot 5 \cdot 13 \cdot 41^2 \cdot 233$
28) $26D$	$3 \cdot 5 \cdot 7 \cdot 19 \cdot$	$\begin{cases} 13 \cdot 509 \\ 577 \cdot 4787 \end{cases}$	$d_1 = 3 \cdot 5 \cdot 7 \cdot 19 \cdot 509$
29) $26D$	$3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 43 \cdot$	$\begin{cases} 101 \cdot 1031 \\ 251 \cdot 1289 \end{cases}$	$d_1 = 3^2 \cdot 7 \cdot 11 \cdot 13 \cdot 43 \cdot 101 \cdot 1289$
30) $28D$	$3^3 \cdot 5^4 \cdot 19 \cdot$	$\begin{cases} 37 \cdot 239 \\ 47 \cdot 9803 \end{cases}$	$d_1 = 3^4 \cdot 5^5 \cdot 19 \cdot 47$
31) $28D$	$2^6 \cdot$	$\begin{cases} 79 \cdot 2699 \\ 367 \cdot 23819 \end{cases}$	$d_1 = 2^6 \cdot 367 \cdot 2699$
32) $30D$	$3^4 \cdot 5 \cdot 11 \cdot$	$\begin{cases} 23 \cdot 11699 \\ 419 \cdot 3167 \end{cases}$	$d_1 = 3^4 \cdot 5 \cdot 11 \cdot 23$
33) $30D$	$2^2 \cdot 11 \cdot 79 \cdot$	$\begin{cases} 17 \cdot 13903 \\ 89 \cdot 112507 \end{cases}$	$d_1 = 2^2 \cdot 11 \cdot 79 \cdot 13903$
34) $31D$	$3^3 \cdot 5^3 \cdot 13 \cdot$	$\begin{cases} 199 \cdot 4751 \\ 271 \cdot 336599 \end{cases}$	$d_1 = 3^3 \cdot 5^3 \cdot 13 \cdot 271$
35) $31D$	$3^2 \cdot 5^2 \cdot 13 \cdot 31 \cdot$	$\begin{cases} 199 \cdot 4751 \\ 271 \cdot 336599 \end{cases}$	$d_1 = 3^2 \cdot 5^2 \cdot 13 \cdot 31 \cdot 271$
36) $37D$	$3^2 \cdot 5 \cdot 13 \cdot 23 \cdot$	$\begin{cases} 19 \cdot 101429 \\ 461 \cdot 22271129 \end{cases}$	$d_1 = 3^2 \cdot 5 \cdot 13 \cdot 19 \cdot 23 \cdot 101429$
37) $41D$	$3^2 \cdot 5 \cdot 11 \cdot 19 \cdot$	$\begin{cases} 131 \cdot 5651377 \\ 509 \cdot 30557 \end{cases}$	$d_1 = 3^2 \cdot 5 \cdot 11 \cdot 19 \cdot 131$
type (3, 1)			
38) $32D$	$3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot$	$\begin{cases} 71 \cdot 83 \cdot 2851 \\ 245519 \end{cases}$	$d_1 = 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 71 \cdot 83$
39) $48D$	$2 \cdot 5 \cdot 11^2 \cdot 241 \cdot$	$\begin{cases} 127 \cdot 181 \cdot 70853 \\ 434571199 \end{cases}$	$d_1 = 2 \cdot 5 \cdot 11^2 \cdot 241$
type (3, 2)			
40) $17D$	$2^3 \cdot$	$\begin{cases} 13 \cdot 23 \cdot 109 \\ 167 \cdot 1231 \end{cases}$	$d_1 = 2^3 \cdot 23 \cdot 167$
41) $19D$	$3^3 \cdot 5^2 \cdot$	$\begin{cases} 11 \cdot 29 \cdot 167 \\ 23 \cdot 449 \end{cases}$	$d_1 = 3^3 \cdot 5^2 \cdot 11 \cdot 23 \cdot 29$
42) $24D$	$2^2 \cdot 11 \cdot$	$\begin{cases} 17 \cdot 439 \cdot 503 \\ 31 \cdot 11087 \end{cases}$	$d_1 = 2^2 \cdot 11 \cdot 439 \cdot 503$
43) $25D$	$3^2 \cdot 5^3 \cdot 13 \cdot$	$\begin{cases} 29 \cdot 59 \cdot 389 \\ 149 \cdot 8423 \end{cases}$	$d_1 = 3^2 \cdot 5^3 \cdot 13 \cdot 29 \cdot 389$
44) $35D$	$3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot$	$\begin{cases} 97 \cdot 109 \cdot 569 \\ 197 \cdot 1667959 \end{cases}$	$d_1 = 3^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 109 \cdot 569$
45) $40D$	$3^5 \cdot 5 \cdot 13 \cdot$	$\begin{cases} 17 \cdot 263 \cdot 67103 \\ 197 \cdot 122039807 \end{cases}$	$d_1 = 3^5 \cdot 5 \cdot 13 \cdot 122039807$

46) $42D$ $2^5 \cdot \begin{cases} 47 \cdot 2137 \cdot 6323 \\ 101 \cdot 642640039 \end{cases}$ $d_1 = 2^5 \cdot 2137 \cdot 6323$

47) $47D$ $2 \cdot 5 \cdot 11 \cdot \begin{cases} 239 \cdot 1051 \cdot 2763119 \\ 31 \cdot 181324319039 \end{cases}$ $d_1 = 2 \cdot 5 \cdot 31 \cdot 239 \cdot 1051 \cdot 2763119$

48) $52D$ $2^2 \cdot 11 \cdot 149 \cdot \begin{cases} 13 \cdot 1297 \cdot 15074627 \\ 167 \cdot 272306275957 \end{cases}$ $d_1 = 2^2 \cdot 11 \cdot 149 \cdot 272306275957$

type (3, 3)

49) $28D$ $2^3 \cdot \begin{cases} 19 \cdot 23 \cdot 47 \\ 79 \cdot 251 \cdot 31727 \end{cases}$ $d_1 = 2^3 \cdot 23 \cdot 79 \cdot 251$

50) $45D$ $2^4 \cdot \begin{cases} 23 \cdot 83 \cdot 227 \\ 191 \cdot 15791 \cdot 10112749 \end{cases}$ $d_1 = 2^4 \cdot 83$

7. The two four-cycles 1), 13) were found by a different type of search: a_1, a_2 were chosen as so-called breeders for amicable number pairs, as studied in [4], [1]. These two new cycles are due to all three authors; the other ones are due to the first two authors. A complete list of all 110 known amicable four-cycles is provided by J. O. M. Pedersen on his internet amicable.adsl.dk pages

`http://amicable.adsl.dk/aliquot/sociable.txt.`

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