

**A NONCONFORMING COMBINATION  
OF THE FINITE ELEMENT AND VOLUME METHODS  
WITH AN ANISOTROPIC MESH REFINEMENT  
FOR A SINGULARLY PERTURBED  
CONVECTION-DIFFUSION EQUATION**

SONG WANG AND ZI-CAI LI

ABSTRACT. In this paper we formulate and analyze a discretization method for a 2D linear singularly perturbed convection-diffusion problem with a singular perturbation parameter  $\varepsilon$ . The method is based on a nonconforming combination of the conventional Galerkin piecewise linear triangular finite element method and an exponentially fitted finite volume method, and on a mixture of triangular and rectangular elements. It is shown that the method is stable with respect to a semi-discrete energy norm and the approximation error in the semi-discrete energy norm is bounded by  $Ch\sqrt{\left|\frac{\ln \varepsilon}{\ln h}\right|}$  with  $C$  independent of the mesh parameter  $h$ , the diffusion coefficient  $\varepsilon$  and the exact solution of the problem.

1. INTRODUCTION

Many phenomena in engineering, physics and finance are governed by a convection-diffusion equation with a diffusion coefficient  $1 > \varepsilon > 0$  which is much smaller than the average magnitude of the convection coefficient function. These equations are convection-dominated or singularly perturbed, and solutions to them normally have sharp boundary or interior layers so that applications of conventional numerical methods to these problems often yield solutions with nonphysical, spurious oscillations. To overcome this stability problem, many methods have been proposed. These include upwind methods (cf., for example, [5, 11, 12, 4, 2]), streamline diffusion methods (cf., for example, [13]) and exponentially fitted methods (cf., for example, [14, 15, 18, 23, 24]). However, no method guarantees, in general, that a numerical solution converges to the exact one uniformly in  $\varepsilon$  on an unstructured triangular partition. Several uniformly convergent schemes (cf., for example, [18, 22]) have been proposed and analyzed based on the Shishkin mesh technique (cf. [16]) and on piecewise uniform, structured partitions. Thus, these schemes can hardly be used for problems with nonrectangular geometries. Work on a least-squares finite element using a Gartland-type mesh is reported in [21]. Recently, Feistauer et al. ([7, 8, 9]) proposed and analyzed some semi-implicit and explicit schemes based on

---

Received by the editor June 7, 2001 and, in revised form, December 28, 2001.  
2000 *Mathematics Subject Classification*. Primary 65N30; Secondary 76M10.

©2003 American Mathematical Society

combinations of the finite element and finite volume methods for nonlinear time-dependent convection-diffusion problems. These methods have been successfully used in solving practical problems. A combination of the finite element and volume methods for a linear, stationary singularly perturbed convection-diffusion problem was also proposed and analyzed in [19]. All these combined methods are based on the idea that the convection and diffusion terms are discretized by a finite volume method and a finite element method, respectively. Although these methods are useful in practice, theoretically they do not guarantee that the errors in the approximate solutions do not deteriorate significantly as the diffusion coefficient goes to zero.

Intuitively, a solution to a singularly perturbed problem is smooth outside the boundary layers, as it is almost equal to the solution of the reduced problem (with  $\varepsilon = 0$ ). Therefore, any classic method should be able to resolve this part of the solution satisfactorily. The solution shows sharp layers in a (small) subregion near part of the boundary of the solution domain, and so a sophisticated method, such as anisotropic mesh refinement, the Shishkin mesh technique, the exponential fitting technique, or a combination of these, needs to be used so as to accurately resolve that part of the solution. This suggests that a combination of classic and other methods can solve a singularly perturbed problem efficiently.

In this paper, we investigate a combination of the conventional piecewise linear triangular finite element method and the exponentially fitted finite element (or volume) method (cf. [14, 15]) with anisotropic meshes for a linear, time-dependent singularly perturbed convection-diffusion problem. In this approach, a solution region is divided into two subregions, of which one does not contain layers and the other does. The equation is then discretized by conventional piecewise linear finite elements on a regular triangulation of the first subregion and by exponentially fitted finite elements on an anisotropic mesh for the second subregion. These two discretizations are then coupled to each other by imposing continuity at the mesh nodes along the intersection of the boundaries of the two subregions. Since the finite element basis functions may not be continuous across the boundaries between the two subregions, the resulting finite element space is nonconforming. The stability of the method is proved, and an  $O(h\sqrt{|\ln \varepsilon / \ln h|})$  upper bound for the error in the approximate solution in a semi-discrete energy norm is established. It is shown that the error bound is almost independent of  $\varepsilon$ . We comment that our approach is completely different from the coupled method in [19]. The latter is based on the discretization of the diffusion term by the finite element method and the convection term by a finite volume method. Also, [19] does not take into consideration the uniformity in  $\varepsilon$ , and thus the resulting error estimate depends strongly on  $\varepsilon$ . Although the problem considered in this paper is linear, the idea can be used for solving nonlinear convection-diffusion problems. This will be one of our future research topics. Theoretically the method needs a priori knowledge of the layer locations. When this information is not available a priori, a mesh adaption technique can be used in conjunction with the method so that layers' locations can be identified approximately from the numerical solution of the previous step.

The rest of the paper is organized as follows.

The continuous problem and some preliminaries are described in the next section. The finite element formulation is discussed in Section 3. This formulation is a combination of those of the conventional Galerkin piecewise linear finite element

and the Petrov-Galerkin exponentially fitted finite element proposed in [14]. The finite element method is globally nonconforming. In Section 4, we will first show that the finite element is uniformly stable by proving that the corresponding bilinear form is coercive. Then, we will show that the approximation error in a semi-discrete energy norm is bounded by  $O(h)$  almost uniformly in  $\varepsilon$ .

## 2. PRELIMINARIES

Consider stationary, linear, convection-diffusion problems of the form

$$(1) \quad -\nabla \cdot \mathbf{f}_u + Gu = F \quad \text{in } \Omega := (0, 1)^2,$$

$$(2) \quad \mathbf{f}_u = \varepsilon \nabla u - \mathbf{a}u,$$

$$(3) \quad u|_{\partial\Omega} = 0,$$

where  $\partial\Omega$  denotes the boundary of  $\Omega$ ,  $\varepsilon > 0$  is a positive parameter,  $\mathbf{a} = (a_1, a_2)$  is a known vector-valued function and  $F$  is a given function.

In what follows  $L^p(S)$  denotes the space of  $p$ -integrable functions on an open and measurable set  $S$  with norm  $\|\cdot\|_{0,p,S}$ , and  $W^{m,p}(\Omega)$  is the usual Sobolev space with norm  $\|\cdot\|_{m,p,S}$  and the  $k$ th order seminorm  $|\cdot|_{k,p,S}$  for any  $1 \leq p < \infty$ , nonnegative integer  $m$  and  $0 \leq k \leq m$ . Obviously  $W^{0,p}(S) = L^p(S)$ . When  $S = \Omega$  we omit the subscript in the above notation. Furthermore, we let  $H^m(\Omega) := W^{m,2}(\Omega)$ ,  $\|\cdot\|_m := \|\cdot\|_{m,2,\Omega}$  and  $|\cdot|_k := |\cdot|_{k,2,\Omega}$ . The inner product on  $L^2(\Omega)$  or on  $L^2(\Omega) := (L^2(\Omega))^2$  is denoted by  $(\cdot, \cdot)$ . We put  $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$ , and the set of functions which together with their derivatives of order  $\leq m$  are continuous on  $\Omega$  (or  $\bar{\Omega}$ ) is denoted by  $C^m(\Omega)$  (or  $C^m(\bar{\Omega})$ ).

For the coefficient functions we assume that  $\mathbf{a} \in (C^1(\bar{\Omega}))^2$ ,  $G \in C(\Omega) \cap H^1(\Omega)$  and  $F \in L^\infty(\Omega)$ . We also assume that  $\mathbf{a}$  satisfies

$$(4) \quad \frac{1}{2} \nabla \cdot \mathbf{a} + G \geq \alpha > 0 \quad \text{in } \Omega$$

for some positive number  $\alpha$ . This condition has been used in many papers and books on uniform convergence analysis, such as [2, 17, 18, 21]. The existence and uniqueness of the solutions to both the continuous and the finite element problem do not need this condition, but it will be used in the proof of the error estimates in Section 5. For simplicity, we assume that each of the components of  $\mathbf{a}$  is bounded below by a positive constant, i.e.,

$$(5) \quad a_1 \geq \alpha_1 > 0, \quad a_2 \geq \alpha_2 > 0 \quad \text{in } \Omega.$$

In this case, the solution to (1)–(3) has two exponential boundary layers of width  $O(\varepsilon)$  at  $x = 1$  and  $y = 1$ . The variational problem corresponding to (1), (2) and (3) is

**Problem 2.1.** Find  $u \in H_0^1(\Omega)$  such that for all  $v \in H_0^1(\Omega)$

$$(6) \quad A(u, v) = (F, v),$$

where  $A(\cdot, \cdot)$  is a bilinear form on  $(H_0^1(\Omega))^2$  defined by

$$(7) \quad A(u, v) = (\varepsilon \nabla u - \mathbf{a}u, \nabla v).$$

Let  $\|\cdot\|_\varepsilon$  be a functional on  $H_0^1(\Omega)$  defined by  $\|v\|_\varepsilon = (A(v, v))^{1/2}$ . Then it is easy to show that (cf., for example, [14])

$$\|v\|_\varepsilon^2 = (\varepsilon \nabla v, \nabla v) + \left( \left( \frac{1}{2} \nabla \cdot \mathbf{a} + G \right) v, v \right).$$

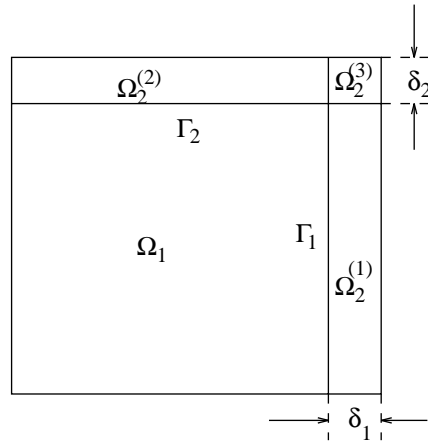


FIGURE 1. Subdomains  $\Omega_1$  and  $\Omega_2 = \Omega_2^{(1)} \cup \Omega_2^{(2)} \cup \Omega_2^{(3)}$ ,  $\Gamma = \Gamma_1 \cup \Gamma_2$ .

Thus,  $\|\cdot\|_\varepsilon$  is a norm on  $H_0^1(\Omega)$ , because  $\frac{1}{2}\nabla \cdot \mathbf{a} + G \geq 0$  by (4) and  $(\nabla u, \nabla v)$  is a norm on  $H_0^1(\Omega)$  by the well-known Poincaré-Friedrichs inequality. Now, from the definition of the norm we have

$$(8) \quad A(u, u) = \|u\|_\varepsilon^2, \quad \forall u \in H_0^1(\Omega).$$

This implies that  $A(\cdot, \cdot)$  is coercive on  $H_0^1(\Omega)$ , and thus, by the well-known Lax-Milgram lemma, Problem 2.1 has a unique solution in  $H_0^1(\Omega)$ .

Because of (5), the solution to Problem 2.1 has two boundary layers of width  $O(\varepsilon)$  at  $x = 1$  and  $y = 1$  respectively. Thus we divide the solution region  $\Omega$  into two parts  $\Omega_1$  and  $\Omega_2$ , given respectively by

$$\Omega_1 = (0, 1 - \delta_1) \times (0, 1 - \delta_2), \quad \Omega_2 = (1 - \delta_1, 1) \times (0, 1) \cup (0, 1 - \delta_1) \times (1 - \delta_2, 1),$$

with  $\delta_1, \delta_2 \in (0, 1)$  (cf. Figure 1). Obviously  $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$ . The region  $\Omega_2$  consists of the three subregions

$$\begin{aligned} \Omega_2^{(1)} &= (1 - \delta_1, 1) \times (0, 1 - \delta_2), \\ \Omega_2^{(2)} &= (0, 1 - \delta_1) \times (1 - \delta_2, 1), \\ \Omega_2^{(3)} &= (1 - \delta_1, 1) \times (1 - \delta_2, 1). \end{aligned}$$

The choice of the transition parameters  $\delta_1$  and  $\delta_2$  is rather arbitrary, but it is required that  $\Omega_2$  contains the boundary layers and  $\delta_1, \delta_2 = O(\varepsilon)$ . One choice is

$$(9) \quad \delta_1 = \frac{\beta}{\alpha_1} \varepsilon \ln(1/\varepsilon) \quad \text{and} \quad \delta_2 = \frac{\beta}{\alpha_2} \varepsilon \ln(1/\varepsilon),$$

where  $\beta \geq 1$  is a positive constant (cf., for example, [18]). We let  $\Gamma_1 = \bar{\Omega}_1 \cap \bar{\Omega}_2^{(1)}$  and  $\Gamma_2 = \bar{\Omega}_1 \cap \bar{\Omega}_2^{(2)}$ , and put  $\Gamma = \Gamma_1 \cup \Gamma_2$ . In the rest of this paper  $\Gamma$  is sometimes regarded as an oriented curve. If  $\Gamma$  is oriented counterclockwise, then it is denoted as  $\Gamma^+$ . Otherwise, we use  $\Gamma^-$  to denote it.

Now we need to make the following assumption on the solution to Problem 2.1:

**Assumption 2.2.** The solution  $u$  to Problem 2.1 has the representation

$$(10) \quad u = U_1 + U_2 + U_3 + U_4,$$

where  $U_1$  satisfies

$$(11) \quad \|U_1\|_{k,\infty,\Omega} \leq C \quad \text{for } k = 0, 1, 2,$$

and  $U_2, U_3$  and  $U_4$  satisfy

$$(12) \quad \left| \frac{\partial^{i+j} U_2}{\partial x^i \partial y^j} \right| \leq C \varepsilon^{-i} \exp\left(-\frac{\alpha_1(1-x)}{\varepsilon}\right),$$

$$(13) \quad \left| \frac{\partial^{i+j} U_3}{\partial x^i \partial y^j} \right| \leq C \varepsilon^{-j} \exp\left(-\frac{\alpha_2(1-y)}{\varepsilon}\right),$$

$$(14) \quad \left| \frac{\partial^{i+j} U_4}{\partial x^i \partial y^j} \right| \leq C \varepsilon^{-(i+j)} \exp\left(-\frac{\alpha_1(1-x)}{\varepsilon}\right) \exp\left(-\frac{\alpha_2(1-y)}{\varepsilon}\right),$$

for  $0 \leq i + j \leq 2$  and some positive constant  $C$ .

This assumption shows that the solution  $u$  is decomposed into 4 parts,  $U_i$  ( $i = 1, 2, 3, 4$ ). The part  $U_1$  is globally smooth and uniformly bounded, while  $U_2, U_3$  and  $U_4$  contain layers in  $\Omega_2^{(1)}, \Omega_2^{(2)}$  and  $\Omega_2^{(3)}$  respectively. Sufficient conditions for the existence of this decomposition have been discussed in various papers and books such as [6], [16] and [22], but necessary and sufficient conditions are unknown. The following theorem shows that  $u$  and all its first and second partial derivatives are uniformly bounded in  $\Omega_1$ .

**Theorem 2.3.** *If  $\beta \geq 3/2$ , then the solution  $u$  to Problem 2.1 satisfies*

$$(15) \quad \|u\|_{i,\Omega_1} \leq C, \quad i = 0, 1, 2,$$

for some positive constants  $C$ , independent of  $u$ . Furthermore, if  $\beta \geq 2$ , then

$$(16) \quad \|u\|_{i,\infty,\Omega_1} \leq C, \quad i = 0, 1, 2.$$

*Proof.* Let  $C$  be a generic positive constant independent of  $u$  and  $\varepsilon$ . We first prove (15). We only show that  $\|u\|_{2,\Omega_1} \leq C$ . The other cases are similar to this one. From (10) we see that  $u$  is decomposed into the sum of  $U_i$  ( $i = 1, 2, 3, 4$ ). Thus it suffices to show that  $\|U_i\|_{2,\Omega_1} \leq C$  for  $i = 1, 2, 3, 4$ . We now prove that  $\|\partial^2 U_2 / \partial x^2\|_{0,\Omega_1} \leq C$ .

From the definition of  $\Omega_1$  and (12) we have

$$\begin{aligned} \|\partial^2 U_2 / \partial x^2\|_{0,\Omega_1}^2 &\leq C \varepsilon^{-4} \int_{\Omega_1} \exp\left(-\frac{2\alpha_1}{\varepsilon}(1-x)\right) dx \\ &= \frac{C(1-\delta_2)}{\varepsilon^4} e^{-2\alpha_1/\varepsilon} \left. \frac{e^{2\alpha_1 x/\varepsilon}}{2\alpha_1/\varepsilon} \right|_0^{1-\delta_1} \\ &= \frac{C(1-\delta_2)}{2\alpha_1 \varepsilon^3} \left( e^{-2\alpha_1 \delta_1/\varepsilon} - e^{-2\alpha_1/\varepsilon} \right) \\ &= C \frac{1-\delta_2}{2\alpha_1 \varepsilon^3} \left( e^{-2\beta \ln \frac{1}{\varepsilon}} - e^{-2\alpha_1/\varepsilon} \right) \\ &= \frac{C(1-\delta_2)}{2\alpha_1} \left( \varepsilon^{2\beta-3} - \varepsilon^{-3} e^{-2\alpha_1/\varepsilon} \right). \end{aligned}$$

Note that  $\varepsilon^{-3}e^{-2\alpha_1/\varepsilon}$  is uniformly bounded above for all  $0 < \varepsilon \leq O(1)$ , since  $\alpha_1 > 0$ . Thus if  $\beta \geq 3/2$ , we have from the above inequality

$$\|\partial^2 U_2 / \partial x^2\|_{0, \Omega_1}^2 \leq C.$$

The proof of boundedness for the first and second seminorms of all other terms are analogous. Therefore we have shown (15).

We now show (16). Again we take  $\partial^2 U_2 / \partial x^2$  for example. Proofs for other terms are similar. For any fixed  $\varepsilon$ , the function in the right side of (12) is increasing in  $x$ . Thus we have

$$\|\partial^2 U_2 / \partial x^2\|_{0, \infty, \Omega_1} \leq C\varepsilon^{-2}e^{-\alpha_1 \delta_1 / \varepsilon} = C\varepsilon^{\beta-2}.$$

The last term in the above is uniformly bounded for all  $\beta \geq 2$ . This completes the proof for (16). □

### 3. THE FINITE ELEMENT FORMULATION

To formulate the finite element method, we first define a mesh for the solution region  $\Omega$  which is a combination of regular triangles in  $\Omega_1$  and rectangles in  $\Omega_2$ . Let  $T_h^{(1)}$  denote a triangular mesh on  $\Omega_1$  with each triangle  $t$  having diameter  $h_t \leq h$ . For all  $0 < h < \dim(\Omega_1)$ ,  $\{T_h^{(1)}\}$  forms a family of triangular meshes on  $\Omega_1$ . For any  $t \in T_h^{(1)}$ , let  $\rho_t$  denote the diameter of the incircle of  $t$ . Then, we assume that this family satisfies

$$\max_{t \in T_h^{(1)}} \frac{h_t}{\rho_t} \leq \gamma \quad \forall h \in (0, \dim(\Omega_1))$$

for some positive constant  $\gamma$ . In this case,  $\{T_h^{(1)}\}$  is said to be a regular family of triangular meshes. Now, the set of vertices of  $T_h^{(1)}$  not on  $\partial\Omega$  is denoted  $\{x_i\}_1^{N_1}$ . We assume that the number of vertices on  $\Gamma$  not on  $\partial\Omega$  is  $N_\Gamma$ . Note that  $\Omega_2$  contains two thin overlapped stripes  $\Omega_2^{(1)} \cup \Omega_2^{(3)}$  and  $\Omega_2^{(2)} \cup \Omega_2^{(3)}$  with widths  $\delta_1$  and  $\delta_2$  respectively. Thus we divide these two strips into rectangles so that the resulting mesh is uniform along the  $x$ -axis with  $M_1$  subintervals in the former subregion and along the  $y$ -axis with  $M_2$  subintervals in the latter (cf. Figure 2). We also require that the mesh points on  $\Gamma$  match those from  $T_h^{(1)}$ . This mesh is denoted by  $T_h^{(2)}$ . Without loss of generality, we assume that the vertices in  $T_h^{(2)}$  not on  $\partial\Omega$  are numbered from  $N_1 + 1$  to  $N_1 + N_2$ . The set of edges of  $T_h^{(2)}$  not on  $\partial\Omega$  is denoted by  $E_h^{(2)}$ . Obviously all rectangles in  $\Omega_2^{(1)}$  and  $\Omega_2^{(2)}$  have lengths  $O(h)$  and widths either  $\delta_1/M_1$  or  $\delta_2/M_2$ , and rectangles in  $\Omega_2^{(3)}$  have length  $\delta_1/M_1$  and width  $\delta_2/M_2$ . The meshes  $T_h^{(1)}$  and  $T_h^{(2)}$  form a conforming mesh on  $\Omega$ , and we denote it by  $T_h$ . A typical case is depicted in Figure 2.

Along with  $T_h^{(2)}$ , we define two meshes dual to it. The first dual mesh, denoted by  $D_h^{(2)}$ , is the Dirichlet tessellation associated with the mesh nodes in  $T_h^{(2)}$ , i.e., the element  $d_i \in D_h^{(2)}$  associated with the node  $x_i$  of  $T_h^{(2)}$  is given by

$$(17) \quad d_i = \{x \in \Omega_2 : |x - x_i| < |x - x_j|, i \neq j\}$$

for any other node  $x_j$  of the mesh  $T_h^{(2)}$ . For each edge in  $T_h^{(2)}$  not on  $\partial\Omega$ , we construct a quadrilateral element by connecting the two end-points of the edge and the mid-points of the rectangles (or rectangle, if the edge is on  $\Gamma$ ) sharing the edge (see Figure 2). All these quadrilaterals form the second dual mesh, denoted by

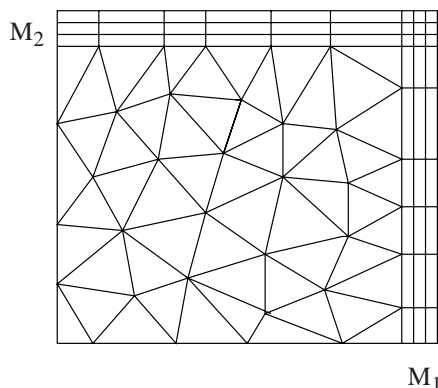


FIGURE 2. A typical hybrid mesh for  $\Omega$ .

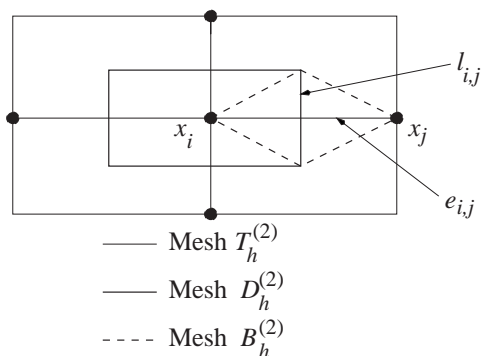


FIGURE 3. Elements and edges associated with the node  $x_i$ .

$B_h^{(2)}$ . An example of these nested meshes is depicted in Figure 3. For a detailed description of the constructions of the dual meshes, we refer to [14], Section 3.

Using the meshes defined above, we now construct finite element trial and test spaces. Let  $U_h^{(1)}$  denote the conventional piecewise linear finite element space of dimension  $N_1$  constructed on the partition  $T_h^{(1)}$ . In our formulation below, we will use  $U_h^{(1)}$  as both trial and test spaces in  $\Omega_1$ . Corresponding to  $T_h^{(2)}$ , we construct a test space and a trial space in the same way as in Section 3 of [14]. The test space is chosen to be  $V_h^{(2)} = \text{span}\{\xi_i\}$ , where  $\xi_i$  is piecewise constant, given by

$$\xi_i = \begin{cases} 1 & \text{on } d_i, \\ 0 & \text{otherwise.} \end{cases}$$

To construct the trial space  $U_h^{(2)}$ , we follow the discussion in [14] based on the idea of exponential fitting proposed independently in [1] and [20]. For each  $e_{i,j} \in E_h^{(2)}$  connecting the two neighbouring nodes  $x_i$  and  $x_j$ , we define an exponential function

$\phi_{i,j}$  on  $e_{i,j}$  by

$$(18) \quad \begin{aligned} \frac{d}{de_{i,j}}(\varepsilon \frac{d\phi_{i,j}}{de_{i,j}} - \bar{a}_{i,j}\phi_{i,j}) &= 0, \\ \phi_{i,j}(x_i) &= 1, \quad \phi_{i,j}(x_j) = 0, \end{aligned}$$

where  $e_{i,j}$  denotes the unit vector from  $x_i$  to  $x_j$  and  $\bar{a}_{i,j}$  is a constant approximation to  $\mathbf{a} \cdot \mathbf{e}_{i,j}$  on  $e_{i,j}$  such that the mapping  $\mathbf{a} \cdot \mathbf{e}_{i,j} \mapsto \bar{a}_{i,j}$  from  $C(e_{i,j}) \mapsto P_0(e_{i,j})$  preserves constants (e.g.,  $\bar{a}_{i,j} = (\mathbf{a}(x_i) + \mathbf{a}(x_j)) \cdot \mathbf{e}_{i,j}/2$ ), where  $C(e_{i,j})$  and  $P_0(e_{i,j})$  denote respectively the spaces of all continuous functions and all 0th order polynomials on  $e_{i,j}$ . The above linear, constant coefficient two-point boundary value problem can be solved exactly, yielding the local 1D basis function  $\phi_{i,j}$  on the edge  $e_{i,j}$ . We then extend  $\phi_{i,j}$  to  $b_{i,j}$  by defining it to be constant along perpendiculars to  $e_{i,j}$ . Using this exponential function, we define a global basis function for  $U_h^{(2)}$  on  $\Omega$  as follows:

$$\phi_i = \begin{cases} \phi_{i,j} & \text{on } b_{i,j} \text{ if } j \in I_i, \\ 0 & \text{otherwise,} \end{cases}$$

where  $b_{i,j}$  denotes the element of  $B_h$  containing  $e_{i,j}$  and

$$(19) \quad I_i = \{j : e_{i,j} \in E_h\}$$

denotes the index set of all neighbour nodes of  $x_i$ . The support of  $\phi_i$  is star-shaped. We put  $U_h^{(2)} = \text{span}\{\phi_i\}_{i=1}^{N'}$ . Obviously we have  $U_h^{(2)} \subset L^2(\Omega)$ , and thus the trial space is nonconforming. This finite element space has the property that for any sufficiently smooth function  $u$ , the projection of the flux of the  $U_h^{(2)}$ -interpolant  $u_I$  of  $u$  on  $e_{i,j}$  satisfies

$$(20) \quad f_{i,j} := \varepsilon \frac{du_I}{de_{i,j}} - \bar{a}_{i,j}u_I = \frac{\varepsilon}{|e_{i,j}|} (B(\frac{\bar{a}_{i,j}|e_{i,j}|}{\varepsilon})u_j - B(-\frac{\bar{a}_{i,j}|e_{i,j}|}{\varepsilon})u_i)$$

on the edge  $e_{i,j}$ , where  $B$  denotes the Bernoulli function defined by

$$(21) \quad B(x) = \begin{cases} x/(e^x - 1), & x \neq 0, \\ 1, & x = 1. \end{cases}$$

Also, the approximation error in  $f_{i,j}$  satisfies

$$(22) \quad \|\mathbf{f}_u \cdot \mathbf{e}_{i,j} - f_{i,j}\|_{\infty, e_{i,j}} \leq C(\|\mathbf{f}_u\|_{1, \infty, e_{i,j}} + \|\mathbf{a}\|_{1, \infty, e_{i,j}} \|u\|_{\infty, e_{i,j}}),$$

where  $C$  is a positive constant independent of  $h, u$  and  $\varepsilon$ . For a detailed discussion, we refer to [14].

Now we choose the global trial and test spaces to be  $U_h = U_h^{(1)} \oplus U_h^{(2)}$  and  $V_h = U_h^{(1)} \oplus V_h^{(2)}$  respectively. Obviously both  $U_h$  and  $V_h$  are nonconforming, as they are not continuous across  $\Gamma$ . To define the discrete problem using  $U_h$  and  $V_h$ , it is convenient to introduce some notation. Let  $l_{i,j} = \partial d_i \cap \partial d_j$  denote the intersection of the boundaries of  $d_i$  and  $d_j$  (cf. Figure 3). Clearly  $|l_{i,j}| = \frac{2|b_{i,j}|}{|e_{i,j}|}$  if  $j \in I_i$ , and  $|l_{i,j}| = 0$  otherwise. Corresponding to each  $l_{i,j}$ , we introduce a unit vector  $\mathbf{l}_{i,j}$  directed so that  $\arg(\mathbf{l}_{i,j}) = \arg(\mathbf{e}_{i,j}) + \pi/2$ . For convenience we let  $\hat{\mathbf{a}}$  be the approximation of  $\mathbf{a}$  defined on  $\bar{\Omega}_2$  such that, for all  $b_{i,j} \in B_h$ ,

$$(23) \quad \hat{\mathbf{a}}|_{b_{i,j}} = \bar{a}_{i,j}\mathbf{e}_{i,j} + \bar{a}_{i,j}^\perp \mathbf{l}_{i,j},$$

where  $\bar{a}_{i,j}$  is the constant used in (18) and  $\bar{a}_{i,j}^\perp = \sup_{x \in e_{i,j}} \mathbf{a}(x) \cdot \mathbf{l}_{i,j}$ . Obviously,  $\hat{\mathbf{a}}$  is a piecewise constant approximation to  $\mathbf{a}$  on  $\Omega_2$ . Note that the component  $\bar{a}_{i,j}^\perp$  will make a contribution to the finite element formulation only on the integrals along

$\Gamma^-$  that appear in the bilinear form  $a_2(\cdot, \cdot)$  defined below. This contribution will be used in the proof of coercivity of the bilinear form in the next section. Before defining the finite element problem, we first introduce the mass lumping operator  $P : C(\bar{\Omega}_2) \mapsto V_h^{(2)}$  such that

$$(24) \quad P(w)(x) = \sum_{i=N_1+1}^{N_2} w(x_i) \xi_i(x), \quad x \in \bar{\Omega}_2,$$

for all  $w \in C(\bar{\Omega}_2)$ . When restricted to  $U_h^{(2)}$ , it is easy to show that the mapping  $P$  is one-to-one from  $U_h^{(2)}$  to  $V_h^{(2)}$ . Using this mapping, we define the following finite element problem:

**Problem 3.1.** Find  $u_h \in U_h$  such that

$$(25) \quad a(u_h, v_h) := a_1(u_h, v_h) + a_2(u_h, v_h) = (F, v_h), \quad \forall v_h \in V_h,$$

where  $a_1(\cdot, \cdot)$  and  $a_2(\cdot, \cdot)$  are bilinear forms on  $\Omega_1$  and  $\Omega_2$  defined by

$$(26) \quad a_1(u_h, v_h) = \int_{\Omega_1} (\varepsilon \nabla u_h - \mathbf{a} u_h) \cdot \nabla v_h dx + (G u_h, v_h)_{\Omega_1},$$

$$(27) \quad a_2(u_h, v_h) = - \sum_{d \in D_h^{(2)}} \int_{\partial d \setminus \partial \Omega_2} (\varepsilon \nabla u_h - \hat{\mathbf{a}} u_h) \cdot \mathbf{n} v_h|_d ds + (P(G u_h), v_h)_{\Omega_2}.$$

Here  $v_h|_d$  denotes the restriction of  $v_h$  to  $d$ .

It is clear that the bilinear form  $a(\cdot, \cdot)$  is a combination of two terms. The former is the standard bilinear form on  $\Omega_1$ , and the latter is a nonstandard Petrov-Galerkin form on  $\Omega_2$  originally defined in [14]. It has been shown in [14] that, using the mass lumping operator  $P$  defined in (24), the second bilinear form  $a_2(u_h, v_h)$  can be transformed into a Bubnov-Galerkin formulation, yielding the following Bubnov-Galerkin problem corresponding to Problem 3.1:

**Problem 3.2.** Find  $u_h \in U_h$  such that

$$(28) \quad b(u_h, v_h) = (F, v_h)_{\Omega_1} + (F, P(v_h))_{\Omega_2}, \quad \forall v_h \in U_h,$$

where  $(\cdot, \cdot)_{\Omega_k}$  denotes the inner product on  $L^2(\Omega_k)$  for  $k = 1, 2$  and  $b(\cdot, \cdot)$  is a bilinear form on  $U_h \times U_h$  defined by

$$(29) \quad b(u_h, v_h) := a_1(u_h, v_h) + a_2(u_h, P(v_h)).$$

#### 4. COERCIVITY OF THE BILINEAR FORM $b(\cdot, \cdot)$

In this section we show that the bilinear form on the right side of (29) is coercive. This implies that Problem 3.2 is uniquely solvable and the finite element formulation is stable, irrespectively of  $\varepsilon$ . The coercivity result is also the foundation for the proof of convergence of the solutions to Problem 3.2. We now consider the two parts of  $a(\cdot, \cdot)$  separately.

Before further discussion, we make the following assumption:

**Assumption 4.1.** Let the mesh  $T_h^{(2)}$  be sufficiently fine such that the inequality

$$(30) \quad \frac{1}{2} \int_{\partial d_i} \hat{\mathbf{a}} \cdot \mathbf{n} ds + G(x_i) |d_i| \geq \alpha_0 > 0$$

holds for all  $d_i \in D_h^{(2)}$ , where  $\hat{\mathbf{a}}$  is the approximation of  $\mathbf{a}$  defined in (23) and  $x_i$  denotes the mesh node contained in  $d_i$ .

We comment that (30) is essentially a discrete analogue of (4). In fact, it can be obtained by integrating (4) over  $d_i \in D_h^{(2)}$ , integrating the first term by parts, and then approximating  $\mathbf{a}$  and  $G$  by  $\hat{\mathbf{a}}$  and  $G(x_i)$ . Since the errors in the quadrature approximations are of order  $O(h)$  and  $\alpha$  in (4) is a positive constant, (30) will always be satisfied when  $h$  is sufficiently small.

Furthermore, since all the mesh lines in  $T_h^{(2)}$  are parallel to one of the axes and  $\mathbf{a}$  satisfies (5), it is obvious that

$$(31) \quad \min_{e_{i,j} \in E_h^{(2)}} |\hat{\mathbf{a}} \cdot \mathbf{e}_{i,j}| = \min_{e_{i,j} \in E_h^{(2)}} |a_{i,j}| \geq \min\{\alpha_1, \alpha_2\}.$$

Note that, when restricted to  $\Omega_1$ , the test and trial spaces are equal to each other, and thus from (26) we have

$$(32) \quad \begin{aligned} a_1(u, u) &= \int_{\Omega_1} (\varepsilon \nabla u - \mathbf{a}u) \cdot \nabla u \, dx + \int_{\Omega_1} Gu^2 \, dx \\ &= \varepsilon \|\nabla u\|_{0,\Omega_1}^2 - \int_{\Omega_1} \mathbf{a}u \cdot \nabla u \, dx + (Gu, u)_{\Omega_1} \end{aligned}$$

for any  $u \in U_h$ . Observing that  $u \nabla u = \frac{1}{2} \nabla(u^2)$ , a direct application of the Gauss-Green-Stokes formula then yields

$$-\frac{1}{2} \int_{\Omega_1} \mathbf{a} \cdot \nabla(u^2) \, dx = -\frac{1}{2} \int_{\partial\Omega_1} u^2 \mathbf{a} \cdot \mathbf{n} \, ds + \frac{1}{2} u^2 \nabla \cdot \mathbf{a} \, dx.$$

Substituting the above into (32), we obtain, using (4),

$$(33) \quad \begin{aligned} a_1(u, u) &= \varepsilon \|\nabla u\|_{0,\Omega_1}^2 + \frac{1}{2} (\nabla \cdot \mathbf{a}u, u)_{\Omega_1} - \frac{1}{2} \int_{\partial\Omega_1} u^2 \mathbf{a} \cdot \mathbf{n} \, ds \\ &= \varepsilon \|\nabla u\|_{0,\Omega_1}^2 + ((\frac{1}{2} \nabla \cdot \mathbf{a} + G)u, u)_{\Omega_1} - \frac{1}{2} \int_{\Gamma^+} u^2 \mathbf{a} \cdot \mathbf{n} \, ds \\ &\geq \varepsilon \|\nabla u\|_{0,\Omega_1}^2 + \alpha \|u\|_{0,\Omega_1}^2 - \frac{1}{2} \int_{\Gamma^+} u^2 \mathbf{a} \cdot \mathbf{n} \, ds, \end{aligned}$$

since  $u = 0$  on  $\partial\Omega_1 \setminus \Gamma^+$ .

Now, for any  $u \in U_h$  it is shown in Section 4 of [14] that

$$(34) \quad \begin{aligned} a_2(u, P(u)) &= - \sum_{d_i \in D_h^{(2)}} \int_{\partial d_i \setminus \partial\Omega_N} (\varepsilon \nabla u_h - \hat{\mathbf{a}}u_h) \cdot \mathbf{n} P(u_h) \, ds + \sum_{d_i \in D_h^{(2)}} G(x_i) u_i^2 |d_i| \\ &= \sum_{e_{i,j} \in E_h^{(2)}} \sigma_{i,j} B(\rho_{i,j}) (1 + e^{\rho_{i,j}}) (u_j - u_i)^2 \frac{|b_{i,j}|}{|e_{i,j}|} \\ &\quad + \sum_{e_{i,j} \in E_h^{(2)}} \frac{\bar{a}_{i,j}}{2} (u_i^2 - u_j^2) |l_{i,j}| + \sum_{d_i \in D_h^{(2)}} G(x_i) u_i^2 |d_i|, \end{aligned}$$

where  $l_{i,j} = \partial d_i \cap \partial d_j$ ,  $\sigma_{i,j} = \varepsilon / |e_{i,j}|$ ,  $\rho_{i,j} = \bar{a}_{i,j} / \sigma_{i,j}$  and  $B(\cdot)$  is the Bernoulli function defined in (21). Transforming from a summation over the edges to a

summation over the nodes of  $T_h^{(2)}$ , the last two terms in the right-hand side of (34) can be written as

$$\begin{aligned}
 & \sum_{e_{i,j} \in E_h^{(2)}} \frac{\bar{a}_{i,j}}{2} (u_i^2 - u_j^2) |l_{i,j}| + \sum_{d_i \in D_h^{(2)}} G(x_i) u_i^2 |d_i| \\
 &= \frac{1}{2} \sum_{d_i \in D_h^{(2)}} u_i^2 \sum_{j \in I_i} \bar{a}_{i,j} |l_{i,j}| + \sum_{d_i \in D_h^{(2)}} G(x_i) u_i^2 |d_i| \\
 &= \frac{1}{2} \sum_{d_i \in D_h^{(2)}} u_i^2 \int_{\partial d_i \setminus \Gamma} \hat{\mathbf{a}} \cdot \mathbf{n} ds + \sum_{d_i \in D_h^{(2)}} G(x_i) u_i^2 |d_i| \\
 (35) \quad &= \sum_{d_i \in D_h^{(2)}} u_i^2 \left( \frac{1}{2} \int_{\partial d_i} \hat{\mathbf{a}} \cdot \mathbf{n} ds + G(x_i) |d_i| \right) \\
 &\quad - \frac{1}{2} \sum_{d_i \in D_h^{(2)}} u_i^2 \int_{\partial d_i \cap \Gamma^-} \hat{\mathbf{a}} \cdot \mathbf{n} ds \\
 &\geq \alpha_0 \sum_{d_i \in D_h^{(2)}} u_i^2 - \frac{1}{2} \sum_{d_i \in D_h^{(2)}} u_i^2 \int_{\partial d_i \cap \Gamma^-} \hat{\mathbf{a}} \cdot \mathbf{n} ds,
 \end{aligned}$$

because of (30).

Using the definition of the Bernoulli function  $B(\cdot)$  in (21), we have

$$\sigma_{i,j} B(\rho_{i,j}) (1 + e^{\rho_{i,j}}) = \bar{a}_{i,j} \frac{e^{\rho_{i,j}} + 1}{e^{\rho_{i,j}} - 1} \geq |\bar{a}_{i,j}|.$$

Using this inequality, (35), and (34), we obtain

$$\begin{aligned}
 a_2(u, P(u)) &\geq C \left( \sum_{e_{i,j} \in E_h^{(2)}} |e_{i,j}| \left( \frac{u_j - u_i}{|e_{i,j}|} \right)^2 |b_{i,j}| + \sum_{d_i \in D_h^{(2)}} u_i^2 \right) \\
 (36) \quad &\quad - \frac{1}{2} \sum_{d_i \in D_h^{(2)}} u_i^2 \int_{\partial d_i \cap \Gamma^-} \mathbf{a} \cdot \mathbf{n} ds.
 \end{aligned}$$

Let  $\|\cdot\|$  be a functional on the finite element space  $U_h$  defined by

$$(37) \quad \|v\| = (\|v\|_{\varepsilon, \Omega_1}^2 + \|v\|_{h, \Omega_2}^2)^{1/2},$$

where

$$(38) \quad \|v\|_{\varepsilon, \Omega_1}^2 = \varepsilon \|\nabla v\|_{0, \Omega_1}^2 + \|v\|_{0, \Omega_1}^2,$$

$$(39) \quad \|v\|_{h, \Omega_2}^2 = \sum_{e_{i,j} \in E_h^{(2)}} |e_{i,j}| \left( \frac{v_j - v_i}{|e_{i,j}|} \right)^2 |b_{i,j}| + \sum_{d_i \in D_h^{(2)}} v_i^2.$$

It is easy to see that  $\|\cdot\|$  is a semi-discrete energy norm on  $U_h$  consisting of a continuous  $H^1$ -norm on  $\Omega_1$  and a discrete  $H^1$ -norm on  $\Omega_2$ . Using this norm, we have the following theorem:

**Theorem 4.2.** *Let Assumption 4.1 be fulfilled. Then, for all  $u \in U_h$ ,*

$$(40) \quad b(u, u) \geq C \|u\|^2,$$

where  $C$  denotes a generic positive constant independent of  $\varepsilon$ ,  $h$  and  $u$ .

*Proof.* For any  $u \in U_h$ , using (33) and (36) we have

$$\begin{aligned}
 b(u, u) &= a_1(u, u) + a_2(u, P(u)) \\
 &\geq \varepsilon \|\nabla u\|_{0, \Omega_1}^2 + \|u\|_{0, \Omega_1}^2 - \frac{1}{2} \int_{\Gamma^+} u^2 \mathbf{a} \cdot \mathbf{n} ds \\
 &\quad + C \left( \sum_{e_{i,j} \in E_h^{(2)}} |e_{i,j}| \left( \frac{u_j - u_i}{|e_{i,j}|} \right)^2 |b_{i,j}| + \sum_{d_i \in D_h^{(2)}} u_i^2 \right) \\
 &\quad - \frac{1}{2} \sum_{d_i \in D_h^{(2)}} u_i^2 \int_{\partial d_i \cap \Gamma^-} \hat{\mathbf{a}} \cdot \mathbf{n} ds \\
 &\geq C \|u\|^2 - \frac{1}{2} \int_{\Gamma^+} u^2 \mathbf{a} \cdot \mathbf{n} ds - \frac{1}{2} \sum_{d_i \in D_h^{(2)}} u_i^2 \int_{\partial d_i \cap \Gamma^-} \hat{\mathbf{a}} \cdot \mathbf{n} ds \\
 &=: C \|u\|^2 + \frac{1}{2} J,
 \end{aligned}$$

where  $J$  denotes the sum of the two line integrals. It remains to show that  $J$  is nonnegative. Let  $\bar{u}(x)$  denote the piecewise constant function on  $\Gamma$  defined by  $\bar{u}(x) = u_i$  if  $x \in \partial d_i \cap \Gamma$ , i.e.,  $\bar{u}$  is the restriction of  $P(u|_{\Omega_2})$  to  $\Gamma$ . Using  $\bar{u}$  and transforming the second term of  $J$  from the summation over the boundaries of Dirichlet tiles to a summation over the edges of triangles, we have

$$J = \sum_{k=1}^{N_\Gamma} \int_{\Gamma_k^+} (\hat{\mathbf{a}} \bar{u}^2 - \mathbf{a} u^2) \cdot \mathbf{n} ds,$$

where  $\Gamma_k^+$  denotes an edge of the mesh on  $\Gamma^+$  and  $N_\Gamma$  is the number of edges on  $\Gamma$  defined in the previous section. Note that  $\Gamma^+$  is the positive side of  $\Gamma$ , and its normal direction  $\mathbf{n}$  is the same as the outward normal direction of  $\Omega_1$ . Thus,  $\mathbf{a} \cdot \mathbf{n} \geq \alpha_1$  or  $\alpha_2$  on  $\Gamma$ . Let us now consider the integral along a typical edge  $\Gamma_k := e_{i,j}$  on  $\Gamma^+$  connecting the two mesh vertices  $x_i$  and  $x_j$ . On  $\Gamma_k$  we have  $\hat{\mathbf{a}} \cdot \mathbf{n} = \bar{a}_{i,j}^\perp = \sup_{x \in e_{i,j}} \mathbf{a} \cdot \mathbf{n}$  by (23), and  $u$  and  $\bar{u}$  are defined by

$$(41) \quad u = u_i + \frac{|x - x_i|}{|x_j - x_i|} (u_j - u_i)$$

and

$$(42) \quad \bar{u} = \begin{cases} u_i, & x_i \leq x < (x_i + x_j)/2, \\ u_j, & (x_i + x_j)/2 < x \leq x_j. \end{cases}$$

Parametrizing  $e_{i,j}$  as  $\{s : 0 \leq s \leq |e_{i,j}|\}$ , we have

$$\begin{aligned}
 J_{i,j} &:= \int_0^{|e_{i,j}|} (\bar{a}_{i,j}^\perp \bar{u}^2 - \mathbf{a} u^2 \cdot \mathbf{n}) ds \\
 (43) \quad &= \bar{a}_{i,j}^\perp \int_0^{|e_{i,j}|} (\bar{u}^2 - u^2) ds + \int_0^{|e_{i,j}|} (a_{i,j}^\perp - \mathbf{a} \cdot \mathbf{n}) u^2 ds \\
 &\geq \bar{a}_{i,j}^\perp \int_0^{|e_{i,j}|} (\bar{u}^2 - u^2) ds,
 \end{aligned}$$

since  $a_{i,j}^\perp \geq \mathbf{a} \cdot \mathbf{n} > 0$  on  $e_{i,j}$ . Using (41) and (42), it is easily seen that  $\int_0^{|e_{i,j}|} \bar{u}^2 = (u_i^2 + u_j^2) \frac{|e_{i,j}|}{2}$  is equal to the numerical integral of  $\int_0^{|e_{i,j}|} u^2 ds$  by the trapezoidal quadrature rule. Therefore, by a standard argument for the trapezoidal rule there exists  $\xi \in (0, |e_{i,j}|)$  such that

$$\int_0^{|e_{i,j}|} (\bar{u}^2 - u^2) ds = \frac{1}{12} \frac{d^2(u^2)}{ds^2} \Big|_{s=\xi} |e_{i,j}|^3 = \frac{|e_{i,j}|}{6} (u_j - u_i)^2 \geq 0,$$

since  $d^2(u^2)/ds^2 = 2(u_j - u_i)^2/|e_{i,j}|^2$  by (41). Combining the above bound with (43), we obtain  $J_{i,j} \geq 0$ , and so

$$J = \sum_{e_{i,j} \subset \Gamma^+} J_{i,j} \geq 0.$$

This completes the proof.  $\square$

## 5. CONVERGENCE

In this section we establish an upper bound for  $\|u_I - u_h\|$ , where  $u_I$  and  $u_h$  denote respectively the  $U_h$ -interpolant of the solution  $u$  to Problem 2.1 and the solution to Problem 3.1. We start this discussion by proving the following lemma, which will be used in the proof of the main result of this section.

**Lemma 5.1.** *Let Assumptions 2.2 be fulfilled. If  $\beta \geq 3$  in (9) and  $M_1 = M_2 = M$ , a positive integer, then, for any element edge  $e_{i,j} \in E_h^{(2)}$ , there exists a positive integer  $C$ , independent of  $h$ ,  $u$  and  $\varepsilon$ , such that*

$$(44) \quad \int_{l_{i,j}} |\mathbf{f}_u \cdot \mathbf{e}_{i,j} - f_{i,j}| ds \leq \begin{cases} C|l_{i,j}|hK_1, & e_{i,j} \subset \bar{\Omega}_2^{(1)} \cup \bar{\Omega}_2^{(2)}, \\ C|l_{i,j}|M^{-1} \ln(1/\varepsilon), & e_{i,j} \subset \Omega_2^{(3)}, \end{cases}$$

where  $\mathbf{f}_u$  and  $f_{i,j}$  are defined in (2) and (20) respectively and

$$(45) \quad K_1 = \max\{1, h^{-1}\varepsilon^{\beta/2M}, h^{-1}\varepsilon \ln \frac{1}{\varepsilon}\}.$$

*Proof.* From the construction of the mesh on  $\Omega_2$  we see that the mesh lines are parallel to either the  $x$ -axis or the  $y$ -axis. We now discuss these two cases separately.

*Case 1:*  $\mathbf{e}_{i,j} = (1, 0)$ . In this case  $e_{i,j}$  and  $l_{i,j}$  are parallel to the  $x$ -axis and  $y$ -axis respectively, and have lengths either  $M^{-1}\varepsilon \ln \frac{1}{\varepsilon}$  and  $O(h)$  respectively (if  $e_{i,j} \subset \Omega_2^{(1)} \cup \Omega_2^{(3)}$ ), or  $O(h)$  and  $M^{-1}\varepsilon \ln \frac{1}{\varepsilon}$  respectively (if  $e_{i,j} \subset \Omega_2^{(2)}$ ).

$$(46) \quad \begin{aligned} & \int_{l_{i,j}} |\mathbf{f}_u \cdot \mathbf{e}_{i,j} - f_{i,j}| ds \\ & \leq \int_{l_{i,j}} [ |(\mathbf{f}_u - \mathbf{f}_u(x_{i,j})) \cdot \mathbf{e}_{i,j}| + |\mathbf{f}_u(x_{i,j}) \cdot \mathbf{e}_{i,j} - f_{i,j}| ] ds \\ & \leq C \left( |l_{i,j}| \int_{l_{i,j}} \left| \frac{\partial \mathbf{f}_u \cdot \mathbf{e}_{i,j}}{\partial y} \right| ds + |l_{i,j}| |e_{i,j}| \cdot \left\| \frac{\partial \mathbf{f}_u \cdot \mathbf{e}_{i,j}}{\partial x} \right\|_{0,b_{i,j},\infty} \right) \\ & \leq C|l_{i,j}| \left[ \int_{l_{i,j}} \left| \varepsilon \frac{\partial^2 u}{\partial x \partial y} - a_1 \frac{\partial u}{\partial y} \right| ds + |e_{i,j}| \cdot \left\| \varepsilon \frac{\partial^2 u}{\partial x^2} - a_1 \frac{\partial u}{\partial x} \right\|_{0,b_{i,j},\infty} \right] \\ & =: C|l_{i,j}|(I_1 + I_2), \end{aligned}$$

where  $x_{i,j} := (x_i + x_j)/2$ . We discuss the two terms  $I_1$  and  $I_2$  in (46). From (1) and (2) we have

$$-\varepsilon \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} + (\nabla \cdot \mathbf{a} + G)u = F.$$

Using this, Assumption 2.2 and the regularity assumptions on  $\mathbf{a}, G$  and  $F$ , we get

$$\begin{aligned} & \left\| \varepsilon \frac{\partial^2 u}{\partial x^2} - a_1 \frac{\partial u}{\partial x} \right\|_{0,b_{i,j},\infty} \\ & \leq \left\| \varepsilon \frac{\partial^2 u}{\partial y^2} - a_2 \frac{\partial u}{\partial y} \right\|_{0,b_{i,j},\infty} + \|(\nabla \cdot \mathbf{a} + G)u\|_{0,b_{i,j},\infty} + \|F\|_{0,b_{i,j},\infty} \leq C, \end{aligned}$$

if  $e_{i,j} \subset \Omega_2^{(1)}$ . By Theorem 2.3,  $\left\| \varepsilon \frac{\partial^2 u}{\partial x^2} - a_1 \frac{\partial u}{\partial x} \right\|_{0,b_{i,j},\infty}$  is also bounded above by  $C$  when  $e_{i,j} \subset \Omega_2^{(2)}$ , because  $b_{i,j}$  is away from the layer in which the partial derivatives of  $u$  with respect to  $x$  are bounded. When  $e_{i,j} \subset \Omega_2^{(3)}$ , from Assumption 2.2 we have

$$\left\| \varepsilon \frac{\partial^2 u}{\partial x^2} - a_1 \frac{\partial u}{\partial x} \right\|_{0,b_{i,j},\infty} \leq C(\varepsilon\varepsilon^{-2} + \varepsilon^{-1}) = C\varepsilon^{-1}.$$

Therefore, combining all the above estimates for  $\left\| \varepsilon \frac{\partial^2 u}{\partial x^2} - a_1 \frac{\partial u}{\partial x} \right\|_{0,b_{i,j},\infty}$  on  $\Omega_2^{(k)}$ ,  $k = 1, 2, 3$ , and noticing that  $|e_{i,j}| = O(M^{-1}\varepsilon \ln(1/\varepsilon))$  in  $\Omega_2^{(1)}$  and  $\Omega_2^{(3)}$  and  $O(h)$  in  $\Omega_2^{(2)}$ , we obtain

$$(47) \quad I_2 = |e_{i,j}| \cdot \left\| \varepsilon \frac{\partial^2 u}{\partial x^2} - a_1 \frac{\partial u}{\partial x} \right\|_{0,b_{i,j},\infty} \leq \begin{cases} CM^{-1}\varepsilon \ln(1/\varepsilon), & e_{i,j} \subset \Omega_2^{(1)}, \\ Ch, & e_{i,j} \subset \Omega_2^{(2)}, \\ CM^{-1} \ln(1/\varepsilon), & e_{i,j} \subset \Omega_2^{(3)}. \end{cases}$$

We now consider  $I_1$ . When  $e_{i,j} \subset \Omega_2^{(1)}$  we have  $|l_{i,j}| = O(h)$  and so, from Assumption 2.2,

$$I_1 \leq C|l_{i,j}|(\varepsilon\varepsilon^{-1} + 1) \leq Ch.$$

If  $e_{i,j} \subset \Omega_2^{(2)}$ , then  $l_{i,j}$  is perpendicular to the boundary segment  $y = 1$  and has length  $\frac{\beta}{\alpha_2}M^{-1}\varepsilon \ln(1/\varepsilon)$ . So after a proper parametrization and using Assumption 2.2 we have

$$\begin{aligned} (48) \quad I_1 & \leq C\varepsilon^{-1} \int_{y_*}^{y_*+|l_{i,j}|} e^{-\alpha_2(1-y)/\varepsilon} dy \\ & = C\varepsilon^{-1} e^{-\alpha_2/\varepsilon} \frac{e^{\alpha_2 y/\varepsilon}}{\alpha_2/\varepsilon} \Big|_{y_*}^{y_*+|l_{i,j}|} \\ & \leq C e^{-\alpha(1-y_*)/\varepsilon} \left( e^{\alpha|l_{i,j}|/\varepsilon} - 1 \right), \end{aligned}$$

where  $y_*$  denotes the lower end point of  $l_{i,j}$ . From the construction of the mesh we see that  $1 - y_* \geq 3|l_{i,j}|/2$ , i.e., the  $y$ -coordinate of the segment  $l_{i,j}$  closest to  $y = 1$  ranges from  $1 - 3|l_{i,j}|/2$  to  $1 - |l_{i,j}|/2$ , where  $|l_{i,j}| = \delta_2/M$ . Therefore, from (48)

we have

$$\begin{aligned} I_1 &\leq C e^{-3\alpha|l_{i,j}|/(2\varepsilon)} \left( e^{\alpha|l_{i,j}|/\varepsilon} - 1 \right) \\ &\leq C \left( e^{-\alpha|l_{i,j}|/(2\varepsilon)} - e^{-3\alpha|l_{i,j}|/(2\varepsilon)} \right) \\ &\leq C \left( \varepsilon^{\beta/2M} - \varepsilon^{3\beta/2M} \right) \\ &\leq C \varepsilon^{\beta/2M}. \end{aligned}$$

When  $e_{i,j} \subset \Omega_2^{(3)}$ , from Assumption 2.2 we have

$$I_1 \leq C(\varepsilon\varepsilon^{-2} + \varepsilon^{-1})|l_{i,j}| \leq C\varepsilon^{-1}|l_{i,j}| \leq CM^{-1} \ln(1/\varepsilon).$$

Combining the three cases, we obtain

$$(49) \quad I_1 \leq \begin{cases} Ch, & e_{i,j} \subset \Omega_2^{(1)}, \\ C\varepsilon^{\beta/2M}, & e_{i,j} \subset \Omega_2^{(2)}, \\ CM^{-1} \ln(1/\varepsilon), & e_{i,j} \subset \Omega_2^{(3)}. \end{cases}$$

Finally, substituting (49) and (47) into (46), we obtain

$$(50) \quad \int_{l_{i,j}} |\mathbf{f}_u \cdot \mathbf{e}_{i,j} - f_{i,j}| ds \leq \begin{cases} C|l_{i,j}|(h + M^{-1}\varepsilon \ln(1/\varepsilon)), & e_{i,j} \subset \bar{\Omega}_2^{(1)}, \\ C|l_{i,j}|(\varepsilon^{\beta/2M} + h), & e_{i,j} \subset \bar{\Omega}_2^{(2)}, \\ C|l_{i,j}|M^{-1} \ln(1/\varepsilon), & e_{i,j} \subset \Omega_2^{(3)}, \end{cases} \\ \leq \begin{cases} C|l_{i,j}|hK_1, & e_{i,j} \subset \bar{\Omega}_2^{(1)} \cup \bar{\Omega}_2^{(2)}, \\ C|l_{i,j}|M^{-1} \ln(1/\varepsilon), & e_{i,j} \subset \Omega_2^{(3)}, \end{cases}$$

where  $K_1$  is defined in (45).

*Case 2:*  $\mathbf{e}_{i,j} = (0, 1)$ . From the symmetry of the problem it is easily seen that, by the same argument as in Case 1, the estimate (44) is also satisfied. This completes the proof.  $\square$

The error estimate for the method is established in the following theorem.

**Theorem 5.2.** *Let Assumptions 2.2 and 4.1 be fulfilled. If  $\beta \geq 3$  in (9) and  $M_1 = M_2 = M$ , a positive integer, then there exists a positive integer  $C$ , independent of  $h$ ,  $u$  and  $\varepsilon$ , such that*

$$(51) \quad \|u_I - u_h\| \leq Ch \left( K_1^{1/2} + K_2 \right),$$

where  $u_I$  and  $u_h$  denote respectively the  $U_h$ -interpolation of the solution  $u$  to Problem 2.1 and the solution to Problem 3.1,  $K_1$  is defined in (45) and  $K_2$  is defined as

$$(52) \quad K_2 = \max\{M^{1/2}K_1, M^{-1/2}\sqrt{\varepsilon} \ln \frac{1}{\varepsilon}\}.$$

*Proof.* Note that Problems 3.1 and 3.2 are equivalent to each other. For any  $v_h \in V_h$ , subtracting  $a(u_I, v_h)$  from both sides of (25), we have

$$(53) \quad a(u_h - u_I, v_h) = (F, v_h) - a(u_I, v_h).$$

For  $k = 1, 2$ , let  $v_h^{(k)} = v_h|_{\Omega_k}$ . Clearly,  $v_h^{(1)}$  and  $v_h^{(2)}$  are in  $V_h^{(1)}$  and  $V_h^{(2)}$  respectively. Multiplying (1) by  $v_h$  and integrating by parts, we get

$$\begin{aligned} (F, v_h) &= (\mathbf{f}_u, \nabla v_h^{(1)})_{\Omega_1} + (Gu, v_h^{(1)})_{\Omega_1} - \sum_{d \in D_h^{(2)}} \int_{\partial d \setminus \Gamma^-} v_h^{(2)} \mathbf{f}_u \cdot \mathbf{n} ds \\ &\quad + (Gu, v_h^{(2)})_{\Omega_2} - \int_{\Gamma^-} \mathbf{f}_u \cdot \mathbf{n} (v_h^{(2)} - v_h^{(1)}) ds \\ &= a(u, v_h) + (Gu - P(Gu), v_h^{(2)})_{\Omega_2} - \int_{\Gamma^-} \mathbf{f}_u \cdot \mathbf{n} (v_h^{(2)} - v_h^{(1)}) ds, \end{aligned}$$

where  $P$  is the mapping defined in (24). Substituting the above into (53) gives

$$\begin{aligned} (54) \quad &a(u_h - u_I, v_h) \\ &= a(u - u_I, v_h) + (Gu - P(Gu), v_h^{(2)})_{\Omega_2} - \int_{\Gamma^-} \mathbf{f}_u \cdot \mathbf{n} (v_h^{(2)} - v_h^{(1)}) ds \\ &\leq \left| \int_{\Omega_1} (\mathbf{f}_u - \mathbf{f}_{u_I}) \cdot \nabla v_h^{(1)} dx + (Gu - Gu_I, v_h^{(1)})_{\Omega_1} \right| \\ &\quad + \left| \int_{\Gamma^-} \mathbf{f}_u \cdot \mathbf{n} (v_h^{(2)} - v_h^{(1)}) ds \right| \\ &\quad + \left| \sum_{d \in D_h^{(2)}} v_h^{(2)} (\mathbf{f}_u - \mathbf{f}_{u_I}) \cdot \mathbf{n} ds \right| + |(Gu - P(Gu), v_h^{(2)})_{\Omega_2}| \\ &=: R_1 + R_2 + R_3 + R_4. \end{aligned}$$

We now consider the error terms  $R_1, R_2, R_3$  and  $R_4$  separately. Notice that  $u_I|_{\Omega_1}$  is the piecewise linear interpolant of  $u$  on the triangular mesh  $T_h^{(1)}$ . Using the standard argument, we have

$$\begin{aligned} (55) \quad R_1 &= \left| \int_{\Omega_1} (\varepsilon \nabla(u - u_I) - \mathbf{a}(u - u_I)) \cdot \nabla v_h^{(1)} dx + (Gu - Gu_I, v_h^{(1)})_{\Omega_1} \right| \\ &\leq C(h\varepsilon + h^2) \|u\|_{2, \Omega_1} \|\nabla v_h^{(1)}\|_{0, \Omega_1} + Ch^2 \|u\|_{2, \Omega_1} \|v_h^{(1)}\|_{0, \Omega_1} \\ &\leq Ch \left( \varepsilon \|\nabla v_h^{(1)}\|_{0, \Omega_1} + \|v_h^{(1)}\|_{0, \Omega_1} \right) \|u\|_{2, \Omega_1} \\ &\leq Ch \left( \sqrt{\varepsilon} \|\nabla v_h^{(1)}\|_{0, \Omega_1} + \|v_h^{(1)}\|_{0, \Omega_1} \right) \\ &= Ch \|v_h^{(1)}\|_{\varepsilon, \Omega_1}. \end{aligned}$$

In the above we used the inequality  $h \|\nabla v_h^{(1)}\|_{0, \Omega_1} \leq C \|v_h^{(1)}\|_{0, \Omega_1}$  and (15).

Let us consider  $R_2$ . This term is due to the discontinuity or nonconformity of the test space across  $\Gamma$ . For this term we have

$$(56) \quad R_2 = \left| \int_{\Gamma^-} \mathbf{f}_u \cdot \mathbf{n} (v_h^{(2)} - v_h^{(1)}) ds \right| \\ = \left| \sum_{e_{i,j} \subset \Gamma} \int_{e_{i,j}} \mathbf{f}_u \cdot \mathbf{n} (v_h^{(2)} - v_h^{(1)}) ds \right|,$$

where  $e_{i,j} \in E_h$  denotes the edge connecting  $x_i$  and  $x_j$ . On  $e_{i,j}$ ,  $v_h^{(1)}$  is linear and  $v_h^{(2)}$  is piecewise constant. They are in the form of (41) and (42) respectively. For clarity, we let  $g(x) = \mathbf{f}_u \cdot \mathbf{n}$  and  $G(x) = \int g dx$ . We also put  $v_i := v_h^{(1)}(x_i) = v_h^{(2)}(x_i)$  and  $v_j := v_h^{(1)}(x_j) = v_h^{(2)}(x_j)$ . Parametrizing  $e_{i,j}$  as  $\{s : 0 \leq s \leq |e_{i,j}|\}$  and integrating by parts, we have

$$(57) \quad \int_{e_{i,j}} \mathbf{f}_u \cdot \mathbf{n} (v_h^{(2)} - v_h^{(1)}) ds = \int_0^{\frac{|e_{i,j}|}{2}} g(v_h^{(2)} - v_h^{(1)}) ds + \int_{\frac{|e_{i,j}|}{2}}^{|e_{i,j}|} g(v_h^{(2)} - v_h^{(1)}) ds \\ = G\left(\frac{|e_{i,j}|}{2}\right) \left(\frac{v_i + v_j}{2} - v_i\right) - \frac{v_j - v_i}{|e_{i,j}|} \int_0^{\frac{|e_{i,j}|}{2}} G ds \\ - G\left(\frac{|e_{i,j}|}{2}\right) \left(\frac{v_i + v_j}{2} - v_j\right) - \frac{v_j - v_i}{|e_{i,j}|} \int_{\frac{|e_{i,j}|}{2}}^{|e_{i,j}|} G ds \\ = \frac{v_j - v_i}{|e_{i,j}|} \left[ \int_0^{|e_{i,j}|} G(s) ds - |e_{i,j}| G\left(\frac{|e_{i,j}|}{2}\right) \right].$$

The last term of (57) is essentially the error in the numerical integration of  $G$  by the mid-point quadrature rule. Using Taylor expansion, it is easy to show that

$$\left| \int_0^{|e_{i,j}|} G(s) ds - |e_{i,j}| G\left(\frac{|e_{i,j}|}{2}\right) \right| \leq C \|G\|_{2,\infty} |e_{i,j}|^3 \leq C \|\mathbf{f}_u\|_{1,\infty} |e_{i,j}|^3.$$

Substituting the above into (57) and then the result into (56), we obtain

$$(58) \quad R_2 \leq C \sum_{e_{i,j} \subset \Gamma} |v_j - v_i| |e_{i,j}|^3 \|\mathbf{f}_u\|_{1,\infty} \leq Ch^2 \|v_h\|_\infty,$$

since  $\|\mathbf{f}_u\|_{1,\infty} \leq C$  on  $\Gamma$ .

We now consider  $R_3$ . Analogously to the derivation of (35), transforming from a summation over the edges to a summation over the nodes of  $T_h^{(2)}$  gives

$$(59) \quad R_3 = \left| \sum_{e_{i,j} \in E_h^{(2)}} (v_i - v_j) \int_{l_{i,j}} (\mathbf{f}_u - \mathbf{f}_{u_I}) \cdot \mathbf{n} ds \right| \\ \leq \sum_{e_{i,j} \in E_h^{(2)}} |v_i - v_j| \int_{l_{i,j}} |\mathbf{f}_u \cdot \mathbf{n} - f_{i,j}| ds.$$

Combining (44) and (59) and noting that  $|l_{i,j}| = 2|b_{i,j}|/|e_{i,j}|$ , we have

$$\begin{aligned}
 R_3 &\leq \sum_{e_{i,j} \in E_h^{(2)}} |v_i - v_j| \int_{l_{i,j}} |\mathbf{f}_u \cdot \mathbf{e}_{i,j} - f_{i,j}| ds \\
 &\leq ChK_1 \sum_{e_{i,j} \subset \bar{\Omega}_2^{(1)} \cup \bar{\Omega}_2^{(2)}} |v_i - v_j| \frac{|b_{i,j}|}{|e_{i,j}|} + \frac{C}{M} \ln \frac{1}{\varepsilon} \sum_{e_{i,j} \subset \bar{\Omega}_2^{(3)}} |v_i - v_j| \frac{|b_{i,j}|}{|e_{i,j}|} \\
 &\leq ChK_1 \left( \sum_{e_{i,j} \in \bar{\Omega}_2^{(1)} \cup \bar{\Omega}_2^{(2)}} \left( \frac{v_i - v_j}{|e_{i,j}|} \right)^2 |b_{i,j}| \right)^{1/2} \left( \sum_{e_{i,j} \subset \bar{\Omega}_2^{(1)} \cup \bar{\Omega}_2^{(2)}} |b_{i,j}| \right)^{1/2} \\
 &\quad + \frac{C}{M} \ln \frac{1}{\varepsilon} \left( \sum_{e_{i,j} \subset \bar{\Omega}_2^{(3)}} \left( \frac{v_i - v_j}{|e_{i,j}|} \right)^2 |b_{i,j}| \right)^{1/2} \left( \sum_{e_{i,j} \subset \bar{\Omega}_2^{(3)}} |b_{i,j}| \right)^{1/2} \\
 (60) \quad &\leq ChK_1 \sqrt{\varepsilon} \ln^{1/2} \frac{1}{\varepsilon} \left( \sum_{e_{i,j} \in \bar{\Omega}_2^{(1)} \cup \bar{\Omega}_2^{(2)}} \left( \frac{v_i - v_j}{|e_{i,j}|} \right)^2 |b_{i,j}| \right)^{1/2} \\
 &\quad + \frac{C}{M} \varepsilon \ln^{3/2} \frac{1}{\varepsilon} \left( \sum_{e_{i,j} \subset \bar{\Omega}_2^{(3)}} \left( \frac{v_i - v_j}{|e_{i,j}|} \right)^2 |b_{i,j}| \right)^{1/2} \\
 &\leq ChK_2 \left( \sum_{e_{i,j} \in E_h^{(2)}} |e_{i,j}| \left( \frac{v_i - v_j}{|e_{i,j}|} \right)^2 |b_{i,j}| \right)^{1/2} \\
 &= ChK_2 \|v\|_{h, \Omega_2},
 \end{aligned}$$

where  $\|\cdot\|_{h, \Omega_2}$  is the discrete norm defined by (39) and  $K_2$  is defined in (52). In (60) we used the facts that  $|e_{i,j}| \geq \frac{C}{M} \varepsilon \ln \frac{1}{\varepsilon}$ ,  $(\sum_{e_{i,j} \subset \bar{\Omega}_2^{(1)} \cup \bar{\Omega}_2^{(2)}} |b_{i,j}|)^{1/2} = O(\sqrt{\varepsilon \ln \frac{1}{\varepsilon}})$  and  $(\sum_{e_{i,j} \subset \bar{\Omega}_2^{(3)}} |b_{i,j}|)^{1/2} = O(\varepsilon \ln \frac{1}{\varepsilon})$ .

Now, we consider  $R_4$  in (46). Let  $G_i = G(x_i)$ . Since  $v_h = 0$  on  $\partial\Omega_2 \cap \partial\Omega$ , the term  $R_4$  can be estimated as

$$\begin{aligned}
 R_4 &= \sum_{d_i \in D_h^{(2)}} \int_{d_i} |(Gu - P(Gu))v_h| dx \\
 &\leq \sum_{d_i \in D_h^{(2)}} |G_i v_i| \int_{d_i} |u - u_i| |v_h| dx \\
 (61) \quad &\quad + \sum_{d_i \in D_h^{(2)}} |v_i| \|u\|_{0, \infty, d_i} \int_{d_i} |G - G_i| dx \\
 &\leq \|G\|_{\infty} \|v_h\|_{\infty} \sum_{d_i \in D_h^{(2)}} \int_{d_i} |u - u_i| dx + Ch \|G\|_{1, \infty} \|v_h\|_{\infty} \varepsilon \ln \frac{1}{\varepsilon},
 \end{aligned}$$

since  $\sum_{d_i \in D_h^{(2)}} |d_i| = O(\varepsilon \ln \frac{1}{\varepsilon})$ . Similarly to the discussion of  $R_3$ , using Assumption 2.2, the sum of the integrals in (61) can be estimated as

$$\begin{aligned} \sum_{d_i \in D_h^{(2)}} \int_{d_i} |u - u_i| dx &\leq Ch \sum_{d_i \in D_h^{(2)}} \int_{d_i} \left( \left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right| \right) dx dy \\ &\leq Ch \int_0^{1-\delta_2/M} \int_{1-\delta_1}^{1-\delta_1/2M} \left( \left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right| \right) dx dy \\ &\quad + Ch \int_0^{1-\delta_1} \int_{1-\delta_2}^{1-\delta_2/2M} \left( \left| \frac{\partial u}{\partial x} \right| + \left| \frac{\partial u}{\partial y} \right| \right) dy dx \\ &\leq Ch \left( \varepsilon \ln \frac{1}{\varepsilon} + \varepsilon^{\beta/2M} \right), \end{aligned}$$

where  $\delta_1$  and  $\delta_2$  are defined in (9). Substituting the above estimate into (61), we obtain

$$\begin{aligned} (62) \quad R_4 &\leq C \|G\|_{\infty} \|v_h\|_{\infty} \left( \varepsilon \ln \frac{1}{\varepsilon} + \varepsilon^{\beta/2M} \right) + Ch |G|_{1,\infty} \|v_h\|_{0,\infty} \varepsilon \ln \frac{1}{\varepsilon} \\ &\leq Ch \|v_h\|_{\infty} \left( \varepsilon \ln \frac{1}{\varepsilon} + \varepsilon^{\beta/2M} \right). \end{aligned}$$

Finally, substituting (55), (58), (60) and (62) into (54), we have

$$\begin{aligned} a(u_h - u_I, v_h) &\leq Ch \|v\|_{\varepsilon, \Omega_1} + Ch K_2 \|v\|_{h, \Omega_2} \\ &\quad + C \left[ h^2 + h \left( \varepsilon \ln \frac{1}{\varepsilon} + \varepsilon^{\beta/2M} \right) \right] \|v_h\|_{\infty} \\ &\leq C \left[ h K_2 \|v\| + h^2 K_1 \|v_h\|_{\infty} \right], \end{aligned}$$

where  $K_1$  and  $K_2$  are defined in (45) and (52). Choosing  $v_h = P(u_h - u_I)$  and using (40), we obtain

$$\|u_h - u_I\|^2 \leq C \left[ h K_2 \|u_h - u_I\| + h^2 K_1 \right].$$

This is of the form

$$y^2 \leq CK_2 h y + CK_1 h^2 \quad \text{or} \quad \left( y - \frac{1}{2} CK_2 h \right)^2 \leq CK_1 h^2 + \frac{(CK_2)^2}{4} h^2,$$

which reduces to

$$y \leq h \sqrt{CK_1 + \frac{(CK_2)^2}{4}} + \frac{CK_2}{2} h \leq Ch (K_2 + K_1^{1/2}).$$

Replacing  $y$  with  $\|u_h - u_I\|$ , we obtain

$$\|u_h - u_I\| \leq Ch (K_1^{1/2} + K_2).$$

This completes the proof of the theorem.  $\square$

Theorem 5.2 yields the following corollary.

**Corollary 5.3.** *Let the assumptions in Theorem 5.2 be fulfilled. If  $\varepsilon, h, M$  and  $\beta$  are such that  $h^{-1} \varepsilon^{\beta/2M} \leq O(1)$  and  $h^{-1} \varepsilon \ln(1/\varepsilon) \leq O(1)$ , then*

$$(63) \quad \|u_I - u_h\| \leq Ch M^{1/2}.$$

Furthermore, if we choose  $h^{-1} \varepsilon^{\beta/2M} = O(1)$ , then

$$(64) \quad \|u_I - u_h\| \leq Ch \sqrt{\left| \frac{\ln \varepsilon}{\ln h} \right|}.$$

*Proof.* Since  $h^{-1}\varepsilon^{\beta/2M} \leq O(1)$  and  $h^{-1}\varepsilon \ln(1/\varepsilon) \leq O(1)$ , it follows that  $K_1 = 1$  and  $K_2 \leq M^{1/2}$ . Substituting these estimates into (51) yields (63).

We now prove (64). Consider  $h^{-1}\varepsilon^{\beta/2M} = 1$ . Solving this equation for  $M$ , we get  $M = \frac{\beta}{2} |\ln \varepsilon / \ln h|$ , from which we see that if  $h^{-1}\varepsilon^{\beta/2M} = O(1)$ , then  $M = O(|\ln \varepsilon / \ln h|)$ . Substituting this into (63), we obtain (64).  $\square$

We comment that obviously the above corollary implies that if  $\varepsilon \ll h$ , then  $\|u_I - u_h\|$  converges to zero at the rate of  $h$  almost uniformly in  $\varepsilon$ . This improves the results  $O(h \ln^{1/2} \frac{1}{\varepsilon})$  in [3] and [10]. Unlike most of the previous results, the present method uses a mixture of triangular and rectangular elements. Thus, the results can be easily extended to singularly perturbed problems with nonrectangular, polygonal regions, if the a priori estimates corresponding to those in Assumption 2.2 can be established on those regions. Moreover, unlike most of the previous cases, the discrete part of the energy norm  $\|\cdot\|$  is weighted by the local mesh size  $|e_{i,j}|$ , though some of them are of length  $O(\varepsilon \ln \frac{1}{\varepsilon})$ .

#### ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for their valuable contributions toward the improvement of this paper.

#### REFERENCES

- [1] D.N. de G. Allen, R.V. Southwell, "Relaxation methods applied to determine the motion, in two dimensions, of a viscous fluid past a fixed cylinder", *Quart. J. Mech. Appl. Math.*, **8** (1955) 129–145. MR **16**:1171a
- [2] L. Angermann, "Error estimates for the finite-element solution of an elliptic singularly perturbed problem", *IMA J. Num. Anal.*, **15** (1995) 161–196. MR **96c**:65177
- [3] T. Apel and G. Lube, "Anisotropic mesh refinement in stabilized Galerkin methods", *Numer. Math.*, **74** (1996) 261–282. MR **95a**:65158
- [4] Bank R.E., Bürgler J.F., Fitchner W. and Smith R.K., "Some upwinding techniques for finite element approximations of convection-diffusion equations", *Numer. Math.*, **58** (1990) 185–202. MR **91i**:65175
- [5] I. Christie, D.F. Griffiths, A.R. Mitchell, and O.C. Zienkiewicz, "Finite element methods for second order differential equations with significant first derivatives", *Int. J. Num. Meth. Engng.* **10** (1976) 1389–1396. MR **56**:4178
- [6] M. Dobrowolski and H.-G. Roos, "A priori estimates for the solution of convection-diffusion problems and interpolation on Shishkin meshes", *Zeitschrift für Analysis und Anwendungen*, **16** (1997) 1001–1012. MR **99e**:65154
- [7] M. Feistauer, J. Felcman and M. Lukáčová-Medvid'ová, "Combined finite-element volume solution of compressible flow", *J. Comp. Appl. Math.*, **63** (1995) 179–199. MR **96k**:76071
- [8] M. Feistauer and J. Felcman, "On the convergence of a combined finite volume-finite element method for nonlinear convection-diffusion problems", *Num. Methods PDEs*, **13** (1997) 163–190. MR **98a**:65123
- [9] M. Feistauer, J. Slavik and P. Stupka, "On the convergence of a combined finite volume-finite element method for nonlinear convection-diffusion problems. Explicit schemes", *Num. Methods PDEs*, **13** (1999) 215–235. MR **2000a**:65107
- [10] R. Hangleiter and G. Lube, "Boundary layer-adapted grids and domain decomposition in stabilized Galerkin methods for elliptic problems", *CWI Quarterly*, **10**, No.3&4 (1998) 1–24. MR **95d**:65346
- [11] J.C. Heinrich, P.S. Huyakorn, A.R. Mitchell, and O.C. Zienkiewicz, "An upwind finite element scheme for two-dimensional convective transport equations", *Internat. J. Num. Meth. Engng.* **11** (1977) 131–143.
- [12] J.T.R. Hughes and A.N. Brooks, "A Multidimensional Upwind Scheme with no Crosswind Diffusion", *Finite element methods for convection dominated flows*, (ed. T.J.R. Hughes) AMD Vol. 34, Amer. Soc. of Mech. Eng., New York (1979), 19–35. MR **81f**:76040

- [13] C. Johnson “Streamline diffusion methods for problems in fluids” in *Finite elements in fluids, vol. VI*, R.H. Gallagher et al. (eds.) John Wiley and Sons, London (1986) 251-261.
- [14] J.J.H. Miller, S. Wang, “A new nonconforming Petrov-Galerkin finite element method with triangular element for a singularly perturbed advection-diffusion problem”, *IMA J. Numer. Anal.*, **14** (1994) 257–276. MR **95a**:65190
- [15] J.J.H. Miller, S. Wang, “An exponentially fitted finite element volume method for the numerical solution of 2D unsteady incompressible flow problems”, *J. Comput. Phys.*, **115**, No.1 (1994) 56–64. MR **95f**:76081
- [16] J.J.H. Miller, E. O’Riordan, G.I. Shishkin, *Fitted numerical methods for singular perturbation problems*, World Scientific, Singapore (1996). MR **98c**:65002
- [17] H.-G. Roos, M. Stynes, L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations*, Springer-Verlag, Berlin-Heidelberg (1996). MR **99a**:65134
- [18] H.-G. Roos, D. Adam, A. Felgenhauer, “A novel nonconforming uniformly convergent finite element method in two dimensions”, *J. Math. Anal. Appl.*, **201** (1996) 711–755. MR **97k**:65260
- [19] M. Sardella, “On a coupled finite element-finite volume method for convection-diffusion problems”, *IMA J. Numer. Anal.*, **20** (2000) 281–301. MR **2001b**:65108
- [20] D. Scharfetter, H.K. Gummel, Large-signal analysis of a silicon read diode oscillator, *IEEE Trans. Elec. Dev.*, **ED-16**, 64–77 (1969) 64–77.
- [21] T. Skalický, H.-G. Roos, D., “Galerkin/least-squared finite element method for convection-diffusion problems on Gartland meshes”, *Report MATH-NM-12-98*, Technical University of Dresden (1998).
- [22] M. Stynes, E. O’Riordan “A uniform convergent Galerkin method on a Shishkin mesh for a convection-diffusion problem”, *J. Math. Anal. Appl.*, **214** (1997) 34–54. MR **99f**:65177
- [23] S. Wang, “A novel exponentially fitted triangular finite element method for an advection-diffusion problem with boundary layers”, *J. Comp. Phys.*, **134** (1997) 253–260. MR **98d**:76111
- [24] J. Xu and L. Zikatanov, “A monotone finite element scheme for convection-diffusion equations”, *Math. Comp.*, **68**, No.228 (1999) 1429–1446. MR **99m**:65228

DEPARTMENT OF MATHEMATICS & STATISTICS, THE UNIVERSITY OF WESTERN AUSTRALIA, 35 STIRLING HIGHWAY, CRAWLEY, WESTERN AUSTRALIA 6009, AUSTRALIA  
*E-mail address:* swang@maths.uwa.edu.au

DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL SUN YAT-SEN UNIVERSITY, KAOHSIUNG, TAIWAN 80424  
*E-mail address:* zcli@math.nsysu.edu.tw