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EVALUATION FORMULAS FOR TORNHEIM'S TYPE OF ALTERNATING DOUBLE SERIES

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Abstract. In this paper, we give some evaluation formulas for Tornheim's type of alternating series by an elementary and combinatorial calculation of the uniformly convergent series. Indeed, we list several formulas for them by means of Riemann's zeta values at positive integers.

1. Introduction

In [4], Tornheim considered the double series $T(r,s,t)$ defined by

$$T(r,s,t)=\sum_{m,n=1}^{\infty}\frac{1}{m^r n^s (m+n)^t}, \tag{1.1}$$

where r,s,t are nonnegative integers with $r+t>1,\ s+t>1$ and $r+s+t>2$. He showed that $T(r,s,N-r-s)$ is a polynomial in $\{\zeta(j)|\ 2\le j\le N\}$ with rational coefficients when N is odd and $N\ge 3$. Recently Huard, Williams and Zhang Nan-Yue gave an explicit formula for $T(r,s,N-r-s)$ as a rational linear combination of the products $\zeta(2j)\zeta(N-2j)$ $(0\le j\le (N-3)/2)$ when N is odd, $N\ge 3$, and r,s are nonnegative integers satisfying $1\le r+s\le N-1$, $r\le N-2$ and $s\le N-2$ (see [2]). On the other hand, it is an open problem to determine an explicit formula for $T(r,s,N-r-s)$ when N is even. In a previous paper ([5]), we give some relation formulas for $T(r,s,N-r-s)$ when N is even. These can be regarded as generalizations of the Subbarao and Sitaramachandrarao formula given in [3].

Alternating analogues of (1.1)

$$S(r,s,t)=\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^r n^s (m+n)^t}, \tag{1.2}$$

$$R(r,s,t)=\sum_{m,n=1}^{\infty}\frac{(-1)^{n}}{m^r n^s (m+n)^t} \tag{1.3}$$

were also considered in [3]. Subbarao and Sitaramachandrarao posed the problem to evaluate $S(r,r,r)$ and $R(r,r,r)$ for any positive integer r. As a partial answer to their problem, we gave an evaluation formula for $S(r,r,r)$ for any positive *odd*

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integer r (see [5], Corollary 3), and for $R(r,r,r)$ for any positive *odd* integer r (see [6], Theorem 3.6).

The purpose of this paper is to give an evaluation formula for $S(r,s,t)$ for positive integers r,s,t when $r+s+t$ is odd (see Theorem 3.4). In order to evaluate $S(r,s,t)$, we make use of the same method that we introduced in [5], which is an elementary and primitive calculation of the uniformly convergent series. By this method, we can write $S(r,s,t)$ as a rational linear combination of products of Riemann's zeta values at positive integers. This result corresponds to the formula for $T(r,s,t)$ given by Huard, Williams and Zhang Nan-Yue mentioned above.

2. Preliminaries

We use the same notation as those in [5]. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, and ρ the field of real numbers. Throughout this paper we fix $\delta \in \rho$ with $\delta > 0$. For $u \in \rho$ with $1 \le u \le 1+\delta$ and $s \in \rho$, we define

$$\phi(s;u) := \sum_{m=1}^{\infty} \frac{(-u)^{-m}}{m^s}. \tag{2.1}$$

If $u > 1$, then $\phi(s;u)$ is convergent for any $s \in \mathbb{Z}$. In the case when $u = 1$, let $\phi(s) := \phi(s;1) = (2^{1-s}-1)\zeta(s)$. Corresponding to $\phi(s;u)$, we define a set of numbers $\{\mathcal{E}_m(u)\}$ by

$$F(x;u) := \frac{(1+u)e^x}{e^x+u} = \sum_{m=0}^{\infty} \mathcal{E}_m(u)\ \frac{x^m}{m!}. \tag{2.2}$$

In particular when $u = 1$, we have $\mathcal{E}_m(1) = E_m(1)$ for $m \in \mathbb{N}_0$, where $E_m(X)$ is the m-th Euler polynomial (see, e.g., [1]). Hence

$$\mathcal{E}_{2j}(1) = E_{2j}(1) = 0 \qquad (j \in \mathbb{N}). \tag{2.3}$$

It follows from (2.2) that if $u \in [1, 1+\delta]$, then

$$\liminf_{m\to\infty} \left(\frac{|\mathcal{E}_m(u)|}{m!}\right)^{-1/m} \ge \pi. \tag{2.4}$$

When $u \in (1, 1+\delta]$, by (2.2), we have

$$\phi(-k;u) = -\frac{1}{1+u}\mathcal{E}_k(u) \qquad (k \in \mathbb{N}_0) \tag{2.5}$$

(see [5], Lemma 1). We define

$$\imath_p(\theta;k;u) := \sum_{m=1}^{\infty} \frac{(-u)^{-m}\sin^{(p)}(m\theta)}{m^k}, \tag{2.6}$$

for $p \in \mathbb{N}_0$, $k \in \mathbb{N}$, $\theta \in [-\pi,\pi]$ and $u \in [1, 1+\delta]$, where we denote the l-th derivative of a function $f(\theta)$ by $f^{(l)}(\theta)$. It is well known that

$$\sin^{(p)}(\theta) = i^{p-1} \sum_{n=0}^{\infty} \lambda_{p+1+n} \frac{(i\theta)^n}{n!}, \tag{2.7}$$

where $\lambda_j := (1+(-1)^j)/2$ and $i = \sqrt{-1}$. Suppose $u \in (1, 1+\delta]$. Then

$$\imath_p(\theta;k;u) = i^{p-1} \sum_{n=0}^{\infty} \phi(k-n;u)\lambda_{p+1+n} \frac{(i\theta)^n}{n!}. \tag{2.8}$$

By (2.4) and (2.5), we see that (2.8) is uniformly convergent with respect to $u \in (1, 1+\delta]$ when $\theta \in (-\pi, \pi)$. Furthermore we define

$$\jmath_p(\theta; k; u) := \imath_p(\theta; k; u) - i^{p-1} \sum_{j=0}^{k} \phi(k-j; u) \lambda_{p+1+j} \frac{(i\theta)^j}{j!}. \tag{2.9}$$

When $u \in (1, 1+\delta]$, by (2.5), we have

$$\jmath_p(\theta; k; u) = -\frac{i^{p-1}}{1+u} \sum_{n=1}^{\infty} \mathcal{E}_n(u) \lambda_{p+1+n+k} \frac{(i\theta)^{n+k}}{(n+k)!}.$$

This is also uniformly convergent with respect to $u \in (1, 1+\delta]$ when $\theta \in (-\pi, \pi)$. So it follows from (2.3) that

$$\jmath_{k+1}(\theta; k; u) \to 0 \qquad (u \to 1;\ \theta \in (-\pi, \pi)). \tag{2.10}$$

Corresponding to (1.2), we define

$$S(r, s, t; u) := \sum_{m,n=1}^{\infty} \frac{(-u)^{-m-n}}{m^r n^s (m+n)^t} \qquad (r, s \in \mathbb{N};\ t \in \mathbb{Z}), \tag{2.11}$$

for $u \in [1, 1+\delta]$. We also define

$$\begin{aligned} \beta_n(k, l; u) :=& S(k, l, -n; u) \\ &- \sum_{j=0}^{k} \binom{n}{j} \phi(k-j; u) \lambda_{k+j} \phi(l+j-n; u) \\ &- \sum_{j=0}^{l} \binom{n}{j} \phi(l-j; u) \lambda_{l+j} \phi(k+j-n; u), \end{aligned} \tag{2.12}$$

for $n \in \mathbb{Z}$, when $u \in (1, 1+\delta]$. In particular when $n \le -1$, we define $\beta_n(k, l; 1)$ by (2.12) with $u = 1$. Note that $\phi(0; 1) = -\frac{1}{2}$.

Lemma 2.1. *For $k, l \in \mathbb{N}$ and $u \in (1, 1+\delta]$,*

$$\begin{aligned} &\jmath_{k+1}(\theta; k; u) \imath_l(\theta; l; u) + \imath_k(\theta; k; u) \jmath_{l+1}(\theta; l; u) \\ &\quad = i^{k+l-1} \sum_{n=0}^{\infty} \beta_n(k, l; u) \lambda_{k+l+1+n} \frac{(i\theta)^n}{n!}. \end{aligned} \tag{2.13}$$

Proof. It is well known that

$$\sin^{(k+1)}(\alpha) \cdot \sin^{(l)}(\beta) + \sin^{(k)}(\alpha) \cdot \sin^{(l+1)}(\beta) = \sin^{(k+l)}(\alpha + \beta).$$

Hence by (2.6), (2.7) and (2.11), we have

$$\begin{aligned} &\imath_{k+1}(\theta; k; u) \imath_l(\theta; l; u) + \imath_k(\theta; k; u) \imath_{l+1}(\theta; l; u) \\ &\quad = \sum_{m,n=1}^{\infty} \frac{(-u)^{-m-n}}{m^k n^l} \sin^{(k+l)}((m+n)\theta) \\ &\quad = i^{k+l-1} \sum_{N=0}^{\infty} S(k, l, -N; u) \lambda_{k+l+1+N} \frac{(i\theta)^N}{N!}. \end{aligned}$$

On the other hand, by (2.8) and (2.9), we have

$$\left\{-i^k\sum_{j=0}^{k}\phi(k-j;u)\lambda_{k+j}\frac{(i\theta)^j}{j!}\right\}\mathrm{l}_l(\theta;l;u)$$
$$=-i^{k+l-1}\sum_{n=0}^{\infty}\left\{\sum_{j=0}^{k}\binom{n}{j}\phi(k-j;u)\lambda_{k+j}\phi(l+j-n;u)\right\}\lambda_{k+l+1+n}\frac{(i\theta)^n}{n!},$$

because $\lambda_{p+r}\lambda_{q+r}=\lambda_{p+r}\lambda_{p+q}$. So we obtain the proof of Lemma 2.1. □

By the above consideration, we see that (2.13) is uniformly convergent with respect to $u\in(1,1+\delta]$ when $\theta\in(-\pi,\pi)$. Hence by (2.10), we have the following.

Lemma 2.2. *For $k,l\in\mathbb{N}$,*

$$\liminf_{m\to\infty}\left(\frac{|\beta_m(k,l;u)|}{m!}\lambda_{k+l+1+m}\right)^{-1/m}\geq\pi\qquad(u\in(1,1+\delta]) \tag{2.14}$$

and

$$\lim_{u\to1}\ \beta_m(k,l;u)\lambda_{k+l+1+m}=0\qquad(m\in\mathbb{N}_0). \tag{2.15}$$

3. Evaluation formulas

We begin by proving the following proposition.

Proposition 3.1. *For $k,l,d\in\mathbb{N}$ and $u\in(1,1+\delta]$,*

$$\begin{aligned}
&i\sum_{m,n=1}^{\infty}\frac{(-u)^{-m-n}\sin((m+n)\theta)}{m^kn^l(m+n)^d}\\
&\quad-i\sum_{j=0}^{k}\phi(k-j;u)(-1)^j\lambda_{k+j}\\
&\qquad\cdot\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^\nu}{\nu!}\mathrm{l}_\nu(\theta;d+j+l-\nu;u)\\
&\quad-i\sum_{j=0}^{l}\phi(l-j;u)(-1)^j\lambda_{l+j}\\
&\qquad\cdot\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^\nu}{\nu!}\mathrm{l}_\nu(\theta;d+j+k-\nu;u)\\
&=\sum_{r=-d}^{\infty}\beta_r(k,l;u)\lambda_{r+d+1}\frac{(i\theta)^{r+d}}{(r+d)!}.
\end{aligned}\tag{3.1}$$

Proof. By (2.7) and (2.11), we have

$$\begin{aligned}
i\sum_{m,n=1}^{\infty}\frac{(-u)^{-m-n}\sin((m+n)\theta)}{m^kn^l(m+n)^d}&=\sum_{n=0}^{\infty}S(k,l,d-n;u)\lambda_{n+1}\frac{(i\theta)^n}{n!}\\
&=\sum_{r=-d}^{\infty}S(k,l,-r;u)\lambda_{r+d+1}\frac{(i\theta)^{r+d}}{(r+d)!}.
\end{aligned}\tag{3.2}$$

On the other hand, in the same way as in Lemma 6 of [5], we obtain

$$\sum_{\nu=0}^{b}\binom{a-1+b-\nu}{b-\nu}\frac{(-\theta)^{\nu}}{\nu!}\frac{\sin^{(\nu+p)}(\theta x)}{x^{a+b+c-\nu}} = i^{p-1}\sum_{N=0}^{\infty}(-1)^{b}\binom{N-a}{b}\frac{(i\theta)^{N}}{N!}\lambda_{p-1+N}\ x^{-a-b-c+N}, \tag{3.3}$$

by (2.7) and using the well-known relation

$$\binom{-X}{j} = (-1)^{j}\binom{X+j-1}{j}.$$

Using (2.8) and (3.3), we have

$$\begin{aligned} &-i\sum_{j=0}^{k}\phi(k-j;u)(-1)^{j}\lambda_{k+j} \\ &\qquad\cdot\sum_{\nu=0}^{j}\binom{d-1+j-\nu}{j-\nu}\frac{(-\theta)^{\nu}}{\nu!}\mathrm{l}_{\nu}(\theta;d+j+l-\nu;u) \\ &= -\sum_{j=0}^{k}\phi(k-j;u)\lambda_{k+j} \\ &\qquad\cdot\sum_{N=0}^{\infty}\binom{N-d}{j}\ \phi(d+j+l-N;u)\lambda_{N+1}\frac{(i\theta)^{N}}{N!} \\ &= -\sum_{r=-d}^{\infty}\sum_{j=0}^{k}\binom{r}{j}\phi(k-j;u)\lambda_{k+j}\phi(l+j-r;u)\lambda_{r+d+1}\frac{(i\theta)^{r+d}}{(r+d)!}. \end{aligned} \tag{3.4}$$

Combining (2.12), (3.2) and (3.4), we obtain the proof. □

Proposition 3.2. *For $k,l\in\mathbb{N}$ and $m\in\mathbb{N}_0$ with $m\equiv k+l \pmod 2$,*

$$\begin{aligned} &(-1)^{k}\sum_{\rho=0}^{[k/2]}\phi(2\rho)\sum_{\mu=0}^{[(k-2\rho-1)/2]}\binom{m+k-2\rho-2\mu}{k-2\rho-2\mu-1} \\ &\qquad\cdot\zeta(k+l+m-2\rho-2\mu+1)\frac{(i\pi)^{2\mu}}{(2\mu+1)!} \\ &+(-1)^{l}\sum_{\rho=0}^{[l/2]}\phi(2\rho)\sum_{\mu=0}^{[(l-2\rho-1)/2]}\binom{m+l-2\rho-2\mu}{l-2\rho-2\mu-1} \\ &\qquad\cdot\zeta(k+l+m-2\rho-2\mu+1)\frac{(i\pi)^{2\mu}}{(2\mu+1)!} \\ &=\sum_{\mu=0}^{[m/2]}\beta_{2\mu-1-m}(k,l;1)\frac{(i\pi)^{2\mu}}{(2\mu+1)!}. \end{aligned} \tag{3.5}$$

Proof. For $k,l\in\mathbb{N}$, we take $d\in\mathbb{N}$ with $d\geq 2$ and $d\equiv k+l \pmod 2$. Then $\lambda_{r+d+1}=\lambda_{k+l+1+r}$ for any $r\in\mathbb{N}_0$. By (2.14), we see that (3.1) is uniformly convergent with respect to $u\in(1,1+\delta]$ when $d\geq 2$ and $\theta\in[-\pi,\pi]$. Hence we can

let $\theta = \pi$ and $u \to 1$ in both sides of (3.1). Putting $m = d-2$, and using (2.15), we obtain the proof. $\square$

We recall the following lemma which can be proved by the same method as that in Lemma 8 of [5].

Lemma 3.3. *Suppose $\{P_m\}$ and $\{Q_m\}$ are sequences which satisfy the relation*

$$\sum_{j=0}^{[m/2]} P_{m-2j} \frac{(i\pi)^{2j}}{(2j+1)!} = Q_m,$$

for any $m \in \mathbb{N}_0$. Then the relation

$$P_m = -2 \sum_{\nu=0}^{m} \phi(m-\nu) \lambda_{m-\nu} Q_\nu$$

holds for any $m \in \mathbb{N}_0$.

By (3.5), we can apply this lemma with $P_m = \beta_{-m-1}(k,l;1)\lambda_{k+l+m}$ and

$$\begin{aligned} Q_m = \Bigg\{ &(-1)^k \sum_{\rho=0}^{[k/2]} \phi(2\rho) \sum_{\mu=0}^{[(k-2\rho-1)/2]} \binom{m+k-2\rho-2\mu}{k-2\rho-2\mu-1} \\ &\qquad \cdot \zeta(k+l+m-2\rho-2\mu+1) \frac{(i\pi)^{2\mu}}{(2\mu+1)!} \\ &+ (-1)^l \sum_{\rho=0}^{[l/2]} \phi(2\rho) \sum_{\mu=0}^{[(l-2\rho-1)/2]} \binom{m+l-2\rho-2\mu}{l-2\rho-2\mu-1} \\ &\qquad \cdot \zeta(k+l+m-2\rho-2\mu+1) \frac{(i\pi)^{2\mu}}{(2\mu+1)!} \Bigg\} \lambda_{k+l+m}, \end{aligned}$$

for $m \in \mathbb{N}_0$. Then we have

$$\begin{aligned} &\beta_{-m-1}(k,l;1)\lambda_{k+l+m} \\ &\quad = -2 \sum_{\nu=0}^{m} \phi(m-\nu) \lambda_{m+\nu} \lambda_{k+l+\nu} \\ &\qquad \cdot \Bigg\{ (-1)^k \sum_{\rho=0}^{[k/2]} \phi(2\rho) \sum_{\mu=0}^{[(k-2\rho-1)/2]} \binom{\nu+k-2\rho-2\mu}{k-2\rho-2\mu-1} \\ &\qquad\qquad \cdot \zeta(k+l+\nu-2\rho-2\mu+1) \frac{(i\pi)^{2\mu}}{(2\mu+1)!} \\ &\qquad + (-1)^l \sum_{\rho=0}^{[l/2]} \phi(2\rho) \sum_{\mu=0}^{[(l-2\rho-1)/2]} \binom{\nu+l-2\rho-2\mu}{l-2\rho-2\mu-1} \\ &\qquad\qquad \cdot \zeta(k+l+\nu-2\rho-2\mu+1) \frac{(i\pi)^{2\mu}}{(2\mu+1)!} \Bigg\}. \end{aligned} \tag{3.6}$$

When $k+l+m \equiv 0 \pmod 2$, it holds that $\lambda_{k+l+m}=1$ and $\lambda_{m+\nu}\lambda_{k+l+\nu}=\lambda_{m+\nu}$ in (3.6). On the other hand, it follows from (2.12) with $u=1$ that

$$\begin{aligned}
(3.7)\qquad & \beta_{-m-1}(k,l;1)\\
&= S(k,l,m+1)\\
&\quad -\sum_{j=0}^{k}\binom{-m-1}{j}\phi(k-j)\lambda_{k+j}\phi(l+j+m+1)\\
&\quad -\sum_{j=0}^{l}\binom{-m-1}{j}\phi(l-j)\lambda_{l+j}\phi(k+j+m+1)\\
&= S(k,l,m+1;1)\\
&\quad -(-1)^k\sum_{\rho=0}^{[k/2]}\binom{m+k-2\rho}{k-2\rho}\phi(2\rho)\phi(k+l+m+1-2\rho)\\
&\quad -(-1)^l\sum_{\rho=0}^{[l/2]}\binom{m+l-2\rho}{l-2\rho}\phi(2\rho)\phi(k+l+m+1-2\rho).
\end{aligned}$$

Combining (3.6) with (3.7) and putting $h=m+1$, we obtain the following.

Theorem 3.4. *For $k,l,h\in\mathbb{N}$ with $k+l+h\equiv 1 \pmod 2$,*

$$\begin{aligned}
S(k,l,h) &= (-1)^k\sum_{\rho=0}^{[k/2]}\binom{k+h-2\rho-1}{k-2\rho}\phi(2\rho)\phi(k+l+h-2\rho)\\
&\quad +(-1)^l\sum_{\rho=0}^{[l/2]}\binom{l+h-2\rho-1}{l-2\rho}\phi(2\rho)\phi(k+l+h-2\rho)\\
&\quad -2\sum_{j=0}^{[(h-1)/2]}\phi(2j)\\
&\quad \cdot\Bigg\{(-1)^k\sum_{\rho=0}^{[k/2]}\phi(2\rho)\sum_{\mu=0}^{[(k-2\rho-1)/2]}\binom{k+h-2j-2\rho-2\mu-1}{k-2\rho-2\mu-1}\\
&\qquad\qquad \cdot\zeta(k+l+h-2j-2\rho-2\mu)\frac{(i\pi)^{2\mu}}{(2\mu+1)!}\\
&\quad +(-1)^l\sum_{\rho=0}^{[l/2]}\phi(2\rho)\sum_{\mu=0}^{[(l-2\rho-1)/2]}\binom{l+h-2j-2\rho-2\mu-1}{l-2\rho-2\mu-1}\\
&\qquad\qquad \cdot\zeta(k+l+h-2j-2\rho-2\mu)\frac{(i\pi)^{2\mu}}{(2\mu+1)!}\Bigg\}.
\end{aligned}$$

Example. We list several evaluation formulas for $S(k,l,h)$ deduced from Theorem 3.4. Note that we replace $\phi(s)$ with $(2^{1-s}-1)\zeta(s)$:

$S(1,1,1)=\frac{1}{4}\zeta(3)$

$S(1,1,3)=\frac{1}{6}\zeta(3)\pi^2-\frac{29}{16}\zeta(5)$

$$S(1,2,2) = \tfrac{1}{16}\zeta(3)\pi^2 - \tfrac{17}{32}\zeta(5)$$
$$S(1,3,1) = -\tfrac{1}{16}\zeta(3)\pi^2 + \tfrac{17}{16}\zeta(5)$$
$$S(2,2,1) = \tfrac{1}{8}\zeta(3)\pi^2 - \tfrac{17}{16}\zeta(5)$$

$$S(1,1,5) = \tfrac{7}{360}\zeta(3)\pi^4 + \tfrac{1}{6}\zeta(5)\pi^2 - \tfrac{251}{64}\zeta(7)$$
$$S(1,2,4) = -\tfrac{17}{192}\zeta(5)\pi^2 + \tfrac{61}{64}\zeta(7)$$
$$S(1,3,3) = \tfrac{19}{192}\zeta(5)\pi^2 - \tfrac{115}{128}\zeta(7)$$
$$S(1,4,2) = \tfrac{7}{960}\zeta(3)\pi^4 + \tfrac{15}{64}\zeta(5)\pi^2 - \tfrac{387}{128}\zeta(7)$$
$$S(1,5,1) = -\tfrac{7}{960}\zeta(3)\pi^4 - \tfrac{5}{64}\zeta(5)\pi^2 + \tfrac{129}{64}\zeta(7)$$
$$S(2,2,3) = -\tfrac{17}{96}\zeta(5)\pi^2 + \tfrac{61}{32}\zeta(7)$$
$$S(2,3,2) = -\tfrac{5}{64}\zeta(5)\pi^2 + \tfrac{129}{128}\zeta(7)$$
$$S(2,4,1) = \tfrac{7}{960}\zeta(3)\pi^4 + \tfrac{5}{32}\zeta(5)\pi^2 - \tfrac{129}{64}\zeta(7)$$
$$S(3,3,1) = -\tfrac{5}{32}\zeta(5)\pi^2 + \tfrac{129}{64}\zeta(7)$$

$$S(1,1,7) = \tfrac{31}{15120}\zeta(3)\pi^6 + \tfrac{7}{360}\zeta(5)\pi^4 + \tfrac{1}{6}\zeta(7)\pi^2 - \tfrac{1529}{256}\zeta(9)$$
$$S(1,2,6) = -\tfrac{7}{360}\zeta(5)\pi^4 - \tfrac{193}{768}\zeta(7)\pi^2 + \tfrac{2289}{512}\zeta(9)$$
$$S(1,3,5) = \tfrac{7}{180}\zeta(5)\pi^4 + \tfrac{389}{768}\zeta(7)\pi^2 - \tfrac{571}{64}\zeta(9)$$
$$S(1,4,4) = \tfrac{7}{768}\zeta(5)\pi^4 + \tfrac{9}{128}\zeta(7)\pi^2 - \tfrac{799}{512}\zeta(9)$$
$$S(1,5,3) = -\tfrac{7}{256}\zeta(5)\pi^4 - \tfrac{41}{128}\zeta(7)\pi^2 + \tfrac{387}{64}\zeta(9)$$
$$S(1,6,2) = \tfrac{31}{40320}\zeta(3)\pi^6 + \tfrac{7}{256}\zeta(5)\pi^4 + \tfrac{105}{256}\zeta(7)\pi^2 - \tfrac{3845}{512}\zeta(9)$$
$$S(1,7,1) = -\tfrac{31}{40320}\zeta(3)\pi^6 - \tfrac{7}{768}\zeta(5)\pi^4 - \tfrac{21}{256}\zeta(7)\pi^2 + \tfrac{769}{256}\zeta(9)$$
$$S(2,2,5) = -\tfrac{7}{180}\zeta(5)\pi^4 - \tfrac{193}{384}\zeta(7)\pi^2 + \tfrac{2289}{256}\zeta(9)$$
$$S(2,3,4) = \tfrac{1}{256}\zeta(7)\pi^2 + \tfrac{5}{256}\zeta(9)$$
$$S(2,4,3) = \tfrac{7}{768}\zeta(5)\pi^4 + \tfrac{19}{256}\zeta(7)\pi^2 - \tfrac{789}{512}\zeta(9)$$
$$S(2,5,2) = -\tfrac{7}{384}\zeta(5)\pi^4 - \tfrac{63}{256}\zeta(7)\pi^2 + \tfrac{2307}{512}\zeta(9)$$
$$S(2,6,1) = \tfrac{31}{40320}\zeta(3)\pi^6 + \tfrac{7}{768}\zeta(5)\pi^4 + \tfrac{21}{128}\zeta(7)\pi^2 - \tfrac{769}{256}\zeta(9)$$
$$S(3,3,3) = \tfrac{1}{128}\zeta(7)\pi^2 + \tfrac{5}{128}\zeta(9)$$
$$S(3,4,2) = \tfrac{7}{768}\zeta(5)\pi^4 + \tfrac{21}{256}\zeta(7)\pi^2 - \tfrac{769}{512}\zeta(9)$$
$$S(3,5,1) = -\tfrac{7}{768}\zeta(5)\pi^4 - \tfrac{21}{128}\zeta(7)\pi^2 + \tfrac{769}{256}\zeta(9)$$
$$S(4,4,1) = \tfrac{7}{384}\zeta(5)\pi^4 + \tfrac{21}{128}\zeta(7)\pi^2 - \tfrac{769}{256}\zeta(9)$$

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