ASYMPTOTICALLY EXACT A POSTERIORI ESTIMATORS FOR THE POINTWISE GRADIENT ERROR ON EACH ELEMENT IN IRREGULAR MESHES.
PART II: THE PIECEWISE LINEAR CASE

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Abstract. We extend results from Part I about estimating gradient errors elementwise a posteriori, given there for quadratic and higher elements, to the piecewise linear case. The key to our new result is to consider certain technical estimates for differences in the error, \( e(x_1) - e(x_2) \), rather than for \( e(x) \) itself. We also give a posteriori estimators for second derivatives on each element.

1. Introduction

As in Part I, we consider a second order elliptic partial differential equation with a natural homogeneous Neumann conormal boundary condition. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with a smooth boundary and, for simplicity of presentation at certain points in our present arguments, we now assume it is also convex. The bilinear form on \( W^{1,2}_0(\Omega) \) associated with the partial differential equation,

\[
A(v, w) = \int_{\Omega} \left( \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + \sum_{i=1}^{N} b_i(x) \frac{\partial v}{\partial x_i} w + c(x) vw \right) dx,
\]

is assumed to have smooth coefficients on \( \overline{\Omega} \) and, again for simplicity of presentation, to be coercive. I.e., there is \( c_{\text{coer}} > 0 \) such that \( c_{\text{coer}} \|v\|_{W^{1,2}_0(\Omega)}^2 \leq A(v, v) \), for all \( v \in W^{1,2}_0(\Omega) \).

Now consider approximation of the solution \( u \) to the problem \( A(u, \varphi) = (f, \varphi) \equiv \int_{\Omega} f \varphi dx \), for all \( \varphi \in W^{1,2}_0(\Omega) \). For \( 0 < h < 1 \), let \( S_h \) be the subspace of \( W^{1,2}_0(\Omega) \) consisting of continuous piecewise linear functions defined on globally quasi-uniform and globally shape-regular simplicial triangulations of \( \Omega \) that fit \( \partial \Omega \) exactly. Thus, elements with curved faces are allowed at the boundary. Let \( u_h \in S_h \) be the standard Galerkin finite element approximation of \( u \) defined by \( A(u_h, \varphi) = (f, \varphi) \), for all \( \varphi \in S_h \), so that

\[
(1.1) \quad A(u - u_h, \varphi) = 0, \quad \text{for all } \varphi \in S_h.
\]

Our primary aim is to study asymptotically exact a posteriori estimators for \( \| \nabla e \|_{L^\infty(\tau)} \), \( e = u - u_h \), the maximum norm of the gradient error on any given
element. The problem of estimating second derivatives of \( u \) will also be studied. Our estimators for the gradient error will be of the form
\[
\mathcal{E}(\tau) = \|\nabla u_h - \mathcal{G}_H u_h\|_{L_\infty(\tau)},
\]
where \( \mathcal{G}_H v \) is an averaging operator that will be defined in terms of a domain \( d_H \) which includes \( \tau \) and is of diameter \( H \), for some \( H \geq 2h \). We shall assume that \( \mathcal{G}_H \) has the following properties:
\[
\mathcal{G}_H 1 = 0, \quad \text{and} \quad \|\nabla v - \mathcal{G}_H v\|_{L_\infty(\tau)} \leq C_\mathcal{G} H^2\|v\|_{W^2_2(d_H)}, \quad \text{for} \quad v \in C^3(\overline{d}_H),
\]
and
\[
\|\mathcal{G}_H v\|_{L_\infty(\tau)} \leq C_\mathcal{G} H^{-1}\|v\|_{L_\infty(d_H)}, \quad \text{for} \quad v \in C(\overline{d}_H).
\]
The inequality in (1.3) says that \( \mathcal{G}_H v \) is locally a second order (in \( H \)) approximation to the gradient, and (1.4) may be interpreted as a smoothing property. We note that, for a given \( d_H \), any element \( \tau \) in it will work. I.e., it is not necessary to change \( d_H \) for each and every \( \tau \). We shall give three examples of operators satisfying these properties. The verification that they hold is essentially given in [3].

**Example 1.** Let \( d_H \subseteq \Omega \) be such that \( d_H \) contains a ball \( \mathcal{B} \) of radius \( C_1 \), \( C_1 > 0 \) and is contained in a concentric ball \( \mathcal{B}' \) of radius \( C_1 H \), and where \( \text{meas}(\partial d_H) \leq C_1 H^{N-1} \). In particular \( d_H \) could be a mesh domain. Let \( \Pi_1(d_H) \) be the space of first degree, affine polynomials restricted to \( \overline{d}_H \). We define \( \mathcal{G}_H v = P^1_H \nabla v \), where \( P^1_H \) is the componentwise \( L_2 \)-projection into \( \Pi_1(d_H) \).

**Example 2.** Let \( d_H \) be as in Example 1 let \( \Pi_2(d_H) \) be the quadratic polynomials, and let \( \mathcal{G}_H v = \nabla P^2_H v \). In this example we could replace the \( L_2 \)-projection \( P^2_H \) by a suitable approximation, such as interpolating \( v \) into \( \Pi_2(d_H) \) at \( N^2/2 + 3N/2 + 1 \) appropriately placed points or using a discrete \( L_2 \)-projection at a greater number of points.

**Example 3.** For each \( x \in \tau \), let \( \mathcal{G}_H v(x) = (Q^H_1 v(x), \ldots, Q^H_N v(x)) \), where each \( Q^H_i \) is a second order accurate difference approximation to \( \frac{\partial}{\partial x_i} \). If \( \text{dist}(\tau, \partial \Omega) \geq C_2 H \), then we may take each \( Q^H_i v(x) = \frac{v(x + H e_i) - v(x - H e_i)}{2H} \),
the standard second order accurate centered difference approximation to \( \frac{\partial v}{\partial x_i} \). Here, \( e_i \) is the unit vector in the positive \( x_i \) direction. Near the boundary, one-sided differences may be employed, but we shall not give details.

Our main result, which is an extension of Theorem 2.1 of [3] to the piecewise linear case, is as follows.

**Theorem 1.1.** Fix \( 0 < \varepsilon < 1 \). Let \( \mathcal{G}_H \) satisfy (1.3) and (1.4). There exists a constant \( C_1 \) such that with
\[
m := C_1 \left( \left( \frac{H}{h} \right)^2 h^\varepsilon + \left( \frac{h}{H} \right)^\varepsilon \ln \left( \frac{H}{h} \right) \right),
\]
and \( u \) and \( u_h \in S_h \) satisfying (1.1), one of the following two alternatives holds for each element \( \tau \).
Combining this with the estimate above, we see that condition (1.3) says that the best that we can expect even asymptotically exact for the gradient error on the element.

Furthermore, if \( H = H(h) \) is chosen so that \( m < 1 \), then our estimator given in (1.2) is equivalent to the real gradient error on the element,

\[
\frac{1}{1+m} \mathcal{E}(\tau) \leq \| \nabla e \|_{L_\infty(\tau)} \leq \frac{1}{1-m} \mathcal{E}(\tau).
\]

If \( H(h) \) is chosen so that \( m \to 0 \) as \( h \to 0 \), our estimator is asymptotically exact for the gradient error on the element.

Alternative II. Suppose that (1.5) does not hold, i.e.,

\[
\| \nabla e \|_{L_\infty(\tau)} < h^{1-\varepsilon} \| u \|_{W^3(\Omega)}.
\]

In this case \( \| \nabla e \|_{L_\infty(\tau)} \) is “small” with

\[
\| \nabla e \|_{L_\infty(\tau)} \leq C_1 h^{2-\varepsilon} \| u \|_{W^3(\Omega)},
\]

and our error indicator is similarly “small”,

\[
\mathcal{E}(\tau) \leq (C_1 + m) h^{2-\varepsilon} \| u \|_{W^3(\Omega)}.
\]

In the above, \( C_1 \) depends on \( N, \Omega, c_{\text{coer}}, a_{ij}, b_i, c \), constants of quasi-uniformity and shape-regularity for the meshes, \( C_G \), and \( \varepsilon \).

Remark 1.1. In the case that \( \mathcal{G}_H u_h \) gives an asymptotically exact estimator for \( \nabla u \), it is a better approximation to \( \nabla u \) than \( \nabla u_h \) is.

Remark 1.2. For a discussion of how results of this type relate to the previous literature on a posteriori estimates, and for a fuller description of the general framework of the methods considered here, see Part I [5].

Remark 1.3. Here we shall make two comments that may give some insight into the role that condition (1.5) plays towards insuring that the locally defined estimator \( \mathcal{E}(\tau) \) is asymptotically exact. First of all, it follows from (1.5), and Lemmas 2.1 and 2.3 below that, for \( h \) sufficiently small,

\[
C_* h \| u \|_{W^3(\Omega)} \leq \| \nabla e \|_{L_\infty(\tau)} \leq C^* h \| u \|_{W^3(\tau)}.
\]

This says that the finite element gradient error on \( \tau \) has a similar type of local behavior as the interpolation error. So it is plausible that a locally defined estimator may be effective. Secondly, asymptotic exactness follows if we can show that, roughly speaking, \( \mathcal{G}_H u_h \) is a better approximation to \( \nabla u \) than is \( \nabla u_h \). Now condition (1.5) says that the best that we can expect even \( \mathcal{G}_H u \) to approximate \( \nabla u \) is \( O(H^2) \), or roughly \( O(h^2) \) (since we roughly want \( H \) to be only slightly larger than \( h \)). The “worst” case of condition (1.5) occurs when \( \| u \|_{W^3(\Omega)} = h^{1-\varepsilon} \| u \|_{W^3(\Omega)} \). Combining this with the estimate above, we see that

\[
\| \nabla e \|_{L_\infty(\tau)} \leq C h^{2-\varepsilon} \| u \|_{W^3(\Omega)}.
\]

Thus, at least for \( \varepsilon \) small, we are at a point past which we have no reason to expect that \( \mathcal{G}_H u_h \) would be much closer to \( \nabla u \) than \( \nabla u_h \) is.
In general, if (1.3) is violated, it may happen that $|u|_{W^2_2(\tau)} \leq h|u|_{W^2_2(\Omega)}$. In such a situation, Lemma 2.1 actually gives
\[ \|\nabla e\|_{L^\infty(\tau)} \leq C'h^{2-\epsilon'}|u|_{W^2_2(\Omega)}, \]
for any $\epsilon' > 0$. In this case, surely we have no reason to expect that $G_H u_h$ would be a much better approximation.

We now turn to estimates for the second derivatives of $u$ on an element $\tau$. Of course, since the second derivatives of the piecewise linear function $u_h$ are zero (the second derivatives being regarded in an elementwise fashion), here we are not speaking about estimating errors, but merely about the size itself of second derivatives of $u$. Let $D^\beta u$, $|\beta| = 2$, denote any second order derivative, and let $G_H^{(\beta)} u_h$ denote the analogue coming from taking a derivative, elementwise, of a component of $G_H u_h$. (For the mixed derivatives, two choices are possible.)

To be precise, let $|u|_{W^2_2(\tau)} = \max_{|\beta|=2} \|D^\beta u\|_{L^\infty(\tau)}$, and similarly let
\[ E^{(2)}(\tau) = \max_{|\beta|=2} \|G_H^{(\beta)} u_h\|_{L^\infty(\tau)}. \]

We assume that
\[ E = C_H |\nabla u|_{W^2_2(\Omega)} \leq C_H |\nabla u|_{L^\infty(\Omega)}, \]
and
\[ E \leq C_H |\nabla u|_{W^2_2(\Omega)}, \]
for $v \in C^2((d_H))$.

It is easy to check that the operators in Examples 1.3 satisfy (1.12) and (1.13). (The verification of (1.13) in the case of Example 2 uses that $G_H 1 = 0$.)

We now have:

**Theorem 1.2.** Fix $0 < \epsilon < 1$. Assume that (1.12) and (1.13) hold. There exists a constant $C_2$ such that with
\[ \bar{m} := C_2 \left( \frac{H}{h} \right)^\epsilon + \frac{h}{H} \]
and $u$ and $u_h$ satisfying (1.1), one of the following two alternatives holds for each element $\tau$.

**Alternative I.** Suppose (1.13) holds. In this case,
\[ \|D^\beta u - G_H^{(\beta)} u_h\|_{L^\infty(\tau)} \leq \bar{m}|u|_{W^2_2(\tau)}, \]
for each $|\beta| = 2$.

If $H = H(h)$ is chosen so that $\bar{m} < 1$, then our estimator given in (1.11) is an equivalent estimator,
\[ \frac{1}{1 + \bar{m}} E^{(2)}(\tau) \leq |u|_{W^2_2(\tau)} \leq \frac{1}{1 - \bar{m}} E^{(2)}(\tau). \]

If $H(h)$ is chosen so that $\bar{m} \to 0$ as $h \to 0$, then the estimator is asymptotically exact.

**Alternative II.** Suppose (1.8) holds. Then, of course, $|u|_{W^2_2(\tau)} \leq h^{1-\epsilon}|u|_{W^3_2(\Omega)}$ is already “small”. We then assert that our estimator is similarly “small”,
\[ E^{(2)}(\tau) \leq (1 + \bar{m})h^{1-\epsilon}|u|_{W^3_2(\Omega)}. \]
Remark 1.4. For simplicity of presentation, we have only considered estimators for second derivatives of $u$ of the form: take an elementwise derivative of $G_H u_h$. Certainly, instead of using straight differentiation, one could use iterated variants of $G_H$, cf., e.g., Eriksson and Johnson [2]. Results similar to Theorem 1.2 are readily derived.

Remark 1.5. In the case that (1.5) holds and $\bar{m} \to 0$ as $h \to 0$, then (1.14) says that $G_H^{(\beta)} u_h$ converges to $D^\beta u$ on $\tau$.

An outline of the rest of this note is as follows. In Section 2 we collect two a priori estimates, following Schatz [4] and Schatz and Wahlbin [5], and some other preliminary material. In particular, following Demlow [1], we elucidate why the piecewise linear case was not included in Part I of this paper. In Section 3 we prove Theorems 1.1 and 1.2.

2. SOME PRELIMINARES

From [4] we have the following asymptotic error expansion inequality.

Lemma 2.1. For any $\varepsilon > 0$, there exists a constant $C$ such that

$$|e(x)| + |\nabla e(x)| \leq Ch \left( \max_{|\beta|=2} |D^\beta u(x)| + h^{1-\varepsilon} \|u\|_{W^3_\infty(\Omega)} \right).$$

A key estimate used in [3] was a similar expansion inequality for $e(x)$ alone, proven in [4] for piecewise quadratics or higher order elements. This estimate is of the form

$$|e(x)| \leq C h^r \left( \max_{|\alpha|=r} |\nabla^\alpha u(x)| + h^{1-\varepsilon} \|u\|_{W^{r+1}_\infty(\Omega)} \right),$$

for $r \geq 3$, where $r = 3$ corresponds to piecewise quadratics, etc. In [1] it has been shown, via a simple example in one space dimension, that such an estimate is impossible in the piecewise linear case, $r = 2$. As a substitute, we shall instead use the following recent result from [5].

For $x_1, x_2 \in \Omega$, let $\rho = h + |x_2 - x_1|$ and $\overline{\rho} = (x_1 + x_2)/2$.

Lemma 2.2. For any $\varepsilon > 0$, there exists a constant $C$ such that

$$|e(x_2) - e(x_1)| \leq C h^2 \left( 1 + \ln(\rho/h) \right) \left( \max_{|\beta|=2} |D^\beta u(\overline{\rho})| + \rho^{1-\varepsilon} \|u\|_{W^3_\infty(\Omega)} \right).$$

We next record a trivial fact which however hints at how Lemma 2.2 will come into play.

Lemma 2.3. Let $G_H$ satisfy (1.3) and (1.4). Then for any point $z \in d_H$,

$$\|G_H v\|_{L_\infty(\tau)} \leq \frac{C}{H} \|v(\cdot) - v(z)\|_{L_\infty(d_H)}.$$  

Proof. Since $G_H 1 = 0$, this follows from (1.4).

Finally, essentially from approximation theory, there is a lower bound for gradient errors on an element; see [3, Lemma 3.3], for a proof.

Lemma 2.4. There exists a constant $c > 0$ such that

$$c (h |u|_{W^2_\infty(\tau)} - h^2 \|u\|_{W^3_\infty(\tau)}) \leq \|\nabla e\|_{L_\infty(\tau)}.$$
3. Proofs of the theorems

Proof of Theorem 1.1. We have, by use of (1.3) and Lemma 2.3 with any \( z \in d_H, \)
\[
\|\nabla u - G_H u_h\|_{L^\infty(\tau)} \leq \|\nabla u - G_H u\|_{L^\infty(\tau)} + \|G_H e\|_{L^\infty(\tau)} \\
\leq CH^2\|\nu\|_{W^2_d(\Omega)} + \frac{C}{H}\|e(\cdot) - e(z)\|_{L^\infty(d_H)}.
\]
(3.1)

Let \( x_0 \) be a point where \( |e(\cdot) - e(z)|_{L^\infty(d_H)} \) is taken on, and let \( \bar{x} = (x_0 + z)/2. \)
Then, by Lemma 2.2 and the mean-value theorem, since \( \text{dist}(\bar{x}, \tau) \leq H, \)
\[
|e(\cdot) - e(z)|_{L^\infty(d_H)} \leq C\epsilon(h/\epsilon)\left(\max_{|\beta| \leq 2} |D^\beta u(\bar{x})| + H^{1-\epsilon}\|u\|_{W^2_d(\Omega)}\right) \\
\leq C\epsilon(h/\epsilon)((u\nu)_{W^2_d(\tau)} + H^{1-\epsilon}\|u\|_{W^2_d(\Omega)}).
\]
Thus, from (3.1),
\[
\|\nabla u - G_H u_h\|_{L^\infty(\tau)} \leq CH^2\|\nu\|_{W^2_d(\Omega)} + C\frac{h^2}{H}\left(\ln(H/h)\|u\|_{W^2_d(\tau)}
\]
\[
+ C\frac{h^2}{H\epsilon}\ln(H/h)\|\nu\|_{W^2_d(\Omega)}).
\]
(3.2)

In case of Alternative I, \( \|u\|_{W^2_d(\Omega)} \leq h^{-1+\epsilon}\|u\|_{W^2_d(\sigma)}, \) we hence obtain
\[
\|\nabla u - G_H u_h\|_{L^\infty(\tau)} \leq C\left(\frac{H^2}{h^{1-\epsilon}} + \frac{h^2}{H}\left(\ln(H/h)\|u\|_{W^2_d(\tau)}
\]
\[
+ C\frac{h^2}{H\epsilon}\ln(H/h)\|\nu\|_{W^2_d(\Omega)}\right)\|\nu\|_{W^2_d(\tau)}.
\]
(3.3)

From Lemma 2.3 in our present Alternative I, for \( h \) small, \( cH\|u\|_{W^2_d(\tau)} \leq \|\nabla u\|_{L^\infty(\tau)}, \)
and hence from (3.3),
\[
\|\nabla u - G_H u_h\|_{L^\infty(\tau)} \leq C\left(\left(\frac{H}{h}\right)^2 \|\nabla u\|_{L^\infty(\tau)} \right)\|\nu\|_{W^2_d(\tau)}.
\]
This is (1.10). Obviously, (1.7) follows from this by the triangle inequality.

In the case of Alternative II, \( |u\|_{W^2_d(\Omega)} \leq h^{-1-\epsilon}\|u\|_{W^2_d(\Omega)}, \) we have from Lemma 2.1
\[
\|\nabla e\|_{L^\infty(\tau)} \leq C\epsilon(h/\epsilon)\|u\|_{W^2_d(\Omega)},
\]
which is (1.9). From (3.2) we now get
\[
\|\nabla u - \nabla G_H u_h\|_{L^\infty(\tau)} \leq C\left(\frac{H^2}{h^{1-\epsilon}} + \frac{1}{H}\|\nabla u\|_{L^\infty(\tau)} \right)\|\nu\|_{W^2_d(\Omega)}
\]
\[
= C\left(\frac{H^2}{h^{1-\epsilon}} + \frac{1}{H}\|\nabla u\|_{L^\infty(\tau)} \right)\|\nu\|_{W^2_d(\Omega)}
\]
\[
\leq mh^{2-\epsilon}\|\nu\|_{W^2_d(\Omega)},
\]
and hence (1.10) also follows. This completes the proof of Theorem 1.1. \( \square \)

Proof of Theorem 1.2. We have, using (1.12) and (1.13),
\[
\|D^\beta u - G_H^\beta u_h\|_{L^\infty(\tau)} \leq \|D^\beta u - G_H^\beta u\|_{L^\infty(\tau)} + \|G_H e\|_{L^\infty(\tau)} \\
\leq CH\|\nu\|_{W^2_d(\tau)} + C\frac{h^2}{H}\|\nu\|_{W^2_d(d_H)}.
\]
(3.4)
From Lemma 2.1, using the mean-value theorem, we find that
\[ \|e\|_{W^1_2(dh)} \leq Ch\left(\|u\|_{W^2_2(\tau)} + H^{1-\varepsilon}\|u\|_{W^3_2(\Omega)}\right). \]
Hence, from (3.4),
\[ (3.5) \quad \|D^\beta u - G_H^{(\beta)} u_h\|_{L^\infty(\tau)} \leq CH\|u\|_{W^2_2(\Omega)} + C\frac{h}{H}\|u\|_{W^3_2(\tau)} + C\frac{H}{h}\varepsilon\|u\|_{W^3_2(\Omega)}. \]
Thus, in case of Alternative I,
\[ \|u\|_{W^2_2(\Omega)} \leq h^{-1+\varepsilon}\|u\|_{W^3_2(\tau)}, \]
and Theorem 1.2, (1.15) and the asymptotic equivalence, follows in this case.
In Alternative II, \( |u|_{W^2_2(\tau)} \leq h^{1-\varepsilon}\|u\|_{W^3_2(\Omega)} \), and (3.5) gives
\[ \|D^\beta u - G_H^{(\beta)} u_h\|_{L^\infty(\tau)} \leq \tilde{m}\|u\|_{W^3_2(\Omega)} \]
Via the triangle inequality, this proves (1.16) and completes the proof of Theorem 1.2.

\section*{References}
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