

ASYMPTOTICALLY EXACT A POSTERIORI ESTIMATORS
FOR THE POINTWISE GRADIENT ERROR
ON EACH ELEMENT IN IRREGULAR MESHES.
PART II: THE PIECEWISE LINEAR CASE

ALFRED H. SCHATZ AND LARS B. WAHLBIN

ABSTRACT. We extend results from Part I about estimating gradient errors elementwise a posteriori, given there for quadratic and higher elements, to the piecewise linear case. The key to our new result is to consider certain technical estimates for *differences* in the error, $e(x_1) - e(x_2)$, rather than for $e(x)$ itself. We also give a posteriori estimators for second derivatives on each element.

1. INTRODUCTION

As in Part I, [3], we consider a second order elliptic partial differential equation with a natural homogeneous Neumann conormal boundary condition. Let Ω be a bounded domain in R^N with a smooth boundary and, for simplicity of presentation at certain points in our present arguments, we now assume it is also convex. The bilinear form on $W_2^1(\Omega)$ associated with the partial differential equation,

$$A(v, w) = \int_{\Omega} \left(\sum_{i,j=1}^N a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + \sum_{i=1}^N b_i(x) \frac{\partial v}{\partial x_i} w + c(x)vw \right) dx,$$

is assumed to have smooth coefficients on $\overline{\Omega}$ and, again for simplicity of presentation, to be coercive. I.e., there is $c_{\text{coer}} > 0$ such that $c_{\text{coer}} \|v\|_{W_2^1(\Omega)}^2 \leq A(v, v)$, for all $v \in W_2^1(\Omega)$.

Now consider approximation of the solution u to the problem $A(u, \varphi) = (f, \varphi) \equiv \int_{\Omega} f\varphi dx$, for all $\varphi \in W_2^1(\Omega)$. For $0 < h < 1$, let S_h be the subspace of $W_2^1(\Omega)$ consisting of continuous piecewise linear functions defined on globally quasi-uniform and globally shape-regular simplicial triangulations of Ω that fit $\partial\Omega$ exactly. Thus, elements with curved faces are allowed at the boundary. Let $u_h \in S_h$ be the standard Galerkin finite element approximation of u defined by $A(u_h, \varphi) = (f, \varphi)$, for all $\varphi \in S_h$, so that

$$(1.1) \quad A(u - u_h, \varphi) = 0, \quad \text{for all } \varphi \in S_h.$$

Our primary aim is to study asymptotically exact a posteriori estimators for $\|\nabla e\|_{L_{\infty}(\tau)}$, $e = u - u_h$, the maximum norm of the gradient error on any given

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element. The problem of estimating second derivatives of u will also be studied. Our estimators for the gradient error will be of the form

$$(1.2) \quad \mathcal{E}(\tau) = \|\nabla u_h - \mathcal{G}_H u_h\|_{L_\infty(\tau)},$$

where $\mathcal{G}_H v$ is an averaging operator that will be defined in terms of a domain d_H which includes τ and is of diameter H , for some $H \geq 2h$. We shall assume that \mathcal{G}_H has the following properties:

$$(1.3) \quad \mathcal{G}_H 1 = 0, \text{ and } \|\nabla v - \mathcal{G}_H v\|_{L_\infty(\tau)} \leq C_G H^2 \|v\|_{W_\infty^3(d_H)}, \text{ for } v \in C^3(\bar{d}_H),$$

and

$$(1.4) \quad \|\mathcal{G}_H v\|_{L_\infty(\tau)} \leq C_G H^{-1} \|v\|_{L_\infty(d_H)}, \text{ for } v \in C(\bar{d}_H).$$

The inequality in (1.3) says that $\mathcal{G}_H v$ is locally a second order (in H) approximation to the gradient, and (1.4) may be interpreted as a smoothing property. We note that, for a given d_H , any element τ in it will work. I.e., it is not necessary to change d_H for each and every τ . We shall give three examples of operators satisfying these properties. The verification that they hold is essentially given in [3].

Example 1. Let $d_H \subseteq \Omega$ be such that d_H contains a ball \underline{B} of radius $\underline{C}_1 H$, $\underline{C}_1 > 0$ and is contained in a concentric ball \bar{B} of radius $\bar{C}_1 H$, and where $\text{meas}(\partial d_H) \leq \bar{C}_1 H^{N-1}$. In particular d_H could be a mesh domain. Let $\Pi_1(d_H)$ be the space of first degree, affine polynomials restricted to \bar{d}_H . We define $\mathcal{G}_H v = P_H^1 \nabla v$, where P_H^1 is the componentwise L_2 -projection into $\Pi_1(d_H)$.

Example 2. Let d_H be as in Example 1, let $\Pi_2(d_H)$ be the quadratic polynomials, and let $\mathcal{G}_H v = \nabla P_H^2 v$. In this example we could replace the L_2 -projection P_H^2 by a suitable approximation, such as interpolating v into $\Pi_2(d_H)$ at $N^2/2 + 3N/2 + 1$ appropriately placed points or using a discrete L_2 -projection at a greater number of points.

Example 3. For each $x \in \tau$, let $\mathcal{G}_H v(x) = (Q_1^H v(x), \dots, Q_N^H v(x))$, where each Q_i^H is a second order accurate difference approximation to $\frac{\partial v}{\partial x_i}$. If $\text{dist}(\tau, \partial\Omega) \geq \bar{C}_2 H$, then we may take each

$$Q_i^H v(x) = \frac{v(x + H e_i) - v(x - H e_i)}{2H},$$

the standard second order accurate centered difference approximation to $\frac{\partial v}{\partial x_i}$. Here, e_i is the unit vector in the positive x_i direction. Near the boundary, one-sided differences may be employed, but we shall not give details.

Our main result, which is an extension of Theorem 2.1 of [3] to the piecewise linear case, is as follows.

Theorem 1.1. Fix $0 < \varepsilon < 1$. Let \mathcal{G}_H satisfy (1.3) and (1.4). There exists a constant C_1 such that with

$$m := C_1 \left(\left(\frac{H}{h} \right)^2 h^\varepsilon + \left(\frac{h}{H} \right)^\varepsilon \ln \left(\frac{H}{h} \right) \right),$$

and u and $u_h \in S_h$ satisfying (1.1), one of the following two alternatives holds for each element τ .

Alternative I. Suppose that on the element τ , the function u satisfies

$$(1.5) \quad |u|_{W_\infty^2(\tau)} \geq h^{1-\varepsilon} \|u\|_{W_\infty^3(\Omega)}.$$

In this case

$$(1.6) \quad \|\nabla u - \mathcal{G}_H u_h\|_{L_\infty(\tau)} \leq m \|\nabla e\|_{L_\infty(\tau)}.$$

If $H = H(h)$ is chosen so that $m < 1$, then our estimator given in (1.2) is equivalent to the real gradient error on the element,

$$(1.7) \quad \frac{1}{1+m} \mathcal{E}(\tau) \leq \|\nabla e\|_{L_\infty(\tau)} \leq \frac{1}{1-m} \mathcal{E}(\tau).$$

Furthermore, if $H(h)$ is chosen so that $m \rightarrow 0$ as $h \rightarrow 0$, our estimator is asymptotically exact for the gradient error on the element.

Alternative II. Suppose that (1.5) does not hold, i.e.,

$$(1.8) \quad |u|_{W_\infty^2(\tau)} < h^{1-\varepsilon} \|u\|_{W_\infty^3(\Omega)}.$$

In this case $\|\nabla e\|_{L_\infty(\tau)}$ is “small” with

$$(1.9) \quad \|\nabla e\|_{L_\infty(\tau)} \leq C_1 h^{2-\varepsilon} \|u\|_{W_\infty^3(\Omega)},$$

and our error indicator is similarly “small”,

$$(1.10) \quad \mathcal{E}(\tau) \leq (C_1 + m) h^{2-\varepsilon} \|u\|_{W_\infty^3(\Omega)}.$$

In the above, C_1 depends on $N, \Omega, c_{\text{coer}}, a_{ij}, b_i, c$, constants of quasi-uniformity and shape-regularity for the meshes, C_G , and ε .

Remark 1.1. In the case that $\mathcal{G}_H u_h$ gives an asymptotically exact estimator for ∇u , it is a better approximation to ∇u than ∇u_h is.

Remark 1.2. For a discussion of how results of this type relate to the previous literature on a posteriori estimates, and for a fuller description of the general framework of the methods considered here, see Part I [3].

Remark 1.3. Here we shall make two comments that may give some insight into the role that condition (1.5) plays towards insuring that the locally defined estimator $\mathcal{E}(\tau)$ is asymptotically exact. First of all, it follows from (1.5), and Lemmas 2.1 and 2.4 below that, for h sufficiently small,

$$C_* h |u|_{W_\infty^2(\tau)} \leq \|\nabla e\|_{L_\infty(\tau)} \leq C^* h |u|_{W_\infty^2(\tau)}.$$

This says that the finite element gradient error on τ has a similar type of local behavior as the interpolation error. So it is plausible that a locally defined estimator may be effective. Secondly, asymptotic exactness follows if we can show that, roughly speaking, $\mathcal{G}_H u_h$ is a better approximation to ∇u than is ∇u_h . Now condition (1.3) says that the best that we can expect even $\mathcal{G}_H u$ to approximate ∇u is $O(H^2)$, or roughly $O(h^2)$ (since we roughly want H to be only slightly larger than h). The “worst” case of condition (1.5) occurs when $|u|_{W_\infty^2(\tau)} = h^{1-\varepsilon} \|u\|_{W_\infty^3(\Omega)}$. Combining this with the estimate above, we see that

$$\|\nabla e\|_{L_\infty(\tau)} \leq C h^{2-\varepsilon} \|u\|_{W_\infty^3(\Omega)}.$$

Thus, at least for ε small, we are at a point past which we have no reason to expect that $\mathcal{G}_H u_h$ would be much closer to ∇u than ∇u_h is.

In general, if (1.5) is violated, it may happen that $|u|_{W_\infty^2(\tau)} \leq h\|u\|_{W_\infty^3(\Omega)}$. In such a situation, Lemma 2.1 actually gives

$$\|\nabla e\|_{L_\infty(\tau)} \leq C'h^{2-\varepsilon'}\|u\|_{W_\infty^3(\Omega)},$$

for any $\varepsilon' > 0$. In this case, surely we have no reason to expect that $\mathcal{G}_H u_h$ would be a much better approximation.

We now turn to estimates for the second derivatives of u on an element τ . Of course, since the second derivatives of the piecewise linear function u_h are zero (the second derivatives being regarded in an elementwise fashion), here we are not speaking about estimating errors, but merely about the size itself of second derivatives of u . Let $D^\beta u$, $|\beta| = 2$, denote any second order derivative, and let $\mathcal{G}_H^{(\beta)} u_h$ denote the analogue coming from taking a derivative, elementwise, of a component of $\mathcal{G}_H u_h$. (For the mixed derivatives, two choices are possible.)

To be precise, let $|u|_{W_\infty^2(\tau)} = \max_{|\beta|=2} \|D^\beta u\|_{L_\infty(\tau)}$, and similarly let

$$(1.11) \quad \mathcal{E}^{(2)}(\tau) = \max_{|\beta|=2} \|\mathcal{G}_H^{(\beta)} u_h\|_{L_\infty(\tau)}.$$

We assume that

$$(1.12) \quad \|\nabla v - \mathcal{G}_H v\|_{W_\infty^1(\tau)} \leq C_G H \|v\|_{W_\infty^3(d_H)}, \quad \text{for } v \in C^3(\bar{d}_H),$$

and

$$(1.13) \quad \|\nabla \mathcal{G}_H v\|_{L_\infty(\tau)} \leq C_G H^{-1} \|v\|_{W_\infty^1(d_H)}, \quad \text{for } v \in W_\infty^1(\bar{d}_H).$$

It is easy to check that the operators in Examples 1–3 satisfy (1.12) and (1.13). (The verification of (1.13) in the case of Example 2 uses that $\mathcal{G}_H 1 = 0$.)

We now have:

Theorem 1.2. *Fix $0 < \varepsilon < 1$. Assume that (1.12) and (1.13) hold. There exists a constant C_2 such that with*

$$\tilde{m} := C_2 \left(\left(\frac{H}{h} \right) h^\varepsilon + \left(\frac{h}{H} \right)^\varepsilon \right)$$

and u and u_h satisfying (1.1), one of the following two alternatives holds for each element τ .

Alternative I. Suppose that (1.5) holds. In this case,

$$(1.14) \quad \|D^\beta u - \mathcal{G}_H^{(\beta)} u_h\|_{L_\infty(\tau)} \leq \tilde{m} |u|_{W_\infty^2(\tau)}, \quad \text{for each } |\beta| = 2.$$

If $H = H(h)$ is chosen so that $\tilde{m} < 1$, then our estimator given in (1.11) is an equivalent estimator,

$$(1.15) \quad \frac{1}{1 + \tilde{m}} \mathcal{E}^{(2)}(\tau) \leq |u|_{W_\infty^2(\tau)} \leq \frac{1}{1 - \tilde{m}} \mathcal{E}^{(2)}(\tau).$$

If $H(h)$ is chosen so that $\tilde{m} \rightarrow 0$ as $h \rightarrow 0$, then the estimator is asymptotically exact.

Alternative II. Suppose (1.8) holds. Then, of course, $|u|_{W_\infty^2(\tau)} \leq h^{1-\varepsilon} \|u\|_{W_\infty^3(\Omega)}$ is already “small”. We then assert that our estimator is similarly “small”,

$$(1.16) \quad \mathcal{E}^{(2)}(\tau) \leq (1 + \tilde{m}) h^{1-\varepsilon} \|u\|_{W_\infty^3(\Omega)}.$$

Remark 1.4. For simplicity of presentation, we have only considered estimators for second derivatives of u of the form: take an elementwise derivative of $\mathcal{G}_H u_h$. Certainly, instead of using straight differentiation, one could use iterated variants of \mathcal{G}_H , cf., e.g., Eriksson and Johnson [2]. Results similar to Theorem 1.2 are readily derived.

Remark 1.5. In the case that (1.5) holds and $\tilde{m} \rightarrow 0$ as $h \rightarrow 0$, then (1.14) says that $\mathcal{G}_H^{(\beta)} u_h$ converges to $D^\beta u$ on τ .

An outline of the rest of this note is as follows. In Section 2 we collect two a priori estimates, following Schatz [4] and Schatz and Wahlbin [5], and some other preliminary material. In particular, following Demlow [1], we elucidate why the piecewise linear case was not included in Part I of this paper. In Section 3 we prove Theorems 1.1 and 1.2.

2. SOME PRELIMINARES

From [4] we have the following asymptotic error expansion inequality.

Lemma 2.1. *For any $\varepsilon > 0$, there exists a constant C such that*

$$|e(x)| + |\nabla e(x)| \leq Ch \left(\max_{|\beta|=2} |D^\beta u(x)| + h^{1-\varepsilon} \|u\|_{W_\infty^3(\Omega)} \right).$$

A key estimate used in [3] was a similar expansion inequality for $e(x)$ alone, proven in [4] for piecewise quadratics or higher order elements. This estimate is of the form

$$|e(x)| \leq Ch^r \left(\max_{|\alpha|=r} |D^\alpha u(x)| + h^{1-\varepsilon} \|u\|_{W_\infty^{r+1}(\Omega)} \right), \quad \text{for } r \geq 3,$$

where $r = 3$ corresponds to piecewise quadratics, etc. In [1] it has been shown, via a simple example in one space dimension, that such an estimate is *impossible* in the piecewise linear case, $r = 2$. As a substitute, we shall instead use the following recent result from [5].

For $x_1, x_2 \in \Omega$, let $\rho = h + |x_2 - x_1|$ and $\bar{x} = (x_1 + x_2)/2$.

Lemma 2.2. *For any $\varepsilon > 0$, there exists a constant C such that*

$$|e(x_2) - e(x_1)| \leq Ch^2 (1 + \ln(\rho/h)) \left(\max_{|\beta|=2} |D^\beta u(\bar{x})| + \rho^{1-\varepsilon} \|u\|_{W_\infty^3(\Omega)} \right).$$

We next record a trivial fact which however hints at how Lemma 2.2 will come into play.

Lemma 2.3. *Let \mathcal{G}_H satisfy (1.3) and (1.4). Then for any point $z \in d_H$,*

$$\|\mathcal{G}_H v\|_{L_\infty(\tau)} \leq \frac{C}{H} \|v(\cdot) - v(z)\|_{L_\infty(d_H)}.$$

Proof. Since $\mathcal{G}_H 1 = 0$, this follows from (1.4). \square

Finally, essentially from approximation theory, there is a lower bound for gradient errors on an element; see [3, Lemma 3.3], for a proof.

Lemma 2.4. *There exists a constant $c > 0$ such that*

$$c(h|u|_{W_\infty^2(\tau)} - h^2\|u\|_{W_\infty^3(\tau)}) \leq \|\nabla e\|_{L_\infty(\tau)}.$$

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. We have, by use of (1.3) and Lemma 2.3, with any $z \in d_H$,

$$(3.1) \quad \begin{aligned} \|\nabla u - \mathcal{G}_H u_h\|_{L_\infty(\tau)} &\leq \|\nabla u - \mathcal{G}_H u\|_{L_\infty(\tau)} + \|\mathcal{G}_H e\|_{L_\infty(\tau)} \\ &\leq CH^2 \|u\|_{W_\infty^3(\Omega)} + \frac{C}{H} \|e(\cdot) - e(z)\|_{L_\infty(d_H)}. \end{aligned}$$

Let x_0 be a point where $\|e(\cdot) - e(z)\|_{L_\infty(d_H)}$ is taken on, and let $\bar{x} = (x_0 + z)/2$. Then, by Lemma 2.2, and the mean-value theorem, since $\text{dist}(\bar{x}, \tau) \leq H$,

$$\begin{aligned} \|e(\cdot) - e(z)\|_{L_\infty(d_H)} &\leq Ch^2 (\ln(H/h)) \left(\max_{|\beta|=2} |D^\beta u(\bar{x})| + H^{1-\varepsilon} \|u\|_{W_\infty^3(\Omega)} \right) \\ &\leq Ch^2 (\ln H/h) (\|u\|_{W_\infty^2(\tau)} + H^{1-\varepsilon} \|u\|_{W_\infty^3(\Omega)}). \end{aligned}$$

Thus, from (3.1),

$$(3.2) \quad \begin{aligned} \|\nabla u - \mathcal{G}_H u_h\|_{L_\infty(\tau)} &\leq CH^2 \|u\|_{W_\infty^3(\Omega)} + C \frac{h^2}{H} \ln(H/h) |u|_{W_\infty^2(\tau)} \\ &\quad + C \frac{h^2}{H^\varepsilon} \ln(H/h) \|u\|_{W_\infty^3(\Omega)}. \end{aligned}$$

In case of Alternative I, $\|u\|_{W_\infty^3(\Omega)} \leq h^{-1+\varepsilon} |u|_{W_\infty^2(\tau)}$, we hence obtain

$$(3.3) \quad \begin{aligned} \|\nabla u - \mathcal{G}_H u_h\|_{L_\infty(\tau)} &\leq C \left(\frac{H^2}{h^{1-\varepsilon}} + \frac{h^2}{H} \ln(H/h) + \frac{h^{1+\varepsilon}}{H^\varepsilon} \ln(H/h) \right) |u|_{W_\infty^2(\tau)} \\ &\leq C \left(\frac{H^2}{h^{1-\varepsilon}} + \frac{h^{1+\varepsilon}}{H^\varepsilon} \ln(H/h) \right) |u|_{W_\infty^2(\tau)}. \end{aligned}$$

From Lemma 2.4, in our present Alternative I, for h small, $ch|u|_{W_\infty^2(\tau)} \leq \|\nabla e\|_{L_\infty(\tau)}$, and hence from (3.3),

$$\|\nabla u - \mathcal{G}_H u_h\|_{L_\infty(\tau)} \leq C \left(\left(\frac{H}{h} \right)^2 h^\varepsilon + \left(\frac{h}{H} \right)^\varepsilon \ln(H/h) \right) \|\nabla e\|_{L_\infty(\tau)}.$$

This is (1.6). Obviously, (1.7) follows from this by the triangle inequality.

In the case of Alternative II, $|u|_{W_\infty^2(\tau)} \leq h^{1-\varepsilon} \|u\|_{W_\infty^3(\Omega)}$, we have from Lemma 2.1,

$$\|\nabla e\|_{L_\infty(\tau)} \leq Ch^{2-\varepsilon} \|u\|_{W_\infty^3(\Omega)},$$

which is (1.9). From (3.2) we now get

$$\begin{aligned} \|\nabla u - \nabla \mathcal{G}_H u_h\|_{L_\infty(\tau)} &\leq C \left(H^2 + \frac{h^{3-\varepsilon}}{H} \ln(H/h) + \frac{h^2}{H^\varepsilon} \ln(H/h) \right) \|u\|_{W_\infty^3(\Omega)} \\ &= C \left(\left(\frac{H}{h} \right)^2 h^\varepsilon + \frac{h}{H} \ln(H/h) + \left(\frac{h}{H} \right)^\varepsilon \ln(H/h) \right) h^{2-\varepsilon} \|u\|_{W_\infty^3(\Omega)} \\ &\leq mh^{2-\varepsilon} \|u\|_{W_\infty^3(\Omega)}, \end{aligned}$$

and hence (1.10) also follows. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We have, using (1.12) and (1.13),

$$(3.4) \quad \begin{aligned} \|D^\beta u - \mathcal{G}_H^{(\beta)} u_h\|_{L_\infty(\tau)} &\leq \|D^\beta u - \mathcal{G}_H^{(\beta)} u\|_{L_\infty(\tau)} + \|\mathcal{G}_H^{(\beta)} e\|_{L_\infty(\tau)} \\ &\leq CH \|u\|_{W_\infty^3(\Omega)} + \frac{C}{H} \|e\|_{W_\infty^1(d_H)}. \end{aligned}$$

From Lemma 2.1, using the mean-value theorem, we find that

$$\|e\|_{W_\infty^1(d_H)} \leq Ch(|u|_{W_\infty^2(\tau)} + H^{1-\varepsilon}\|u\|_{W_\infty^3(\Omega)}).$$

Hence, from (3.4),

$$(3.5) \quad \|D^\beta u - \mathcal{G}_H^{(\beta)} u_h\|_{L_\infty(\tau)} \leq CH\|u\|_{W_\infty^3(\Omega)} + C\frac{h}{H}|u|_{W_\infty^2(\tau)} + C\frac{h}{H^\varepsilon}\|u\|_{W_\infty^3(\Omega)}.$$

Thus, in case of Alternative I, $\|u\|_{W_\infty^3(\Omega)} \leq h^{-1+\varepsilon}|u|_{W_\infty^2(\tau)}$,

$$\|D^\beta u - \mathcal{G}_H^{(\beta)} u_h\|_{L_\infty(\tau)} \leq \tilde{m}|u|_{W_\infty^2(\tau)},$$

and Theorem 1.2, (1.15) and the asymptotic equivalence, follows in this case.

In Alternative II, $|u|_{W_\infty^2(\tau)} \leq h^{1-\varepsilon}\|u\|_{W_\infty^3(\Omega)}$, and (3.5) gives

$$\begin{aligned} \|D^\beta u - \mathcal{G}_H^{(\beta)} u_h\|_{L_\infty(\tau)} &\leq C\left(H + \frac{h^{2-\varepsilon}}{H} + \frac{h}{H^\varepsilon}\right)\|u\|_{W_\infty^3(\Omega)} \\ &= C\left(\left(\frac{H}{h}\right)h^\varepsilon + \frac{h}{H} + \left(\frac{h}{H}\right)^\varepsilon\right)h^{1-\varepsilon}\|u\|_{W_\infty^3(\Omega)} \\ &\leq \tilde{m}h^{1-\varepsilon}\|u\|_{W_\infty^3(\Omega)}. \end{aligned}$$

Via the triangle inequality, this proves (1.16) and completes the proof of Theorem 1.2. \square

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DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853
E-mail address: schatz@math.cornell.edu

DEPARTMENT OF MATHEMATICS, CORNELL UNIVERSITY, ITHACA, NEW YORK 14853
E-mail address: wahlbin@math.cornell.edu