CONVERGENCE OF NONCONFORMING $V$-CYCLE AND $F$-CYCLE MULTIGRID ALGORITHMS
FOR SECOND ORDER ELLIPTIC BOUNDARY VALUE PROBLEMS

SUSANNE C. BRENNER

Abstract. The convergence of $V$-cycle and $F$-cycle multigrid algorithms with a sufficiently large number of smoothing steps is established for nonconforming finite element methods for second order elliptic boundary value problems.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain. Consider the variational problem of finding $u \in H^1_0(\Omega)$ such that
\begin{equation}
(1.1) \quad a(u, v) = F(v) \quad \forall \, v \in H^1_0(\Omega),
\end{equation}
where $F \in H^{-1}(\Omega)$ and
\begin{equation}
(1.2) \quad a(v, w) = \int_{\Omega} \left[ \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + r(x) vw \right] \, dx.
\end{equation}

We assume that $a_{ij}, r \in C^1(\Omega)$, $a_{12} = a_{21}$, $r \geq 0$ on $\Omega$, and
\begin{equation}
(1.3) \quad \sum_{i,j=1}^{2} a_{ij}(x) \xi_i \xi_j \geq c(\xi_1^2 + \xi_2^2) \quad \forall \, x \in \overset{\circ}{\Omega}, \xi_1, \xi_2 \in \mathbb{R},
\end{equation}
where $c$ is a positive constant. Under these conditions the bilinear form $a(\cdot, \cdot)$ on $H^1_0(\Omega) \times H^1_0(\Omega)$ is bounded and coercive, and (1.1) has a unique solution.

It is well known (cf. §5.C and §14.A of [32]) that there exists $\alpha \in (\frac{1}{2}, 1]$ such that the solution $u$ of (1.1) belongs to $H^{1+\alpha}(\Omega) \cap H^1_0(\Omega)$ whenever $F \in H^{-1+\alpha}(\Omega)$ and
\begin{equation}
(1.4) \quad \|u\|_{H^{1+\alpha}(\Omega)} \leq C_\Omega \|F\|_{H^{-1+\alpha}(\Omega)}.
\end{equation}

Approximate solutions of the variational problem (1.1) can be obtained by the finite element method (cf. [30, 25]). The resulting symmetric positive definite systems are sparse and can be solved efficiently by multigrid algorithms (cf. [36], [39], [7], [16], [49]). For the symmetric $V$-cycle algorithm with equal numbers of pre-smoothing and post-smoothing steps, the classical result by Braess and Hackbusch...
(cf. [35, 34, 33, 39, 9]) states that, in the case where $\alpha = 1$ (i.e., when $\Omega$ is convex),
\[ \gamma_k \leq \frac{C}{C + m} \quad \text{for } m = 1, 2, \ldots, \]
where $\gamma_k$ is the contraction number of the $k$-th level $V$-cycle algorithm in the norm
\[ \| \cdot \|_a = \sqrt{a(\cdot, \cdot)}, \]
$m$ is the number of pre-smoothing and post-smoothing steps, and $C$ (with or without subscripts) henceforth denotes a generic positive constant which is independent of $k$ and $m$.

The case where $\frac{1}{2} < \alpha < 1$ (i.e., when $\Omega$ has re-entrant corners) is more subtle. It was not until the early nineties, after a multiplicative theory for multilevel methods (cf. [13, 12]) had been developed, that the following result was established (cf. [57, 53, 10, 56, 11, 34, 41]):
\[ \gamma_k \leq \delta, \]
where $\delta \in (0, 1)$ is independent of $k$. The asymptotic behavior of $\gamma_k$ with respect to $m$ was studied in [22] by an additive convergence theory. It was shown that, for the $P_1$ finite element and with the Richardson relaxation scheme as smoother,
\[ \gamma_k \leq \frac{C}{m^\alpha} \quad \text{for } m \geq m_0, \]
where the positive integer $m_0$ is independent of $k$. It then follows easily from (1.6) and (1.7) that
\[ \gamma_k \leq \frac{C}{C + m^\alpha} \quad \text{for } m = 1, 2, \ldots. \]

In other words, a complete generalization of the result of Braess and Hackbusch to the case of less than full elliptic regularity has been obtained. The results in [22] were generalized to include other smoothers in [23].

In this paper we extend the theory in [22] to nonconforming finite elements and establish the same estimate (1.7). Since the $V$-cycle algorithm for nonconforming finite elements in general does not converge uniformly for $m = 1$, the estimate (1.7) is the best possible general estimate for such methods. As a by-product we also obtain similar estimates for nonconforming $F$-cycle algorithms (cf. [17, 40, 51]). As far as we know, this is the first convergence result for nonconforming $F$-cycle algorithms, even though it has been known for some time (cf. [50]) that these algorithms perform very well in practice. The theory developed in this paper can also be applied to fourth order problems (cf. [58]).

We also note in passing that most of the general convergence results for nonconforming multigrid algorithms were obtained for the $W$-cycle algorithm and the variable $V$-cycle preconditioner (cf. [18, 14, 21] and the references therein, and also [43, 47] for the cascadic multigrid algorithm). The only existent nonconforming $V$-cycle convergence results involve either special elements (cf. [24, 45, 16]), conforming finite element spaces on coarse grids (cf. [52, 55]), or the suboptimal Galerkin approach (cf. [29]).

The rest of the paper is organized as follows. In Section 2 we set up $V$-cycle and $F$-cycle algorithms for nonconforming finite elements in an abstract setting. The assumptions for the convergence theory are stated in Section 3. We then prove a strengthened Cauchy-Schwarz inequality in Section 4. The estimate (1.7) for $V$-cycle and $F$-cycle algorithms is established in Section 5. In Section 6 we show
that the assumptions in Section 3 can be verified within an abstract framework for nonconforming multigrid methods. Applications to concrete nonconforming finite elements are then given in Section 7.

For future reference we state here two simple inequalities:

(1.8a) \[ 2ab \leq (\theta a)^2 + (\theta^{-1}b)^2 \] for all \( a, b \in \mathbb{R}, \theta \in (0, 1) \),

(1.8b) \[ (a + b)^2 \leq (1 + \theta^2)a^2 + (1 + \theta^{-2})b^2 \] for all \( a, b \in \mathbb{R}, \theta \in (0, 1) \).

2. Nonconforming V-cycle and F-cycle multigrid algorithms

Let \( T_k \) be a triangulation of \( \Omega \) and let the triangulations \( T_k \), for \( k = 2, 3, \ldots \), be obtained by successive regular subdivisions. The mesh size \( h_k \) of \( T_k \) therefore satisfies the relation

\[ h_k = 2h_{k+1} \quad \text{for } k = 1, 2, \ldots. \]

The discontinuous energy space \( H^1(T_k) \) associated with \( T_k \) is defined by

\[ H^1(T_k) = \{ v \in L^2(\Omega) : v|_T \in H^1(T) \; \forall T \in T_k \}. \]

We define the nonconforming variational form \( a_k(\cdot, \cdot) \) on \( H^1(T_k) \) by

\[ a_k(v, w) = \sum_{T \in T_k} \int_T \left( \sum_{i,j=1}^2 a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_j} + r(x)vw \right) \, dx \quad \forall v, w \in H^1(T_k), \]

and the corresponding nonconforming energy (semi-)norm \( \| \cdot \|_{a_k} \) by

\[ \|v\|_{a_k} = \sqrt{a_k(v, v)} \quad \forall v \in H^1(T_k). \]

Note that \( H^1(T_1) \subset H^1(T_2) \subset \cdots \) and

\[ \|v\|_{a_{k-1}} = \|v\|_{a_k} \quad \forall v \in H^1(T_{k-1}). \]

Moreover, it follows from the boundedness and coercivity of \( a(\cdot, \cdot) \) and the Poincaré inequality that

\[ \|\zeta\|_{a_k} = \|\zeta\|_{a} \approx \|\zeta\|_{H^1(\Omega)} \approx \|\zeta\|_{H^1(\Omega)} \quad \forall \zeta \in H^1_0(\Omega). \]

Let \( V_k \subset H^1(T_k) \) be a nonconforming finite element space associated with \( T_k \) such that \( a_k(\cdot, \cdot) \) is positive definite on \( V_k \), and let \( (\cdot, \cdot)_k \) be a discrete inner product on \( V_k \). We can then represent \( a_k(\cdot, \cdot) \) by the operator \( A_k : V_k \rightarrow V_k \) defined by

\[ (A_kv_1, v_2)_k = a_k(v_1, v_2) \quad \forall v_1, v_2 \in V_k. \]

Note that \( A_k \) is symmetric positive definite with respect to \( (\cdot, \cdot)_k \) and the following relation holds:

\[ a_k(A_k^sv_1, v_2) = a_k(v_1, A_s^tv_2) \quad \forall v_1, v_2 \in V_k, s \in \mathbb{R}. \]

2.1. V-cycle and F-cycle multigrid algorithms. The \( k \)-th level multigrid V-cycle and F-cycle algorithms are multilevel iterative methods for the equation

\[ A_k z = g. \]

We assume that the finite element space \( V_{k-1} \) is connected to \( V_k \) by the (linear) intergrid transfer operator \( I_{k-1}^k : V_{k-1} \rightarrow V_k \) and denote its transpose with respect to the discrete inner products by \( I_{k-1}^k : V_k \rightarrow V_{k-1} \), i.e.,

\[ (I_{k-1}^k v, w)_{k-1} = (v, I_{k-1}^k w)_k \quad \forall v \in V_k, w \in V_{k-1}. \]
The operator $P_k^{1-k} : V_k \rightarrow V_{k-1}$ is the transpose of $I_k^{1-k}$ with respect to the nonconforming variational forms, i.e.,
\begin{equation}
(2.11) \quad a_{k-1}(P_k^{1-k} v, w) = a_k(v, I_k^{1-k} w) \quad \forall v \in V_k, w \in V_{k-1}.
\end{equation}
These operators satisfy the well-known relation
\begin{equation}
(2.12) \quad I_k^{1-k} A_k = A_{k-1} P_k^{1-k}.
\end{equation}
Finally we take $A_k$ to be a number dominating the spectral radius $\rho(A_k)$ of $A_k$.

**Algorithm 2.1** (The $V$-cycle algorithm).

The $k$-th level symmetric $V$-cycle algorithm produces $MG_V(k, g, z_0, m)$ as an approximate solution for (2.9) with initial guess $z_0$, where $m$ denotes the number of pre-smoothing and post-smoothing steps.

For $k = 1$ we define $MG_V(1, g, z_0, m) = A_1^{-1}g$.

For $k \geq 2$ the approximate solution $MG_V(k, g, z_0, m)$ is computed recursively in three steps:

**Pre-smoothing.** For $j = 1, \ldots, m$, compute $z_j$ by
\[
z_j = z_{j-1} + A_k^{-1}(g - A_k z_{j-1}).
\]

**Coarse grid correction.** Let $r_{k-1} = I_{k-1}^{1-k}(g - A_k z_m)$ and compute $z_{m+1}$ by
\[
z_{m+1} = z_m + I_{k-1}^{1-k} MG_V(k-1, r_{k-1}, 0, m).
\]

**Post-smoothing.** For $j = m + 2, \ldots, 2m + 1$, compute $z_j$ by
\[
z_j = z_{j-1} + A_k^{-1}(g - A_k z_{j-1}).
\]

We then define $MG_V(k, g, z_0, m) = z_{2m+1}$.

**Algorithm 2.2.** (The $F$-cycle algorithm)

The $k$-th level $F$-cycle algorithm (associated with the symmetric $V$-cycle algorithm) produces $MG_F(k, g, z_0, m)$ as an approximate solution for (2.9).

For $k = 1$, we define $MG_F(1, g, z_0, m) = A_1^{-1}g$.

For $k \geq 2$, we define $MG_F(k, g, z_0, m)$ recursively in three steps:

**Pre-smoothing.** For $j = 1, \ldots, m$, compute $z_j$ by
\[
z_j = z_{j-1} + A_k^{-1}(g - A_k z_{j-1}).
\]

**Coarse grid correction.** Let $r_{k-1} = I_{k-1}^{1-k}(g - A_k z_m)$ and compute $z_{m+1}$ by
\[
z_{m+1} = z_m + I_{k-1}^{1-k} MG_V(k-1, r_{k-1}, 0, m),
\]
\[
z_{m+1} = z_m + I_{k-1}^{1-k} MG_V(k-1, r_{k-1}, z_{m+1}, m).
\]

**Post-smoothing.** For $j = m + 1, \ldots, 2m + 1$, compute $z_j$ by
\[
z_j = z_{j-1} + A_k^{-1}(g - A_k z_{j-1}).
\]

We then define $MG_F(k, g, z_0, m) = z_{2m+1}$.

**Remark 2.3.** We use the Richardson relaxation scheme as the smoother for simplicity. The theory in this paper can be applied to other smoothers if the definition of the mesh-dependent norms in Section 3 is modified appropriately (cf. [23]).
2.2. Error representations. Let $E_{k,m} : V_k \rightarrow V_k$ be the operator relating the initial error and the final error of the multigrid V-cycle algorithm applied to the equation (2.9), i.e.,

$$E_{k,m}(z - z_0) = z - MG_F(k, g, z_0, m).$$

The operator $E_{k,m}$ can be described in terms of the operators $I_j^i$ and $P_j^{i-1}$ ($2 \leq j \leq k$) and the operators $R_j : V_j \rightarrow V_j$ defined by

$$R_j = Id_j - \Lambda_j^{-1} A_j,$$

where $Id_j : V_j \rightarrow V_j$ is the identity operator. Clearly we have

$$a_j(R_j v, w) = a_j(v, R_j w) \quad \forall v, w \in V_j. \tag{2.14}$$

The following relations (cf. [49]) are well known:

$$E_{k,m} = R_k^m \left[ (Id_k - I_{k-1}^k P_{k-1}^{k-1}) + I_{k-1}^k E_{k-1,m} P_{k-1}^{k-1} \right] R_k^m \quad \text{for } k \geq 2, \tag{2.15}$$

$$E_{1,m} = 0. \tag{2.16}$$

Using (2.15) and (2.16), we obtain an additive expression for $E_{k,m}$:

$$E_{k,m} = R_k^m \left[ (Id_k - I_{k-1}^k P_{k-1}^{k-1}) + I_{k-1}^k E_{k-1,m} P_{k-1}^{k-1} \right] R_k^m$$

$$+ R_k^m I_{k-1}^k P_{k-1}^{k-1} \left[ (Id_{k-1} - I_{k-2}^k P_{k-2}^{k-1}) + I_{k-2}^k E_{k-2,m} P_{k-2}^{k-1} \right] R_{k-1}^{k-1} P_{k-1}^{k-1} R_k^m$$

$$= \sum_{j=2}^{k} T_{k,j,m} R_j^m (Id_j - I_{j-1}^{j-1} P_j^{j-1}) R_j^m T_{j,k,m}, \tag{2.17}$$

where

$$T_{k,k,m} = Id_k,$$

and for $j < k$, $T_{j,k,m} : V_k \rightarrow V_j$ and $T_{k,j,m} : V_j \rightarrow V_k$ are defined by

$$T_{j,k,m} = P_{j+1}^j R_{j+1}^m \cdots P_k^k R_k^m, \tag{2.19}$$

$$T_{k,j,m} = R_k^m I_{k-1}^j \cdots P_{j+1}^j R_{j+1}^m. \tag{2.20}$$

Note that for $1 \leq j \leq k \leq \ell$ the following relations are valid:

$$T_{j,\ell,m} = T_{j,k,m} T_{k,\ell,m} \quad \text{and} \quad T_{\ell,j,m} = T_{\ell,k,m} T_{k,j,m}. \tag{2.21}$$

Also (2.11) and (2.14) imply

$$a_j(T_{j,k,m} v, w) = a_k(v, T_{k,j,m} w) \quad \forall v \in V_k, w \in V_j. \tag{2.22}$$

Let $\tilde{E}_{k,m} : V_k \rightarrow V_k$ be the operator relating the initial error and the final error of the F-cycle algorithm applied to the equation (2.9), i.e.,

$$\tilde{E}_{k,m}(z - z_0) = z - MG_F(k, g, z_0, m).$$

The following relations (cf. [49]) are also well known:

$$\tilde{E}_{1,m} = 0, \tag{2.23}$$

$$\tilde{E}_{k,m} = R_k^m \left[ (Id_k - I_{k-1}^k P_{k-1}^{k-1}) + I_{k-1}^k E_{k-1,m} \tilde{E}_{k-1,m} P_{k-1}^{k-1} \right] R_k^m, \quad k \geq 2. \tag{2.24}$$
3. Assumptions

In this section we state the assumptions for the convergence theory and derive some of their immediate consequences.

First we introduce a scale of mesh-dependent norms (cf. [4]) on each $V_k$:

\begin{equation}
\|v\|_{s,k} = \sqrt{(A^k_s v, v)_k} \quad \forall s \in \mathbb{R}, \ v \in V_k.
\end{equation}

It is clear from (2.4), (2.7) and (3.1) that

\begin{equation}
\|v\|_{0,k} = \sqrt{(v, v)_k} \quad \forall v \in V_k,
\end{equation}

\begin{equation}
\|v\|_{1,k} = \sqrt{a_k(v, v)} = \|v\|_{a_k} \quad \forall v \in V_k,
\end{equation}

\begin{equation}
\|A^k_s v\|_{t,k} = \|v\|_{t+2s,k} \quad \forall v \in V_k, s, t \in \mathbb{R}.
\end{equation}

Moreover the Cauchy-Schwarz inequality implies

\begin{equation}
\|v\|_{1+t,k} = \sup_{w \in V_k \setminus \{0\}} \frac{a_k(v, w)}{\|w\|_{1-t,k}} \quad \forall t \in \mathbb{R}, \ v \in V_k.
\end{equation}

To avoid the proliferation of constants we henceforth use the notation $A \lesssim B$ to represent the inequality $A \leq (\text{constant}) \times B$, where the constant is independent of both the mesh (i.e., independent of the mesh size and the mesh level) and the number of smoothing steps. The statement $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

Assumptions on $V_k$. We assume that

\begin{equation}
(v, v)_k \approx \|v\|^2_{L^2(\Omega)} \quad \forall v \in V_k,
\end{equation}

\begin{equation}
\|v\|_{a_k} \lesssim h^{-1}_k \|v\|_{L^2(\Omega)} \quad \forall v \in V_k.
\end{equation}

Assumptions on $I^k_{k-1}$ and $P^{k-1}_k$. We assume that $I^k_{k-1}$ and $P^{k-1}_k$ have the following properties:

\begin{equation}
\|I^k_{k-1} v\|^2_{1,k-1} \leq (1 + \theta^2) \|v\|^2_{1,k-1} + C_1 \theta^{-2} h^{2\alpha}_k \|v\|^2_{1+\alpha,k-1}
\end{equation}

\begin{equation}
\forall v \in V_{k-1}, \ \theta \in (0, 1),
\end{equation}

\begin{equation}
\|I^k_{k-1} v\|^2_{1-\alpha,k} \leq (1 + \theta^2) \|v\|^2_{1-\alpha,k-1} + C_2 \theta^{-2} h^{2\alpha}_k \|v\|^2_{1,k-1}
\end{equation}

\begin{equation}
\forall v \in V_{k-1}, \ \theta \in (0, 1),
\end{equation}

\begin{equation}
\|P^{k-1}_k v\|^2_{1-\alpha,k} \leq (1 + \theta^2) \|v\|^2_{1-\alpha,k} + C_3 \theta^{-2} h^{2\alpha}_k \|v\|^2_{1,k}
\end{equation}

\begin{equation}
\forall v \in V_k, \ \theta \in (0, 1),
\end{equation}

where $\alpha$ is the index of elliptic regularity in (1.4) and the positive constants $C_1$, $C_2$ and $C_3$ are mesh-independent.

Remark 3.1. The estimates in [22] corresponding to (3.9) and (3.10) involve an index $\beta \in (0, \frac{1}{2})$ instead of $\alpha$. Using the tools developed in Section 6 of this paper, one can also replace $\beta$ by $\alpha$ in [22].
Assumptions on $I_{k-1}^kP_{k-1}^k$ and $P_{k-1}^{k-1}I_{k-1}^k$. We assume the nonconforming finite element spaces have the following approximation properties:

\begin{align}
&\| (I_{k-1}^k - I_{k-1}^kP_{k-1}^k) v \|_{1-\alpha,k} \lesssim h_k^{2\alpha} \| v \|_{1+\alpha,k} \quad \forall v \in V_k, \tag{3.11} \\
&\| (I_{k-1}^k - P_{k-1}^{k-1}I_{k-1}^k) v \|_{1-\alpha, k-1} \lesssim h_k^{2\alpha} \| v \|_{1,k-1} \quad \forall v \in V_{k-1}. \tag{3.12}
\end{align}

We now derive some simple consequences of the assumptions above.

First we note that (3.6) and (3.7) imply

\begin{equation}
\rho(A_k) \lesssim h_k^{-2}. \tag{3.13}
\end{equation}

It follows easily (cf. [4, 36]) from (2.13), (3.1) and (3.13) that

\begin{align}
&\| v \|_{s,k} \lesssim h_k^{s} \| v \|_{t,k} \quad \forall v \in V_k, \ 0 \leq t \leq s \leq 2, \tag{3.14} \\
&\| R_k v \|_{s,k} \lesssim \| v \|_{s,k} \quad \forall v \in V_k, \ s \in \mathbb{R}, \tag{3.15} \\
&\| R_k^n v \|_{s,k} \lesssim h_k^{t-s} m^{(t-s)/2} \| v \|_{t,k} \quad \forall v \in V_k, \ 0 \leq t \leq s \leq 2, \ m \geq 1. \tag{3.16}
\end{align}

The estimate (3.8) and (3.12) imply through (2.11), (3.5) and (3.13) three additional estimates:

\begin{align}
&\| I_{k-1}^k v \|_{1,k} \lesssim \| v \|_{1,k-1} \quad \forall v \in V_{k-1}, \tag{3.17} \\
&\| P_{k-1}^{k-1} v \|_{1,k-1} \lesssim \| v \|_{1,k} \quad \forall v \in V_k, \tag{3.18} \\
&\| (I_{k-1}^k - P_{k-1}^{k-1}I_{k-1}^k) v \|_{1,k-1} \lesssim h_k^{2\alpha} \| v \|_{1+\alpha,k-1} \quad \forall v \in V_{k-1}. \tag{3.19}
\end{align}

Again, to avoid the proliferation of constants, we henceforth say that an estimate holds for $m$ sufficiently large if it is valid for $m \geq m_*$, where the positive integer $m_*$ is mesh-independent.

**Lemma 3.2.** Given any number $\omega \in (0,1)$, the following estimates hold for $m$ sufficiently large:

\begin{align}
&\| I_{k-1}^kR_{k-1}^m v \|_{1,k} \leq (1 + \omega) \| v \|_{1,k-1} \quad \forall v \in V_{k-1}, \tag{3.20} \\
&\| P_{k-1}^{k-1}R_{k-1}^m v \|_{1-\alpha,k-1} \leq (1 + \omega) \| v \|_{1-\alpha,k} \quad \forall v \in V_k, \tag{3.21} \\
&\| R_{k-1}^m P_{k-1}^{k-1} v \|_{1+\alpha,k-1} \leq (1 + \omega) \| v \|_{1+\alpha,k-1} \quad \forall v \in V_{k-1}. \tag{3.22}
\end{align}

**Proof.** From (2.1), (3.8), (3.15) and (3.16) we have

\[
\| I_{k-1}^kR_{k-1}^m v \|_{1,k}^2 \leq (1 + \omega^2) \| R_{k-1}^m v \|_{1,k-1}^2 + C_1 \omega^{-2} h_k^{2\alpha} \| R_{k-1}^m v \|_{1+\alpha,k-1}^2 \\
\leq (1 + \omega^2) \| v \|_{1,k-1}^2 + C_1' \omega^{-2} h_k^{-2} \| v \|_{1,k-1}^2,
\]

where the positive constant $C_1'$ is independent of the mesh and the number of smoothing steps. The estimate (3.20) then follows if

\[
m \geq \frac{C_1' \omega^{-3}}{2}.
\]

Similarly we obtain (3.21) using (3.10), (3.15) and (3.13). The estimate (3.22) then follows from (2.11), (2.14), (3.6) and (3.21). \qed
4. A strengthened Cauchy-Schwarz inequality

In this section we derive a strengthened Cauchy-Schwarz inequality which takes into account the effect of smoothing. We begin by estimating the bounds of the operator $T_{j,k,m}$ with respect to various mesh-dependent norms. For brevity we will sometimes suppress the parameter $m$ and write $T_{j,k}$ instead of $T_{j,k,m}$.

From (2.18)–(2.20), (3.21) and (3.22) we immediately have the following lemma.

**Lemma 4.1.** Let $j \leq k$. Given any $\omega \in (0,1)$, the following estimates hold for $m$ sufficiently large:

\begin{align}
(4.1) \quad & \|T_{j,k,m} v\|_{1-\alpha,j} \leq (1 + \omega)^{k-j} \|v\|_{1-\alpha,k} \quad \forall v \in V_k, \\
(4.2) \quad & \|T_{k,j,m} v\|_{1+\alpha,k} \leq (1 + \omega)^{k-j} \|v\|_{1+\alpha,j} \quad \forall v \in V_j.
\end{align}

The next three lemmas and one corollary are preparatory for the crucial estimate involving $T_{K,k,m} T_{k,K,m}$ for $k \leq K$.

**Lemma 4.2.** Let $k \leq K$. Then the estimate

\begin{equation}
(4.3) \quad \|T_{K,k,m} v\|_{1,K} \lesssim \|v\|_{1,k} \quad \forall v \in V_k
\end{equation}

holds for $m$ sufficiently large.

**Proof.** Let $v \in V_k$ be arbitrary. Given any $\omega \in (0,1)$, we have, from (1.8a), (2.11), (2.18), (2.20), (5.1), (3.3), (3.15), (3.19), (4.2) and the Cauchy-Schwarz inequality, that

\begin{align}
\|T_{K,k,v}\|_{1,K}^2 &= a_K(T_{K,k,v}, T_{K,k,v}) \\
&= a_K(R^p_{K} I_{K-1} T_{K-1,k,v}, R^p_{K} I_{K-1} T_{K-1,k,v}) \\
&\leq a_K(T_{K-1,k,v}, I_{K-1} T_{K-1,k,v}) \\
&= a_{K-1}(T_{K-1,k,v})^2 \\
&= a_{K-1}(P_{K-1} I_{K-1} - Id_{K-1}) T_{K-1,k,v} T_{K-1,k,v} \\
&\leq \|T_{K-1,k,v}\|_{1,K-1}^2 \\
&\leq (1 + \theta_{K}^2) \|T_{K-1,k,v}\|_{1,K-1}^2 + C_{\alpha} \theta_{K}^{-2} h_{K}^{2p} \|T_{K-1,k,v}\|_{1+\alpha,K-1}^2 \\
&\leq (1 + \theta_{K}^2) \|T_{K-1,k,v}\|_{1,K-1}^2 + C_{\alpha} \theta_{K}^{-2} (1 + \omega)^{2(K-k)} h_{K}^{2p} \|v\|_{1+\alpha,k}
\end{align}

for all $v \in V_k$, provided $m$ is sufficiently large. Note that $\theta_{K} \in (0,1)$ is arbitrary, and the positive constant $C_{\alpha}$ is independent of $\alpha$, the number of smoothing steps and the parameters $\omega$ and $\theta_{K}$.

Iterating (4.3), we find

\begin{equation}
(4.4) \quad \|T_{K,k,v}\|_{1,K}^2 \leq \prod_{k+1 \leq q \leq K} (1 + \theta_q^2) \|v\|_{1,k}^2
\end{equation}

and the result follows with

\begin{equation}
(4.5) \quad \|T_{K,k,v}\|_{1,K}^2 \leq \prod_{k+1 \leq q \leq K} (1 + \theta_q^2) \|v\|_{1,k}^2 + C_{\alpha} \sum_{k+1 \leq p \leq K} \left( \prod_{p+1 \leq q \leq K} (1 + \theta_q^2) \theta_p^{-2} (1 + \omega)^{2(K-k)} h_{K}^{2p} \right) \|v\|_{1+\alpha,k}^2,
\end{equation}

where $\theta_{k+1}, \ldots, \theta_{K}$ are arbitrary numbers in $(0,1)$. \hfill $\square$
Note that \( h_p = 2^{k-p} h_k \) by (2.1). We now choose
\[
\omega = \left( \frac{4}{3} \right)^{\alpha/2} - 1
\]
so that
\[
(4.6) \quad (1 + \omega)^{2(p-k)} h_p^{2\alpha} = \left[ (1 + \omega)^{2(4-\alpha)} \right]^{p-k} h_k^{2\alpha} = 3^{-\alpha(p-k)} h_k^{2\alpha},
\]
and then we take
\[
(4.7) \quad \theta_q = \left( \frac{2}{3} \right)^{\alpha(q-k)/2} \quad \text{for } k + 1 \leq q \leq K.
\]
Combining (4.5)–(4.7), we have
\[
(4.8) \quad \| T_{K,k,v} \|_{1,K}^2 \leq \rho_1 \| v \|_{1,k}^2 + C_2 \rho_1 \rho_2 h_k^{2\alpha} \| v \|_{1+\alpha,k}^2,
\]
where
\[
\rho_1 = \prod_{j=1}^\infty \left[ 1 + \left( \frac{2}{3} \right)^{\alpha j} \right] < \infty \quad \text{and} \quad \rho_2 = \sum_{j=1}^\infty \left( \frac{1}{2} \right)^{\alpha j} < \infty.
\]
The estimate (4.3) follows from (3.14) and (4.8).

Corollary 4.3. Let \( k \leq K \). Then the estimate
\[
(4.9) \quad \| T_{K,m,v} \|_{1,K} \lesssim \| v \|_{1,K} \quad \forall v \in V_K
\]
holds for \( m \) sufficiently large.

Lemma 4.4. Let \( k \leq K \). Then the estimate
\[
(4.10) \quad \| T_{K,m,v} \|_{1-\alpha,K} \lesssim \| v \|_{1-\alpha,K} \quad \forall v \in V_k
\]
holds for \( m \) sufficiently large.

Proof. Let \( v \in V_k \) be arbitrary. It follows from (2.18), (2.20), (3.14) and (4.3) that, for \( m \) sufficiently large,
\[
\| T_{K,k,v} \|_{1-\alpha,K}^2 = \| R_{K}^{K-1} I_{K-1} T_{K-1,k,v} \|_{1-\alpha,K}^2 \\
\leq \| I_{K-1} T_{K-1,k,v} \|_{1-\alpha,K}^2 \\
\leq (1 + \theta_K^2) \| T_{K-1,k,v} \|_{1-\alpha,K-1}^2 + C_2 \theta_K^{-2} h_k^{2\alpha} \| T_{K-1,k,v} \|_{1,K-1}^2 \\
\leq (1 + \theta_K^2) \| T_{K-1,k,v} \|_{1-\alpha,K-1}^2 + C_1 \theta_K^{-2} h_k^{2\alpha} \| v \|_{1,k}^2,
\]
for any \( \theta_K \in (0,1) \), where the positive constant \( C_1 \) is independent of the mesh, the number of smoothing steps and the parameter \( \theta_K \). Iterating (4.11), we find
\[
(4.12) \quad \| T_{K,k,v} \|_{1-\alpha,K}^2 \leq \left[ \prod_{k+1 \leq q \leq K} (1 + \theta_q^2) \right] \| v \|_{1-\alpha,k}^2 + C_1 \left[ \sum_{k+1 \leq p \leq K} \left( \prod_{p+1 \leq q \leq K} (1 + \theta_q^2) \right) \theta_p^{-2} h_p^{2\alpha} \right] \| v \|_{1,k}^2,
\]
where \( \theta_{k+1}, \ldots, \theta_K \) are arbitrary numbers in \( (0,1) \).

As in the proof of Lemma 4.2 by choosing
\[
(4.13) \quad \theta_q = \left( \frac{1}{2} \right)^{\alpha(q-k)/2} \quad \text{for } k + 1 \leq q \leq K,
\]
we can deduce from (2.14) and (4.12) that

\[(4.14)\]
\[\|T_{K,k} v\|_{1^-\alpha,K}^2 \lesssim \|v\|_{1^-\alpha,k}^2 + h_k^{2\alpha}\|v\|_{1,k}^2.\]

The estimate (4.10) now follows from (3.14) and (4.14). \(\square\)

**Lemma 4.5.** Let \(k \leq K\). Then the estimate

\[(4.15)\]
\[\|T_{k,k}v\|_{1^-\alpha,k}^2 \lesssim \|v\|_{1^-\alpha,k}^2 + h_k^{2\alpha}\|v\|_{1,k}^2 \quad \forall v \in V_K\]

holds for \(m\) sufficiently large.

**Proof.** Let \(v \in V_K\) be arbitrary. From (2.19), (3.10), (3.15) and (4.9) we have, for \(m\) sufficiently large,

\[(4.16)\]
\[\|T_{k,k}v\|_{1^-\alpha,k}^2 = \|T_{k+1,k}^R R_{k+1}^T T_{k+1,K}v\|_{1^-\alpha,k}^2 \]
\[\leq (1 + \theta_{k+1}^2)\|T_{k+1,K}v\|_{1^-\alpha,k+1}^2 + C_v\theta_{k+1}^2 h_k^{2\alpha}\|T_{k+1,K}v\|_{1,k+1}^2 \]
\[\leq (1 + \theta_{k+1}^2)\|T_{k+1,K}v\|_{1^-\alpha,k+1}^2 + C_v\theta_{k+1}^2 h_k^{2\alpha}\|T_{k+1,K}v\|_{1,k+1}^2 ,\]

where \(\theta_{k+1} \in (0,1)\) is arbitrary and the positive constant \(C_v\) is independent of the mesh, the number of smoothing steps and the parameter \(\theta_{k+1}\).

Iterating (4.16), we find

\[(4.17)\]
\[\|T_{k,k}v\|_{1^-\alpha,k}^2 \leq \prod_{k+1 \leq q \leq K} (1 + \theta_q^2)\|v\|_{1^-\alpha,k}^2 \]
\[+ C_v\sum_{k+1 \leq p \leq K} \left( \prod_{k+1 \leq q \leq p-1} (1 + \theta_q^2) \right) \theta_p^{-2\alpha} h_p^{2\alpha}\|v\|_{1,k}^2 .\]

Choosing \(\theta_q\) by formula (4.13), we deduce (4.15) from (2.1) and (4.17) as in the proof of Lemma 4.2. \(\square\)

We can now establish a crucial estimate.

**Lemma 4.6.** Let \(k \leq K\). Then the estimate

\[(4.18)\]
\[\|T_{k,k}T_{k,k,m}v\|_{1^-\alpha,k} \lesssim \|v\|_{1^-\alpha,k} \quad \forall v \in V_k\]

holds for \(m\) sufficiently large.

**Proof.** It follows from (3.14), Lemma 4.2, Lemma 4.4 and Lemma 4.5 that

\[\|T_{k,k}T_{k,k,m}v\|_{1^-\alpha,k} \lesssim \|T_{k,k,m}v\|_{1^-\alpha,k} + h_k^{2\alpha}\|T_{k,k,m}v\|_{1,k} \]
\[\lesssim \|v\|_{1^-\alpha,k} + h_k^{2\alpha}\|v\|_{1,k} \lesssim \|v\|_{1^-\alpha,k} ,\]

provided \(m\) is sufficiently large. \(\square\)

The following proposition is the main result of this section.

**Proposition 4.7** (Strengthened Cauchy-Schwarz inequality with smoothing). Let \(1 \leq j \leq k \leq K\). Given any \(\omega \in (0,1)\), the estimate

\[(4.19)\]
\[a_K(T_{k,j}R_j^\omega v_j, T_{k,k,m}R_k^\omega v_k) \]
\[\lesssim q^{-\alpha} \left( \frac{1 + \omega}{2\alpha} \right)^{k-j} (h_j^{-\alpha}\|v_j\|_{1^-\alpha,j})(h_k^{-\alpha}\|v_k\|_{1^-\alpha,k}) \]

holds for all \(v_j \in V_j\) and \(v_k \in V_k\), provided \(m\) is sufficiently large.
Proof. Given any \( \omega \in (0,1) \), from (4.1) and Lemma 5.1, we obtain
\[
a_k(T_{k,j}R^q_k v_j, T_K R^q_k v_k) = a_j(R^q_k v_j, T_{j,k} T_K R^q_k v_k) \\
\leq \|R^q_k v_j\|_{1+\alpha, j} \|T_{j,k} T_K R^q_k v_k\|_{1-\alpha, j} \\
\lesssim h_j^{-2\alpha} \|v_j\|_{1-\alpha, j} (1 + \omega)^{k-j} \|v_k\|_{1-\alpha, k} \\
= \frac{(1 + \omega)^{k-j}}{2^\alpha} \left( \frac{h_k}{h_j} \right)^\alpha (h_j^{-\alpha} \|v_j\|_{1-\alpha, j}) (h_k^{-\alpha} \|v_k\|_{1-\alpha, k}) \\
= q^{-\alpha} \left( \frac{1 + \omega}{2^\alpha} \right)^{k-j} (h_j^{-\alpha} \|v_j\|_{1-\alpha, j}) (h_k^{-\alpha} \|v_k\|_{1-\alpha, k}),
\]
provided \( m \) is sufficiently large. \( \square \)

Corollary 4.8. Let \( v_k \in V_k \) for \( 1 \leq k \leq K \). Then the estimate
\[
a_k \left( \sum_{k=1}^K T_{k,k,m} R^q_k v_k, \sum_{k=1}^K T_{k,k,m} R^q_k v_k \right) \lesssim q^{-\alpha} \sum_{k=1}^K h_k^{-2\alpha} \|v_k\|^2_{1-\alpha, k}
\]
holds for \( m \) sufficiently large.

Proof. It follows from Proposition 4.7 that, given any \( \omega \in (0,1), \)
\[
a_k \left( \sum_{k=1}^K T_{k,k,m} R^q_k v_k, \sum_{k=1}^K T_{k,k,m} R^q_k v_k \right) = \sum_{j,k=1}^K a_k(T_{k,j} R^q_k v_j, T_{k,k} R^q_k v_k) \\
\lesssim q^{-\alpha} \sum_{j,k=1}^K \left( \frac{1 + \omega}{2^\alpha} \right)^{|k-j|} (h_j^{-\alpha} \|v_j\|_{1-\alpha, j}) (h_k^{-\alpha} \|v_k\|_{1-\alpha, k})
\]
for \( m \) sufficiently large. We now choose \( \omega \) so that \( (1 + \omega)2^{-\alpha} < 1 \). The estimate (4.20) then follows from (4.21) and a discrete Young’s inequality (cf. 37). \( \square \)

5. Convergence of V-cycle and F-cycle algorithms

In this section we establish the asymptotic behavior of the contraction numbers of \( E_{k,m} \) and \( E_{k,m} \) \( (K \geq 2) \) with respect to \( m \).

The analysis in 22 for conforming V-cycle multigrid methods uses the fact that
\[
[Id_k - I_{k-1} P^{k-1}_k] = Id_k - I_{k-1} P^{k-1}_k
\]
in the case \( V_{k-1} \subset V_k \subset H^1(\Omega) \). Since (5.1) does not hold for nonconforming finite element spaces, we need to consider first the contraction property of an auxiliary operator \( \delta_{k,m} : V_K \rightarrow V_K \) defined by
\[
\delta_{k,m} = \sum_{k=2}^K T_{k,k,m} \left[ R^m_k (Id_k - I_{k-1} P^{k-1}_k) h_k^{-2\alpha} A_k^{-\alpha} \right] \times (Id_k - I_{k-1} P^{k-1}_k) R^m_k T_{k,k,m}.
\]

Lemma 5.1. The estimate
\[
\|\delta_{k,m} v\|_{1,K} \lesssim m^{-\alpha} \|v\|_{1,K} \quad \forall v \in V_K
\]
holds for \( m \) sufficiently large.
Proof. Let \( v \in V_K \) be arbitrary and

\[
(5.4) \quad v_k = (Id_k - I_{k-1}^k P_{k-1}^k) h_k^{-2a} A_k^{-\alpha}\left(Id_k - I_{k-1}^k P_{k-1}^k\right) R_k^m T_{k,K,m} v.
\]

We have, by (5.2), (5.4) and Corollary 4.8,

\[
(5.5) \quad a_K(\delta_{K,m,v}, \delta_{k,v}) = a_K\left(\sum_{k=2}^{K} T_{K,K,m} R_k^m v_k, \sum_{k=2}^{K} T_{K,K,m} R_k^m v_k\right)
\]

\[
\lesssim m^{-\alpha} \sum_{k=2}^{K} h_k^{-2a} \|v_k\|_{1-k}^{-\alpha}
\]

for \( m \) sufficiently large. Moreover, (3.1), (3.11) and (5.4) imply

\[
(5.6) \quad \|v_k\|_{1-\alpha,k} \lesssim \|A_k^{-\alpha}(Id_k - I_{k-1}^k P_{k-1}^k) R_k^m T_{K,K,m} v\|_{1+\alpha,k}
\]

\[
= \|A_k^{-\alpha/2}(Id_k - I_{k-1}^k P_{k-1}^k) R_k^m T_{K,K,m} v\|_{1,k}.
\]

Let \( w_k = (Id_k - I_{k-1}^k P_{k-1}^k) R_k^m T_{K,K,m} v \). It follows from (2.8), (2.11), (2.14), (2.22), (3.1) and (5.2) that

\[
(5.7) \quad \sum_{k=2}^{K} h_k^{-2a} \|A_k^{-\alpha/2}(Id_k - I_{k-1}^k P_{k-1}^k) R_k^m T_{K,K,m} v\|_{1,k}^2
\]

\[
= \sum_{k=2}^{K} h_k^{-2a} a_k\left(A_k^{-\alpha/2} w_k, A_k^{-\alpha/2}(Id_k - I_{k-1}^k P_{k-1}^k) R_k^m T_{K,K,m} v\right)
\]

\[
= \sum_{k=2}^{K} a_k\left(T_{K,K,m} R_k^m (Id_k - I_{k-1}^k P_{k-1}^k) h_k^{-2a} A_k^{-\alpha} w_k, v\right)
\]

\[
= a_K(\delta_{K,m,v}, v).
\]

Combining (3.3), (5.5), (5.7), and the Cauchy-Schwarz inequality, we find

\[
\|\delta_{K,m,v}\|_{1,K}^2 = a_K(\delta_{K,m,v}, \delta_{K,m,v}) \lesssim m^{-\alpha} a_K(\delta_{K,m,v}, v) \leq m^{-\alpha} \|\delta_{K,m,v}\|_{1,K} \|v\|_{1,K}
\]

and (5.3) follows. \( \square \)

We can now prove the convergence of the symmetric V-cycle algorithm.

**Theorem 5.2** (Convergence of the symmetric V-cycle algorithm). There exist positive mesh-independent constants \( C \) and \( m_* \) such that

\[
(5.8) \quad \|E_{K,m} v\|_{1,K} \leq \frac{C}{m^\alpha} \|v\|_{1,K} \quad \forall v \in V_K, \ K \geq 1, \ m \geq m_*.
\]

**Proof.** The case where \( K = 1 \) is trivial. Let \( v \in V_K \) \( (K \geq 2) \) be arbitrary and

\[
(5.9) \quad v_k = (Id_k - I_{k-1}^k P_{k-1}^k) R_k^m T_{K,K,m} v.
\]

Then (2.17), (5.9) and Corollary 4.8 imply that

\[
(5.10) \quad a_K(E_{K,m}, v) = a_K\left(\sum_{k=2}^{K} T_{K,K,m} R_k^m v_k, \sum_{k=2}^{K} T_{K,K,m} R_k^m v_k\right)
\]

\[
\lesssim m^{-\alpha} \sum_{k=2}^{K} h_k^{-2a} \|v_k\|_{1-\alpha,k}^2
\]

for \( m \) sufficiently large.
In other words we have, sufficiently large, 

\[\|v_k\|_{1,\alpha,k} = \|A_k^{-\alpha/2} ((I_{k} - I_{k-1}^{k})R_{k}^{m}T_{k,K,m}v)\|_{1,k}.\]

Combining (3.3), (5.7), (5.10)–(5.11), Lemma 5.1 and the Cauchy-Schwarz inequality, we find

\[\|E_{K,m}v\|_{1,K}^2 = a_K(E_{K,m}v, E_{K,m}v) \leq m^{-\alpha} a_K(E_{K,m}v, v) \leq m^{-\alpha} \|E_{K,m}v\|_{1,K} \|v\|_{1,K} \leq m^{-2\alpha} \|v\|_{1,K}^2\]

and (5.8) follows. \(\square\)

Finally we prove the convergence of the F-cycle algorithm.

**Theorem 5.3** (Convergence of the F-cycle algorithm). There exist positive mesh-independent constants \(C\) and \(m_*\) such that

\[\|\tilde{E}_{K,m}v\|_{1,K} \leq \frac{C}{m^\alpha} \|v\|_{1,K} \quad \forall v \in V_K, K \geq 1, m \geq m_*\]

**Proof.** Suppose that

\[\|\tilde{E}_{k-1,m}v\|_{1,k-1} \leq \eta_{k-1} \|v\|_{1,k-1} \quad \forall v \in V_{k-1}\]

From (2.24), (3.11), (3.15)–(3.18), (5.13) and Theorem 5.2 we have, for \(m\) sufficiently large,

\[\|\tilde{E}_{k,m}v\|_{1,k} \leq \|R_{k}^{m}(I_{k} - I_{k-1}^{k})R_{k}^{m}v\|_{1,k} + \|I_{k-1}^{k}\tilde{E}_{k-1,m}R_{k}^{m}v\|_{1,k} + \|\tilde{E}_{k-1,m}R_{k}^{m}v\|_{1,k-1} \leq m^{-\alpha/2} \|R_{k}^{m}v\|_{1,\alpha,k} + m^{-\alpha} \|\tilde{E}_{k-1,m}R_{k}^{m}v\|_{1,k-1} \leq m^{-\alpha}(1 + \eta_{k-1}) \|v\|_{1,k}\]

In other words we have,

\[\|\tilde{E}_{k,m}v\|_{1,k} \leq \eta_k \|v\|_{1,k} \quad \forall v \in V_k, m \geq m_+\]

where

\[\eta_k = C_1 m^{-\alpha}(1 + \eta_{k-1})\]

and the positive constants \(m_+\) and \(C_1\) are mesh-independent.

In view of (2.23) and (5.13)–(5.14), we can obtain by mathematical induction

\[\|\tilde{E}_{k,m}v\|_{1,k} \leq \frac{C_1}{m^\alpha - C_1^*} \|v\|_{1,k} \quad \forall v \in V_k, k \geq 1, m \geq m_*\]

provided \(m_* \geq \max(m_+ C_1^* m^\alpha)\). The estimate (5.12) follows immediately from (5.16). \(\square\)

### 6. An abstract framework for nonconforming multigrid methods

In order to apply the convergence results in Section 5 to a specific nonconforming multigrid method, one must verify the assumptions (3.0)–(3.12). This can be accomplished through the framework developed in [21]. Indeed the standard discrete estimate (3.6) and inverse estimate (3.7) (cf. [30], [25]) are the assumptions (P) and (I) in [21], and (3.11) is established in Lemma 4.2 there.

The truly new ingredients among the assumptions in Section 3 are therefore the estimates (3.8)–(3.10) and (3.12). We will show in this section that they can be derived using the framework in [21] for second order problems and four additional
6.1. Results from [21] and new conditions. A key ingredient of the theory in [21] is the relation between the nonconforming finite element space \( V_k \) and a conforming finite element space \( \tilde{V}_k \subset H^1_0(\Omega) \) (referred to as a conforming relative of \( V_k \) in [21]). These spaces are connected by the linear maps \( E_k : V_k \to \tilde{V}_k \) and \( F_k : \tilde{V}_k \to V_k \). Two of the properties of these maps are (cf. (FE) and Lemma 3.1 in [21]):

\[
(6.1) \quad F_k \circ E_k = Id_k ,
\]

\[
(6.2) \quad \|F_k \tilde{v}\|_{L^2(\Omega)} \lesssim \|\tilde{v}\|_{L^2(\Omega)} \quad \text{and} \quad \|F_k \tilde{v}\|_{ak} \lesssim \|\tilde{v}\|_{H^1(\Omega)} \quad \forall \tilde{v} \in \tilde{V}_k .
\]

Let \( \zeta \in H^{1+\alpha}(\Omega) \cap H^1_0(\Omega) \), \( \zeta_k \in V_k \) and \( \zeta_{k-1} \in V_{k-1} \) be related by

\[
a(\zeta, E_k v) = a_k(\zeta_k, v) \quad \forall v \in V_k ,
\]

\[
a(\zeta, E_{k-1} v) = a_{k-1}(\zeta_{k-1}, v) \quad \forall v \in V_{k-1} .
\]

Then the following estimates are valid within the framework in [21] (cf. Theorem 3.5 and Lemma 3.7 there):

\[
(6.3) \quad \|\zeta - \zeta_k\|_{ak} \lesssim h_k^\alpha \|\zeta\|_{H^{1+\alpha}(\Omega)} ,
\]

\[
(6.4) \quad \|\Pi_k \zeta - \zeta_k\|_{L^2(\Omega)} \lesssim h_k^\alpha \|\zeta\|_{H^{1+\alpha}(\Omega)} ,
\]

\[
(6.5) \quad \|\zeta_{k-1} - P_k^{k-1} \zeta_{k-1}\|_{L^2(\Omega)} \lesssim h_k^\alpha \|\zeta\|_{H^{1+\alpha}(\Omega)} ,
\]

where \( \Pi_k : H^1_0(\Omega) \to V_k \) is an interpolation operator.

The following estimates concerning \( I_{k-1}^k \) and \( \Pi_k \) are also established within the framework in [21] (cf. Lemma 3.3, (II-1), (II-2) and (I-2) there):

\[
(6.6) \quad \|I_{k-1}^k v\|_{s,k} \lesssim \|v\|_{s,k-1} \quad \forall v \in V_{k-1} , \ 0 \leq s \leq 1 ,
\]

\[
(6.7) \quad \|\zeta - \Pi_k \zeta\|_{L^2(\Omega)} \lesssim h_k \|\zeta\|_{H^{1}(\Omega)} \quad \forall \zeta \in H^1_0(\Omega) ,
\]

\[
(6.8) \quad \|\zeta - \Pi_k \zeta\|_{ak} \lesssim h_k^\alpha \|\zeta\|_{H^{1+\alpha}(\Omega)} \quad \forall \zeta \in H^{1+\alpha}(\Omega) \cap H^1_0(\Omega) ,
\]

\[
(6.9) \quad \|\Pi_k \zeta - I_{k-1}^k \Pi_{k-1} \zeta\|_{1-\alpha,k} \lesssim h_k^\alpha \|\zeta\|_{H^{1}(\Omega)} \quad \forall \zeta \in H^{1+\alpha}(\Omega) \cap H^1_0(\Omega) .
\]

We will derive the estimates (6.8)–(6.10) and (6.12) using (6.1)–(6.9) and four additional conditions imposed on the intergrid transfer operator \( I_{k-1}^k \) and the interpolation operator \( \Pi_k \).

Four additional conditions. We assume that, in addition to the conditions (I-1) and (I-2) in [21],

\[
(6.10) \quad \|I_{k-1}^k v\|_{0,k}^2 \leq (1 + \theta^2) \|v\|_{0,k-1}^2 + C_0 \theta^{-2} h_k^2 \|v\|_{\alpha,k-1}^2
\]

for all \( v \in V_{k-1} \) and \( \theta \in (0, 1) \), where the positive constant \( C_0 \) is mesh-independent.

Furthermore, the interpolation operator \( \Pi_{k-1} : H^1_0(\Omega) \to V_{k-1} \) actually maps the larger space \( H^1_0(\Omega) + V_k \) into \( V_{k-1} \) and satisfies, in addition to (II-1) and (II-2) in [21], the estimates

\[
(6.11) \quad \|\Pi_{k-1} v\|_{ak} \lesssim \|v\|_{ak} \quad \forall v \in H^1_0(\Omega) + V_k ,
\]

\[
(6.12) \quad \|\Pi_{k-1} v - v\|_{L^2(\Omega)} \lesssim h_k \|v\|_{ak} \quad \forall v \in V_k ,
\]

\[
(6.13) \quad \|\Pi_{k-1} v\|_{0,k-1}^2 \leq (1 + \theta^2) \|v\|_{0,k-1}^2 + C_0 \theta^{-2} h_k^2 \|v\|_{\alpha,k}^2 \quad \forall v \in V_k , \ \theta \in (0, 1) ,
\]

where the positive constant \( C_0 \) is mesh-independent.
6.2. Derivation of \((3.8)-(3.10)\) and \((3.12)\). We begin by introducing an operator \(J_k\). Let \(Q_k : L_2(\Omega) \rightarrow V_k\) be the \(L_2\)-orthogonal projection operator. Then we define

\[
J_k \phi = F_k Q_k \phi \quad \forall \phi \in L_2(\Omega).
\]

**Lemma 6.1.** The operator \(J_k\) satisfies

\[
J_k E_k v = v \quad \forall v \in V_k,
\]

\[
\|J_k \zeta\|_{1-\alpha,k} \lesssim \|\zeta\|_{H^{1-\alpha}(\Omega)} \quad \forall \zeta \in H^{1-\alpha}_0(\Omega).
\]

**Proof.** The relation \((6.15)\) follows immediately from \((6.1)\) and \((6.14)\).

From \((3.3), (3.6), (6.2)\) and standard properties of \(Q_k\) (cf. [15]) we have

\[
\|J_k \zeta\|_{0,k} \approx \|F_k Q_k \zeta\|_{L_2(\Omega)} \lesssim \|Q_k \zeta\|_{L_2(\Omega)} \lesssim \|\zeta\|_{L_2(\Omega)} \quad \forall \zeta \in L_2(\Omega),
\]

\[
\|J_k \zeta\|_{1,k} = \|F_k Q_k \zeta\|_{a_k} \lesssim \|Q_k \zeta\|_{H^1(\Omega)} \lesssim \|\zeta\|_{H^1(\Omega)} \quad \forall \zeta \in H^1_0(\Omega).
\]

Then we have

\[
a(\zeta, E_k v) = a_k (\zeta_k, v) \quad \forall v \in V_k,
\]

\[
|\zeta|_{H^1(\Omega)} \lesssim \|\zeta_k\|_{1,k},
\]

\[
\|\zeta\|_{H^{1+\alpha}(\Omega)} \lesssim \|\zeta_k\|_{1+\alpha,k},
\]

\[
\|\Pi_k \zeta\|_{a_k} \lesssim \|\zeta_k\|_{1,k}.
\]

**Proof.** The relation \((6.20)\) follows immediately from \((6.15)\) and \((6.19)\).

From \((3.3), (6.18), (6.19)\) and the Cauchy-Schwarz inequality we have

\[
a(\zeta, \zeta) = a_k (\zeta_k, J_k \zeta) \leq \|\zeta_k\|_{a_k} \|J_k \zeta\|_{a_k} \lesssim \|\zeta_k\|_{1,k} \|\zeta\|_{H^1(\Omega)}.
\]

which implies \((6.21)\) in view of \((2.9)\). The estimate \((6.23)\) then follows immediately from \((2.0), (6.11)\) and \((6.21)\).

Using \((3.5), (6.16)\) and \((6.19)\), we find

\[
a_k (\zeta_k, J_k \phi) \leq \|\zeta_k\|_{1+\alpha,k} \|J_k \phi\|_{1-\alpha,k} \lesssim \|\zeta_k\|_{1+\alpha,k} \|\phi\|_{H^{1-\alpha}(\Omega)} \quad \forall \phi \in H^1_0(\Omega).
\]

Thus the right-hand side of \((6.19)\) defines a linear functional \(F\) on \(H^1_0(\Omega)\) which actually belongs to \(H^{-1+\alpha}(\Omega)\) and

\[
\|F\|_{H^{-1+\alpha}(\Omega)} \lesssim \|\zeta_k\|_{1+\alpha,k}.
\]

The estimate \((6.22)\) follows from \((1.3)\) and \((6.24)\).  

We are now ready to derive the estimates \((3.8)-(3.10)\) and \((3.12)\). In the following derivations we use \(C\) to denote a generic mesh-independent positive constant that is also independent of the parameter \(\theta\).

**Lemma 6.3.** The estimate \((3.8)\) holds.
Proof. Let $\zeta_{k-1} \in V_{k-1}$ be arbitrary and $\zeta \in H^1_0(\Omega)$ be defined by
\begin{equation}
\begin{aligned}
a(\zeta, \phi) &= a_{k-1}(\zeta_{k-1}, J_{k-1} \phi) \quad \forall \phi \in H^1_0(\Omega).
\end{aligned}
\end{equation}

Then it follows from (1.83), (2.11), (2.5), (3.1), (6.3), (6.6), (6.8), (6.9), (6.25) and Lemma 6.2 that
\begin{equation}
\begin{aligned}
\|I_{k-1}^k \zeta_{k-1}\|_{1,k}^2 &\leq (\|\zeta_{k-1}\|_{a_k} + \|\zeta_{k-1} - I_{k-1}^k \zeta_{k-1}\|_{a_k})^2 \\
&\leq (1 + \theta^2)\|\zeta_{k-1}\|_{a_k}^2 + C\theta^{-2}(\|\zeta_{k-1} - \zeta\|_{a_k}^2 + \|\zeta - \Pi_k \zeta\|_{a_k}^2) \\
&\quad + \|\Pi_k \zeta - I_{k-1}^k \Pi_k \zeta\|_{a_k}^2 + \|I_{k-1}^k (\Pi_{k-1} \zeta - \zeta_{k-1})\|_{a_k}^2 \\
&\leq (1 + \theta^2)\|\zeta_{k-1}\|_{1,k-1}^2 + C\theta^{-2}h_k^{2\alpha} \|\zeta_{k-1}\|_{1+a,k-1}^2.
\end{aligned}
\end{equation}

Lemma 6.4. The estimate (3.4) holds.

Proof. Let $C_\epsilon$ be a constant that is greater than or equal to the constants $C_0$ and $C_1$ in (6.10) and (3.3), and define, for any $\theta \in (0,1), 
\begin{equation}
\begin{aligned}
\langle v_1, v_2 \rangle_{k-1, \theta} &= (1 + \theta^2)\langle v_1, v_2 \rangle_{k-1} + C_\epsilon \theta^{-2}h_k^{2\alpha} \langle A_{k-1}^0 v_1, v_2 \rangle_{k-1}
\end{aligned}
\end{equation}
for all $v_1, v_2 \in V_{k-1}$. Note that $A_{k-1}$ is symmetric positive definite with respect to the inner product $\langle \cdot, \cdot \rangle_{k-1, \theta}$.

It follows from (3.1), (3.8), (6.10) and (6.26) that
\begin{equation}
\begin{aligned}
\|I_{k-1}^k v\|_{0,k}^2 &\leq \langle A_{k-1}^0 v, v \rangle_{k-1, \theta} \quad \forall v \in V_{k-1}, \\
\|I_{k-1}^k v\|_{1,k}^2 &\leq \langle A_{k-1}^0 v, v \rangle_{k-1, \theta} \quad \forall v \in V_{k-1},
\end{aligned}
\end{equation}
which imply through (3.1) and interpolation between Hilbert scales
\begin{equation}
\begin{aligned}
\|I_{k-1}^k v\|_{1+a,k}^2 &\leq \langle A_{k-1}^{1-a} v, v \rangle_{k-1, \theta} = (1 + \theta^2)\|v\|_{1+a,k}^2 + C_\epsilon \theta^{-2}h_k^{2\alpha} \|v\|_{1,k-1}^2.
\end{aligned}
\end{equation}

Lemma 6.5. It holds that
\begin{equation}
\begin{aligned}
\|\Pi_{k-1} v\|_{k-1}^2 &\leq (1 + \theta^2)\|v\|_{1,k}^2 + C_\epsilon \theta^{-2}h_k^{2\alpha} \|v\|_{1+a,k}^2
\end{aligned}
\end{equation}
for all $v \in V_k$ and $\theta \in (0,1)$, where the positive constant $C_\epsilon$ is mesh-independent.

Proof. Let $\zeta_k \in V_k$ be arbitrary and define $\zeta \in H^1_0(\Omega)$ by (6.19).

From (1.83), (2.11), (2.5), (3.1), (6.3), (6.6), (6.8), (6.11) and Lemma 6.2 we have
\begin{equation}
\begin{aligned}
\|\Pi_{k-1} \zeta\|_{1,k-1}^2 &\leq (\|\zeta\|_{a_k} + \|\zeta_{k-1} - \Pi_{k-1} \zeta\|_{a_k})^2 \\
&\leq (1 + \theta^2)\|\zeta\|_{1,k}^2 + C\theta^{-2}(\|\zeta_{k-1} - \zeta\|_{a_k}^2 + \|\zeta - \Pi_{k-1} \zeta\|_{a_k}^2) \\
&\quad + \|\Pi_{k-1} (\zeta - \zeta_{k-1})\|_{a_k}^2 \\
&\leq (1 + \theta^2)\|\zeta\|_{1,k}^2 + C\theta^{-2}h_k^{2\alpha} \|\zeta\|_{1+a,k}^2.
\end{aligned}
\end{equation}

The next lemma follows from (6.13), (6.27) and interpolation between Hilbert scales, as in the proof of Lemma 6.4

Lemma 6.6. It holds that
\begin{equation}
\begin{aligned}
\|\Pi_{k-1} v\|_{1+a,k-1}^2 &\leq (1 + \theta^2)\|v\|_{1+a,k}^2 + C_\epsilon \theta^{-2}h_k^{2\alpha} \|v\|_{1,k-1}^2
\end{aligned}
\end{equation}
for all $v \in V_k$ and $\theta \in (0,1)$, where the positive constant $C_\epsilon$ is mesh-independent.
We need one more estimate for the derivation of (3.10).

**Lemma 6.7.** The following estimate holds:

\[(6.29) \quad \| \Pi_{k-1} \zeta - \Pi_k \zeta \|_{L^2(\Omega)} \lesssim h_k \| \zeta \|_{H^1(\Omega)} \quad \forall \zeta \in H^1_0(\Omega).\]

**Proof.** From (2.1), (2.6), (6.7), (6.11) and (6.12) we have
\[
\| \Pi_{k-1} \zeta - \Pi_k \zeta \|_{L^2(\Omega)} \leq \| \Pi_{k-1} \zeta - \zeta \|_{L^2(\Omega)} + \| \zeta - \Pi_k \zeta \|_{L^2(\Omega)} + \| \Pi_k \zeta - \Pi_{k-1} \Pi_k \zeta \|_{L^2(\Omega)} \\
\lesssim h_k \| \zeta \|_{H^1(\Omega)} + h_k \| \Pi_k \zeta \|_{a_k} \lesssim h_k \| \zeta \|_{H^1(\Omega)}.
\]

\[\blacksquare\]

**Lemma 6.8.** The estimate \((3.10)\) holds.

**Proof.** Let \(\zeta_\kappa \in V_k\) be arbitrary. Define \(\zeta \in H^1_0(\Omega)\) by \((6.19)\) and \(\zeta_{k-1} \in V_{k-1}\) by
\[
a_{k-1}(\zeta_{k-1}, v) = a(\zeta, E_{k-1} v) \quad \forall v \in V_{k-1}.
\]

We have, by \((1.8)\),
\[
P_{k}^{-1} \zeta_{k-1} \|_{\alpha-k-1} \leq (\| \Pi_{k-1} \zeta \|_{1-\alpha,k-1} + \| \Pi_k \zeta - P_{k}^{-1} \zeta_k \|_{1-\alpha,k-1})^2 \\
\leq (1 + \theta^2) \| \Pi_{k-1} \zeta \|_{1-\alpha,k-1}^2 + C \theta^{-2} \| P_{k}^{-1} \zeta_k \|_{1-\alpha,k-1}^2.
\]

The first term on the right-hand side of \((6.30)\) can be estimated using \((1.8)\), \((3.3)\), \((\ref{eq:Lemma6.8})\), \((\ref{eq:Lemma6.8})\), \((\ref{eq:Lemma6.8})\), \((\ref{eq:Lemma6.8})\) and Lemma \((6.2)\)
\[
\| \Pi_{k-1} \Pi_k \zeta \|_{1-\alpha,k-1} \leq (1 + \theta^2) \| \Pi_k \zeta \|_{1-\alpha,k}^2 + C \theta^{-2} h_k \| \Pi_k \zeta \|_{1,k}^2 \\
\leq (1 + \theta^2) \| \Pi_k \zeta \|_{1-\alpha,k} + \| \Pi_k \zeta - \zeta \|_{1-\alpha,k} + C \theta^{-2} h_k \| \Pi_k \zeta \|_{1,k}^2 \\
\leq (1 + \theta^2) \| \Pi_k \zeta \|_{1-\alpha,k} + C \theta^{-2} h_k \| \Pi_k \zeta \|_{1,k}^2 \\
\leq (1 + \theta^2) \| \Pi_k \zeta \|_{1-\alpha,k} + C \theta^{-2} h_k \| \Pi_k \zeta \|_{1,k}^2.
\]

Similarly the second term on the right-hand side of \((6.30)\) can be estimated using \((2.1)\), \((\ref{eq:Lemma6.8})\), \((\ref{eq:Lemma6.8})\), \((\ref{eq:Lemma6.8})\), \((\ref{eq:Lemma6.8})\), \((\ref{eq:Lemma6.8})\) \((\ref{eq:Lemma6.8})\), and Lemma \((6.2)\)
\[
\| \Pi_{k-1} \Pi_k \zeta - P_{k}^{-1} \zeta_k \|_{1-\alpha,k-1} \leq (\| \Pi_{k-1} \Pi_k \zeta - \zeta \|_{1-\alpha,k-1} + \| \Pi_{k-1} \zeta - \zeta_{k-1} \|_{1-\alpha,k-1})^2 \\
+ \| \zeta_{k-1} - P_{k}^{-1} \zeta_k \|_{1-\alpha,k-1} \\
\leq C \| \Pi_{k-1} \Pi_k \zeta - \zeta \|_{1-\alpha,k-1}^2 + h_k \| \Pi_k \zeta \|_{1,k}^2.
\]

Combining \((6.30)\) \((\ref{eq:Lemma6.8})\), we have
\[
\| P_{k}^{-1} \zeta_k \|_{1-\alpha,k-1} \leq (1 + \theta^2) ^3 \| \zeta_k \|_{1-\alpha,k} + C \theta^{-2} h_k \| \zeta_k \|_{1,k}^2
\]
which implies \((3.10)\) since \(\theta \in (0,1)\) is arbitrary. \(\blacksquare\)

We now turn to the derivation of \((3.12)\) by first noting that the estimate
\[(6.33) \quad \| P_{k}^{-1} v \|_{1-\alpha,k-1} \lesssim \| v \|_{1-\alpha,k} \quad \forall v \in V_k
\]
follows from \((3.10)\) and \((\ref{eq:Lemma6.8})\).
Lemma 6.9. The estimate \((7.2)\) holds.

Proof. Let \(\zeta_{k-1} \in V_{k-1}\) be arbitrary and define \(\zeta \in H_0^1(\Omega)\) and \(\zeta_k \in V_k\) by \((6.25)\) and \((6.29)\), respectively. Then it follows from \((2.1)\), \((3.1)\), \((3.5)\), \((3.14)\), \((6.4)\), \((6.6)\), \((6.9)\), \((6.33)\) and Lemma \(6.2\) that

\[
a_{k-1}((Id_{k-1} - P_{k-1}^k)\zeta_{k-1}, w) = a_{k-1}(\zeta_{k-1} - P_{k-1}^k\zeta_k, w) - a_{k-1}(P_{k-1}^k(\Pi_{k-1}^k\zeta_{k-1} - \zeta_k), w)
\leq \|\zeta_{k-1} - P_{k-1}^k\zeta_k\|_{1-\alpha, k-1}\|w\|_{1+\alpha, k-1}
+ \|P_{k-1}^k(\Pi_{k-1}^k\zeta_{k-1} - \zeta_k)\|_{1-\alpha, k-1}\|w\|_{1+\alpha, k-1}
\leq h_{k}^{2\alpha}\|\zeta\|_{H^{1+\alpha}(\Omega)}\|w\|_{1+\alpha, k-1}
+ (\|P_{k-1}^k(\Pi_{k-1}^k\zeta_{k-1} - \Pi_{k-1}^k\zeta)\|_{1-\alpha, k} + \|\Pi_{k}^k\zeta - \zeta_k\|_{1-\alpha, k})\|w\|_{1+\alpha, k-1}
\leq h_{k}^{2\alpha}\|\zeta\|_{H^{1+\alpha}(\Omega)}\|w\|_{1+\alpha, k-1}
\leq h_{k}^{2\alpha}\|\zeta_{k-1}\|_{1, k-1}\|w\|_{1+\alpha, k-1}
\leq h_{k}^{2\alpha}\|\zeta_{k-1}\|_{1, k-1}\|w\|_{1+\alpha, k-1}
\]

for all \(w \in V_{k-1}\), which implies \((6.12)\) because of \((6.5)\).

Remark 6.10. The proof of Lemma \(6.9\) actually establishes the stronger estimate

\[
\|(Id_{k-1} - P_{k-1}^k)w\|_{1-\alpha, k-1} \leq h_{k}^{2\alpha}\|w\|_{1+\alpha, k-1} \quad \forall w \in V_{k-1}.
\]

7. Applications

In this section we apply the theory developed in Sections \(6-8\) to two nonconforming finite element methods for the variational problem \((1.1)\). As demonstrated in \(21\), both of these methods satisfy the assumptions of the framework developed in that paper. Therefore, from the discussion in Section \(6\), it only remains for us to check the additional conditions \((6.10)\), \((6.13)\) for these examples.

7.1. The nonconforming \(P_1\) finite element method. Let \(e\) be (a segment of) an edge of \(T \in \mathcal{T}_k\). We define

\[
M_{e,T}(v) = \frac{1}{|e|} \int_e v|_T \, ds \quad \text{for } v \in H^1(T), \quad \ell \geq 1.
\]

Remark 7.1. In the case where \(\ell > k\) and \(e\) is an edge of \(T\), the integral on the right-hand side of \((7.1)\) should be interpreted as

\[
\int_e v|_T \, ds = \sum_{j=1}^{2^{k-1}} \int_{e_j} v|_{T_j} \, ds
\]

where \(e_1, \ldots, e_{2^{k-1}}\) are the edges from \(\mathcal{T}_k\) that form a partition of \(e\) and \(T_j \subset T\) is a triangle in \(\mathcal{T}_k\) with \(e_j\) as an edge (cf. Figure \(7.1\) for the case \(k = 2\)).

The nonconforming \(P_1\) finite element space \(V_k\) is defined by (cf. \(91\))

\[
V_k = \{ v \in L_2(\Omega) : v|_T \in \langle 1, x_1, x_2 \rangle \forall T \in \mathcal{T}_k, M_{e,T}(v) = M_{e,T'}(v) \text{ if } e \text{ is the common edge of } T \text{ and } T' \in \mathcal{T}_k, M_{e,T}(v) = 0 \text{ if } e \subset \partial \Omega \text{ is an edge of } T \}
\]
Henceforth we will denote $M_{e,T}(v)$ by $M_e(v)$ if there is no ambiguity about $T$ or if $v$ has an unambiguous mean value on $e$.

Let $\mathcal{E}_k$ be the set of the internal edges of $T_k$. We define $(\cdot,\cdot)_k$ by

$$
(v_1, v_2)_k = h_k^2 \sum_{e \in \mathcal{E}_k} M_e(v_1)M_e(v_2) \quad \forall v_1, v_2 \in V_k.
$$

Let $\sum_{k=1}^{\infty} H^1(T_k)$ be the subspace of $L^2(\Omega)$ whose members are finite sums of functions from the spaces $H^1(T_1), H^1(T_2), \ldots$. The weak interpolation operator $\Pi_k : \sum_{k=1}^{\infty} H^1(T_k) \to V_k$ is defined by

$$
M_e(\Pi_k v) = \frac{1}{2} \sum_{e \subset \partial T, T \subset T_k} M_{e,T}(v) \quad \forall e \in \mathcal{E}_k.
$$

Note that there are exactly two triangles in the sum and the mean value of $\Pi_k v$ on $e$ is just the average of the mean values of $v$ on $e$ from these two triangles.

The restriction of $\Pi_k$ to $H^1_0(\Omega)$ is precisely the weak interpolation operator used in [21] (cf. also [19], [5]), and the intergrid transfer operator $I^{k}_{k-1} : V_{k-1} \to V_k$ in [21] is just the restriction of $\Pi_k$ to $V_{k-1}$.

The following lemma, a simple consequence of the Bramble-Hilbert lemma (cf. [3]) and scaling, is the key to the estimates (6.10)–(6.13).

**Lemma 7.2.** Let $e_1, e_2$ be two of the edges of $T \in T_k$. Then we have

$$
|M_{e_1}(v) - M_{e_2}(v)| \lesssim |v|_{H^1(T)} \quad \forall v \in H^1(T).
$$

**Verification of (6.11).** Let $e \in \mathcal{E}_{k-1}$ be the common edge of two triangles $T$ and $T'$ in $T_{k-1}$, and let $e_1$ (resp. $e_2$) be the half of the edge $e$ neighboring the two triangles $T_1$ and $T_2$ (resp. $T_3$ and $T_4$) in $T_k$ (cf. Figure 7.2).

**Figure 7.2.** Two neighboring triangles in $T_{k-1}$
Then for \( v \in H^1_0(\Omega) + V_k \), we have
\[
(7.5) \quad M_{e,T}(v) = \frac{1}{2} [M_{e_1}(v) + M_{e_2}(v)] = M_{e,T'}(v).
\]
In view of (7.4) and (7.5), the interpolation operator \( \Pi_{k-1} : H^1_0(\Omega) + V_k \to V_{k-1} \) can be analyzed locally on each triangle of \( T_{k-1} \).

Let \( T \) be a triangle in \( T_{k-1} \) subdivided into four triangles \( T_1, \ldots, T_4 \) and let \( e_1, \ldots, e_9 \) be the edges from \( T_k \) that are on \( \partial T \) (cf. Figure 7.3). For any \( v \in H^1_0(\Omega) + V_k \) we obtain from (7.4), (7.5) and scaling
\[
(7.6) \quad |\Pi_{k-1} v|_{H^1(T)}^2 \lesssim \sum_{i,j=1}^6 |M_{e_i}(v) - M_{e_j}(v)|^2.
\]
Note that Lemma 7.2 implies
\[
(7.7) \quad |M_{e_i}(v) - M_{e_j}(v)|^2 \lesssim \sum_{\ell=1}^4 |v|_{H^1(T_\ell)}^2 \quad \forall v \in H^1(T) + V_k, \ 1 \leq i,j \leq 9.
\]
For example, we have (cf. Figure 7.3)
\[
|M_{e_1}(v) - M_{e_4}(v)|^2 \lesssim |M_{e_1}(v) - M_{e_7}(v)|^2 + |M_{e_7}(v) - M_{e_9}(v)|^2
+ |M_{e_9}(v) - M_{e_4}(v)|^2
\lesssim |v|_{H^1(T_1)}^2 + |v|_{H^1(T_4)}^2 + |v|_{H^1(T_2)}^2.
\]
It follows from (7.6) and (7.7) that
\[
(7.8) \quad |\Pi_{k-1} v|_{H^1(T)}^2 \lesssim \sum_{\ell=1}^4 |v|_{H^1(T_\ell)}^2 \quad \forall v \in H^1(T) + V_k.
\]
We obtain by summing (7.8) over all the triangles in \( T_{k-1} \)
\[
(7.9) \quad \sum_{T \in T_{k-1}} \int_T |\nabla (\Pi_{k-1} v)|^2 \, dx \lesssim \sum_{T \in T_k} \int_T |\nabla v|^2 \, dx \quad \forall v \in H^1_0(\Omega) + V_k.
\]
The estimate (6.11) then follows from (1.3), (2.3), (2.4), (7.9) and the Poincaré inequality for nonconforming \( P_1 \) finite element functions (cf. [24] and the references therein).

Let \( T \) be a generic triangle in \( T_{k-1} \) subdivided into four triangles \( T_1, \ldots, T_4 \in T_k \) (cf. Figure 7.3). Henceforth we will denote by \( e_1, \ldots, e_9 \) the edges of \( T_1, \ldots, T_4 \) (cf.

![Figure 7.3. A divided triangle in \( T_{k-1} \)](image-url)
Figure 7.3) and denote by $e_j$ ($1 \leq j \leq 3$) the edge of $T$ that contains the edges $e_{2j-1}$ and $e_{2j}$ from $T_k$.

**Verification of (6.12).** Let $T$ be a generic triangle in $T_{k-1}$. For any $v \in V_k$, we have by (7.4) and (7.5)

\begin{equation}
M_{e_j}(\Pi_{k-1}v) = \frac{1}{2} (M_{e_{2j-1}}(v) + M_{e_{2j}}(v)). 
\end{equation}

Observe also that

\begin{equation}
M_{\sim e_j}(w) = \frac{1}{2} (M_{e_{2j-1}}(w) + M_{e_{2j}}(w)) \quad \forall w \in H^1(T),
\end{equation}

which together with (7.7) implies

\begin{equation}
|M_{e_j}(w) - M_{e_i}(w)| \lesssim |w|_{H^1(T)} \quad \forall w \in H^1(T), \quad 1 \leq j \leq 3, \quad 1 \leq i \leq 9.
\end{equation}

We have by scaling

\begin{equation}
\|\Pi_{k-1}v - v\|^2_{L^2(T)} \lesssim h^2_k \sum_{j=1}^9 \|M_{e_j}(\Pi_{k-1}v) - M_{e_j}(v)\|^2,
\end{equation}

and it follows from (7.7), (8.9), (7.10) and (7.11) that

\begin{equation}
\sum_{j=1}^9 \|M_{e_j}(\Pi_{k-1}v) - M_{e_j}(v)\|^2 \lesssim \sum_{j=1}^4 |v|_{H^1(T_j)}^2 + \|\Pi_{k-1}v\|_{H^1(T)}^2 \lesssim \sum_{j=1}^4 |v|_{H^1(T_j)}^2.
\end{equation}

Combining (7.12) and (7.13), we find

\begin{equation}
\|\Pi_{k-1}v - v\|^2_{L^2(T)} \lesssim h^2_k \sum_{j=1}^4 |v|_{H^1(T_j)}^2.
\end{equation}

In view of (1.3), (2.3) and (2.4), we obtain the estimate (6.12) by summing up the preceding estimate over all the triangles in $T_{k-1}$. \hfill \Box

**Verification of (6.13).** Denote by $E_T$ the set of the three edges of $T$. Then we have by (7.2) and (7.3)

\begin{equation}
(v, v)_k = \frac{h_k^2}{2} \sum_{T \in T_k} \sum_{e \in E_T} [M_e(v)]^2 \quad \forall v \in V_k.
\end{equation}
The relation (7.14) allows us to focus on a generic $T \in \mathcal{T}_{k-1}$ in the derivation of (6.13). From (1.8), (7.7) and (7.10) we have

$$
\sum_{j=1}^{3} |M_{\varepsilon_{j}}(\Pi_{k-1}v)|^{2} \leq \frac{1}{2} \sum_{i=1}^{6} |M_{\varepsilon_{i}}(v)|^{2}
$$

(7.15)

$$
= \frac{1}{4} \sum_{i=1}^{6} |M_{\varepsilon_{i}}(v)|^{2} + \frac{1}{4} \sum_{j=1}^{3} \left\{ (M_{\varepsilon_{j+6}}(v) - [M_{\varepsilon_{j+6}}(v) - M_{\varepsilon_{j-1}}(v))]^{2}
\right. \\
+ (M_{\varepsilon_{j+6}}(v) - [M_{\varepsilon_{j+6}}(v) - M_{\varepsilon_{j}}(v))]^{2}\}
$$

$$
= \frac{1}{4} \sum_{i=1}^{6} |M_{\varepsilon_{i}}(v)|^{2} + \frac{(1 + \theta^{2})}{2} \sum_{i=7}^{9} |M_{\varepsilon_{i}}(v)|^{2} + C\theta^{-2} \sum_{\ell=1}^{4} |v|_{H^{1}(T)}^{2}
$$

for all $v \in V_{k}$ and $\theta \in (0, 1)$, where the positive constant $C$ is mesh-independent.

Summing up the estimate (7.15) over all the triangles in $\mathcal{T}_{k-1}$ and using (1.3), (2.1), (2.3), (2.4), (3.2), (3.3), (7.2), (7.3) and (7.14), we arrive at

$$
\|\Pi_{k-1}v\|_{0,k-1}^{2} = \frac{h_{k-1}^{2}}{2} \sum_{T \in \mathcal{T}_{k-1}} \sum_{e \in E_{T}} |M_{\varepsilon}(v)|^{2}
$$

(7.16)

$$
\leq (1 + \theta^{2}) \frac{h_{k}^{2}}{4} \sum_{e \in E_{k}} |M_{\varepsilon}(v)|^{2} + C\theta^{-2}h_{k}^{2} \sum_{T \in \mathcal{T}_{k}} |v|_{H^{1}(T)}^{2}
$$

$$
\leq (1 + \theta^{2}) (v,v)_{k} + C\theta^{-2}h_{k}^{2} \|v\|_{0,k}^{2}
$$

$$
= (1 + \theta^{2}) \|v\|_{0,k}^{2} + C\theta^{-2}h_{k}^{2} \|v\|_{1,k}^{2}.
$$

The estimate (6.14) follows from (7.14) and (7.16). □

**Verification of (6.14).** Let $e$ be an edge of $\mathcal{T}_{k}$ which is on the common boundary of $T, T' \in \mathcal{T}_{k-1}$ (cf. Figure 7.2). Then we have by (1.8a) and (7.4)

$$
[M_{\varepsilon}(I_{k-1}^{e}v)]^{2} = \frac{1}{4} [M_{\varepsilon,T}(v) + M_{\varepsilon,T'}(v)]^{2} \leq \frac{1}{2} [M_{\varepsilon,T}(v)]^{2} + \frac{1}{2} [M_{\varepsilon,T'}(v)]^{2}.
$$

(7.17)

For $T \in \mathcal{T}_{k-1}$, let $E_{k}^{b}$ (resp. $E_{k}^{b}$) denote the set of the edges from $\mathcal{T}_{k}$ that are on $\partial T$ (resp. interior to $T$). We obtain from (7.3) and (7.17) the estimate

$$
(I_{k-1}^{e}v, I_{k-1}^{e}v)_{k} \leq h_{k}^{2} \sum_{T \in \mathcal{T}_{k-1}} \left( \frac{1}{2} \sum_{e \in E_{k}^{b}} |M_{\varepsilon,T}(v)|^{2} + \sum_{e \in E_{k}^{b}} |M_{\varepsilon}(v)|^{2} \right)
$$

(7.18)

for all $v \in V_{k-1}$.

Again the estimate (7.18) allows us to focus on a generic $T \in \mathcal{T}_{k-1}$. It follows from (1.8b) and (7.11) that

$$
\frac{1}{2} \sum_{i=1}^{6} |M_{\varepsilon_{i}}(v)|^{2} + \frac{1}{2} \sum_{i=7}^{9} |M_{\varepsilon_{i}}(v)|^{2} \leq 2(1 + \theta^{2}) \sum_{j=1}^{3} |M_{\varepsilon_{j}}(v)|^{2} + C\theta^{-2} |v|_{H^{1}(T)}^{2}
$$

(7.19)

for all $v \in H^{1}(T)$ and $\theta \in (0, 1)$, where the positive constant $C$ is mesh-independent.
Summing up the estimate (7.18) over all the triangles in $T_{k-1}$, we find by (7.13), (2.1), (2.3), (2.4), (3.2), (3.3), (7.14) and (7.18)

$$
\|I_{k-1}^h v\|_{0,k}^2 = (I_{k-1}^h v, I_{k-1}^h v)_k
$$

(7.20)

$$
\leq h_k^2 \sum_{T \in T_{k-1}} \left( \frac{1}{2} \sum_{e \in E_T^h} |M_e(T(v)|^2 + \sum_{e \in E_T} |M_e(v)|^2 \right)
$$

$$
\leq (1 + \theta^2) \frac{h_k^2}{2} \sum_{T \in T_{k-1}} \sum_{e \in E_T} |M_e(v)|^2 + C\theta^{-2} h_k^2 \sum_{T \in T_{k-1}} |v|^2_{H^1(T)}
$$

$$
\leq (1 + \theta^2) \|v\|_{0,k-1}^2 + C\theta^{-2} h_k^2 \|v\|_{1,k-1}^2
$$

for all $v \in V_{k-1}$ and $\theta \in (0,1)$.

The estimate (6.10) follows from 3.14 and (7.20). \qed

7.2. The rotated Q$_1$ finite element method. We shall adopt the notation in subsection 7.1 with obvious modifications. For simplicity we assume that $T_k$ (and hence any $T_j$) is a triangulation of $\Omega$ by rectangles whose sides are parallel to the coordinate axes. The rotated Q$_1$ finite element space (cf. (42)) is defined by

$$
V_k = \{ v \in L_2(\Omega) : |v|_R \in \{1, x_1, x_2, x_1^2 - x_2^2\} \quad \forall R \in T_k, \quad M_{e,R}(v) = 0 \quad \text{if } e \text{ is the common edge of } R \text{ and } R' \text{ in } T_k, \quad M_{e,R}(v) = M_{e,R}(v) = M_{e,R}(v)
$$

(7.21)

$$
= 0 \quad \text{if } e \subset \partial \Omega \text{ is an edge of } R \}.\n$$

We define the inner product $(\cdot, \cdot)_k$ by

$$
(v_1, v_2)_k = h_k^2 \sum_{e \in E_k} M_e(v_1) M_e(v_2) \quad \forall v \in V_k.
$$

(7.22)

The weak interpolation operator $\Pi_k : \sum_{i=1}^{\infty} H^1(T_i) \rightarrow V_k$ is defined by

$$
M_e(\Pi_k v) = \frac{1}{2} \sum_{e \subset \partial R, R \in T_k} M_e(v) \quad \forall e \in E_k.
$$

(7.23)

The restriction of $\Pi_k$ to $H^1_0(\Omega)$ is precisely the weak interpolation operator used in (21), and the intergrid transfer operator $I_{k-1}^h : V_{k-1} \rightarrow V_k$ used in (21) is just the restriction of $\Pi_k$ to $V_{k-1}$.

The following analog of Lemma 7.2 is again a simple consequence of the Bramble-Hilbert lemma and scaling.

Lemma 7.3. Let $e_1$ and $e_2$ be two edges of the rectangle $R \in T_k$. Then we have

$$
|M_{e_1}(v) - M_{e_2}(v)| \lesssim |v|_{H^1(R)} \quad \forall v \in H^1(R).
$$

Using Lemma 7.3 the verification of (6.10) - (6.13) proceeds as in subsection 7.1 and we therefore omit the details.

Remark 7.4. The results of this paper can also be applied to other nonconforming quadrilateral elements (cf. [33], [27], [28], [26]).

Remark 7.5. The nonconforming P$_1$ and the rotated Q$_1$ finite elements are equivalent to the lowest order triangular and rectangular Raviart-Thomas mixed finite elements (cf. [43], [2], [1]). The results of this paper can therefore be applied to multigrid methods for the lowest order triangular and rectangular Raviart-Thomas mixed methods (cf. [20], [1]).
References


Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208

E-mail address: brenner@math.sc.edu