MATHEMATICS OF COMPUTATION
Volume 73, Number 247, Pages 1525–1546
S 0025-5718(03)01586-2
Article electronically published on August 26, 2003

STARK'S CONJECTURE OVER COMPLEX CUBIC NUMBER FIELDS

DAVID S. DUMMIT, BRETT A. TANGEDAL, AND PAUL B. VAN WAMELEN

ABSTRACT. Systematic computation of Stark units over nontotally real base fields is carried out for the first time. Since the information provided by Stark's conjecture is significantly less in this situation than the information provided over totally real base fields, new techniques are required. Precomputing Stark units in relative quadratic extensions (where the conjecture is already known to hold) and coupling this information with the Fincke-Pohst algorithm applied to certain quadratic forms leads to a significant reduction in search time for finding Stark units in larger extensions (where the conjecture is still unproven). Stark's conjecture is verified in each case for these Stark units in larger extensions and explicit generating polynomials for abelian extensions over complex cubic base fields, including Hilbert class fields, are obtained from the minimal polynomials of these new Stark units.

1. INTRODUCTION

Stark's conjecture ([St1]–[St6], [Ta]) posits a remarkable connection between L-function values at $s = 0$ and the arithmetic of global fields. The original formulation of the conjecture evolved over a period of several years and versions of the conjecture both more general and more precise continue to be elucidated as more significant examples of the conjecture are examined. Stark himself was at least partially guided by a series of computations demonstrating that, with certain refinements, the conjecture was sharp enough to produce generating polynomials of abelian extensions over specific types of algebraic number fields using only information from the base field. The computations that Stark carried out do not prove his conjecture, but once the generating polynomial is produced, an independent check can be made to prove that the correct abelian extension is generated. The effective use of Stark's conjecture to produce generating polynomials of abelian extensions in particular required the development (in [DST], [DT]—see also [CR], [R1]–[R2]) of a provably accurate method of computing L-function values to high precision and is based upon a formula of Lavrik [L] and Friedman [F].

The computation of generating polynomials of abelian extensions via Stark's conjecture has thus far only been carried out systematically over totally real base fields (of signature $[m, 0]$ and what Stark [St4] refers to as type I base fields). However, as Stark [St4] was careful to point out, his rank one abelian conjecture

Received by the editor November 14, 2000 and, in revised form, January 3, 2003.
2000 *Mathematics Subject Classification.* Primary 11R42; Secondary 11Y40, 11R37, 11R16.
Key words and phrases. Algebraic number fields, Stark's conjecture.
The first author was supported in part by NSF Grant DMS-9624057 and NSA Grant MDA-9040010024.

applies also over base fields of degree m over $\mathbb{Q}$ with signature $[m-2,1]$ (type II base fields). Stark [St4] proved his conjecture over complex quadratic base fields using elliptic modular functions but otherwise no general progress has been made in proving the conjecture over type II base fields (see [DST] for the current status of the conjecture). The purpose of the present paper is to initiate a systematic study of Stark's conjecture over type II base fields by focusing in particular upon the case where $m = 3$. To the authors' knowledge, only one example of this type has been computed previously (see [Da]). The computations over type II base fields are more subtle than those over type I fields because Stark's conjecture provides information on the absolute value of the "Stark unit" with respect to a distinguished complex embedding ($K \hookrightarrow K^{(2)}$ in Section 2), as opposed to a real embedding, and the argument of the embedded unit is left completely unspecified. In addition to their use in the construction of abelian extensions over type II base fields, it is important to compute the Stark units for these types of fields in order to investigate the arguments of these units with respect to the distinguished complex embedding $K \hookrightarrow K^{(2)}$ and to gain an understanding of the extent to which, as Stark puts it, one can "get inside the absolute value sign" in his conjecture.

The paper is organized as follows. Stark's conjecture over a complex cubic base field is presented in Section 2. We construct explicit Stark units for certain relative quadratic extensions (where the conjecture is known to hold) in Section 3 and use these units later to help find Stark units in larger extension fields (containing these relative quadratic extensions) where the conjecture has not been proved to date. Stark's conjecture for an important class of relative abelian extensions is discussed in Section 4. In Section 5, we discuss the computation of the special values of L-functions and partial zeta-functions arising in Stark's conjecture. We specialize to the case where the complex cubic base field has class number 3 in Section 6 and give a detailed example showing how the new Stark units in Theorem 1 below were found. We also prove Theorem 1 via the example in Section 6. We give tables summarizing our computations in Section 7.

Our principal result is the following

Theorem 1. *Let k be a complex cubic number field with class number $h_k = 3$ and discriminant $|d_k| \le 4300$. Let H_k denote the Hilbert class field of k and let K be the abelian extension of k defined by $K = H_k(\sqrt{-\eta})$, where η is a fundamental unit of k chosen to be positive at the unique real place of k. Then there is a unit ε_0 in K with $K = k(\varepsilon_0)$ and $H_k = k(\varepsilon_0 + 1/\varepsilon_0)$ such that ε_0 satisfies parts (1) and (3) of Stark's conjecture (see Section 2) and part (2) of the conjecture to a proved numerical accuracy of at least 10^{-28}. The irreducible polynomials $f(x), h(x) \in k[x]$ having ε_0 and $\varepsilon_0 + 1/\varepsilon_0$, respectively, as roots are directly obtainable from the tables in Section 7.*

2. Stark's conjecture over a complex cubic number field

For ease of presentation, we specialize the statement of Stark's conjecture to the situation studied in detail here (cf. [Ta]). Let k be a complex cubic number field. We denote the unique real place of k by $\mathfrak{p}_{\mathrm{R}}$. Let K/k be a relative abelian extension of degree n with Galois group $G = \mathrm{Gal}(K/k)$. We assume that $\mathfrak{p}_{\mathrm{R}}$ ramifies in K/k and so $2 \mid n$. The conductor $\mathfrak{f}(K/k)$ then has the form $\mathfrak{p}_{\mathrm{R}}\mathfrak{f}$, where $\mathfrak{f}$ is an integral ideal in k whose prime factorization involves precisely the finite ramified primes in K/k. The absolute norm of $\mathfrak{f}$ satisfies $\mathrm{N}\mathfrak{f} > 1$ since the Hilbert class field and narrow Hilbert class field coincide over a complex cubic base field. For each $\sigma \in G$, a corresponding partial zeta-function is defined for $\mathrm{Re}(s) > 1$ by the sum

$$\zeta_{\mathfrak{f}}(s,\sigma) = \sum_{\sigma_{\mathfrak{a}}=\sigma} \frac{1}{\mathrm{N}\mathfrak{a}^s} \tag{1}$$

over all integral ideals $\mathfrak{a}$ of k relatively prime to $\mathfrak{f}$ having Artin symbol $(K/k,\mathfrak{a}) = \sigma_{\mathfrak{a}} = \sigma$. For $\mathrm{Re}(s) > 1$, we define an L-function

$$L_{\mathfrak{f}}(s,\chi) = \prod_{(\mathfrak{p},\mathfrak{f})=1} \left(1 - \frac{\chi(\sigma_{\mathfrak{p}})}{\mathrm{N}\mathfrak{p}^s}\right)^{-1} \tag{2}$$

for each character $\chi : G \to \mathbb{C}^\times$ of G. We have

$$\zeta_{\mathfrak{f}}(s,\sigma) = \frac{1}{n} \sum_{\chi \in \widehat{G}} \overline{\chi}(\sigma) L_{\mathfrak{f}}(s,\chi) \tag{3}$$

or, equivalently,

$$L_{\mathfrak{f}}(s,\chi) = \sum_{\sigma \in G} \chi(\sigma)\zeta_{\mathfrak{f}}(s,\sigma). \tag{4}$$

For the trivial character χ_0, the function $L_{\mathfrak{f}}(s,\chi_0)$ has a meromorphic continuation to all of $\mathbb{C}$ with exactly one simple pole at $s = 1$. If t denotes the number of prime ideals dividing $\mathfrak{f}$, then $L_{\mathfrak{f}}(s,\chi_0)$ has a zero of order $r_1 + r_2 + t - 1$ at $s = 0$. Since k has signature $[r_1, r_2] = [1, 1]$ and $t \geq 1$, we have

$$L'_{\mathfrak{f}}(0,\chi_0) = 0. \tag{5}$$

If $\chi \neq \chi_0$, the function $L_{\mathfrak{f}}(s,\chi)$ can be analytically continued to an entire function and it has a zero at $s = 0$ of order at least r_2 (see [DT]). We conclude from equation (3) that $\zeta_{\mathfrak{f}}(s,\sigma)$ is analytic at $s = 0$ and has at least a first order zero at $s = 0$.

For the complex cubic number field k, there is a unique real embedding of k into $\mathbb{R}$, denoted by $\alpha \mapsto \alpha^{(1)}$ for $\alpha \in k$, corresponding to the real place $\mathfrak{p}_{\mathrm{R}}$. The image of this embedding is denoted by $k^{(1)}$, and k is isomorphic to $k^{(1)}$ via the embedding. Fix a complex embedding $k \hookrightarrow k^{(2)} \subset \mathbb{C}$ of k as well. Let K/k be a relative abelian extension such that $\mathfrak{p}_{\mathrm{R}}$ ramifies in K/k. This assumption is made because Stark's conjecture holds trivially if both infinite places of k split in K/k (see [Ta], p. 91, Prop. 3.1) and the complex place of k always splits in any relative extension K/k. Let $K \hookrightarrow K^{(1)} \subset \mathbb{C}$ be an embedding of K extending the real embedding $k \hookrightarrow k^{(1)} \subset \mathbb{R}$, and let $K \hookrightarrow K^{(2)} \subset \mathbb{C}$ be an embedding of K that extends the fixed complex embedding $k \hookrightarrow k^{(2)} \subset \mathbb{C}$. Let U_K denote the unit group of K, let μ_K denote the subgroup containing all roots of unity, and let $w_K = \#\mu_K$. The symbol $|\ |$ denotes the usual absolute value on $\mathbb{C}$. Stark's conjecture ([St4], [Ta]) for K/k is the following

Conjecture. *There exists a unit $\varepsilon \in U_K$ such that*

(1) $|\varepsilon^{(1)}| = 1$,
(2) $\log |\sigma(\varepsilon)^{(2)}|^2 = -w_K \zeta'_{\mathfrak{f}}(0,\sigma)$ *for all* $\sigma \in G$,
(3) $K(\varepsilon^{1/w_K})$ *is an abelian extension of* k.

The unit ε, if it exists, is unique up to multiplication by an element of μ_K and is referred to as the "Stark unit" for K/k.

3. Explicit Stark units for certain relative quadratic extensions

Stark's conjecture has been proven in general for relative quadratic extensions (see [Ta], p. 104). In this section we make this explicit for a specific relative quadratic extension M/k of the complex cubic number field k. These explicit Stark units will be used in the following sections to find Stark units in larger extensions K containing M.

Let d_k denote the discriminant of k and let h_k denote the class number of k. Let η be a fundamental unit of k, chosen to be positive at the real place $\mathfrak{p}_{\mathrm{R}}$ of k, and set $M = k(\sqrt{-\eta})$ (the two possible choices of η give the same quadratic extension). By construction, the place $\mathfrak{p}_{\mathrm{R}}$ ramifies in M/k and so $\mathfrak{p}_{\mathrm{R}}$ divides the conductor $\mathfrak{f}(M/k)$. For a relative quadratic extension, the finite part $\mathfrak{f}$ of the conductor $\mathfrak{f}(M/k)$ is equal to the relative discriminant $\mathfrak{d}(M/k)$. Note that $\mathrm{N}\mathfrak{f} = 2^b > 1$ since M/k is unramified outside 2 and at least one finite prime does ramify. This implies that $d_M = -2^b d_k^2$ ([Lo], p. 82). Let $\mathrm{Gal}(M/k) = \{\mathbf{1}, \tau\}$ and let ψ denote the nontrivial character on this group. For a prime ideal $\mathfrak{p}$ in k, unramified in M/k, the value $\psi(\sigma_{\mathfrak{p}})$ equals 1 if $\mathfrak{p}$ splits and equals -1 if $\mathfrak{p}$ is inert. The Dedekind ζ-function of M can thus be written as $\zeta_M(s) = \zeta_k(s) L_{\mathfrak{f}}(s, \psi)$.

The analytic class number formula at $s = 0$ for a number field F gives

$$\frac{\zeta_F^{(r_F)}(0)}{r_F!} = -\frac{h_F R_F}{w_F}, \tag{6}$$

where r_F denotes the rank of the unit group and R_F denotes the regulator of F. In our case, $r_M = 2$, $r_k = 1$, and $w_k = 2$ since k has a real embedding.

Lemma 1. *For M as defined above, $w_M = 2$.*

Proof. The possible values of w_M for a field M of degree 6 over $\mathbb{Q}$ are $2, 4, 6, 14, 18$. Values of $w_M = 14$ or 18 would imply that M is abelian over $\mathbb{Q}$, which is not possible since M contains k. If $w_M = 4$, then by necessity $M = k(\sqrt{-1})$. But this would imply that η is a square in k which is clearly false. If $w_M = 6$, then we would have $M = k(\sqrt{-3})$. The prime 3 will always be divisible by a prime ideal in k with odd ramification index. Such a prime ideal will ramify in $k(\sqrt{-3})/k$ ([H], p. 134), in contradiction to $\mathrm{N}\mathfrak{f} = 2^b$. We conclude that $w_M = 2$. □

It follows from above that $L'_{\mathfrak{f}}(0, \psi) = h_M R_M / h_k R_k$. By use of (3) and (5), we have $\zeta'_{\mathfrak{f}}(0, \mathbf{1}) = \frac{1}{2} L'_{\mathfrak{f}}(0, \psi)$, and so $-w_M \zeta'_{\mathfrak{f}}(0, \mathbf{1}) = -h_M R_M / h_k R_k$. Likewise, $-w_M \zeta'_{\mathfrak{f}}(0, \tau) = h_M R_M / h_k R_k$. Since the extension M/k is totally ramified, we know that $h_k \,|\, h_M$ ([Wa], Prop. 4.11). Let $h = h_M / h_k$. In order to prove Stark's conjecture for M/k, we need to work directly with the units in M. Let $\eta_1 = \sqrt{-\eta}$, which is an element of the unit group U_M of M. Note that $\mathrm{N}_{M/k}(\eta_1) = \eta$, and therefore there exists $\eta_2 \in U_M$ such that η_1, η_2 is a fundamental pair of units in M. We can assume without loss of generality that $\mathrm{N}_{M/k}(\eta_2) = 1$ (if $\mathrm{N}_{M/k}(\eta_2) = \eta^c$, then $\eta_1, \eta_1^{-c}\eta_2$ is still a fundamental pair and $\mathrm{N}_{M/k}(\eta_1^{-c}\eta_2) = 1$).

Proposition 1. *With notation as above, $\upsilon = \eta_2^h$ is the Stark unit for the relative quadratic extension M/k.*

Proof. The field M is totally complex with $[M : \mathbb{Q}] = 6$ and we specify three non-complex-conjugate embeddings $M \hookrightarrow M^{(i)} \subset \mathbb{C}$, $i = 1, 2, 3$, reserving $i = 1$ to denote the embedding extending the real embedding $k \hookrightarrow k^{(1)} \subset \mathbb{R}$. For $\alpha \in M$

we note that $|\alpha^{(1)}|^2 = (\mathrm{N}_{M/k}(\alpha))^{(1)}$ since algebraic conjugation is equivalent to complex conjugation over the real place $\mathfrak{p}_{\mathrm{R}}$ of k. This implies that $|\eta_2^{(1)}| = 1$. Since η_2 is a unit and not a root of unity, $|\eta_2^{(1)}||\eta_2^{(2)}||\eta_2^{(3)}| = 1$ and either $|\eta_2^{(2)}| < 1$ or $|\eta_2^{(2)}| > 1$. We reorder the embeddings, if necessary, so that $|\eta_2^{(2)}| < 1$, and then define the embedding $i = 3$ such that $\alpha^{(3)} = \tau(\alpha)^{(2)}$ for all $\alpha \in M$. The regulator R_M is given by the absolute value of the determinant

$$\begin{vmatrix} \log|\eta_1^{(1)}|^2 & \log|\eta_1^{(2)}|^2 \\ \log|\eta_2^{(1)}|^2 & \log|\eta_2^{(2)}|^2 \end{vmatrix}. \tag{7}$$

By our comments above,

$$R_M = |\log\eta^{(1)}|\,|\log|\eta_2^{(2)}|^2| = R_k(-\log|\eta_2^{(2)}|^2). \tag{8}$$

We finally set $\upsilon = \eta_2^h$ and verify that υ is the Stark unit for M/k. Equation (8) implies that $\log|\upsilon^{(2)}|^2 = h\log|\eta_2^{(2)}|^2 = -w_M\zeta_{\mathfrak{f}}'(0,\mathbf{1})$. This verifies part (2) of the conjecture for the trivial automorphism. Since $\upsilon\tau(\upsilon) = 1$ by construction, and $\zeta_{\mathfrak{f}}'(0,\tau) = -\zeta_{\mathfrak{f}}'(0,\mathbf{1})$, we see that part (2) of the conjecture is satisfied for the nontrivial automorphism τ as well. Part (1) is also satisfied by construction. Part (3) requires that $M(\sqrt{\upsilon})$ be an abelian extension of k. If $M(\sqrt{\upsilon}) = M$, we are done, so assume that $[M(\sqrt{\upsilon}) : k] = 4$. Since $\upsilon\tau(\upsilon) = 1$ and $\upsilon \notin k$, we see that the conjugates of $\sqrt{\upsilon}$ with respect to the extension $M(\sqrt{\upsilon})/k$ are $\pm\sqrt{\upsilon}$ and $\pm 1/\sqrt{\upsilon}$. This implies that $M(\sqrt{\upsilon})$ is Galois over k, and since the relative degree is 4, $M(\sqrt{\upsilon})$ is an abelian extension of k. □

4. Special abelian extensions

Let k be a given complex cubic number field. In Sections 4–6, a detailed outline is given of how to use a precomputed Stark unit υ corresponding to a relative quadratic extension L/k, coupled with the high precision computation of first derivatives at $s = 0$ of properly chosen L-functions, to find new Stark units in larger extension fields K where $k \subset L \subset K$. The general situation may be described as follows. We assume that $[L : k] = 2$ and that the real place $\mathfrak{p}_{\mathrm{R}}$ of k ramifies in L/k. We also assume we are given an extension field F with $[F : k] > 1$ such that $\mathfrak{p}_{\mathrm{R}}$ splits in F/k. Clearly $L \not\subseteq F$. The goal is to find the conjectured Stark unit $\varepsilon \in K$ corresponding to the composite field $K = LF$ of degree $2\,[F : k]$ over k. There are three important obstacles to overcome to apply the method outlined below. First is explicitly computing the Stark unit υ corresponding to L/k. Second is explicitly computing the precise ray class group characters corresponding to the extensions L/k and F/k via class field theory in order to compute the proper L-function values in Stark's conjecture for the extension K/k. Third is computing the precise number of roots of unity w_K in K without having an explicit generating polynomial for the field K itself.

In order to give a more straightforward presentation, we will make specific choices for the fields L, F, and K below where all of the essential elements of the general method will become apparent. In particular, we choose $L = M = k(\sqrt{-\eta})$, where we already solved the first obstacle above, giving the Stark unit υ corresponding to M/k in Proposition 1, Section 3. We choose $F = H_k$, the Hilbert class field of k, which enables us to easily handle the second obstacle, as we will see below, and assume $[H_k : k] = h_k > 1$. By definition, $K = H_k(\sqrt{-\eta})$, where η is a fundamental unit of k, chosen to be positive at $\mathfrak{p}_{\mathrm{R}}$. We leave the determination of w_K to Lemma 2 in Section 6. The version of Stark's conjecture in Section 2 applies to K/k since the place $\mathfrak{p}_{\mathrm{R}}$ ramifies in K/k.

Let $G = \mathrm{Gal}(K/k)$ and for a subfield L of K containing k let $J = \mathrm{Gal}(K/L)$. We denote the set of characters on the abelian group $G/J \cong \mathrm{Gal}(L/k)$ by X_L:

$$\mathrm{X}_L = \{\chi \in \mathrm{X}_K \mid \chi(\sigma) = 1, \forall \sigma \in J\}. \tag{9}$$

We have $\#\mathrm{X}_K = 2h_k$, and the subgroup X_{H_k} is of index 2 in X_K and consists precisely of all characters having trivial conductor. Let M/k and ψ be the same as in Section 3. Recall that $\mathfrak{f}(M/k) = \mathfrak{f}(\psi) = \mathfrak{p}_{\mathrm{R}}\mathfrak{f}$ with $\mathrm{N}\mathfrak{f} = 2^b$ and $b \geq 1$. We have $\psi \in \mathrm{X}_K$, and every character in the coset $\psi\mathrm{X}_{H_k}$ has conductor $\mathfrak{f}(M/k)$. Therefore $\mathfrak{f}(K/k) = \mathfrak{f}(M/k)$, and the relative discriminant of K/k is equal to $\mathfrak{f}^{h_k}$. The field K is totally complex by construction. Fix an embedding $K \hookrightarrow K^{(1)}$ (resp., $K \hookrightarrow K^{(2)}$) that extends the embedding $M \hookrightarrow M^{(1)}$ (resp., $M \hookrightarrow M^{(2)}$) in Section 3. If $\tau \in G$ denotes the nontrivial automorphism fixing H_k, then τ restricts to the nontrivial automorphism of M over k and it follows that $\tau(\alpha)^{(1)} = \overline{\alpha^{(1)}}$ for every $\alpha \in K$ since the embedding $K \hookrightarrow K^{(1)} \subset \mathbb{C}$ restricts to a real embedding of H_k.

Suppose now that a unit $\varepsilon \in U_K$ satisfies parts (1) and (2) of Stark's conjecture. These conjectured properties of the Stark unit for K/k place very strong restrictions upon the form of ε as we now show, and they help us to make a reasonable search for it. Since $\varepsilon^{(1)}\overline{\varepsilon^{(1)}} = 1$ by part (1), we see that $\varepsilon^{(1)} \cdot \tau(\varepsilon)^{(1)} = 1$ and since $K \hookrightarrow K^{(1)}$ is an embedding, we have $\varepsilon \cdot \tau(\varepsilon) = 1$. Since G is abelian, it follows that

$$\tau(\sigma(\varepsilon)) = 1/\sigma(\varepsilon) \quad \text{for all } \sigma \in G. \tag{10}$$

As an immediate consequence, we have

$$|\sigma(\varepsilon)^{(1)}| = 1 \quad \text{for all } \sigma \in G. \tag{11}$$

Stark units also satisfy a norm compatability property that we wish to exploit. For example, the relative norm of ε from K to M is equal to a power of the Stark unit υ for M/k found in Section 3 times a root of unity in M. More precisely,

$$\mathrm{N}_{K/M}(\varepsilon) = \pm\, \upsilon^{w_K/2} \tag{12}$$

since $w_M = 2$ (see [Ta] p. 92). Equation (12) helped motivate our choice of the particular relative quadratic extension M/k. The ambiguity in equation (12) is minimal and we have very restricted ramification in the tower $k \subset M \subset K$ (only primes above 2).

From now on we set $J = \mathrm{Gal}(K/M)$. Let $\mathcal{O}_M$ denote the ring of integers in M. Following Dasgupta [Da] we attempt to compute the coefficients of the polynomial

$$g(x) = \prod_{\sigma\in J}(x - \sigma(\varepsilon)) = x^{h_k} - Bx^{h_k-1} + \cdots + (-1)^{h_k} D \in \mathcal{O}_M[x], \tag{13}$$

where $B = \mathrm{Tr}_{K/M}(\varepsilon) = \sum_{\sigma\in J}\sigma(\varepsilon)$ and $D = \mathrm{N}_{K/M}(\varepsilon)$. The point of considering this polynomial, as opposed to the polynomial satisfied by ε over the Hilbert class field H_k as in the original work of Stark and subsequent authors, is that the coefficient D is already essentially determined through equation (12) by Stark's conjecture for this subfield. More generally, a systematic use of Stark's conjecture in all intermediate subfields significantly diminishes the computation required for the Stark unit in K. This sort of "bootstrap" use of Stark's conjecture may prove effective in a wider variety of settings than those considered here (cf., for example, the questions raised in [DT] regarding computations in higher-rank Stark conjecture situations).

As just noted, we already essentially know the coefficient D. We may use Stark's conjecture to place bounds upon all of the other coefficients of $g(x)$, in all complex embeddings of M. As an example, we compute bounds for the coefficient B. By use of equation (11) we find

$$|B^{(1)}| = \left|\sum_{\sigma\in J} \sigma(\varepsilon)^{(1)}\right| \le \sum_{\sigma\in J} |\sigma(\varepsilon)^{(1)}| = h_k. \tag{14}$$

Part (2) of Stark's conjecture gives

$$|B^{(2)}| \le \sum_{\sigma\in J} |\sigma(\varepsilon)^{(2)}| = \sum_{\sigma\in J} \exp\left(\frac{-w_K}{2}\zeta_{\mathfrak{f}}'(0,\sigma)\right). \tag{15}$$

Recalling that $\alpha^{(3)} = \tau(\alpha)^{(2)}$ for all $\alpha \in M$ (see Section 3) and then applying equation (10) gives

(16)
$$|B^{(3)}| = |\tau(B)^{(2)}| \le \sum_{\sigma\in J} |\tau\sigma(\varepsilon)^{(2)}| = \sum_{\sigma\in J} |\sigma(\varepsilon)^{(2)}|^{-1} = \sum_{\sigma\in J} \exp\left(\frac{w_K}{2}\zeta_{\mathfrak{f}}'(0,\sigma)\right).$$

We employ these bounds in the following way. Let $\omega_1,\dots,\omega_6$ be an integral basis for the ring of integers $\mathcal{O}_M$. Let $\alpha \in \mathcal{O}_M$, and assume $|\alpha^{(i)}| \le C_i$ for $i = 1,2,3$. For $\alpha = \sum_{j=1}^{6} a_j\omega_j$, $a_j \in \mathbb{Z}$, we have

$$\alpha^{(1)}\overline{\alpha^{(1)}} + \alpha^{(2)}\overline{\alpha^{(2)}} + \alpha^{(3)}\overline{\alpha^{(3)}} \le C_1^2 + C_2^2 + C_3^2 = C. \tag{17}$$

The left-hand side of equation (17) gives a positive definite quadratic form

$$Q(\mathbf{a}) = \sum_{1\le i,j\le 6} q_{ij}a_ia_j,$$

where the coefficient q_{ij} is equal to the real part of $\sum_{k=1}^{3} \omega_i^{(k)}\overline{\omega_j^{(k)}}$. There are only a finite number of integer vectors $\mathbf{a} = (a_1,\dots,a_6)$ such that $Q(\mathbf{a}) \le C$ and we may employ the Fincke-Pohst algorithm ([PZ], p. 190) to determine them.

5. Computation of the L-function values arising in Stark's conjecture

Even though the list of possible candidates for $g(x)$ in equation (13) is finite, as a practical matter the search for $g(x)$ can be somewhat lengthy. For one thing, we do not assume an explicit knowledge of K from the outset. All we know from class field theory is the existence of K and its conductor $\mathfrak{f}(K/k)$. We work inside the base field k, using the ray class group mod $\mathfrak{f}(K/k)$ to compute the relevant partial zeta and L-function values at $s = 0$. These values are used in conjunction with Stark's conjecture to compute $g(x)$ in Section 6. The package PARI/GP [GP] was relied upon for all of our computations.

Let $k = \mathbb{Q}(\gamma)$ be a complex cubic number field generated by the algebraic integer γ. Let η be a fundamental unit of k, chosen to be positive at the real place $\mathfrak{p}_{\mathrm{R}}$ of k and set $M = k(\sqrt{-\eta})$. If $\mathfrak{d}(M/k)$ is the relative discriminant, H_k is the Hilbert class field of k, and $K = H_k(\sqrt{-\eta})$, then $\mathfrak{f}(K/k) = \mathfrak{f}(M/k) = \mathfrak{p}_{\mathrm{R}}\mathfrak{d}(M/k) = \mathfrak{p}_{\mathrm{R}}\mathfrak{f}$, as we saw in Sections 3 and 4. The ray class group mod $\mathfrak{f}(K/k)$, which we denote by $\mathrm{Cl}(\mathfrak{p}_{\mathrm{R}}\mathfrak{f})$, contains at least $2h_k$ elements since K is contained in the corresponding ray class field. We compute the structure of $\mathrm{Cl}(\mathfrak{p}_{\mathrm{R}}\mathfrak{f})$ and the conductor of each character defined on this group. The characters in X_{H_k} are easy to identify since they are exactly the characters with trivial conductor. For the characters χ of conductor $\mathfrak{p}_{\mathrm{R}}\mathfrak{f}$ we compute $L'_{\mathfrak{f}}(0,\chi)$ using the method described below. There are potentially several quadratic characters of conductor $\mathfrak{p}_{\mathrm{R}}\mathfrak{f}$ defined on the ray class group, but the quadratic character ψ associated to the extension M/k was always identifiable by the value $L'_{\mathfrak{f}}(0,\psi) = h_M R_M / h_k R_k$ (see Section 3). With X_{H_k} and ψ in hand, we know exactly the characters in $\mathrm{X}_K = \mathrm{X}_{H_k} \cup \psi \mathrm{X}_{H_k}$. We define $\mathrm{X}_K^{\perp} = \{\sigma^* \in \mathrm{Cl}(\mathfrak{p}_{\mathrm{R}}\mathfrak{f}) \,|\, \chi(\sigma^*) = 1, \forall \chi \in \mathrm{X}_K\}$ and the corresponding quotient group $G^* = \mathrm{Cl}(\mathfrak{p}_{\mathrm{R}}\mathfrak{f})/\mathrm{X}_K^{\perp}$. By class field theory, G^* and $G = \mathrm{Gal}(K/k)$ are isomorphic via the Artin map. Let $\{\sigma_0^* = \mathbf{1}, \ldots, \sigma_{2h_k-1}^*\}$ denote a set of coset representatives of G^*. Corresponding to equation (3), we compute

$$\zeta'_{\mathfrak{f}}(0,\sigma_j^*) = \frac{1}{2h_k} \sum_{\chi \in \mathrm{X}_K} \overline{\chi}(\sigma_j^*) L'_{\mathfrak{f}}(0,\chi) \tag{18}$$

for $0 \le j < 2h_k$. The set of values we compute in equation (18) match up exactly with the set of values $\{\zeta'_{\mathfrak{f}}(0,\sigma) \,|\, \sigma \in G\}$, but we have no means to make an element-wise correspondence until we find a generating polynomial for K. On the other hand, a subgroup to subgroup correspondence can be made explicit. In order to find $g(x) = \prod_{\sigma \in J}(x - \sigma(\varepsilon))$, where $J = \mathrm{Gal}(K/M)$, we need to compute the values $\{\zeta'_{\mathfrak{f}}(0,\sigma) \,|\, \sigma \in J\}$ to high precision. The coset representatives corresponding to J consist of the subset J^* of $\{\sigma_0^*, \ldots, \sigma_{2h_k-1}^*\}$ on which ψ is trivial. This allows us to identify the set of values $\{\zeta'_{\mathfrak{f}}(0,\sigma) \,|\, \sigma \in J\}$.

We now sketch our method for computing the values $L'_{\mathfrak{f}}(0,\chi)$ appearing in equation (18). We refer to [DT] for a fuller treatment. We already noted in equation (5) that $L'_{\mathfrak{f}}(0,\chi_0) = 0$. In general, for $\chi \in \mathrm{X}_K$, we have

$$L'_{\mathfrak{f}}(0,\chi) \ne 0 \Leftrightarrow \mathfrak{f}(\chi) = \mathfrak{p}_{\mathrm{R}}\mathfrak{f} \Leftrightarrow \chi \in \psi \mathrm{X}_{H_k}. \tag{19}$$

Let χ be a character of conductor $\mathfrak{p}_{\mathrm{R}}\mathfrak{f}$. For $\mathrm{Re}(s) > 1$, we have

$$L_{\mathfrak{f}}(s,\chi) = \sum_{(\mathfrak{a},\mathfrak{f})=1} \frac{\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s} = \sum_{n \ge 1} \frac{a_n}{n^s}, \tag{20}$$

where a_n is equal to the sum $\sum \chi(\mathfrak{a})$ over all integral ideals in k relatively prime to $\mathfrak{f}$ of norm n. Let $A(\mathfrak{f}) = \frac{1}{2\pi^{3/2}}\sqrt{|d_k|\mathrm{N}\mathfrak{f}}$. The Lavrik-Friedman formula gives

$$L'_{\mathfrak{f}}(0,\chi) = \frac{1}{\sqrt{\pi}} \sum_{n \ge 1} \left[a_n\, f\!\left(\frac{A(\mathfrak{f})}{n}, 0\right) + W(\chi)\, \overline{a_n}\, f\!\left(\frac{A(\mathfrak{f})}{n}, 1\right) \right], \tag{21}$$

where $f(x,s)$ denotes the line integral

$$f(x,s) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} x^z\, \Gamma\!\left(\frac{z+1}{2}\right) \Gamma(z)\, \frac{dz}{z-s} \tag{22}$$

for any $\delta > 1$. The Artin root number $W(\chi)$ has absolute value 1. We refer to [DT] for a method to compute $W(\chi)$ based upon a global formula dating back to Landau [La] (see also [Co], p. 307). Using Legendre's duplication formula allows us to rewrite $f(x,s)$ in the form

$$f(x,s) = \frac{1}{4\pi^{3/2} i} \int_{\delta - i\infty}^{\delta + i\infty} (2x)^z\, \Gamma\!\left(\frac{z+1}{2}\right)^2 \Gamma\!\left(\frac{z}{2}\right) \frac{dz}{z-s}. \tag{23}$$

We have already computed this *exact* type of integral in [DST] in order to find Stark units over totally real cubic fields. The integrals in both (22) and (23) can be computed by shifting the line of integration to the left and summing up the resulting residues. See Section 3 in [DST] for a detailed discussion and error analysis.

6. Class number = 3

We assume throughout this section that the complex cubic base field k has class number $h_k = 3$. Let η be a fundamental unit of k chosen to be positive at the unique real place of k, and set $M = k(\sqrt{-\eta})$, $H_k =$ the Hilbert class field of k, and $K = H_k(\sqrt{-\eta})$. Let $\tau \in \mathrm{Gal}(K/k)$ denote the nontrivial automorphism fixing H_k and note that τ restricts to the nontrivial automorphism of M over k.

Lemma 2. *For K as defined above, $w_K = 2$.*

Proof. The possible values of w_K for a field K of degree 18 over $\mathbb{Q}$ are $2, 4, 6, 14, 18$, 38, or 54. If $w_K = 4$, then $M(\sqrt{-1})$ is a subfield of K. Since $w_M = 2$ by Lemma 1, we have $[M(\sqrt{-1}) : M] = 2$, in contradiction to $[K : M] = 3$. A similar argument with $M(\sqrt{-3})$ eliminates the possibility of $w_K = 6$ or $w_K = 18$. If $w_K = 38$ or 54, then K would be abelian over $\mathbb{Q}$ and so would not contain a complex cubic subfield. Finally, if $w_K = 14$, then $F = \mathbb{Q}\,(e^{2\pi i/7}) \subset K$, and since $[K : F] = 3$, $d_F^3 \mid d_K$, and 7 would divide d_K at least to the 15th power. But K/M is unramified, so we have $d_K = (d_M)^3 = -2^{3b} d_k^6$ (see Section 3 for the second equality). Since 7 divides d_k to at most the 2nd power, it divides d_K to at most to the 12th power. Thus $w_K = 2$. □

Throughout this section, we write the polynomial $g(x) \in \mathcal{O}_M[x]$ in equation (13) in the form

$$g(x) = x^3 - Bx^2 + Cx - D. \tag{24}$$

If υ is the (precomputed) Stark unit for M/k, we have $D = \pm\upsilon$ by equation (12) and Lemma 2. Since the Stark unit ε for K/k is only defined up to sign, we may, by changing the sign of ε if necessary, assume that $D = \upsilon$. It is easy to see that $C = \tau(B) \cdot D$. It follows that the polynomial $g(x)$ is determined once the single coefficient B has been computed, and Stark's conjecture was used to derive bounds on B in Section 4. We now present a detailed example showing how we find the Stark unit for the extension K/k and prove Theorem 1.

Example (Third entry in the tables of Section 7)**.** Let $k = \mathbb{Q}(\gamma)$, where γ satisfies $t(\gamma) = \gamma^3 - \gamma^2 - 4\gamma + 12 = 0$, and note that k is a complex cubic field of class number $h_k = 3$ and discriminant $d_k = -676$. The element $\eta = -\frac{1}{2}\gamma^2 - \frac{1}{2}\gamma + 2$ is a fundamental unit which is positive at $\mathfrak{p}_{\mathrm{R}}$. The field $M = k(\sqrt{-\eta})$ is generated over $\mathbb{Q}$ by $\beta = \sqrt{-\eta}$ which satisfies the polynomial $m(x) = x^6 + x^4 + 9x^2 + 1$. We obtain $m(x)$ as the polynomial resultant $R(t(\gamma), x^2 + \eta)$ with respect to γ (we think of γ as an indeterminate in this resultant computation). The nontrivial automorphism τ of M/k is obtained by sending $\beta \mapsto -\beta$. A unit $\eta_2 \in U_M$ which satisfies the conditions in Section 3 is given by $\eta_2 = \beta^4 - 3\beta^3 + 5\beta^2 - 3\beta + 1$. The embedding $M \hookrightarrow M^{(1)}$ defined by sending $\beta \mapsto i\ 0.335195\ldots$ extends the real embedding of k. If we define $M \hookrightarrow M^{(2)}$ and $M \hookrightarrow M^{(3)}$ by sending β to $1.126834\ldots + i\ 1.309036\ldots$ and to $-1.126834\ldots - i\ 1.309036\ldots$, respectively, then $|\eta_2^{(2)}| < 1$ and $\alpha^{(3)} = \tau(\alpha)^{(2)}$ for all $\alpha \in M$. We have $h_M = 6$ and so the Stark unit υ for M/k is equal to η_2^2 by

Proposition 1, Section 3. We set $D = \upsilon$. The prime 2 factors in k as $(2) = \mathfrak{p}_1\mathfrak{p}_2^2$. The relative discriminant $\mathfrak{d}(M/k)$ is equal to $\mathfrak{p}_1^2\mathfrak{p}_2^4$, so $\mathfrak{f}(K/k) = \mathfrak{p}_{\mathrm{R}}\mathfrak{p}_1^2\mathfrak{p}_2^4$. The ray class group $\mathrm{Cl}(\mathfrak{p}_{\mathrm{R}}\mathfrak{p}_1^2\mathfrak{p}_2^4)$ has structure $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/6\mathbb{Z}$. The group X_K is cyclic of order 6. Let $\mathrm{X}_K = <\chi>$ for a fixed character χ. Applying the Lavrik-Friedman formula to our particular choice of χ produces the value $L'_{\mathfrak{f}}(0,\chi) = -0.325065\ldots - i\ 9.987117\ldots$. We have $L'_{\mathfrak{f}}(0,\chi^i) = 0$ for $i = 0, 2, 4$, $L'_{\mathfrak{f}}(0,\chi^3) = h_M R_M / h_k R_k = 13.749462\ldots$, and $L'_{\mathfrak{f}}(0,\chi^5) = \overline{L'_{\mathfrak{f}}(0,\chi)}$. We choose $\sigma_j^* \in \mathrm{Cl}(\mathfrak{p}_{\mathrm{R}}\mathfrak{p}_1^2\mathfrak{p}_2^4)$ such that $\chi(\sigma_j^*) = \exp(2\pi i j/6)$ for $j = 0, \ldots, 5$. Equation (18) specializes to

$$\zeta'_{\mathfrak{f}}(0,\sigma_j^*) = \frac{1}{6}\sum_{i=0}^{5}\overline{\chi^i(\sigma_j^*)}L'_{\mathfrak{f}}(0,\chi^i).$$

The real numbers $\{-2\zeta'_{\mathfrak{f}}(0,\sigma_0^*), -2\zeta'_{\mathfrak{f}}(0,\sigma_2^*), -2\zeta'_{\mathfrak{f}}(0,\sigma_4^*)\}$ match the set of values $\{-w_K\zeta'_{\mathfrak{f}}(0,\sigma) \mid \sigma \in J\}$ in some order (see Section 5). Placing these values in ascending order, we obtain the vector

$$\mathbf{v}_1 = (-10.457574\ldots, -4.366444\ldots, 1.074556\ldots).$$

The bounds for the coefficient $B \in \mathcal{O}_M$ can now be computed using equations (14), (15), and (16). We find that

$$|B^{(1)}| \le C_1 = 3,$$
$$|B^{(2)}| \le C_2 = 1.829380\ldots,$$
$$|B^{(3)}| \le C_3 = 196.025577\ldots. \tag{25}$$

With respect to the quadratic form defined in equation (17), there are approximately 4.34×10^{11} algebraic integers B such that $\sum_{i=1}^{3} B^{(i)}\overline{B^{(i)}} \le C_1^2 + C_2^2 + C_3^2$. The vast majority of these will not simultaneously satisfy all three bounds in (25) and therefore this choice of quadratic form is far from optimal. We found in general that the positive definite quadratic form defined on the left-hand side of

$$\frac{\alpha^{(1)}\overline{\alpha^{(1)}}}{C_1^2} + \frac{\alpha^{(2)}\overline{\alpha^{(2)}}}{C_2^2} + \frac{\alpha^{(3)}\overline{\alpha^{(3)}}}{C_3^2} \le 3 \tag{26}$$

is a far better choice. This is well illustrated in the present example by the fact that there are only 238,616 algebraic integers B that satisfy (26), and of these, 53,122 satisfy the bounds in (25). For the B which satisfy all three bounds in (25), we find the complex roots z_1, z_2, z_3 of the polynomial $g(x)^{(2)} = x^3 - B^{(2)}x^2 + (\tau(B)D)^{(2)}x - D^{(2)} \in \mathbb{C}[x]$. Ordering the roots so that $|z_1| \le |z_2| \le |z_3|$, we form the vector $\mathbf{v}_2 = (\log|z_1|^2, \log|z_2|^2, \log|z_3|^2)$. The Stark unit ε for the extension K/k satisfies $g(\varepsilon) = 0$ for the "right" polynomial $g(x) \in \mathcal{O}_M[x]$. Part (2) of Stark's conjecture predicts that $\log|\sigma(\varepsilon)^{(2)}|^2 = -w_K\zeta'_{\mathfrak{f}}(0,\sigma)$ for all $\sigma \in J$ and thus the "right" polynomial $g(x)$ corresponds to a very small L^2-norm for $||\mathbf{v}_1 - \mathbf{v}_2||^2$. The bound $\delta = .00001$ for $||\mathbf{v}_1 - \mathbf{v}_2||^2 < \delta$ is sufficient to single out a unique candidate for $g(x)$, which is

$$x^3 - (2\beta^5 - 10\beta^4 + 18\beta^3 - 16\beta^2 - 3)x^2 + (52\beta^5 - 40\beta^4 - 12\beta^3 + 224\beta^2 + 27)x$$
$$- (-30\beta^5 + 18\beta^4 + 18\beta^3 - 144\beta^2 - 17).$$

The actual value of $||\mathbf{v}_1 - \mathbf{v}_2||^2$ with this choice of $g(x)$ is on the order of 10^{-36} when computing to 28 significant digits. In all of our computations, there was always a *unique* choice of $g(x)$ for which this L^2-norm was much smaller than any other choice. The "right" $g(x)$ was therefore always easy to identify and these polynomials are given explicitly for all 58 examples we computed in the tables in Section 7.

If Stark's conjecture is true, then $K = k(\varepsilon)$ (see Theorem 1 of [St3]). Therefore, $f(x) = \prod_{\sigma \in G}(x - \sigma(\varepsilon))$ should be a generating polynomial for the abelian extension K/k. Note that $f(x) = g(x) \cdot \tau(g(x))$ since $g(x) = \prod_{\sigma \in J}(x - \sigma(\varepsilon))$ and $G = J \cup \tau J$. For the present example, we find $f(x) \in k[x]$ to be

$$x^6 + (16\gamma^2 - 24\gamma - 158)x^5 + (256\gamma^2 + 352\gamma - 721)x^4 + (480\gamma^2 + 880\gamma - 804)x^3 + (256\gamma^2 + 352\gamma - 721)x^2 + (16\gamma^2 - 24\gamma - 158)x + 1.$$

The constant coefficient in $f(x)$ is equal to $\mathrm{N}_{M/k}(D)$, which is 1 in all of our examples since D is a power of the unit η_2 by construction and $\mathrm{N}_{M/k}(\eta_2) = 1$ (see Section 3). Let ε_0 be a root of $g(x)$ (and therefore also of $f(x)$), and set $\widetilde{K} = M(\varepsilon_0)$. We next make an independent check to see that $\widetilde{K} = K$. We first form a new polynomial $f_{\varepsilon_0}(x) \in \mathbb{Q}[x]$ equal to the polynomial resultant $R(m(\beta), g(x))$ with respect to β (we think of β as an indeterminate in this resultant computation). The polynomial $f_{\varepsilon_0}(x)$ is the norm of $g(x) \in M[x]$ (see [PZ], p. 346) and also of $f(x) \in k[x]$. The multiplicativity of the norm implies that if $f_{\varepsilon_0}(x)$ is irreducible in $\mathbb{Q}[x]$, then $g(x)$ and $f(x)$ are irreducible in $M[x]$ and $k[x]$, respectively. For the present example, we find that $f_{\varepsilon_0}(x)$ is equal to

$$x^{18} - 354x^{17} + 33113x^{16} + 142032x^{15} + 2828340x^{14} + 18863304x^{13} + 62598980x^{12} + 133013936x^{11} + 203040558x^{10} + 232702004x^9 + 203040558x^8 + 133013936x^7 + 62598980x^6 + 18863304x^5 + 2828340x^4 + 142032x^3 + 33113x^2 - 354x + 1,$$

which is easily verified to be irreducible in $\mathbb{Q}[x]$. We conclude that $\widetilde{K}$ is generated over $\mathbb{Q}$ by the unit ε_0, consistent with Theorem 1 of [St3]. We also conclude that $[\widetilde{K} : M] = 3$. We can prove that $K = \widetilde{K}$ by showing that $H_k \subset \widetilde{K}$ since $M \subset \widetilde{K}$ and K is the composite of H_k and M. We begin by finding six roots of the polynomial $f_{\varepsilon_0}(x)$ in the field $\widetilde{K}$ (the PARI command **nfgaloisconj** was used for this) and verify that they are roots of $f(x)$ as well and have the form $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 = 1/\varepsilon_0$, $\varepsilon_4 = 1/\varepsilon_1$, $\varepsilon_5 = 1/\varepsilon_2$, consistent with equation (10). By Theorem 1 of [St3], we expect the Hilbert class field to be generated by the polynomial

$$h(x) = (x - (\varepsilon_0 + \varepsilon_3))(x - (\varepsilon_1 + \varepsilon_4))(x - (\varepsilon_2 + \varepsilon_5)) = x^3 - bx^2 + cx - d \in k[x].$$

If $f(x)$ has the form $x^6 + b_1x^5 + c_1x^4 + d_1x^3 + c_1x^2 + b_1x + 1$, a little computation shows that $b = -b_1$, $c = c_1 - 3$, and $d = 2b_1 - d_1$. For the present situation, we find

$$h(x) = x^3 - (-16\gamma^2 + 24\gamma + 158)x^2 + (256\gamma^2 + 352\gamma - 724)x - (-448\gamma^2 - 928\gamma + 488).$$

We may use the norm of $h(x)$, as we did above for $f(x)$, to check that $h(x)$ is irreducible in $k[x]$. We can verify that a root α_0 of $h(x)$ generates the Hilbert class field H_k by checking that there is no ramification in the relative extension $k(\alpha_0)/k$. The relative extension $k(\alpha_0)/k$ is then abelian since otherwise its normal closure would be a sextic unramified extension with an unramified quadratic subfield over k, in contradiction to $h_k = 3$. This shows that once $g(x)$ has been found, a generating polynomial for the Hilbert class field can be readily constructed and tested. Clearly $\varepsilon_0 + 1/\varepsilon_0 \in \widetilde{K}$, and so $H_k \subset \widetilde{K}$. With this established, we know that $f(x)$ is a generating polynomial for the abelian extension K/k and that K is generated over $\mathbb{Q}$ by the unit ε_0 satisfying $f_{\varepsilon_0}(x)$.

Proof of Theorem 1. We have $K = k(\varepsilon_0)$ and $H_k = k(\varepsilon_0 + 1/\varepsilon_0)$, which verifies the first statement in Theorem 1, and we next prove that the unit $\varepsilon_0 \in U_K$ satisfies parts (1) and (3) of Stark's conjecture and part (2) to within the numerical accuracy of our computations.

We define the embedding $K \hookrightarrow K^{(2)} \subset \mathbb{C}$ by sending ε_0 to the unique root $z = 0.013815\ldots - i\, 0.111827\ldots$ of $g(x)^{(2)} \in \mathbb{C}[x]$ satisfying $\log|z|^2 = -2\zeta_{\mathfrak{f}}'(0, \sigma_0^*)$ to the accuracy of our computations. The roots of $f(x)$ computed above (with **nfgaloisconj**) are the six Galois conjugates of ε_0 in K/k, and we number them in such a way that

$$\log|\varepsilon_j^{(2)}|^2 = -2\zeta_{\mathfrak{f}}'(0, \sigma_j^*) \quad \text{for } j = 0, \ldots, 5.$$

Defining the map $\sigma : K \to K$ by $\sigma(\varepsilon_0) = \varepsilon_1$, we make an algebraic check that $\sigma^j(\varepsilon_0) = \varepsilon_j$ for $j = 0, \ldots, 5$. In accordance with equation (10), we also check that $\varepsilon_{i+3} = 1/\varepsilon_i$ for $i = 0, 1, 2$. Since $\tau = \sigma^3$, we have $\tau(\varepsilon_0) = 1/\varepsilon_0$, and recalling that $\tau(\varepsilon_0)^{(1)} = \overline{\varepsilon_0^{(1)}}$ (see Section 4), we see that $\varepsilon_0^{(1)} \cdot \overline{\varepsilon_0^{(1)}} = 1$, which proves that ε_0 satisfies part (1) of the conjecture. In order to complete the numerical proof of part (2), we still need to verify that $\zeta_{\mathfrak{f}}'(0, \sigma_j^*) = \zeta_{\mathfrak{f}}'(0, \sigma^j)$ for $j = 0, \ldots, 5$. We first find a prime ideal $\mathfrak{p}$ in the ray class σ_1^*. If we can prove that $\sigma_{\mathfrak{p}} = \sigma$, we are done by the multiplicativity of the Artin map. The Frobenius automorphism $\sigma_{\mathfrak{p}}$ is an element of order 6 in $\mathrm{Gal}(K/k)$ and therefore must equal either σ or σ^5. It suffices to simply eliminate σ^5 as a possibility. The prime 11 splits in k as a product of two prime ideals with inertial degrees 1 and 2. Let $\mathfrak{p}_{11}$ denote the prime ideal of norm 11. It is easy to check that $\mathfrak{p}_{11} \in \sigma_1^*$. There is a single prime ideal $\mathfrak{P}_{11}$ in K above $\mathfrak{p}_{11}$ since $f(\mathfrak{P}_{11}/\mathfrak{p}_{11}) = 6$ (= order of σ_1^* in the ray class group). A quick check shows that $\varepsilon_5 - \varepsilon_0^{11}$ is not divisible by $\mathfrak{P}_{11}$. This proves that $\sigma_{\mathfrak{p}_{11}} = \sigma$.

The proof of part (3) is purely algebraic. Recall from Lemma 2 that $w_K = 2$ and so part (3) asserts that $K(\sqrt{\varepsilon_0})$ is an abelian extension of k.

Lemma 3. *If $\varepsilon_0^{\sigma-1} = \alpha^2$ for some nonzero $\alpha \in K$, then $K(\sqrt{\varepsilon_0})/k$ is abelian.*

Proof. (Adapted from p. 85 of [Ta]) If $K(\sqrt{\varepsilon_0}) = K$, we are done, so assume that $[K(\sqrt{\varepsilon_0}) : K] = 2$. Let ρ be an extension to $K(\sqrt{\varepsilon_0})$ of the generating automorphism $\sigma \in \mathrm{Gal}(K/k)$. We note that $(\sqrt{\varepsilon_0})^\rho = \pm\alpha\sqrt{\varepsilon_0}$ follows from $\varepsilon_0^{\sigma-1} = \alpha^2$ and conclude that $K(\sqrt{\varepsilon_0})$ is Galois over k. Set $\Gamma = \mathrm{Gal}(K(\sqrt{\varepsilon_0})/k)$. If ν is the nontrivial element of Γ fixing K, then $\nu\rho = \rho\nu$ (just check their respective actions on $\sqrt{\varepsilon_0}$), and therefore Γ is abelian since ν and ρ generate Γ. □

We prove part (3) as follows. Let $s(x)$ be the minimal polynomial of $\varepsilon_1/\varepsilon_0$ in $\mathbb{Q}[x]$. From a verification that $s(x^2)$ factors nontrivially in $\mathbb{Q}[x]$, it follows that $\varepsilon_0^{\sigma-1}$ is a square in K and we are done by Lemma 3.

This completes the proof of Theorem 1 for this example and the other 57 examples in the tables in Section 7 are proved in the same way. □

Computing the correct coefficient B in the polynomial $g(x)$ is the most time-consuming part of computing the Stark unit. We already noticed above how much time can be saved by making a judicious choice of the positive definite quadratic form we use. Even with the superior choice made in equation (26), the search for B can be prohibitively long in some cases. In many of the most difficult examples,

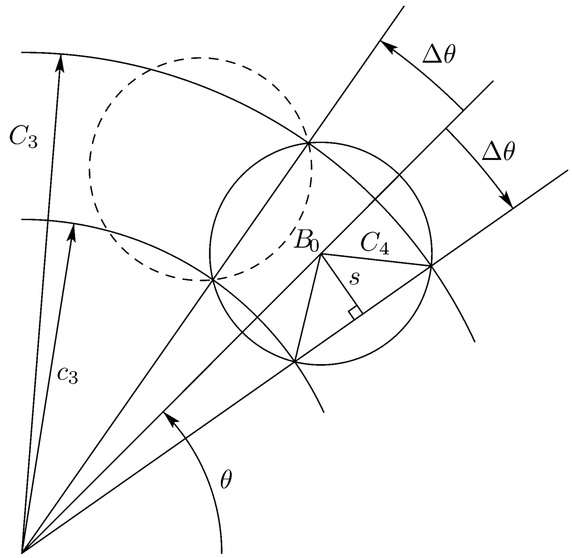

FIGURE 1.

one of the values $\{\exp(\zeta'_{\mathfrak{f}}(0,\sigma)) \mid \sigma \in J\}$ is much larger than the other two and a significant improvement can be made in this case using the following technique. Assume, for example, that $\exp(\zeta'_{\mathfrak{f}}(0,\sigma_0))$ is much larger than $\exp(\zeta'_{\mathfrak{f}}(0,\sigma_2))$ and $\exp(\zeta'_{\mathfrak{f}}(0,\sigma_4))$. By equations (10) and (16),

$$|B^{(3)}| = |1/\sigma_0(\varepsilon)^{(2)} + 1/\sigma_2(\varepsilon)^{(2)} + 1/\sigma_4(\varepsilon)^{(2)}|,$$

and therefore

$$\begin{aligned} |B^{(3)}| &\geq |1/\sigma_0(\varepsilon)^{(2)}| - |1/\sigma_2(\varepsilon)^{(2)}| - |1/\sigma_4(\varepsilon)^{(2)}| \\ &= \exp\left(\zeta'_{\mathfrak{f}}(0,\sigma_0)\right) - \exp\left(\zeta'_{\mathfrak{f}}(0,\sigma_2)\right) - \exp\left(\zeta'_{\mathfrak{f}}(0,\sigma_4)\right) := c_3, \end{aligned}$$

giving a good lower bound on $|B^{(3)}|$. Assume that $c_3 \leq |B^{(3)}| \leq C_3$, and also $|\arg(B^{(3)}) - \theta| \leq \Delta\theta$ for some θ and $\Delta\theta$. If we set $s = \frac{1}{2}(c_3 + C_3)\tan(\Delta\theta)$ (see Figure 1), and

$$B_0 = e^{i\theta}\sqrt{\left(\frac{c_3 + C_3}{2}\right)^2 + s^2},$$

then

$$|B^{(3)} - B_0| \leq \sqrt{\left(\frac{C_3 - c_3}{2}\right)^2 + s^2} = C_4.$$

This inequality, combined with $|B^{(1)}| \leq C_1$ and $|B^{(2)}| \leq C_2$, will have only a small number of solutions which we determine using the Fincke-Pohst algorithm applied to a quadratic form in seven variables. Let $\alpha = \sum_{i=1}^{6} a_i\omega_i$, and introduce a seventh variable a_7. We consider the quadratic form defined by

$$\frac{\alpha^{(1)}\overline{\alpha^{(1)}}}{C_1^2} + \frac{\alpha^{(2)}\overline{\alpha^{(2)}}}{C_2^2} + \frac{(\alpha^{(3)} - a_7B_0)\overline{(\alpha^{(3)} - a_7B_0)}}{C_4^2}, \tag{27}$$

and we wish to determine all integer vectors $(a_1, \ldots, a_6, 1)$ which give a value to the form less than or equal to 3. The first step in the Fincke-Pohst algorithm ([PZ], p. 190) is to complete the square, rewriting the quadratic form as

$$\sum_{i=1}^{7} q_{ii} \left(a_i + \sum_{j=i+1}^{7} q_{ij} a_j \right)^2 .$$

Some care is required since the quadratic form above is not positive definite; however, the only coefficient q_{ii} equal to 0 is q_{77}. Now start the Fincke-Pohst algorithm with bound 3, $a_7 = 1$, $i = 6$, and terminate as soon as i becomes 7 (note that we never divide by q_{77}). In this way we find all $B \in \mathcal{O}_M$ such that

$$\frac{B^{(1)}\overline{B^{(1)}}}{C_1^2} + \frac{B^{(2)}\overline{B^{(2)}}}{C_2^2} + \frac{(B^{(3)} - B_0)\overline{(B^{(3)} - B_0)}}{C_4^2} \leq 3. \tag{28}$$

A search for all possible B is carried out by letting θ take on successively the values $\Delta\theta, 3\Delta\theta, \ldots, \pi - \Delta\theta$ (roots of $g(x)^{(2)}$ are computed with both B and $-B$). A careful choice of $\Delta\theta$ can give a dramatic reduction in the search. For example, for field number 56 in the tables, there are approximately 4.12×10^9 algebraic integers B that satisfy (26). Choosing $\Delta\theta = \pi/22000$, there is a total of 2,120,184 algebraic integers B satisfying equation (28) for all values of θ, and of these, 201,350 satisfy the inequalities $|B^{(1)}| \leq C_1$, $|B^{(2)}| \leq C_2$, and $c_3 \leq |B^{(3)}| \leq C_3$. The technique described here was used with greatest effect upon fields 17, 38, 44, 48, 56, 57, and 58 in the tables in Section 7.

To the authors' knowledge, the only computation of a Stark unit over a complex cubic base field previous to our own work was done by Dasgupta [Da]. He considered the extension K/k, where k is the first field in our tables ($d_k = -588$) and K is the composite of H_k and $M = k(\sqrt{-3})$. The norm compatability expressed in equation (12) (here $w_M = 6$) was not taken advantage of and thus a simultaneous search on both B and D was required. His search method was also much less efficient than the quadratic form methods employed here.

7. Tables

This section contains tables summarizing our computations over complex cubic number fields of class number 3, all of which are in full agreement with Stark's conjecture up to the accuracy of our computations. Throughout, k denotes a complex cubic base field with class number $h_k = 3$, and $\mathcal{O}_M$ denotes the ring of integers in a number field M. The tables comprise the complete list of complex cubic fields with class number 3 and discriminant $|d_k| \leq 4300$, numbered as they appear on the PARI ftp site `ftp://megrez.math.u-bordeaux.fr/pub/numberfields/`. For each field k the following data is provided:

(1) The discriminant d_k of $k = \mathbb{Q}(\gamma)$ together with the minimal polynomial $t(x)$ for γ over $\mathbb{Q}$.

(2) The unit $\eta \in k$ defining $M = k(\sqrt{-\eta})$ in the form $\eta = a_2\gamma^2 + a_1\gamma + a_0$ (recall that η is a fundamental unit of k chosen to be positive at the unique real place of k). The minimal polynomial $m(x)$ over $\mathbb{Q}$ for $\beta = \sqrt{-\eta}$ with $M = \mathbb{Q}(\beta)$ is then given by the polynomial resultant $R(t(y), x^2 + (a_2y^2 + a_1y + a_0))$ with respect to the variable y.

(3) The coefficient $D = \eta_2^{h_M/3} \in \mathcal{O}_M$ in the polynomial $g(x) = x^3 - Bx^2 + Cx - D$ in the form $e_5\beta^5 + \cdots + e_0$ (recall that D is the Stark unit for the extension M/k). The embedding $M \hookrightarrow M^{(2)} \subset \mathbb{C}$ in Section 3 is defined by mapping β to the unique root of $m(x)$ with positive imaginary part such that $|D^{(2)}| < 1$.

(4) The coefficient $B \in \mathcal{O}_M$ in the polynomial $g(x)$ in the form $j_5\beta^5 + \cdots + j_0$. The coefficient C in $g(x)$ is given by $\tau(B) \cdot D$ (where $\tau \in \mathrm{Gal}(M/k)$ is always defined by $\tau(\beta) = -\beta$). If $K = H_k(\sqrt{-\eta})$, where H_k is the Hilbert class field of k, then $g(x)$ is the explicit generating polynomial for the relative extension K/M having the Stark unit ε_0 of Theorem 1 as a root.

(5) The unique complex root z of $g(x)^{(2)}$ satisfying $\log|z|^2 = -2\zeta'_{\mathfrak{f}}(0, \mathbf{1})$ (to within the accuracy of the computations), giving the Stark unit $\varepsilon_0^{(2)} = z$ with respect to the distinguished complex embedding $K \hookrightarrow K^{(2)} \subset \mathbb{C}$ extending the embedding $M \hookrightarrow M^{(2)}$ above.

With regard to the question of "getting inside the absolute value sign" of the conjecture brought up in Section 1, we note that the arguments of the 58 complex roots z in the tables appear to be uniformly distributed, although just 58 data points are insufficient to make any reliable predictions.

Using the data provided, it is straightforward to determine an explicit generating polynomial $h(x)$ for H_k over k, as well as the minimal polynomials $f(x)$ and $f_{\varepsilon_0}(x)$, over k and $\mathbb{Q}$, respectively, for the computed Stark unit ε_0 associated to the extension K/k. We indicate the computations required in the following example.

Example. Consider the first field in the tables, of discriminant -588: $k = \mathbb{Q}(\gamma)$, where $\gamma^3 - \gamma^2 + 5\gamma + 1 = 0$.

<table>
<tr><td>num</td><td>disc</td><td>η</td><td>z</td></tr>
<tr><td colspan="2">$t(x)$</td><td colspan="2">D
B</td></tr>
<tr><td>1</td><td>-588</td><td>$-\gamma$</td><td>$-0.0047943 + 0.000868081\, i$</td></tr>
<tr><td colspan="2">$x^3 - x^2 + 5x + 1$</td><td colspan="2">$3\beta^5 - 5\beta^4 + 2\beta^3 + 4\beta^2 + \beta$
$-13\beta^5 + 17\beta^4 + 2\beta^3 - 28\beta^2 + \beta - 8$</td></tr>
</table>

In this case $M = k(\sqrt{-\eta}) = k(\sqrt{\gamma})$. The minimal polynomial $m(x)$ defining M is given by the resultant $R(y^3 - y^2 + 5y + 1, x^2 - y)$ with respect to y, which in this case gives the polynomial $m(x) = x^6 - x^4 + 5x^2 + 1$, so $M = \mathbb{Q}(\beta)$ with $\beta^6 - \beta^4 + 5\beta^2 + 1 = 0$.

The embedding $M \hookrightarrow M^{(2)}$ is defined by mapping β to the root $1.20\ldots + i\,0.919\ldots$ of $m(x)$ with positive imaginary part and for which $D = 3\beta^5 - 5\beta^4 + 2\beta^3 + 4\beta^2 + \beta$ has absolute value less than one.

The coefficient C is given by the product of D and $\tau(B) = 13\beta^5 + 17\beta^4 - 2\beta^3 - 28\beta^2 - \beta - 8$, which gives the polynomial $g(x)$ defining K over M explicitly in this case as:

$$x^3 - (-13\beta^5 + 17\beta^4 + 2\beta^3 - 28\beta^2 + \beta - 8)x^2 + (2\beta^5 + 8\beta^4 - 24\beta^3 + 28\beta^2 - 6\beta + 7)x - (3\beta^5 - 5\beta^4 + 2\beta^3 + 4\beta^2 + \beta).$$

The minimal polynomial $f(x) \in k[x]$ having ε_0 as a root and defining K over k explicitly is given by $f(x) = g(x) \cdot \tau(g(x))$. It has the general form $f(x) = x^6 + b_1x^5 + c_1x^4 + d_1x^3 + c_1x^2 + b_1x + 1$ in all of our examples and in this example we have $b_1 = -34\gamma^2 + 56\gamma + 16$, $c_1 = -80\gamma^2 - 104\gamma - 9$, and $d_1 = 36\gamma^2 - 128\gamma - 16$. These coefficients also give us an explicit polynomial $h(x) = x^3 + b_1x^2 + (c_1 - 3)x + (d_1 - 2b_1)$ generating the Hilbert class field H_k over k (see Section 6).

The irreducible polynomial $f_{\varepsilon_0}(x) \in \mathbb{Q}[x]$ satisfied by the Stark unit ε_0 for K/k can be computed by taking the polynomial resultant $R(m(y), g_y(x))$ with respect to y, where $g_y(x)$ is obtained from $g(x)$ by substituting y for β. In this example $f_{\varepsilon_0}(x)$ is given explicitly by

$$x^{18} + 410x^{17} + 44601x^{16} + 264272x^{15} + 861524x^{14} + 1881368x^{13} + 3659428x^{12} + 6401840x^{11} + 9213390x^{10} + 10429340x^9 + 9213390x^8 + 6401840x^7 + 3659428x^6 + 1881368x^5 + 861524x^4 + 264272x^3 + 44601x^2 + 410x + 1.$$

We have $K = \mathbb{Q}(\varepsilon_0)$, and the embedding $K \hookrightarrow K^{(2)}$ is defined by sending ε_0 to the unique root $z = -0.00479\ldots + i\ 0.000868\ldots$ of $g(x)^{(2)} \in \mathbb{C}[x]$ satisfying $\log|z|^2 = -2\zeta_{\mathfrak{f}}'(0, \mathbf{1})$ indicated in the table.

num	disc	η	z
$t(x)$		D B	
1	-588	$-\gamma$	$-0.0047943 + 0.000868081\,i$
$x^3 - x^2 + 5x + 1$		$3\beta^5 - 5\beta^4 + 2\beta^3 + 4\beta^2 + \beta$ $-13\beta^5 + 17\beta^4 + 2\beta^3 - 28\beta^2 + \beta - 8$	
2	-648	$\frac{\gamma^2-\gamma-4}{2}$	$-0.0743044 - 0.411254\,i$
$x^3 - 3x - 10$		$\frac{\beta^5-\beta^4-2\beta^3+2\beta^2+7\beta-3}{4}$ $\frac{2\beta^5+\beta^4-6\beta^3+8\beta+3}{2}$	
3	-676	$\frac{-\gamma^2-\gamma+4}{2}$	$0.0138158 - 0.1118277\,i$
$x^3 - x^2 - 4x + 12$		$-30\beta^5 + 18\beta^4 + 18\beta^3 - 144\beta^2 - 17$ $2\beta^5 - 10\beta^4 + 18\beta^3 - 16\beta^2 - 3$	
4	-891	γ	$0.0101587 - 0.158227\,i$
$x^3 + 6x - 1$		$2\beta^5 + 7\beta^4 + 11\beta^3 + 8\beta^2 + 3\beta + 2$ $\frac{-\beta^5-\beta^4+\beta^3+5\beta^2+3\beta-1}{2}$	
5	-931	$\gamma - 1$	$0.352044 - 0.311576\,i$
$x^3 - x^2 + 5x - 6$		$\frac{\beta^5+\beta^4+\beta^2+5\beta}{2}$ $\frac{-\beta^5+3\beta^3+\beta^2-6\beta+1}{2}$	
6	-980	$-\gamma + 2$	$-0.39494 + 0.0609113\,i$
$x^3 - x^2 + 5x - 13$		$4\beta^5 - 5\beta^4 - 2\beta^3 - 40\beta^2 - 2\beta - 4$ $4\beta^5 + 17\beta^4 + 30\beta^3 + 40\beta^2 - 2\beta + 4$	

num	disc	η	z	$t(x)$	D	B
7	-1083	$\frac{\gamma^2+3\gamma+4}{2}$	$0.176504 + 0.0380955\,i$	$x^3 - x^2 - 6x - 12$	$\frac{4\beta^5+3\beta^4+57\beta^3+42\beta^2+23\beta+3}{6}$	$\frac{-7\beta^5+\beta^4-96\beta^3+15\beta^2+13\beta+14}{6}$
8	-1176	$\gamma^2 - 3\gamma + 1$	$-0.0836886 - 0.0543439\,i$	$x^3 - x^2 - 2x - 6$	$\frac{4\beta^5+56\beta^4+151\beta^3+287\beta^2+\beta}{7}$	$\frac{-5\beta^5-11\beta^4-26\beta^3+10\beta^2-31\beta-29}{28}$
9	-1228	$\gamma - 2$	$-0.000029021 + 0.000160199\,i$	$x^3 - x^2 + x - 7$	$323\beta^5+715\beta^4-210\beta^3-1064\beta^2-23\beta-120$	$243\beta^5+513\beta^4-242\beta^3-880\beta^2-23\beta-96$
10	-1228	γ	$0.000321311 - 0.0000551564\,i$	$x^3 - x^2 + 7x - 1$	$49\beta^5 - 121\beta^4 + 174\beta^3 - 66\beta^2 + 25\beta - 8$	$97\beta^5 - 227\beta^4 + 318\beta^3 - 94\beta^2 + 41\beta - 8$
11	-1228	$\gamma - 1$	$0.00010298 + 0.000713224\,i$	$x^3 + 4x - 6$	$3507\beta^5-11255\beta^4+12900\beta^3-4190\beta^2+1677\beta-360$	$19\beta^5 - 77\beta^4 + 116\beta^3 - 66\beta^2 + 13\beta - 8$
12	-1323	$-\gamma + 2$	$-0.5978 - 1.10908\,i$	$x^3 - 7$	$\frac{\beta^5+\beta^4+3\beta^3+5\beta^2+5\beta-1}{2}$	$\beta^5 + \beta^4 + 3\beta^3 + 6\beta^2 + 4\beta + 1$
13	-1356	$\frac{\gamma^2+\gamma-3}{3}$	$10.4065 + 16.4696\,i$	$x^3 - x^2 + x + 21$	$3\beta^5 + 9\beta^4 + 8\beta^3 - 6\beta^2 - 3\beta$	$3\beta^5 + 3\beta^4 - 8\beta^3 - 26\beta^2 - 3\beta$
14	-1356	$-\gamma$	$-0.198335 - 0.368121\,i$	$x^3 - x^2 + 11x + 1$	$\frac{3\beta^5+2\beta^4-8\beta^3-22\beta^2-\beta}{2}$	$\frac{-\beta^5-2\beta^4+6\beta^2+3\beta}{2}$
15	-1356	$\frac{-\gamma^2+2\gamma+1}{2}$	$0.175522 - 0.0940394\,i$	$x^3 - x^2 + 3x - 15$	$\frac{123\beta^5-245\beta^4+556\beta^3+70\beta^2+41\beta}{5}$	$\frac{27\beta^5-55\beta^4+124\beta^3+10\beta^2+9\beta}{5}$
16	-1620	$-2\gamma + 5$	$-0.260757 - 0.234339\,i$	$x^3 - 3x - 8$	$\frac{-\beta^5-2\beta^4-10\beta^3-10\beta^2-9\beta-4}{4}$	$\frac{-\beta^5+2\beta^4-2\beta^3+26\beta^2+31\beta+4}{4}$
17	-1836	$\gamma - 3$	$1.23445\,10^{-6} + 6.83889\,10^{-6}\,i$	$x^3 - 6x - 10$	$29\beta^5 + 51\beta^4 - 178\beta^3 - 340\beta^2 - 13\beta - 16$	$6413\beta^5+14521\beta^4-25298\beta^3-59156\beta^2-1197\beta-2792$
18	-1836	$-\gamma + 1$	$0.000145558 - 0.000124248\,i$	$x^3 + 6x - 6$	$35\beta^5 + 131\beta^4 + 230\beta^3 + 224\beta^2 + 29\beta + 24$	$-141\beta^5-343\beta^4-602\beta^3-328\beta^2-67\beta-32$
19	-1960	$\gamma^2 + \gamma - 1$	$-0.0772755 + 0.128167\,i$	$x^3 - x^2 + 5x + 15$	$\frac{17\beta^5+61\beta^4-138\beta^3-1082\beta^2-1289\beta-33}{116}$	$\frac{-8\beta^5-7\beta^4+82\beta^3+108\beta^2-666\beta+19}{58}$

num	disc	η	z
$t(x)$		D B	
20	-2028	$-\gamma+3$	$-87.5183+67.3119\,i$
x^3-26		$\beta^5+\beta^4+4\beta^3-2\beta^2-5\beta$ $\beta^5-5\beta^4-12\beta^3-30\beta^2-5\beta$	
21	-2075	γ^2-14	$-5.97993+12.5444\,i$
$x^3-10x-15$		$\frac{39\beta^5+115\beta^4-507\beta^3-1485\beta^2+279\beta+5}{30}$ $\frac{13\beta^5+33\beta^4-159\beta^3-384\beta^2+188\beta-27}{15}$	
22	-2075	$\gamma-1$	$0.0335624-0.0240994\,i$
x^3-x^2+7x-8		$891\beta^5-512\beta^4-2907\beta^3+5651\beta^2-368\beta+692$ $\frac{-9\beta^5+25\beta^4-23\beta^3-7\beta^2+7\beta-1}{2}$	
23	-2075	$-\gamma+3$	$1.97387+2.88244\,i$
x^3-x^2-3x-8		$\frac{5\beta^5+9\beta^4+23\beta^3+23\beta^2-7\beta+1}{2}$ $-\beta^3-\beta^2-3\beta+1$	
24	-2188	$\gamma+2$	$-0.070029-0.184207\,i$
$x^3-x^2+3x+17$		$\frac{\beta^5-6\beta^4-4\beta^3-30\beta^2+5\beta}{2}$ $\frac{-3\beta^5+6\beta^4+4\beta^3+46\beta^2+9\beta}{2}$	
25	-2188	$\frac{\gamma^2-2\gamma-6}{2}$	$-58.7501-9.90888\,i$
$x^3-8x-20$		$\beta^5+\beta^4-8\beta^3+16\beta^2+\beta$ $\frac{35\beta^5-71\beta^4+72\beta^3+72\beta^2+19\beta+8}{3}$	
26	-2188	$\gamma-2$	$-0.389841-0.118699\,i$
$x^3-x^2+7x-19$		$\frac{53\beta^5+80\beta^4-186\beta^3-430\beta^2-17\beta-30}{2}$ $\frac{\beta^5-10\beta^3-18\beta^2-5\beta-2}{2}$	
27	-2303	$\frac{-\gamma^2+2\gamma+6}{3}$	$-0.0608005-0.0214484\,i$
$x^3-x^2-2x-27$		$106619\beta^5+121805\beta^4+301012\beta^3-392875\beta^2+14964\beta-20205$ $-85\beta^5-405\beta^4-852\beta^3-1073\beta^2-48\beta-55$	
28	-2548	$-\gamma^2+\gamma+1$	$0.0427048-0.0967754\,i$
$x^3-x^2+12x+8$		$\frac{21\beta^5+337\beta^4+482\beta^3+4442\beta^2+61\beta+17}{8}$ $\frac{-7\beta^5-\beta^4-102\beta^3+38\beta^2-111\beta-1}{8}$	
29	-2695	$-\gamma+1$	$-0.0000623593+4.05409\,10^{-6}\,i$
x^3+7x-7		$1093\beta^5-3591\beta^4+6431\beta^3-5342\beta^2+654\beta-512$ $-83\beta^5+875\beta^4-1589\beta^3+2346\beta^2-166\beta+232$	
30	-2700	$\frac{-\gamma^2+2\gamma+2}{2}$	$-1.45443+0.991771\,i$
x^3-20		$\frac{12377\beta^5-38581\beta^4+78638\beta^3-15456\beta^2+2399\beta-440}{7}$ $\frac{1033\beta^5-47\beta^4-2674\beta^3+16968\beta^2-\beta+528}{7}$	
31	-2835	$2\gamma^2+8\gamma+9$	$3.14912-2.05322\,i$
$x^3-12x-19$		$\frac{49\beta^5-3\beta^4+3668\beta^3-228\beta^2-377\beta-213}{24}$ $\frac{11\beta^5+6\beta^4+826\beta^3+450\beta^2+107\beta+16}{4}$	
32	-2888	$-2\gamma-1$	$0.0110234-0.0367296\,i$
$x^3-x^2+13x+7$		$\frac{3\beta^5+\beta^4-26\beta^3-62\beta^2+59\beta+1}{8}$ $\frac{-\beta^5-3\beta^4-6\beta^3-14\beta^2-53\beta-11}{16}$	

num	disc	η	z
$t(x)$		D	B
33	-2891	$-\gamma-1$	$0.0316372+0.0189916\,i$
x^3-x^2+5x+8		$808\beta^5-1521\beta^4-899\beta^3+3412\beta^2-97\beta+342$	$\frac{-17\beta^5+19\beta^4+61\beta^3-115\beta^2+7\beta-13}{2}$
34	-2891	$\frac{-2\gamma^2-10\gamma-9}{3}$	$0.267743-0.459433\,i$
$x^3-x^2-9x+36$		$\frac{3\beta^5+7\beta^4-108\beta^3-36\beta^2+1257\beta+125}{144}$	$\frac{\beta^4-36\beta^2+72\beta+131}{72}$
35	-2891	$\frac{-\gamma^2+\gamma+8}{2}$	$-0.0517534+0.00584932\,i$
$x^3-x^2-2x-20$		$\frac{2\beta^5+\beta^4+19\beta^3+2\beta^2+59\beta-3}{10}$	$\frac{\beta^4+5\beta^3+12\beta^2+20\beta-3}{5}$
36	-3020	$-2\gamma^2+1$	$-0.0234304+0.0189654\,i$
x^3+8x-6		$\frac{\beta^5-209\beta^4-188\beta^3-3776\beta^2-41\beta-15}{4}$	$\frac{-5\beta^5-141\beta^4-228\beta^3-2448\beta^2-115\beta-3}{12}$
37	-3020	$-\gamma$	$-1.60005\,10^{-6}-9.92933\,10^{-6}\,i$
x^3-x^2+9x+1		$592047\beta^5+2543633\beta^4+4421412\beta^3+3179840\beta^2+477599\beta+318032$	$1871\beta^5-333\beta^4-8412\beta^3-16152\beta^2-945\beta-1768$
38	-3020	$-\gamma-2$	$6.83648\,10^{-6}-8.57399\,10^{-6}\,i$
x^3-x^2-x+11		$69\beta^5+113\beta^4-318\beta^3-598\beta^2-23\beta-40$	$-3339\beta^5-6453\beta^4+11570\beta^3+24934\beta^2+761\beta+1640$
39	-3299	$-\gamma+1$	$0.285571+0.104771\,i$
x^3-x^2+9x-8		$\beta^5+\beta^4-3\beta^3+9\beta^2+\beta$	$\frac{\beta^5-\beta^4+3\beta^2+\beta-4}{2}$
40	-3299	$-\gamma+2$	$-0.272189+0.0798091\,i$
$x^3+2x-11$		$\frac{\beta^5+7\beta^4+12\beta^3+23\beta^2+5\beta}{2}$	$\beta^4+2\beta^3+5\beta^2+3\beta$
41	-3299	$3\gamma-14$	$11.2719+10.9291\,i$
$x^3-16x-27$		$\frac{\beta^5+\beta^4-37\beta^3-19\beta^2+349\beta+7}{18}$	$\frac{\beta^5+\beta^4-25\beta^3-22\beta^2+64\beta+13}{9}$
42	-3299	$2\gamma+3$	$-0.0285274+0.0278932\,i$
$x^3-x^2+3x+10$		$\frac{157\beta^5+97\beta^4+786\beta^3-1326\beta^2+33\beta-23}{4}$	$\frac{11\beta^5-61\beta^4-64\beta^3-568\beta^2+33\beta-7}{8}$
43	-3332	$\frac{-\gamma^2-\gamma+4}{2}$	$2.68222-2.15098\,i$
$x^3-x^2+12x-20$		$122\beta^5-654\beta^4+854\beta^3-5872\beta^2+4\beta-81$	$90\beta^5+678\beta^4+950\beta^3+5584\beta^2+36\beta+77$
44	-3564	$\frac{\gamma^2+4\gamma+6}{2}$	$-14.58+9.27875\,i$
$x^3-12x-28$		$997\beta^5+545\beta^4+20422\beta^3+11606\beta^2-7799\beta+4968$	$\frac{-10337\beta^5-463\beta^4-214558\beta^3-10994\beta^2+22027\beta-27880}{3}$
45	-3564	$-\gamma^2-\gamma+3$	$-0.00013479-0.000164733\,i$
$x^3+6x-10$		$\frac{279\beta^5-997\beta^4+3168\beta^3-9770\beta^2-3\beta-88}{3}$	$\frac{183\beta^5-599\beta^4+2064\beta^3-5830\beta^2-3\beta-56}{3}$

num	disc	$t(x)$	η	z	D	B
46	-3675	x^3-35	$\frac{-\gamma^2+10\gamma-22}{3}$	$-0.0702182-0.33894\,i$	$\frac{5\beta^5+\beta^4-125\beta^3+67\beta^2+1305\beta-84}{115}$	$\frac{22\beta^5-3\beta^4-481\beta^3+144\beta^2+5627\beta+367}{230}$
47	-3724	$x^3-x^2-9x+29$	$\gamma^2+2\gamma-6$	$-0.10545-0.0863725\,i$	$\frac{6\beta^5+9\beta^4-104\beta^3+454\beta^2-202\beta-3}{20}$	$\frac{-6\beta^5-9\beta^4+64\beta^3-294\beta^2-198\beta+3}{20}$
48	-3724	$x^3-x^2-9x-13$	$\gamma-4$	$0.0000108752+4.40822\,10^{-6}\,i$	$1945\beta^5-4737\beta^4-9952\beta^3+24678\beta^2-325\beta+792$	$1993\beta^5-4155\beta^4-13488\beta^3+29210\beta^2-437\beta+936$
49	-3807	$x^3+15x-8$	$-4\gamma^2-38\gamma+21$	$-0.0234281-0.0731308\,i$	$\frac{51\beta^5+85\beta^4+10596\beta^3+17660\beta^2+1472553\beta+126655}{126960}$	$\frac{-1131\beta^5+29\beta^4-206976\beta^3+424\beta^2-31694493\beta-74413}{126960}$
50	-3896	$x^3-x^2+8x+20$	$\frac{-\gamma^2-9\gamma-12}{2}$	$0.210551+0.035425\,i$	$\frac{6194\beta^5+9835\beta^4-145377\beta^3-738805\beta^2+583\beta-1905}{125}$	$\frac{-\beta^5-19\beta^4-142\beta^3-498\beta^2-707\beta+67}{100}$
51	-3896	$x^3-x^2+16x-8$	$2\gamma-1$	$-0.00743299+0.0275978\,i$	$\frac{11\beta^4+42\beta^3+74\beta^2-26\beta-1}{4}$	$\frac{-\beta^5+6\beta^3+18\beta^2-33\beta-14}{8}$
52	-3896	$x^3-x^2+9x+19$	$2\gamma+3$	$0.0332829-0.0147042\,i$	$\frac{\beta^5-\beta^4+14\beta^3+6\beta^2+125\beta+7}{16}$	$\frac{2\beta^5-\beta^4+16\beta^3-18\beta^2+118\beta-17}{8}$
53	-3896	$x^3+2x-24$	$-8\gamma^2-\gamma+59$	$-0.00262036+0.00578187\,i$	$\frac{-401212\beta^5-2094945\beta^4-47399709\beta^3-2111361\beta^2-476399\beta-729}{4053}$	$\frac{-6013\beta^5-23837\beta^4-712278\beta^3+646506\beta^2-4849943\beta-94939}{48636}$
54	-4027	$x^3-8x-15$	$2\gamma-7$	$0.00110391-0.00165645\,i$	$\frac{213853\beta^5+635185\beta^4-2646454\beta^3-8135262\beta^2-23031\beta-70679}{4}$	$\frac{135\beta^5+491\beta^4-1080\beta^3-4144\beta^2+29\beta-39}{8}$
55	-4027	$x^3-x^2-7x-12$	$\gamma^2-4\gamma+1$	$0.205885+0.230289\,i$	$\frac{3\beta^5+\beta^4+27\beta^3+45\beta^2+195\beta+20}{27}$	$\frac{2\beta^5+7\beta^4+9\beta^3+72\beta^2+229\beta+113}{54}$
56	-4107	x^3-37	$-3\gamma+10$	$0.0002274-0.000121957\,i$	$\frac{\beta^5+\beta^4+59\beta^3+77\beta^2+769\beta+31}{54}$	$\frac{79\beta^5-11\beta^4+953\beta^3-3178\beta^2-440\beta+1}{27}$
57	-4300	$x^3-x^2+7x-13$	$\gamma^2-\gamma-1$	$-13.444-11.\,i$	$5698979\beta^5+16510459\beta^4-52397634\beta^3-171972644\beta^2-562967\beta-1847448$	$\frac{9461783\beta^5+20394909\beta^4-129278970\beta^3-353327668\beta^2-1388235\beta-3794112}{13}$
58	-4300	$x^3-x^2-13x+27$	$\gamma+4$	$4.74315\,10^{-8}-4.02226\,10^{-8}\,i$	$778613\beta^5+782511\beta^4+5271514\beta^3+3589386\beta^2+123037\beta+83640$	$540485\beta^5+880661\beta^4+3761578\beta^3+4704630\beta^2+87789\beta+109704$

Acknowledgments

The authors would like to thank Harold Stark for some helpful conversations held at the beginning of this project as well as an anonymous referee for constructive remarks and useful suggestions.

References

[Co] Henri Cohen, *Advanced topics in computational number theory*, Springer-Verlag, New York, 2000. MR **2000k:**11144

[CR] Henri Cohen and Xavier-François Roblot, *Computing the Hilbert class field of real quadratic fields*, Math. Comp. **69** (2000), no. 231, 1229–1244. MR **2000j:**11200

[Da] Samit Dasgupta, *Stark's conjectures*, Honors thesis, Harvard University, 1999.

[DST] David S. Dummit, Jonathan W. Sands, and Brett A. Tangedal, *Computing Stark units for totally real cubic fields*, Math. Comp. **66** (1997), no. 219, 1239–1267. MR **97i:**11110

[DT] David S. Dummit and Brett A. Tangedal, *Computing the lead term of an abelian L-function*, Lecture Notes in Comput. Sci. **1423**, Springer, Berlin, 1998, 400–411. MR **2001d:**11110

[F] Eduardo Friedman, *Hecke's integral formula*, Séminaire de Théorie des Nombres Univ. Bordeaux I, Talence (1987–88), Exposé n° 5, 23 pp. MR **90i:**11136

[GP] C. Batut, K. Belabas, D. Bernardi, H. Cohen, and M. Olivier, *User's guide to PARI/GP version 2.0.20*, 2000.

[H] Erich Hecke, *Lectures on the theory of algebraic numbers*, Springer-Verlag, New York, 1981. MR **83m:**12001

[L] A. F. Lavrik, *On functional equations of Dirichlet functions*, English translation in Math. USSR–Izvestija **1** (1967), 421–432. MR **35:**4174

[La] Edmund Landau, *Über Ideale und Primideale in Idealklassen*, Math. Zeit. **2** (1918), 52–154.

[Lo] Robert L. Long, *Algebraic number theory*, Marcel Dekker, Inc., New York, Basel, 1977. MR **57:**9668

[PZ] Michael Pohst and Hans Zassenhaus, *Algorithmic algebraic number theory*, Cambridge University Press, Cambridge, 1989. MR **92b:**11074

[R1] Xavier-François Roblot, *Algorithmes de factorisation dans les extensions relatives et applications de la conjecture de Stark à la construction des corps de classes de rayon*, Thèse, Université Bordeaux I, 1997.

[R2] ———, *Stark's conjectures and Hilbert's twelfth problem*, Experiment. Math. **9** (2000), 251–260. MR **2001g:**11174

[St1] Harold M. Stark, *Values of L-functions at $s = 1$. I. L-functions for quadratic forms*, Advances in Math. **7** (1971), 301–343. MR **44:**6620

[St2] ———, *L-functions at $s = 1$. II. Artin L-functions with rational characters*, Advances in Math. **17** (1975), 60–92. MR **52:**3082

[St3] ———, *L-functions at $s = 1$. III. Totally real fields and Hilbert's twelfth problem*, Advances in Math. **22** (1976), 64–84. MR **55:**10427

[St4] ———, *L-functions at $s = 1$. IV. First derivatives at $s = 0$*, Advances in Math. **35** (1980), 197–235. MR **81f:**10054

[St5] ———, *Class fields for real quadratic fields and L-series at* 1, Algebraic number fields (A. Fröhlich, ed.), Academic Press, London, 1977, 355–375. MR **56:**11963

[St6] ———, *Hilbert's twelfth problem and L-series*, Bull. Amer. Math. Soc. **83** (1977), 1072–1074. MR **56:**314

[Ta] John Tate, *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$*, Birkhäuser, Boston, 1984. Lecture notes edited by Dominique Bernardi and Norbert Schappacher. MR **86e:**11112

[Wa] Lawrence C. Washington, *Introduction to cyclotomic fields*, 2nd Ed., Springer-Verlag, New York, 1997. MR **97h:**11130

Department of Mathematics and Statistics, University of Vermont, Burlington, Vermont 05401-1455

E-mail address: dummit@math.uvm.edu

Department of Mathematics, College of Charleston, Charleston, South Carolina 29424-0001

E-mail address: tangedalb@cofc.edu

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803-4918

E-mail address: wamelen@math.lsu.edu