MATHEMATICS OF COMPUTATION
Volume 73, Number 247, Pages 1089–1105
S 0025-5718(03)01603-X
Article electronically published on October 2, 2003

ANALYSIS OF FINITE ELEMENT APPROXIMATION FOR TIME-DEPENDENT MAXWELL PROBLEMS

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ABSTRACT. We provide an error analysis of finite element methods for solving time-dependent Maxwell problem using Nedelec and Thomas-Raviart elements. We study the regularity of the solution and develop some new error estimates of Nedelec finite elements. As a result, the optimal $\boldsymbol{L}^2$-error bound for the semidiscrete scheme is obtained.

1. INTRODUCTION

Let Ω be a bounded and simply connected polyhedral domain in $\mathbb{R}^3$ with connected boundary $\partial\Omega$ and unit outward normal $\boldsymbol{n}$. The time-dependent Maxwell equations with a volume production of charges [14] state that

$$\varepsilon \boldsymbol{E}_t + \sigma \boldsymbol{E} - \mathbf{curl}\,(\mu^{-1}\boldsymbol{B}) = \boldsymbol{J}, \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$\boldsymbol{B}_t + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0}, \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

where the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$ are unknowns, the known function $\boldsymbol{J}$ specifies the applied current, and the permittivity ε, the permeability μ, and the conductivity σ describe the properties of the medium occupying the domain Ω. We assume that the boundary of Ω is a supraconductive boundary such that

$$\boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} \quad \text{and} \quad \boldsymbol{B} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

We supplement (1.1) and (1.2) with initial conditions

$$\boldsymbol{E}(x,0) = \boldsymbol{E}_0(x) \quad \text{and} \quad \boldsymbol{B}(x,0) = \boldsymbol{B}_0(x) \quad \text{in } \Omega, \tag{1.4}$$

where

$$\operatorname{div} \boldsymbol{B}_0 = 0 \quad \text{in } \Omega. \tag{1.5}$$

Across an interface Σ between two media with different material constants, the equations (1.1)–(1.4) imply the following jump conditoins [13]

$$[\boldsymbol{E} \times \boldsymbol{n}] = 0, \qquad [\varepsilon \boldsymbol{E} \cdot \boldsymbol{n}] = \rho_\Sigma, \tag{1.6}$$

$$[\boldsymbol{B} \cdot \boldsymbol{n}] = 0, \qquad \left[\mu^{-1}\boldsymbol{B} \times \boldsymbol{n}\right] = \boldsymbol{J}_\Sigma, \tag{1.7}$$

where $\boldsymbol{n}$ is the unit outward normal to Σ and ρ_Σ and $\boldsymbol{J}_\Sigma$ are the surface charge and current density. Therefore, these jump conditions will not be part of the

Received by the editor August 3, 2002 and, in revised form, December 17, 2002.
2000 *Mathematics Subject Classification.* Primary 65N30.
Key words and phrases. Maxwell's equations, finite element methods, Nedelec element, Thomas-Raviart element, interface problem.

formation of Maxwell's equations. But they will be used in the analysis frequently and implicitly.

The existence and uniqueness of the solution $\boldsymbol{E}$ and $\boldsymbol{B}$ to (1.1)–(1.4) were studied in [14, 23] in some special cases. In [23], the existence and uniqueness were shown for Maxwell's equations in two space dimensions with $\sigma \equiv 0$ and constant ε and μ using the semigroup theory. In [14], Davaut and Lions studied the problem for more general piecewise constant coefficients using the Galerkin method. However, they assumed that ε and μ are constant in a neighborhood of $\partial\Omega$ and the boundary $\partial\Omega$ is regular [14]. One effort of this paper is to give the existence and uniqueness results without the above constraints. Moreover, with the technique in [12] used to study Maxwell time-harmonic problems, we will study the regularity of the solution $\boldsymbol{E}$ and $\boldsymbol{B}$, which is critical in the analysis of finite element approximations. More precisely, we will reduce the regularity problem of the solution of the system (1.1)–(1.4) to regularity problems of the solution of certain Laplacian equations, which were the subject of [18, 25].

There have been many studies on the finite element methods and the corresponding convergence analysis for the system (1.1)–(1.4). In [22], Nedelec proposed to approximate $\boldsymbol{E}$ and $\boldsymbol{B}$ by Nedelec and Thomas-Raviart finite elements, respectively. For the use of other finite elements, we refer to [4, 20, 23]. One advantage of Nedelec's approach is that the method handles discontinuous material properties (1.6) and (1.7) in a transparent way. Makridakis and Monk [19] analyzed both the semidiscrete scheme and fully discrete schemes of Nedelec's approach for Maxwell problems with smooth coefficients ε and μ. In [11], Ciarlet, Jr. and Zou eliminated the magnetic induction $\boldsymbol{B}$ in (1.1) and (1.2) to obtain an equation of the electric field $\boldsymbol{E}$ and analyzed a fully discrete scheme for the resulting equation using the Nedelec element. It seems that this approach only works for continuous coefficients ε and μ. To deal with the equations with $\sigma \equiv 0$ and discontinuous coefficients ε and μ, Zou et al. [10] presented a mixed finite element approach to the above-mentioned equation of $\boldsymbol{E}$ by introducing a Lagrangian multiplier corresponding to the divergence condition of $\varepsilon\boldsymbol{E}$. The error estimates in [10, 11] were obtained under the assumption that, for all $t \in (0, T)$, $\boldsymbol{E}(t)$ and $\mathbf{curl}\,\boldsymbol{E}(t)$ belong to $\boldsymbol{H}^{\alpha}(\Omega)$ for some $\alpha > 1/2$.

In this paper, we study the semidiscrete scheme proposed by Nedelec [22] for Maxwell problems with discontinuous coefficients ε and μ. As we will see in the second section, the solutions $\boldsymbol{E}(t)$ and $\boldsymbol{B}(t)$ are likely in $\boldsymbol{H}^{\alpha}(\Omega)$ for some $\alpha < 1/2$ and thus many estimates used in [10, 11, 19] are not valid any more. To overcome this difficulty, we develop some new approximation estimates for vector fields in $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ with low regularity. Essentially we give an error estimate of the operator $\boldsymbol{\pi}_h$ defined in (3.5) and (3.6), which is critical in the error analysis. Based on this estimate, a standard argument [6, 19] then gives the $\boldsymbol{L}^2$-error bound of the semidiscrete scheme.

This paper is organized as follows. In Section 2, we study the regularity of the solution of the problem (1.1)–(1.4) with discontinuous coefficients. After developing some new error estimates in Section 3, we are able to provide the optimal $\boldsymbol{L}^2$-error estimate for the semidiscrete scheme in Section 4.

2. Regularity

We begin this section by introducing some notation. In general, we use boldface type for vector fields, spaces of vector fields, and operators between vector fields. For any domain $\mathcal{D} \subseteq \mathbb{R}^3$, the norm and seminorm in the Sobolev spaces $H^r(\mathcal{D})$ and $\boldsymbol{H}^r(\mathcal{D})$ are both denoted by $\|\cdot\|_{r,\mathcal{D}}$ and $|\cdot|_{r,\mathcal{D}}$, respectively, with the subscript $\mathcal{D}$ dropped if $\mathcal{D} = \Omega$.

We prescribe parameters ε, μ, and σ in (1.1) and (1.2) more precisely. We assume that

$$\varepsilon = \varepsilon_i > 0, \quad \mu = \mu_i > 0 \text{ in } \Omega_i, \quad i = 1, \ldots, q,$$

where Ω_i is a polyhedral domain in Ω and ε_i and μ_i are constants in Ω_i. We assume that σ is nonnegative and bounded above by $\sigma_{\max}$. Let Γ be the set of interfaces, namely

$$\Gamma = \bigcup_{i=1}^{q} \partial\Omega_i \backslash \partial\Omega.$$

Here we do not assume that $\partial\Omega$ and Γ are regular and ε and μ are constant in a neighborhood of $\partial\Omega$.

When $\mathcal{D}$ is the union of $\Omega_i, i = 1, \ldots, q$, we mean by $\boldsymbol{H}^\alpha(\mathcal{D})$ the Sobolev space consisting of functions $\boldsymbol{u}$ such that $\boldsymbol{u}|_{\Omega_i} \in \boldsymbol{H}^\alpha(\Omega_i)$ for $i = 1, \ldots, q$ with norm

$$\|\boldsymbol{u}\|^2_{\alpha,\bigcup\Omega_i} = \sum_{i=1}^{q} \|\boldsymbol{u}\|^2_{\alpha,\Omega_i}.$$

The Hilbert space $\boldsymbol{H}(\mathbf{curl};\Omega)$ consists of vector fields in $\boldsymbol{L}^2(\Omega)$ with square-integrable $\mathbf{curl}$ and $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ is the subspace of vector fields in $\boldsymbol{H}(\mathbf{curl};\Omega)$ satisfying $\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$. The norm in $\boldsymbol{H}(\mathbf{curl};\Omega)$ is defined by

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega)} = \|\boldsymbol{u}\|^2 + \|\mathbf{curl}\,\boldsymbol{u}\|^2.$$

The space $\boldsymbol{H}(\mathrm{div};\Omega)$ consists of vector fields in $\boldsymbol{L}^2(\Omega)$ with square-integrable div and $\boldsymbol{H}_0(\mathrm{div};\Omega)$ is the subspace of vector fields in $\boldsymbol{H}(\mathrm{div};\Omega)$ satisfying $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. The norm in $\boldsymbol{H}(\mathrm{div};\Omega)$ is defined by

$$\|\boldsymbol{v}\|^2_{\boldsymbol{H}(\mathrm{div};\Omega)} = \|\boldsymbol{v}\|^2 + \|\mathrm{div}\,\boldsymbol{v}\|^2.$$

We set $\mathcal{H} = \boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$ and we set the innerproduct in $\mathcal{H}$ to be

$$\left(\begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{v}_1 \end{pmatrix}, \begin{pmatrix} \boldsymbol{u}_2 \\ \boldsymbol{v}_2 \end{pmatrix} \right)_{\mathcal{H}} = (\boldsymbol{u}_1, \boldsymbol{u}_2)_\varepsilon + (\boldsymbol{v}_1, \boldsymbol{v}_2)_{\mu^{-1}} \equiv (\varepsilon\boldsymbol{u}_1, \boldsymbol{u}_2) + (\mu^{-1}\boldsymbol{v}_1, \boldsymbol{v}_2).$$

Since ε and μ are piecewise positive constants, weighted innerproducts $(\cdot,\cdot)_\varepsilon$ and $(\cdot,\cdot)_{\mu^{-1}}$ are equivalent to the usual innerproduct $(\cdot,\cdot)$ in $\boldsymbol{L}^2(\Omega)$.

To derive the existence and uniqueness of the solution to (1.1)–(1.4), we define the operator $\mathcal{A}$ and transfer (1.1) and (1.2) to the operator form. The domain $D(\mathcal{A})$ is given by

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} \in \mathcal{H} \,\middle|\, \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \text{ and } \mathbf{curl}\,\mu^{-1}\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \right\},$$

and we define $\mathcal{A}$ by

$$\mathcal{A}\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} -\varepsilon^{-1}\mathbf{curl}\,\mu^{-1}\boldsymbol{v} + \varepsilon^{-1}\sigma\boldsymbol{u} \\ \mathbf{curl}\,\boldsymbol{u} \end{pmatrix} \quad \text{for all } \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} \in D(\mathcal{A}). \tag{2.1}$$

Using $\mathcal{A}$, we can rewrite (1.1) and (1.2) as

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix} + \mathcal{A}\begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{B} \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\boldsymbol{J} \\ \boldsymbol{0} \end{pmatrix}. \tag{2.2}$$

The following lemma states [28] that the operator $-\mathcal{A}$ is an infinitesimal generator of a semigroup of class (C_0) on $\mathcal{H}$, which is critical in order to apply the semigroup theory [26] on (2.2). Note that, unlike the operator in [23], the operator $\mathcal{A}$ in (2.1) is not skew-symmetric. Following [14], we will use the Galerkin method to show this property.

Lemma 2.1. *The operator $\mathcal{A}$ in (2.2) is a linear operator with the domain $D(\mathcal{A})$ dense in $\mathcal{H}$. For all $\boldsymbol{f} \in \mathcal{H}$ and $\lambda > 0$, there is a unique $\boldsymbol{\Psi} \in D(\mathcal{A})$ such that*

$$(\mathcal{A} + \lambda)\boldsymbol{\Psi} = \boldsymbol{f} \tag{2.3}$$

and

$$\|\boldsymbol{\Psi}\|_{\mathcal{H}} \le \lambda^{-1}\|\boldsymbol{f}\|_{\mathcal{H}}. \tag{2.4}$$

Proof. We define the operator $\mathcal{B}$ by

$$\mathcal{B}\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} -\varepsilon^{-1}\mathbf{curl}\,\mu^{-1}\boldsymbol{v} \\ \mathbf{curl}\,\boldsymbol{u} \end{pmatrix}.$$

Clearly $D(\mathcal{B}) = D(\mathcal{A})$. It is a routine to verify that the domain $D(\mathcal{B})$ is dense in $\mathcal{H}$, $\mathcal{B}$ is closed, $\mathcal{B}^* = -\mathcal{B}$, and $D(\mathcal{B}^*) = D(\mathcal{B})$. For a detailed proof we refer to [29].

Since $\mathcal{B}$ is closed, the space

$$\left\{(\boldsymbol{u}, \boldsymbol{v}, -\varepsilon^{-1}\mathbf{curl}\,\mu^{-1}\boldsymbol{v}, \mathbf{curl}\,\boldsymbol{u}) \;\middle|\; (\boldsymbol{u}, \boldsymbol{v})^T \in D(\mathcal{A})\right\}$$

is a closed subspace of $(\boldsymbol{L}^2(\Omega))^4$. This point of view allows us to identify $D(\mathcal{A})$ as a closed subspace of $(\boldsymbol{L}^2(\Omega))^4$ and thus $D(\mathcal{A})$ is separable. Then let $\{\boldsymbol{\Phi}_j = (\boldsymbol{u}_j, \boldsymbol{v}_j)\}_{j=1}^{\infty}$ be a base for $D(\mathcal{A})$ in the sense that for any positive integer m, $\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_m$ are linearly independent and finite combinations $\sum \alpha_j \boldsymbol{\Phi}_j$, $\alpha_j \in \mathbb{R}$, are dense in $D(\mathcal{A})$.

For any m, let $\boldsymbol{\Psi}_m \equiv (\boldsymbol{x}_m, \boldsymbol{y}_m)^T$ in $\operatorname{span}\{\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_m\}$ satisfy

$$((\mathcal{A} + \lambda)\boldsymbol{\Psi}_m, \boldsymbol{\Phi}_j)_{\mathcal{H}} = (\boldsymbol{f}, \boldsymbol{\Phi}_j)_{\mathcal{H}} \quad \text{for all } 1 \le j \le m. \tag{2.5}$$

The above system in finite dimension has a unique solution. Since $\mathcal{B}$ is skew-symmetric, by (2.5), we have

$$\begin{aligned}(\sigma\boldsymbol{x}_m, \boldsymbol{x}_m) + \lambda\|\boldsymbol{\Psi}_m\|_{\mathcal{H}}^2 &= ((\mathcal{A} + \lambda)\boldsymbol{\Psi}_m, \boldsymbol{\Psi}_m)_{\mathcal{H}} \\ &= (\boldsymbol{f}, \boldsymbol{\Psi}_m)_{\mathcal{H}} \le \|\boldsymbol{f}\|_{\mathcal{H}}\|\boldsymbol{\Psi}_m\|_{\mathcal{H}},\end{aligned}$$

from which and $\sigma \ge 0$ it follows that

$$\|\boldsymbol{\Psi}_m\|_{\mathcal{H}} \le \lambda^{-1}\|\boldsymbol{f}\|_{\mathcal{H}}.$$

Therefore, up to a subsequence we can assume that there is $\boldsymbol{\Psi}_* = (\boldsymbol{x}_*, \boldsymbol{y}_*)$ such that $\boldsymbol{\Psi}_j \to \boldsymbol{\Psi}_*$ weakly in $\mathcal{H}$. Clearly we have

$$\|\boldsymbol{\Psi}_*\|_{\mathcal{H}} \le \lambda^{-1}\|\boldsymbol{f}\|_{\mathcal{H}}.$$

We now verify that $\boldsymbol{\Psi}_*$ is a solution of (2.3). From (2.5), for a fixed $j \le m$, we have

$$(\sigma\boldsymbol{x}_m, \boldsymbol{u}_j) + (\lambda\boldsymbol{\Psi}_m, \boldsymbol{\Phi}_j)_{\mathcal{H}} - (\boldsymbol{\Psi}_m, \mathcal{B}\boldsymbol{\Phi}_j)_{\mathcal{H}} = (\boldsymbol{f}, \boldsymbol{\Phi}_j)_{\mathcal{H}}.$$

Letting $m \to \infty$, we get

$$(\sigma\boldsymbol{x}_*, \boldsymbol{u}_j) + (\lambda\boldsymbol{\Psi}_*, \boldsymbol{\Phi}_j)_{\mathcal{H}} - (\boldsymbol{\Psi}_*, \mathcal{B}\boldsymbol{\Phi}_j)_{\mathcal{H}} = (\boldsymbol{f}, \boldsymbol{\Phi}_j)_{\mathcal{H}}.$$

Therefore, for all $\mathbf{\Phi} = (\boldsymbol{u}, \boldsymbol{v}) \in D(\mathcal{A})$, we have

$$(\sigma \boldsymbol{x}_*, \boldsymbol{u}) + (\lambda \mathbf{\Psi}_*, \mathbf{\Phi})_{\mathcal{H}} - (\mathbf{\Psi}_*, \mathcal{B}\mathbf{\Phi})_{\mathcal{H}} = (\boldsymbol{f}, \mathbf{\Phi})_{\mathcal{H}}. \tag{2.6}$$

Note that

$$|(\sigma \boldsymbol{x}_*, \boldsymbol{u}) + (\lambda \mathbf{\Psi}_*, \mathbf{\Phi})_{\mathcal{H}}| \le (\sigma_{\max} + \lambda)\|\mathbf{\Psi}_*\|_{\mathcal{H}}\|\mathbf{\Phi}\|_{\mathcal{H}}.$$

Equation (2.6) then implies that the linear functional $\mathbf{\Phi} \mapsto (\mathbf{\Psi}_*, \mathcal{B}\mathbf{\Phi})_{\mathcal{H}}$ is continuous on $D(\mathcal{B})$ and thus $\mathbf{\Psi}_*$ belongs to $D(\mathcal{B}^*) = D(\mathcal{B}) = D(\mathcal{A})$. Therefore, by (2.6) again, $\mathbf{\Psi}_*$ satisfies (2.3).

If there is another solution $\mathbf{\Phi}$ of (2.3), then $\mathbf{\Psi}_* - \mathbf{\Phi}$ is a solution of (2.3) with $\boldsymbol{f} = \mathbf{0}$ and thus $\mathbf{\Psi}_* - \mathbf{\Phi} = \mathbf{0}$ by (2.4). This shows that the solution of (2.3) is unique. □

By the above lemma, we have the following theorem [26] on the existence and uniqueness of the solution of Maxwell equations (1.1)–(1.4). We will denote by $C^m([0,T]; \boldsymbol{L}^2(\Omega))$ (or $C^m([0,T]; \mathcal{H})$) the space of m times continuously differentiable functions from $[0,T]$ into the space $\boldsymbol{L}^2(\Omega)$ (or $\mathcal{H}$). In Theorem 2.1 and the remainder of the paper, C, with or without subscript, denotes a generic constant independent of h, the discretization parameter. The value of C may differ at different occurrences.

Theorem 2.1. *Assume that $\boldsymbol{J} \in C^1([0,T]; \boldsymbol{L}^2(\Omega))$ and $(\boldsymbol{E}_0, \boldsymbol{B}_0)^T \in D(\mathcal{A})$. Then (1.1)–(1.4) have a unique solution $(\boldsymbol{E}, \boldsymbol{B})^T \in C^1([0,T]; \mathcal{H})$. Moreover, for each $t \in [0,T]$, $(\boldsymbol{E}(t), \boldsymbol{B}(t))^T$ belongs to $D(\mathcal{A})$ and satisfies*

$$\|\boldsymbol{E}(t)\| + \|\boldsymbol{E}_t(t)\| + \|\boldsymbol{B}(t)\| + \|\boldsymbol{B}_t(t)\| \le C. \tag{2.7}$$

Remark 2.1. Since $\boldsymbol{E}(t)$ belongs to $\boldsymbol{H}_0(\mathbf{curl}; \Omega)$, a consequence of (1.5) and (1.2) is that $\boldsymbol{B}(t) \in \boldsymbol{H}_0(\mathrm{div}; \Omega)$ satisfies $\mathrm{div}\, \boldsymbol{B}(t) = 0$ for all $t \in [0,T]$.

Remark 2.2. If the right-hand side $\boldsymbol{J}$ and the initial data $(\boldsymbol{E}_0, \boldsymbol{B}_0)^T$ are smoother, the solution $(\boldsymbol{E}, \boldsymbol{B})^T$ can be more regular. For example, if $\boldsymbol{J} \in C^m([0,T]; \boldsymbol{L}^2(\Omega))$ for some integer $m > 1$ and, for all $k = 1, \ldots, m$,

$$(-\mathcal{A})^{k-1}(\boldsymbol{E}_0, \boldsymbol{B}_0)^T + \sum_{i=0}^{k-2} (-\mathcal{A})^i (\boldsymbol{J}^{(k-i-2)}(0), \mathbf{0})^T \in D(\mathcal{A}),$$

then $(\boldsymbol{E}(t), \boldsymbol{B}(t))^T$ belongs to $C^m([0,T]; \mathcal{H})$.

To study the regularity of the solution $\boldsymbol{E}$ and $\boldsymbol{B}$ of (1.1)–(1.4), as in [12], we introduce two more spaces $\boldsymbol{X}_N(\Omega; \varepsilon)$ for electric fields and $\boldsymbol{X}_T(\Omega; \mu)$ for magnetic fields, which are given by

$$\boldsymbol{X}_N(\Omega; \varepsilon) = \{\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega) \mid \varepsilon \boldsymbol{u} \in \boldsymbol{H}(\mathrm{div}; \Omega)\}$$

and

$$\boldsymbol{X}_T(\Omega; \mu) = \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}; \Omega) \mid \mu \boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{div}; \Omega)\}.$$

Note that our definition of $\boldsymbol{X}_T(\Omega; \mu)$ is slightly different from the one in [12] and allows us not to assume that μ is constant in a neighborhood of $\partial\Omega$. When ε (or μ) is constant in Ω, we will drop ε (or μ) in the above notation. The norms in both $\boldsymbol{X}_N(\Omega; \xi)$ and $\boldsymbol{X}_T(\Omega; \xi)$ are defined by

$$\|\boldsymbol{u}\|_{\boldsymbol{X}}^2 = \|\boldsymbol{u}\|^2 + \|\mathbf{curl}\, \boldsymbol{u}\|^2 + \|\mathrm{div}\, \xi \boldsymbol{u}\|^2.$$

Note that if $\boldsymbol{u} \in \boldsymbol{X}_N(\Omega; \varepsilon)$ and $\mu^{-1}\boldsymbol{v} \in \boldsymbol{X}_T(\Omega; \mu)$, then $(\boldsymbol{u}, \boldsymbol{v})^T$ belongs to $D(\mathcal{A})$ and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.

By the following theorem, we need only study the regularity of vector fields in $\boldsymbol{X}_N(\Omega;\varepsilon)$ and $\boldsymbol{X}_T(\Omega;\mu)$ in order to study the regularity of the solution $\boldsymbol{E}$ and $\boldsymbol{B}$ of (1.1)–(1.4).

Theorem 2.2. *Suppose* $\boldsymbol{J} \in C^3([0,T];\boldsymbol{L}^2(\Omega))$, $\operatorname{div}\boldsymbol{J} \in C^2([0,T];\boldsymbol{L}^2(\Omega))$, *and* $\sigma(x)/\varepsilon(x) \in H^1(\Omega)$. *Let* $\boldsymbol{g} = (\boldsymbol{E}_0, \boldsymbol{B}_0)^T$. *If* $\boldsymbol{g}$, $-\mathcal{A}\boldsymbol{g} + (\boldsymbol{J}(0),0)^T$ *and* $\mathcal{A}^2\boldsymbol{g} - \mathcal{A}(\boldsymbol{J}(0),0)^T + (\boldsymbol{J}'(0),0)^T$ *belong to* $\boldsymbol{X}_N(\Omega;\varepsilon)\times\mu\boldsymbol{X}_T(\Omega;\mu)$, *the solution* $(\boldsymbol{E},\boldsymbol{B})^T$ *to* (1.1)–(1.4) *is such that* $\exp(\rho(x)t)\boldsymbol{E}$ *and* $\exp(\rho(x)t)\boldsymbol{E}_t$ *belong to* $\boldsymbol{X}_N(\Omega;\varepsilon)$ *and* $\mu^{-1}\mathbf{curl}\,\boldsymbol{E}$, $\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_t$ *and* $\mu^{-1}\boldsymbol{B}$ *belong to* $\boldsymbol{X}_T(\Omega;\mu)$.

Proof. Let $\rho(x) \equiv \sigma(x)/\varepsilon(x)$ and $\boldsymbol{F}(t) = \exp(\rho(x)t)\boldsymbol{E}(t)$. We first show that, for all $t \in (0,T)$, $\boldsymbol{F}(t) \in \boldsymbol{X}_N(\Omega;\varepsilon)$ and $\mu^{-1}\boldsymbol{B}(t) \in \boldsymbol{X}_T(\Omega;\mu)$ satisfy

$$\|\boldsymbol{F}(t)\|_{\boldsymbol{X}} + \|\mu^{-1}\boldsymbol{B}(t)\|_{\boldsymbol{X}} \le C. \tag{2.8}$$

Since $\rho(x) \in H^1(\Omega)$ is bounded, by the chain rule, we have that

$$\nabla\exp(\rho(x)t) = t\exp(\rho(x)t)\nabla\rho(x)$$

and $\exp(\rho(x)t)$ belongs to $H^1(\Omega)$. By (2.7), we have

$$\begin{aligned}
&\|\boldsymbol{F}(t)\| + \|\boldsymbol{B}(t)\| + \|\mathbf{curl}\,\boldsymbol{F}(t)\| + \|\mathbf{curl}\,\mu^{-1}\boldsymbol{B}(t)\| \\
&\qquad \le C\|\boldsymbol{E}(t)\| + \|\boldsymbol{B}(t)\| + C\|\mathbf{curl}\,\boldsymbol{E}(t)\| + \|\varepsilon\boldsymbol{E}_t(t) + \sigma\boldsymbol{E}(t) - \boldsymbol{J}(t)\| \\
&\qquad \le C\|\boldsymbol{E}(t)\| + C\|\boldsymbol{E}_t(t)\| + \|\boldsymbol{B}(t)\| + C\|\boldsymbol{B}_t(t)\| + C\|\boldsymbol{J}(t)\| \le C.
\end{aligned}$$

From (1.1) it follows that $\boldsymbol{F}(t)$ satisfies

$$\varepsilon\boldsymbol{F}_t - \exp(\rho(x)t)\mathbf{curl}\,(\mu^{-1}\boldsymbol{B}) = \exp(\rho(x)t)\boldsymbol{J}.$$

Therefore, taking div on both sides of the above and integrating over $(0,t)$ yield that

$$\begin{aligned}
\|\operatorname{div}\varepsilon(\boldsymbol{F}(t) - \boldsymbol{F}(0))\| &= \int_0^t \|\operatorname{div}\,\exp(\rho(x)t)(\mathbf{curl}\,(\mu^{-1}\boldsymbol{B}) + \boldsymbol{J})\|d\tau \\
&\le C\int_0^t (\|\mathbf{curl}\,(\mu^{-1}\boldsymbol{B})\| + \|\boldsymbol{J}\|_{\boldsymbol{H}(\operatorname{div};\Omega)})d\tau \le C
\end{aligned}$$

and thus $\|\operatorname{div}\varepsilon\boldsymbol{F}(t)\| \le C$. Recall that in Remark 2.1 we get $\operatorname{div}\boldsymbol{B} = 0$. This completes the proof of (2.8).

By Remark 2.2, we know that $(\boldsymbol{E},\boldsymbol{B})^T \in C^3([0,T];\mathcal{H})$. If we differentiate both sides of (2.2) with respect to t and repeat the above argument, we get that $\exp(\rho(x)t)\boldsymbol{E}_t \in \boldsymbol{X}_N(\Omega;\varepsilon)$ and $\mu^{-1}\boldsymbol{B}_t$ $(= \mu^{-1}\mathbf{curl}\,\boldsymbol{E}) \in \boldsymbol{X}_T(\Omega;\mu)$ satisfy (2.8). Similarly, differentiating both sides of (2.2) twice yields that $\mu^{-1}\boldsymbol{B}_{tt}$ $(= \mu^{-1}\mathbf{curl}\,\boldsymbol{E}_t)$ belongs to $\boldsymbol{X}_T(\Omega;\mu)$. □

The regularity of vector fields in $\boldsymbol{X}_N(\Omega;\varepsilon)$ and $\boldsymbol{X}_T(\Omega;\mu)$ has been studied by M. Costable et al. [12]. They began the analysis with the decomposition of vector fields in $\boldsymbol{X}_N(\Omega;\varepsilon)$ and $\boldsymbol{X}_T(\Omega;\mu)$ as a sum of a "regular" part in $\boldsymbol{H}^1(\Omega)$ and a "singular" part in the form of a gradient, which contains, in particular, all the jumps through the interfaces.

Lemma 2.2. *Any vector field* $\boldsymbol{u} \in \boldsymbol{X}_N(\Omega;\varepsilon)$ *admits a decomposition*

$$\boldsymbol{u} = \boldsymbol{w} + \nabla\phi \tag{2.9}$$

where $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_N(\Omega)$ *and* $\phi \in H_0^1(\Omega)$ *satisfy*

$$\|\boldsymbol{w}\|_1 + \|\phi\|_1 \le C\|\boldsymbol{u}\|_{\boldsymbol{X}}. \tag{2.10}$$

Similarly, any vector field $\boldsymbol{v} \in \boldsymbol{X}_T(\Omega;\mu)$ *admits a decomposition* (2.9) *where* $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{X}_T(\Omega)$ *and* $\phi \in H^1(\Omega)/\mathbb{R}$ *satisfy* (2.10).

Proof. The proof is an exact rewriting of the proof of Theorem 3.4 in [12]. However, since our $\boldsymbol{X}_T(\Omega;\mu)$ is different from the one in [12], we sketch the proof here.

Let $\boldsymbol{u}$ be as in the lemma. Since its $\mathbf{curl}$ is a divergence-free field in $\boldsymbol{L}^2(\Omega)$ and Ω is simply connected with one boundary component, we can apply Lemma 3.1 in [12] and find $\boldsymbol{w}$ in $\boldsymbol{H}^1(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u}$ and $\boldsymbol{w}\cdot\boldsymbol{n} = 0$ on $\partial\Omega$. Then, $\boldsymbol{u}-\boldsymbol{w}$ is a curl-free field. Since Ω is simply connected, there exists ϕ in $H^1(\Omega)$ such that $\boldsymbol{v}-\boldsymbol{w} = \nabla\phi$. □

Based on the above lemma, M. Costable et al. related the regularity of vector fields in $\boldsymbol{X}_N(\Omega;\varepsilon)$ and $\boldsymbol{X}_T(\Omega;\mu)$ to the regularity of solutions of certain Laplacian interface problems. For example, for $\boldsymbol{u} \in \boldsymbol{X}_T(\Omega;\mu)$, we have $\boldsymbol{u} = \boldsymbol{w} + \nabla\phi$ where $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ and $\phi \in H^1(\Omega)$ satisfies, for all $\psi \in H^1(\Omega)/\mathbb{R}$,

$$\int_\Omega \mu\nabla\phi\cdot\nabla\psi\,dx = \int_\Omega \mu(\boldsymbol{u}-\boldsymbol{w})\cdot\nabla\psi\,dx \equiv (f,\psi), \tag{2.11}$$

where f belongs to the dual space of $H^1(\Omega)/\mathbb{R}$. Often the solution ϕ to (2.11) is more regular than $H^1(\Omega)$ since the right-hand side f is smoother than functions in the dual space of $H^1(\Omega)/\mathbb{R}$. Indeed, for any $\alpha \in (0,1/2)$, we have

$$\begin{aligned}
|(f,\psi)| &= |\int_\Omega \mu\boldsymbol{u}\cdot\nabla\psi\,dx - \int_\Omega \mu\boldsymbol{w}\cdot\nabla\psi\,dx| \\
&= |\int_\Omega \operatorname{div}(\mu\boldsymbol{u})\psi\,dx - \sum_i \mu_i \int_{\Omega_i} \boldsymbol{w}\cdot\nabla\psi\,dx| \\
&\le C\|\psi\| + C\sum_i \|\boldsymbol{w}\|_{\alpha,\Omega_i}\|\nabla\psi\|_{-\alpha,\Omega_i} \le C\|\psi\|_{1-\alpha},
\end{aligned}$$

and thus f belongs to $H^{-1+\alpha}(\Omega)$.

M. Costable et al. pointed out [12] that the regularity of vector fields in $\boldsymbol{X}_N(\Omega;\varepsilon)$ and $\boldsymbol{X}_T(\Omega;\mu)$ can be very low (near $\boldsymbol{L}^2(\Omega)$). For a detailed description, we refer to [12] and references therein. Throughout this paper we will make the following assumption.

Assumption 2.1. $\boldsymbol{X}_N(\Omega;\varepsilon)$ and $\boldsymbol{X}_T(\Omega;\mu)$ are continuously imbedded in $\boldsymbol{H}^s(\bigcup\Omega_i)$ for some $s \in (0,1]$.

When ε (or μ) is constant, we have the following imbedding result [2].

Lemma 2.3. *There exists a real number* $r > 1/2$ *such that* $\boldsymbol{X}_N(\Omega)$ *and* $\boldsymbol{X}_T(\Omega)$ *are continuously imbedded in* $\boldsymbol{H}^r(\Omega)$.

The main result of this section is the following theorem on the regularity of the solution to the system (1.1)–(1.4).

Theorem 2.3. *Let* s *be as in Assumption 2.1. If* $\sigma(x)/\varepsilon(x)$ *belongs to* $W^1_p(\Omega)$ *for some* $p > 3$, *under Assumption 2.1 and assumptions in Theorem 2.2, we have that the solution* $(\boldsymbol{E},\boldsymbol{B})$ *of the system* (1.1)–(1.4) *satisfies that* $\boldsymbol{E}(t)$, $\mathbf{curl}\,\boldsymbol{E}(t)$, $\boldsymbol{E}_t(t)$, $\mathbf{curl}\,\boldsymbol{E}_t(t)$ *and* $\boldsymbol{B}(t)$ *belong to* $\boldsymbol{H}^s(\bigcup\Omega_i)$ *for all* t *in* $[0,T]$.

Proof. Let $\rho(x) \equiv \sigma(x)/\varepsilon(x)$. A direct consequence of Theorem 2.2 and Assumption 2.1 is that $\exp(\rho(x)t)\boldsymbol{E}$, $\exp(\rho(x)t)\boldsymbol{E}_t$, $\mu^{-1}\mathbf{curl}\,\boldsymbol{E}$, $\mu^{-1}\mathbf{curl}\,\boldsymbol{E}_t$ and $\mu^{-1}\boldsymbol{B}$ belong to $\boldsymbol{H}^s(\bigcup\Omega_i)$. By the chain rule, we have

$$\nabla \exp(-\rho(x)t) = -t\exp(-\rho(x)t)\nabla\rho(x).$$

Since $\rho(x)$ is in $W_p^1(\Omega)$, so is $\exp(-\rho(x)t)$. By Theorem 1.4.4.2 in [16], the vector field $\boldsymbol{E}$, as a product of $\exp(-\rho(x)t)$ and $\exp(\rho(x)t)\boldsymbol{E}$, belongs to $\boldsymbol{H}^s(\bigcup\Omega_i)$. Similarly $\boldsymbol{E}_t$ belongs to $\boldsymbol{H}^s(\bigcup\Omega_i)$. □

3. Approximation of $\boldsymbol{H}(\mathbf{curl};\Omega)$ and $\boldsymbol{H}(\mathrm{div};\Omega)$

In this section, we summarize the construction of Nedelec and Raviart-Thomas finite element spaces and give some approximation estimates. Due to the low regularity of the solution to (1.1)–(1.4), we only introduce the lowest order finite elements.

Let $\mathcal{T}_h$ be a simplicial mesh of Ω that is quasi-uniform and shape-regular. This assumption guarantees that all estimates in this paper do not depend on h, the maximal diameter of the tetrahedra in $\mathcal{T}_h$. We also require that the mesh $\mathcal{T}_h$ be aligned with the interface Γ.

Let S_h be the subspace of $H_0^1(\Omega)$ consisting of piecewise linear polynomials. For the approximation property of S_h in $H_0^1(\Omega)$, we need the following assumption on the geometry of the interface Γ.

Assumption 3.1. There are no points on Γ that belong to more than two $\overline{\Omega}_i$'s.

Based on the above assumption, we have the following lemma.

Lemma 3.1. *Under Assumption 3.1, for $\phi \in H_0^1(\Omega)\cap H^\alpha(\bigcup\Omega_i)$, $1 \le \alpha \le 1/2$, there exists $\phi_h \in S_h$ such that*

$$\|\phi-\phi_h\| + h\|\phi-\phi_h\|_{1,\bigcup\Omega_i} \le Ch^\alpha\|\phi\|_{\alpha,\bigcup\Omega_i}. \tag{3.1}$$

Remark 3.1. Lemma 3.1 appeared in [8] and the proof follows the technique in [9], which requires certain geometry regularity of the interface Γ. This is the only place where we use Assumption 3.1. Moreover, Assumption 3.1 is only necessary for the case $\alpha = 1/2$. Indeed, if $\alpha \in [0, 1/2)$, we can take $\phi_h = P_h\phi$, where P_h is the energy projection onto S_h under the innerproduct $(\nabla\cdot,\nabla\cdot)$. Since the interpolation space between $H_0^1(\Omega)$ and $H_0^1(\Omega)\cap H^2(\Omega)$ is $H_0^1(\Omega)\cap H^{1+\alpha}(\Omega)$ [5], we have

$$|\phi-\phi_h|_{1,\Omega} \le Ch^\alpha\|\phi\|_{1+\alpha},$$

and thus (3.1) follows from the equivalence of $\|\cdot\|_{1+\alpha,\bigcup\Omega_i}$ and $\|\cdot\|_{1+\alpha,\Omega}$ for $\alpha \in [0, 1/2)$.

The Raviart-Thomas finite element space $\overline{\boldsymbol{V}}_h$ is defined by

$$\overline{\boldsymbol{V}}_h = \left\{\boldsymbol{v}_h \in \boldsymbol{H}(\mathrm{div};\Omega)\;\middle|\;\boldsymbol{v}_h = \boldsymbol{a}_\tau + \beta_\tau\boldsymbol{x} \text{ on } \tau,\ \forall\tau\in\mathcal{T}_h\right\},$$

where $\boldsymbol{a}_\tau$ is a constant vector and β_τ is a constant scalar. We define $\boldsymbol{V}_h \equiv \boldsymbol{H}_0(\mathrm{div};\Omega)\cap\overline{\boldsymbol{V}}_h$. The degrees of freedom for the Raviart-Thomas element are given by

$$\int_f \boldsymbol{u}\cdot\boldsymbol{n}\,dS, \tag{3.2}$$

where $\boldsymbol{n}$ is the unit outward normal of the face f on $\tau \in \mathcal{T}_h$. Based on degrees of freedom given above, we we define the interpolant $\boldsymbol{r}_\tau\boldsymbol{v}$ such that $\boldsymbol{r}_\tau\boldsymbol{v}$ and $\boldsymbol{v}$ have

the same degrees of freedom on τ and we define the interpolation operator $\boldsymbol{r}_h$ onto $\overline{\boldsymbol{V}}_h$ by $\boldsymbol{r}_h\boldsymbol{v}|_\tau = \boldsymbol{r}_\tau\boldsymbol{v}$ on all τ in $\mathcal{T}_h$. The operator $\boldsymbol{r}_h$ is well defined for vector fields in $\boldsymbol{H}(\mathrm{div};\Omega)\cap\boldsymbol{L}^p(\Omega)$ for any $p>2$ [27]. It can be shown [1, 11, 17] that for all $\boldsymbol{v}$ in $\boldsymbol{H}(\mathrm{div};\Omega)\cap\boldsymbol{H}^\alpha(\bigcup\Omega_i)$, the interpolation operator $\boldsymbol{r}_h$ satisfies

$$\|\boldsymbol{v}-\boldsymbol{r}_h\boldsymbol{v}\| \le C\begin{cases} h^\alpha|\boldsymbol{v}|_{\alpha,\bigcup\Omega_i} + h\|\mathrm{div}\,\boldsymbol{v}\|, & 0<\alpha\le 1/2,\\ h^\alpha|\boldsymbol{v}|_{\alpha,\bigcup\Omega_i}, & 1/2<\alpha\le 1.\end{cases} \tag{3.3}$$

The Nedelec finite element space $\overline{\boldsymbol{U}}_h$ is defined by

$$\overline{\boldsymbol{U}}_h = \left\{\boldsymbol{u}_h\in\boldsymbol{H}(\mathbf{curl};\Omega)\;\middle|\;\boldsymbol{u}_h=\boldsymbol{a}_\tau+\boldsymbol{b}_\tau\times\boldsymbol{x}\text{ on }\tau,\ \forall\tau\in\mathcal{T}_h\right\},$$

where $\boldsymbol{a}_\tau$ and $\boldsymbol{b}_\tau$ are two constant vectors. We define $\boldsymbol{U}_h\equiv\boldsymbol{H}_0(\mathbf{curl};\Omega)\cap\overline{\boldsymbol{U}}_h$. The degrees of freedom for the Nedelec element are given by

$$\int_e \boldsymbol{u}\cdot\boldsymbol{t}\,ds, \tag{3.4}$$

where $\boldsymbol{t}$ is the unit vector directed along the edge e on $\tau\in\mathcal{T}_h$. There are six degrees of freedom on each tetrahedron. Based on degrees of freedom, we can naturally define the interpolation operator $\boldsymbol{\Pi}_h$ onto $\overline{\boldsymbol{U}}_h$. Because of the dependence on edge moments (3.4), $\boldsymbol{\Pi}_h$ is only well defined for vector fields in $\boldsymbol{H}(\mathbf{curl};\Omega)$ with certain regularity. The following lemma [2] makes the condition specific.

Lemma 3.2. *For any $p>2$ and for any tetrahedron τ, the operator $\boldsymbol{\Pi}_\tau$ is well defined and continuous on the space*

$$\{\boldsymbol{u}\in\boldsymbol{L}^p(\tau)\;\big|\;\mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{L}^p(\tau)\text{ and }\boldsymbol{u}\times\boldsymbol{n}\in L^p(\partial\tau)^2\}.$$

Lemma 3.3. *Suppose that $\boldsymbol{u}\in\boldsymbol{H}(\mathbf{curl};\Omega)$ and that $\boldsymbol{\Pi}_h$ is the interpolation operator onto $\boldsymbol{U}_h$. Then, we have the following estimates.*

(1) *If $\boldsymbol{u}\in\boldsymbol{H}^\alpha(\bigcup\Omega_i)$ for some $1/2<\alpha\le 1$ and $\mathbf{curl}\,\boldsymbol{u}\in\overline{\boldsymbol{V}}_h$, then*

$$\|\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}\|\le Ch^\alpha(|\boldsymbol{u}|_{\alpha,\bigcup\Omega_i}+\|\mathbf{curl}\,\boldsymbol{u}\|).$$

(2) *If $\boldsymbol{u}\in\boldsymbol{H}^1(\bigcup\Omega_i)$ and $\mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{H}^\alpha(\bigcup\Omega_i)$ for some $\alpha>0$, then*

$$\|\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}\|+h\|\mathbf{curl}\,(\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u})\|\le Ch(|\boldsymbol{u}|_{1,\bigcup\Omega_i}+|\mathbf{curl}\,\boldsymbol{u}|_{\alpha,\bigcup\Omega_i}).$$

(3) *If both $\boldsymbol{u}$ and $\mathbf{curl}\,\boldsymbol{u}$ belong to $\boldsymbol{H}^\alpha(\bigcup\Omega_i)$ for some $\alpha\in(1/2,1]$, then*

$$\|\boldsymbol{u}-\boldsymbol{\Pi}_h\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}\le Ch^\alpha(\|\boldsymbol{u}\|_\alpha+\|\mathbf{curl}\,\boldsymbol{u}\|_\alpha).$$

(4) *If $\boldsymbol{u}\in\boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfies $\mathbf{curl}\,\boldsymbol{u}\in\boldsymbol{H}^\alpha(\bigcup\Omega_i)$ for some $\alpha>0$, then*

$$\inf_{\boldsymbol{u}_h\in\boldsymbol{U}_h}\|\mathbf{curl}\,\boldsymbol{u}-\mathbf{curl}\,\boldsymbol{u}_h\|\le Ch^\alpha\|\mathbf{curl}\,\boldsymbol{u}\|_{\alpha,\bigcup\Omega_i}.$$

Proof. Inequality (3) is given in Proposition 5.6 of [1] (see also [11]) and the inequality (4) is given in Theorem 4.8 of [2].

Inequality (1) is an extension of (2.4) in [3] and the proof follows along the same lines there. For completeness, we also give a proof here. First by Lemma 3.2 and the Sobolev imbedding theorem, $\boldsymbol{\Pi}_h$ is well defined for vector fields in $\boldsymbol{H}^\alpha(\Omega)$, $\alpha>1/2$, whose $\mathbf{curl}$ are in $\overline{\boldsymbol{V}}_h$. Secondly, on the reference tetrahedron $\widehat{\tau}$ of unit size, we have

$$\|\widehat{\boldsymbol{u}}-\widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|\le C(\|\widehat{\boldsymbol{u}}\|_\alpha+\|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|_{\boldsymbol{L}^\infty})\le C(\|\widehat{\boldsymbol{u}}\|_\alpha+\|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|),$$

by the equivalence of all norms in $\overline{\boldsymbol{V}}(\widehat{\tau})$. Since $\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}$ vanishes for constant $\widehat{\boldsymbol{u}}$, a Bramble-Hilbert argument yields

$$\|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\| \le C(|\widehat{\boldsymbol{u}}|_{\alpha} + \|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|).$$

Finally, if we scale this estimate to a general tetrahedron using Lemmas 5.2 and 5.5 of [1] and sum over all the tetrahedra in $\mathcal{T}_h$, we get

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}\|^2 &\le C\sum_{\tau} h_\tau \|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|^2_{0,\widehat{\tau}} \le C\sum_{\tau} h_\tau(|\widehat{\boldsymbol{u}}|^2_{\alpha,\widehat{\tau}} + |\mathbf{curl}\,\widehat{\boldsymbol{u}}|^2_{0,\widehat{\tau}})\\ &\le C\sum_{\tau} h_\tau^{1+\alpha}|\boldsymbol{u}|^2_{\alpha,\tau} + h_\tau^2\|\mathbf{curl}\,\boldsymbol{u}\|^2_{0,\tau}\\ &\le Ch^{2\alpha}(|\boldsymbol{u}|^2_{\alpha,\tau} + \|\mathbf{curl}\,\boldsymbol{u}\|^2).\end{aligned}$$

The proof of (2) is very similar. Again by Lemma 3.2 and the Sobolev imbedding theorem, $\widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}$ is well defined for $\boldsymbol{H}^1$ vector fields whose **curl** belongs to $\boldsymbol{H}^{\alpha}$. On the reference tetrahedron $\widehat{\tau}$, we have $\|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}} \le C(\|\widehat{\boldsymbol{u}}\|_{1,\widehat{\tau}} + \|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|_{\alpha,\widehat{\tau}})$. A Bramble-Hilbert argument gives that

$$\|\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}} \le C(|\widehat{\boldsymbol{u}}|_{1,\widehat{\tau}} + |\mathbf{curl}\,\widehat{\boldsymbol{u}}|_{\alpha,\widehat{\tau}}).$$

Similarly, we have

$$\begin{aligned}\|\mathbf{curl}\,(\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}})\|_{0,\widehat{\tau}} &\le \|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}} + \|\mathbf{curl}\,\widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}} \le \|\widehat{\boldsymbol{u}}\|_{1,\widehat{\tau}} + C\|\widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|_{0,\widehat{\tau}}\\ &\le C(\|\widehat{\boldsymbol{u}}\|_{1,\widehat{\tau}} + \|\mathbf{curl}\,\widehat{\boldsymbol{u}}\|_{\alpha,\widehat{\tau}}) \le C(|\widehat{\boldsymbol{u}}|_{1,\widehat{\tau}} + |\mathbf{curl}\,\widehat{\boldsymbol{u}}|_{\alpha,\widehat{\tau}}).\end{aligned}$$

A scaling argument using Lemma 5.2 and 5.5 of [1] gives that

$$\begin{aligned}&\|\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}\|^2 + h^2\|\mathbf{curl}\,(\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u})\|^2\\ &\qquad\le C\sum_{\tau} h_\tau(\|\widehat{\boldsymbol{u}} - \boldsymbol{\Pi}_{\widehat{\tau}}\widehat{\boldsymbol{u}}\|^2_{0,\widehat{\tau}} + \|\mathbf{curl}\,(\widehat{\boldsymbol{u}} - \widehat{\boldsymbol{\Pi}}_{\widehat{\tau}}\widehat{\boldsymbol{u}})\|^2_{0,\widehat{\tau}})\\ &\qquad\le C\sum_{\tau} h_\tau(|\widehat{\boldsymbol{u}}|^2_{1,\widehat{\tau}} + |\mathbf{curl}\,\widehat{\boldsymbol{u}}|^2_{\alpha,\widehat{\tau}})\\ &\qquad\le C\sum_{\tau} h_\tau(h_\tau|\boldsymbol{u}|^2_{1,\widehat{\tau}} + h_\tau^{1+2\alpha}|\mathbf{curl}\,\widehat{\boldsymbol{u}}|^2_{\alpha,\widehat{\tau}})\\ &\qquad\le Ch^2(|\boldsymbol{u}|^2_{1,\bigcup\Omega_i} + |\mathbf{curl}\,\boldsymbol{u}|^2_{\alpha,\bigcup\Omega_i}).\end{aligned}$$

□

Now we introduce the operator $\boldsymbol{\pi}_h$ from $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ to $\boldsymbol{U}_h$, which is called the Fortin operator in [7]. For any $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$, $\boldsymbol{\pi}_h\boldsymbol{u} \in \boldsymbol{U}_h$ satisfies

$$(\mu^{-1}\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{w}_h) = (\mu^{-1}\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{w}_h) \quad \forall\, \boldsymbol{w}_h \in \boldsymbol{U}_h, \tag{3.5}$$

$$(\boldsymbol{\pi}_h\boldsymbol{u}, \nabla\psi_h) = (\boldsymbol{u}, \nabla\psi_h) \quad \forall\, \psi_h \in S_h. \tag{3.6}$$

If μ is constant, this operator has been widely studied (see [7, 19, 21, 11]).

It is shown in [15, 24] that $\boldsymbol{\pi}_h$ is well defined. This is also an application of the general results on mixed finite element methods in [27]. In the proof, one key property [15] is that if $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfies $\mathbf{curl}\,\boldsymbol{u} = 0$, then

$$\boldsymbol{u} = \nabla p \quad \text{for some } p \in H_0^1(\Omega). \tag{3.7}$$

Another key inequality [2] is that if $\boldsymbol{u}_h \in \boldsymbol{U}_h$ satisfies that $(\boldsymbol{u}_h, \nabla p_h) = 0$ for all $p_h \in S_h$, then

$$\|\boldsymbol{u}_h\| \le C\|\mathbf{curl}\,\boldsymbol{u}_h\|. \tag{3.8}$$

Note that both (3.8) and the existence of p in (3.7) are only valid when $\partial\Omega$ is connected.

Remark 3.2. $\boldsymbol{\pi}_h$ is also computable [27]. In fact, $\boldsymbol{\pi}_h\boldsymbol{u} = \boldsymbol{u}_h$ where $(\boldsymbol{u}_h, p_h)$ is the solution of the following problem: Find $\boldsymbol{u}_h \in \boldsymbol{U}_h$ and $p_h \in S_h$ such that

$$\begin{aligned}(\mu^{-1}\mathbf{curl}\,\boldsymbol{u}_h, \mathbf{curl}\,\boldsymbol{w}_h) + (\nabla p_h, \boldsymbol{w}_h) &= (\mu^{-1}\mathbf{curl}\,\boldsymbol{u}, \mathbf{curl}\,\boldsymbol{w}_h) \quad \forall\, \boldsymbol{w}_h \in \boldsymbol{U}_h \\ (\boldsymbol{u}_h, \nabla\psi_h) &= (\boldsymbol{u}, \nabla\psi_h) \quad \forall\, \psi \in S_h.\end{aligned}$$

Remark 3.3. When $\boldsymbol{\pi}_h$ is applied to ∇p for some $p \in H_0^1(\Omega)$, we have the following optimal estimate:

$$\|\nabla p - \boldsymbol{\pi}_h \nabla p\| \le \inf_{p_h \in S_h} \|\nabla p - \nabla p_h\|. \tag{3.9}$$

Indeed, let $q_h \in S_h$ solve

$$(\nabla q_h, \nabla\phi_h) = (\nabla p, \nabla q_h), \quad \text{for all } \phi_h \in S_h.$$

Note that ∇q_h satisfies both equations of the definition of π_h. By the uniqueness, $\nabla q_h = \pi_h \nabla p$. Clearly q_h satisfies

$$\|\nabla p - \nabla q_h\| \le \inf_{p_h \in S_h} \|\nabla p - \nabla p_h\|,$$

from which (3.9) follows.

The following lemma extends the previous results to the case that μ is piecewise constant and $\boldsymbol{u}$ is of low regularity.

Lemma 3.4. *Under Assumptions 2.1 and 3.1, if $\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfies that both $\boldsymbol{u}$ and $\mathbf{curl}\,\boldsymbol{u}$ belong to $\boldsymbol{H}^s(\bigcup\Omega_i)$, we have*

$$\|\boldsymbol{u} - \boldsymbol{\pi}_h\boldsymbol{u}\| + \|\mathbf{curl}\,(\boldsymbol{u} - \boldsymbol{\pi}_h\boldsymbol{u})\| \le Ch^s(\|\boldsymbol{u}\|_{s,\bigcup\Omega_i} + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\bigcup\Omega_i}). \tag{3.10}$$

Proof. Let $\boldsymbol{u}$ be as in Lemma 3.4. From (3.5) and the third estimate of Lemma 3.3, we have that

$$\begin{aligned}\|\mathbf{curl}\,(\boldsymbol{u} - \boldsymbol{\pi}_h\boldsymbol{u})\|_{\mu^{-1}} &= \inf_{\boldsymbol{u}_h \in \boldsymbol{U}_h} \|\mathbf{curl}\,(\boldsymbol{u} - \boldsymbol{u}_h)\|_{\mu^{-1}} \\ &\le Ch^s\|\mathbf{curl}\,\boldsymbol{u}\|_{s,\bigcup\Omega_i}.\end{aligned} \tag{3.11}$$

In the following, we will bound $\|\boldsymbol{u} - \boldsymbol{\pi}_h\boldsymbol{u}\|$. Let $\boldsymbol{u} = \boldsymbol{v} + \nabla\psi$ be the Helmholtz decomposition of $\boldsymbol{u}$ where $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and $\psi \in H_0^1(\Omega)$ satisfy $\operatorname{div}\boldsymbol{v} = 0$ and $\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\psi\|_1 \le C\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. Since $\operatorname{div}\boldsymbol{v} = 0$, equation (3.6) implies that $(\boldsymbol{\pi}_h\boldsymbol{v}, \nabla\psi_h) = 0$ for all $\psi_h \in S_h$. Using (3.8) on $\boldsymbol{\pi}_h\boldsymbol{v}$, we have $\|\boldsymbol{\pi}_h\boldsymbol{v}\| \le C\|\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{v}\|$ and thus

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{\pi}_h\boldsymbol{u}\| &\le \|\boldsymbol{v} - \boldsymbol{\pi}_h\boldsymbol{v}\| + \|\nabla\psi - \boldsymbol{\pi}_h\nabla\psi\| \\ &\le \|\boldsymbol{v}\| + C\|\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{v}\| + \|\nabla\psi\| \\ &\le \|\boldsymbol{v}\| + C\|\mathbf{curl}\,\boldsymbol{v}\| + \|\boldsymbol{u}\| \le C\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)},\end{aligned} \tag{3.12}$$

where we have used (3.9) and the first equality in (3.11) on $\boldsymbol{v}$.

When $s > 1/2$, $\|\boldsymbol{u} - \boldsymbol{\pi}_h\boldsymbol{u}\|$ can be bounded as follows. Using the stability (3.12) of $\boldsymbol{\pi}_h$ on $\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}$ and the third inequality in Lemma 3.3, we have that

$$\begin{aligned}\|\boldsymbol{u} - \boldsymbol{\pi}_h\boldsymbol{u}\| &\le \|\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}\| + \|(\boldsymbol{\Pi}_h - \boldsymbol{\pi}_h)\boldsymbol{u}\| \\ &= \|\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}\| + \|\boldsymbol{\pi}_h(\boldsymbol{\Pi}_h\boldsymbol{u} - \boldsymbol{u})\| \\ &\le C\|\boldsymbol{u} - \boldsymbol{\Pi}_h\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le Ch^s(\|\boldsymbol{u}\|_{s,\bigcup\Omega_i} + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\bigcup\Omega_i}).\end{aligned}$$

The main difficulty comes from the case $s \le 1/2$. Since $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ has zero divergence, by Lemma 2.2, we can decompose $\boldsymbol{v} = \boldsymbol{z} + \nabla\phi$ where $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathbf{curl};\Omega)$ and $\phi \in H_0^1(\Omega)$ satisfy $\|\boldsymbol{z}\|_1 + \|\phi\|_1 \le C\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}$. Therefore, letting $p = \phi + \psi \in H_0^1(\Omega)$, we get a decomposition $\boldsymbol{u} = \boldsymbol{z} + \nabla p$ which satisfies

$$\|\boldsymbol{z}\|_1 + \|p\|_1 \le C\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)}. \tag{3.13}$$

Since $\boldsymbol{u} \in \boldsymbol{H}^s(\bigcup \Omega_i)$ and $\boldsymbol{z} \in \boldsymbol{H}^1(\Omega)$, ∇p belongs to $\boldsymbol{H}^s(\bigcup \Omega_i)$. By (3.9) and Lemma 3.1, we have

$$\begin{aligned} \|\nabla p - \boldsymbol{\pi}_h \nabla p\| &\le \inf_{p_h \in S_h} \|\nabla p - \nabla p_h\| \le Ch^s \|p\|_{1+s,\bigcup\Omega_i} \\ &\le Ch^s(\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\boldsymbol{u}\|_{s,\bigcup\Omega_i}). \end{aligned} \tag{3.14}$$

Once we have shown that

$$\|\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\| \le Ch^s(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\bigcup\Omega_i}), \tag{3.15}$$

the desired estimate for $\|\boldsymbol{u} - \pi_h \boldsymbol{u}\|$ will follow from (3.14), (3.15) and the triangle inequality.

To show (3.15), by Lemma 3.3, we first note that $\boldsymbol{\Pi}_h \boldsymbol{z}$ is well defined and satisfies

$$\|\boldsymbol{z} - \boldsymbol{\Pi}_h \boldsymbol{z}\| \le Ch(\|\boldsymbol{z}\|_1 + \|\mathbf{curl}\,\boldsymbol{z}\|_{s,\bigcup\Omega_i}) \le Ch(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\bigcup\Omega_i}). \tag{3.16}$$

Then we decompose $\boldsymbol{\Pi}_h \boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z} = \boldsymbol{w} + \nabla q$ where $q \in H_0^1(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfies $\operatorname{div}\boldsymbol{w} = 0$. By Lemma 2.3, $\boldsymbol{w}$ belongs to $\boldsymbol{H}^r(\Omega)$ and satisfies

$$\begin{aligned} \|\boldsymbol{w}\|_r &\le C\|\boldsymbol{w}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le C\|\boldsymbol{\Pi}_h \boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \\ &= C\|\boldsymbol{\pi}_h(\boldsymbol{\Pi}_h \boldsymbol{z} - \boldsymbol{z})\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \le C\|\boldsymbol{z} - \boldsymbol{\Pi}_h \boldsymbol{z}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \\ &\le C(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\bigcup\Omega_i}), \end{aligned} \tag{3.17}$$

where we have used (3.11) and (3.12) on $\boldsymbol{z}$ and the second estimate of Lemma 3.3. Since $\mathbf{curl}\,\boldsymbol{w}$ belongs to $\boldsymbol{V}_h$, by Lemma 3.3, $\boldsymbol{\Pi}_h \boldsymbol{w}$ is well defined and satisfies

$$\|\boldsymbol{w} - \boldsymbol{\Pi}_h \boldsymbol{w}\| \le Ch^r(\|\boldsymbol{w}\|_r + \|\mathbf{curl}\,\boldsymbol{w}\|) \le Ch^r(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\bigcup\Omega_i}). \tag{3.18}$$

Note that

$$\boldsymbol{\Pi}_h \boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z} = \boldsymbol{\Pi}_h \boldsymbol{w} + \boldsymbol{\Pi}_h \nabla q = \boldsymbol{\Pi}_h \boldsymbol{w} + \nabla q_h$$

for some $q_h \in S_h$. Therefore, we have, by (3.6),

$$\begin{aligned} \|\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\|^2 &= (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{z} - \boldsymbol{\Pi}_h \boldsymbol{z}) + (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{\Pi}_h \boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}) \\ &= (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{z} - \boldsymbol{\Pi}_h \boldsymbol{z}) + (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{\Pi}_h \boldsymbol{w} + \nabla q_h) \\ &= (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{z} - \boldsymbol{\Pi}_h \boldsymbol{z}) - (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w} - \boldsymbol{\Pi}_h \boldsymbol{w}) + (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w}) \\ &\le \|\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\|(\|\boldsymbol{z} - \boldsymbol{\Pi}_h \boldsymbol{z}\| + \|\boldsymbol{w} - \boldsymbol{\Pi}_h \boldsymbol{w}\|) + (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w}). \end{aligned} \tag{3.19}$$

To estimate the term $(\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w})$, we define $\boldsymbol{t} \in \boldsymbol{H}_0(\mathbf{curl};\Omega)$ satisfying

$$\mathbf{curl}\,(\mu^{-1}\mathbf{curl}\,\boldsymbol{t}) = \boldsymbol{w} \quad \text{and} \quad \operatorname{div}\boldsymbol{t} = 0 \text{ in } \Omega. \tag{3.20}$$

Thanks to $\operatorname{div}\boldsymbol{w} = 0$, $\boldsymbol{t}$ is well defined. Since $\mathbf{curl}\,\boldsymbol{t} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, $\mu^{-1}\mathbf{curl}\,\boldsymbol{t}$ actually belongs to $\boldsymbol{X}_T(\Omega;\mu)$. Thus, by Assumption 2.1, we have

$$\|\mathbf{curl}\,\boldsymbol{t}\|_{s,\bigcup\Omega_i} \le C\|\boldsymbol{w}\|. \tag{3.21}$$

Using (3.20) and (3.5), we have

$$\begin{aligned} (\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w}) &= (\mathbf{curl}\,(\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}), \mathbf{curl}\,\boldsymbol{t})_{\mu^{-1}} \\ &= (\mathbf{curl}\,(\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}), \mathbf{curl}\,(\boldsymbol{t} - \boldsymbol{\pi}_h \boldsymbol{t}))_{\mu^{-1}}. \end{aligned}$$

Since $\mathbf{curl}\,\boldsymbol{z} = \mathbf{curl}\,\boldsymbol{u}$ belongs to $\boldsymbol{H}^s(\bigcup \Omega_i)$, by (3.11) and (3.21), we conclude that

$$\begin{aligned}(\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}, \boldsymbol{w}) &\le Ch^{2s}(\|\mathbf{curl}\,\boldsymbol{z}\|_{s,\bigcup\Omega_i} + \|\mathbf{curl}\,\boldsymbol{t}\|_{s,\bigcup\Omega_i})^2 \\ &\le Ch^{2s}(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\bigcup\Omega_i})^2. \end{aligned} \tag{3.22}$$

Finally, the combination of (3.16), (3.18), (3.19), (3.21) and (3.22) gives that

$$\begin{aligned}\|\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\|^2 \le Ch^s \|\boldsymbol{z} - \boldsymbol{\pi}_h \boldsymbol{z}\|(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\bigcup\Omega_i}) \\ + Ch^{2s}(\|\boldsymbol{u}\| + \|\mathbf{curl}\,\boldsymbol{u}\|_{s,\bigcup\Omega_i})^2,\end{aligned}$$

from which (3.15) follows. □

4. Semidiscrete scheme

Let $(\boldsymbol{E}, \boldsymbol{B})$ in $\boldsymbol{H}_0(\mathbf{curl};\Omega) \times \boldsymbol{H}_0(\mathrm{div};\Omega)$ be the solution to (1.1)–(1.4). Then, $(\boldsymbol{E}, \boldsymbol{B})$ satisfies

$$(\varepsilon \boldsymbol{E}_t + \sigma \boldsymbol{E}, \boldsymbol{u}) - (\mu^{-1}\boldsymbol{B}, \mathbf{curl}\,\boldsymbol{u}) = (\boldsymbol{J}, \boldsymbol{u}) \quad \forall\, \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega), \tag{4.1}$$

$$(\mu^{-1}\boldsymbol{B}_t, \boldsymbol{v}) + (\mathbf{curl}\,\boldsymbol{E}, \mu^{-1}\boldsymbol{v}) = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega). \tag{4.2}$$

On the other hand, the system (4.1), (4.2) and (1.4) is uniquely solvable. Indeed, let $(\boldsymbol{E}, \boldsymbol{B})$ be a solution to the system (4.1) and (4.2) with $\boldsymbol{J} = \boldsymbol{E}_0 = \boldsymbol{B}_0 = \boldsymbol{0}$. Take $\boldsymbol{u} = \boldsymbol{E}$ in (4.1) and $\boldsymbol{v} = \boldsymbol{B}$ in (4.2), add (4.1) and (4.2) together, and we get

$$(\varepsilon \boldsymbol{E}_t, \boldsymbol{E}) + (\sigma \boldsymbol{E}, \boldsymbol{E}) + (\mu^{-1}\boldsymbol{B}_t, \boldsymbol{B}) = 0,$$

from which it follows that

$$\frac{d}{dt}[(\varepsilon \boldsymbol{E}, \boldsymbol{E}) + (\mu^{-1}\boldsymbol{B}, \boldsymbol{B})] = -2(\sigma \boldsymbol{E}, \boldsymbol{E}) \le 0.$$

This shows

$$(\varepsilon \boldsymbol{E}(t), \boldsymbol{E}(t)) + (\mu^{-1}\boldsymbol{B}(t), \boldsymbol{B}(t)) \le (\varepsilon \boldsymbol{E}_0, \boldsymbol{E}_0) + (\mu^{-1}\boldsymbol{B}_0, \boldsymbol{B}_0) = 0,$$

and thus $\boldsymbol{E}(t) = \boldsymbol{B}(t) = 0$ for all t in $(0, T)$.

So far we have shown that the system (4.1) and (4.2) is equivalent to (1.1) and (1.2) under conditions (1.3) and (1.4). Using the Nedelec edge elements and the Thomas-Raviart elements introduced in the second section, we can naturally transfer (4.1) and (4.2) to the semidiscrete scheme of seeking $(\boldsymbol{E}_h(t), \boldsymbol{B}_h(t))$ in $\boldsymbol{U}_h \times \boldsymbol{V}_h$ satisfying, for all $0 < t \le T$,

$$(\varepsilon \boldsymbol{E}_{h,t} + \sigma \boldsymbol{E}_h, \boldsymbol{u}_h) - (\mu^{-1}\boldsymbol{B}_h, \mathbf{curl}\,\boldsymbol{u}_h) = (\boldsymbol{J}, \boldsymbol{u}_h) \quad \forall\, \boldsymbol{u}_h \in \boldsymbol{U}_h, \tag{4.3}$$

$$(\mu^{-1}\boldsymbol{B}_{h,t}, \boldsymbol{v}_h) + (\mathbf{curl}\,\boldsymbol{E}_h, \mu^{-1}\boldsymbol{v}_h) = 0 \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.4}$$

with given initial approximations

$$\boldsymbol{E}_h(0) \approx \boldsymbol{E}_0 \quad \text{and} \quad \boldsymbol{E}_h(0) \approx \boldsymbol{E}_0. \tag{4.5}$$

This scheme is uniquely solvable [19].

Some possible choices of $\boldsymbol{E}_h(0)$ and $\boldsymbol{B}_h(0)$ are as follows. We can define $\boldsymbol{B}_h(0) = \boldsymbol{r}_h \boldsymbol{B}(0)$ since $\boldsymbol{r}_h \boldsymbol{B}(0)$ is well defined according to (3.3). We note that $\boldsymbol{B}_h(t)$ is divergence free for all t in this case by (4.4). We can define $\boldsymbol{E}_h(0) = \boldsymbol{\pi}_h \boldsymbol{E}(0)$. However, if $\boldsymbol{E}(0)$ is smooth enough, we can approximate $\boldsymbol{E}(0)$ by $\boldsymbol{\Pi}_h \boldsymbol{E}(0)$ and avoid solving for $\boldsymbol{\pi}_h \boldsymbol{E}(0)$.

In the following theorem, we give the $\boldsymbol{L}^2$-error estimate for the semidiscrete scheme (4.3) and (4.4)

Theorem 4.1. *Let* $(\boldsymbol{E},\boldsymbol{B})$ *be the solution to* (1.1)–(1.4) *and let* $(\boldsymbol{E}_h,\boldsymbol{B}_h)$ *be the solution to* (4.3)–(4.5). *Suppose that* $\boldsymbol{E}(t)$, $\mathbf{curl}\,\boldsymbol{E}(t)$, $\boldsymbol{E}_t(t)$, $\mathbf{curl}\,\boldsymbol{E}_t(t)$ *and* $\boldsymbol{B}(t)$ *belong to* $\boldsymbol{H}^s(\bigcup\Omega_i)$ *for all* t *in* $[0,T]$, *where* s *is as in Assumption 2.1. Under Assumptions 2.1 and 3.1, we have, for all* t *in* $(0,T)$,

$$\|\boldsymbol{E}(t)-\boldsymbol{E}_h(t)\|+\|\boldsymbol{B}(t)-\boldsymbol{B}_h(t)\|\le C(h^s+\|\boldsymbol{E}_0-\boldsymbol{E}_h(0)\|+\|\boldsymbol{B}_0-\boldsymbol{B}_h(0)\|),$$

where C *depends on*

$$\sup_{t\in(0,T)}\|\boldsymbol{B}(t)\|_{s,\bigcup\Omega_i},\qquad \sup_{t\in(0,T)}\|\boldsymbol{E}(t)\|_{s,\bigcup\Omega_i}+\|\mathbf{curl}\,\boldsymbol{E}(t)\|_{s,\bigcup\Omega_i},$$

and

$$\sup_{t\in(0,T)}\|\boldsymbol{E}_t(t)\|_{s,\bigcup\Omega_i}+\|\mathbf{curl}\,\boldsymbol{E}_t(t)\|_{s,\bigcup\Omega_i}.$$

Proof. Let

$$\boldsymbol{e}_h(t)=\boldsymbol{\pi}_h\boldsymbol{E}(t)-\boldsymbol{E}_h(t)\quad\text{and}\quad \boldsymbol{b}_h(t)=\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}(t)-\boldsymbol{B}_h(t),$$

where $\boldsymbol{Q}_h^{\boldsymbol{Z}}$ is the $\boldsymbol{L}^2_{\mu^{-1}}$-projection from $\boldsymbol{H}_0(\mathrm{div};\Omega)$ to $\mathbf{curl}\,\boldsymbol{U}_h$. Since $\mathrm{div}\,\boldsymbol{B}=0$, we know that $\mathrm{div}\,\boldsymbol{r}_h\boldsymbol{B}=0$ and thus $\boldsymbol{r}_h\boldsymbol{B}\in\mathbf{curl}\,\boldsymbol{U}_h$. Therefore, by (3.3) and the regularity assumption of the solution, we have

$$\|\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}\|\le\|\boldsymbol{r}_h\boldsymbol{B}-\boldsymbol{B}\|\le Ch^s.$$

By Lemma 3.4, we also have

$$\|\boldsymbol{\pi}_h\boldsymbol{E}(t)-\boldsymbol{E}(t)\|\le Ch^s.$$

Therefore, we need only show that, for all t in $(0,T)$,

$$\|\boldsymbol{e}_h(t)\|+\|\boldsymbol{b}_h(t)\|\le C(h^s+\|\boldsymbol{E}_0-\boldsymbol{E}_h(0)\|+\|\boldsymbol{B}_0-\boldsymbol{B}_h(0)\|). \tag{4.6}$$

We follow the strategy in [19] and split $\boldsymbol{B}_h(t)=\boldsymbol{B}_h^+(t)+\boldsymbol{B}_h^\perp(t)$ where $\boldsymbol{B}_h^+(t)$ belongs to $\mathbf{curl}\,\boldsymbol{U}_h$ and $\boldsymbol{B}_h^\perp(t)$ is in the $\boldsymbol{L}^2_{\mu^{-1}}$-orthogonal complement of $\mathbf{curl}\,\boldsymbol{U}_h$ in $\boldsymbol{V}_h$.

Due to (4.3) and (4.4), $(\boldsymbol{E}_h,\boldsymbol{B}_h^+)\in\boldsymbol{U}_h\times\mathbf{curl}\,\boldsymbol{U}_h$ satisfies

$$(\varepsilon\boldsymbol{E}_{h,t}+\sigma\boldsymbol{E}_h,\boldsymbol{u}_h)-(\mu^{-1}\boldsymbol{B}_h^+,\mathbf{curl}\,\boldsymbol{u}_h)=(\boldsymbol{J},\boldsymbol{u}_h)\quad\forall\,\boldsymbol{u}_h\in\boldsymbol{U}_h, \tag{4.7}$$

$$(\mu^{-1}\boldsymbol{B}_{h,t}^+,\boldsymbol{z}_h)+(\mathbf{curl}\,\boldsymbol{E}_h,\mu^{-1}\boldsymbol{z}_h)=0\quad\forall\,\boldsymbol{z}_h\in\mathbf{curl}\,\boldsymbol{U}_h. \tag{4.8}$$

In the same way, we can see that $\boldsymbol{B}_{h,t}^\perp(t)=0$ for any $t\in(0,T)$.

From (4.1), (4.2) and definitions of $\boldsymbol{Q}_h^{\boldsymbol{Z}}$ and $\boldsymbol{\pi}_h$ it follows that, for any $\boldsymbol{u}_h\in\boldsymbol{U}_h$,

$$\begin{aligned}(\mu^{-1}\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}_t,\mathbf{curl}\,\boldsymbol{u}_h)&=(\mu^{-1}\boldsymbol{B}_t,\mathbf{curl}\,\boldsymbol{u}_h)\\&=-(\mathbf{curl}\,\boldsymbol{E},\mu^{-1}\mathbf{curl}\,\boldsymbol{u}_h)=-(\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{E},\mu^{-1}\mathbf{curl}\,\boldsymbol{u}_h).\end{aligned}$$

Note that both $\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}_t$ and $\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{E}$ belong to $\mathbf{curl}\,\boldsymbol{U}_h$. This implies that

$$\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}_t+\mathbf{curl}\,\boldsymbol{\pi}_h\boldsymbol{E}=0.$$

By the above and (4.8), we have

$$(\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}_t-\boldsymbol{B}_{h,t}^+,\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h^+)_{\mu^{-1}}=-(\mathbf{curl}\,\boldsymbol{e}_h,\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}-\boldsymbol{B}_h^+)_{\mu^{-1}}. \tag{4.9}$$

Moreover, by the definition of $\boldsymbol{Q}_h^{\boldsymbol{Z}}$, (4.1), (4.7), and (4.8), we have

$$\begin{aligned}
&(\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_{h,t}, \boldsymbol{u}_h)_\varepsilon + (\boldsymbol{B}_h^+ - \boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}, \mathbf{curl}\,\boldsymbol{u}_h)_{\mu^{-1}} \\
&\quad = (\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_{h,t}, \boldsymbol{u}_h)_\varepsilon + (\boldsymbol{B}_h^+ - \boldsymbol{B}, \mathbf{curl}\,\boldsymbol{u}_h)_{\mu^{-1}} \\
&\quad = (\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_{h,t}, \boldsymbol{u}_h)_\varepsilon - (\boldsymbol{E}_t - \boldsymbol{E}_{h,t}, \boldsymbol{u}_h)_\varepsilon - (\sigma(\boldsymbol{E} - \boldsymbol{E}_h), \boldsymbol{u}_h) \\
&\quad = (\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_t, \boldsymbol{u}_h)_\varepsilon - (\sigma(\boldsymbol{E} - \boldsymbol{E}_h), \boldsymbol{u}_h),
\end{aligned}$$

for any $\boldsymbol{u}_h \in \boldsymbol{U}_h$. In particular, choosing $\boldsymbol{u}_h = \boldsymbol{e}_h$, we get

$$\begin{aligned}
&(\boldsymbol{e}_{h,t}, \boldsymbol{e}_h)_\varepsilon + (\boldsymbol{B}_h^+ - \boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B}, \mathbf{curl}\,\boldsymbol{e}_h)_{\mu^{-1}} \\
&\qquad = (\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_t, \boldsymbol{e}_h)_\varepsilon - (\sigma(\boldsymbol{E} - \boldsymbol{E}_h), \boldsymbol{e}_h).
\end{aligned} \tag{4.10}$$

Adding (4.9) and (4.10) together yields

$$\begin{aligned}
&\frac{d}{dt}\Big[(\boldsymbol{e}_h, \boldsymbol{e}_h)_\varepsilon + (\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B} - \boldsymbol{B}_h^+, \boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B} - \boldsymbol{B}_h^+)_{\mu^{-1}}\Big] \\
&\qquad = 2(\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_t, \boldsymbol{e}_h)_\varepsilon - 2(\sigma(\boldsymbol{E} - \boldsymbol{E}_h), \boldsymbol{e}_h).
\end{aligned}$$

Using $\boldsymbol{B}_{h,t}^\perp = 0$ and the orthogonal property between $\boldsymbol{B}_h^\perp$ and $\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B} - \boldsymbol{B}_h^+$, we have that

$$\frac{d}{dt}(\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B} - \boldsymbol{B}_h^+, \boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B} - \boldsymbol{B}_h^+)_{\mu^{-1}} = \frac{d}{dt}(\boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B} - \boldsymbol{B}_h, \boldsymbol{Q}_h^{\boldsymbol{Z}}\boldsymbol{B} - \boldsymbol{B}_h)_{\mu^{-1}},$$

and thus

$$\frac{d}{dt}\left[(\boldsymbol{e}_h, \boldsymbol{e}_h)_\varepsilon + (\boldsymbol{b}_h, \boldsymbol{b}_h)_{\mu^{-1}}\right] = 2(\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_t, \boldsymbol{e}_h)_\varepsilon - 2(\sigma(\boldsymbol{E} - \boldsymbol{E}_h), \boldsymbol{e}_h). \tag{4.11}$$

Integrating both sides of (4.11) over $[0, t]$ yields

$$\begin{aligned}
(\boldsymbol{e}_h, \boldsymbol{e}_h)_\varepsilon + (\boldsymbol{b}_h, \boldsymbol{b}_h)_{\mu^{-1}} &= (\boldsymbol{e}_h(0), \boldsymbol{e}_h(0))_\varepsilon + (\boldsymbol{b}_h(0), \boldsymbol{b}_h(0))_{\mu^{-1}} \\
&\quad + 2\int_0^t (\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_t, \boldsymbol{e}_h)_\varepsilon\, d\tau - 2\int_0^t (\sigma(\boldsymbol{E} - \boldsymbol{E}_h), \boldsymbol{\pi}_h\boldsymbol{E} - \boldsymbol{E}_h)\, d\tau \\
&\le C(h^{2s} + \|\boldsymbol{E}_0 - \boldsymbol{E}_h(0)\|^2 + \|\boldsymbol{B}_0 - \boldsymbol{B}_h(0)\|^2) \\
&\quad + 2\int_0^t (\boldsymbol{\pi}_h\boldsymbol{E}_t - \boldsymbol{E}_t, \boldsymbol{e}_h)_\varepsilon\, d\tau - 2\int_0^t (\sigma(\boldsymbol{E} - \boldsymbol{E}_h), \boldsymbol{\pi}_h\boldsymbol{E} - \boldsymbol{E})\, d\tau \\
&\le C(h^{2s} + \|\boldsymbol{E}_0 - \boldsymbol{E}_h(0)\|^2 + \|\boldsymbol{B}_0 - \boldsymbol{B}_h(0)\|^2) \\
&\quad + Ch^s\int_0^t (\|\boldsymbol{e}_h\| + \|\boldsymbol{E} - \boldsymbol{E}_h\|)\, d\tau \\
&\le C(h^{2s} + \|\boldsymbol{E}_0 - \boldsymbol{E}_h(0)\|^2 + \|\boldsymbol{B}_0 - \boldsymbol{B}_h(0)\|^2) + Ch^s\int_0^t \|\boldsymbol{e}_h\|\, d\tau.
\end{aligned}$$

In the last inequality we used (3.10) and the triangle inequality. It follows from Gronwall's inequality that

$$\|\boldsymbol{e}_h\| + \|\boldsymbol{b}_h\| \le C(h^s + \|\boldsymbol{E}_0 - \boldsymbol{E}_h(0)\| + \|\boldsymbol{B}_0 - \boldsymbol{B}_h(0)\|).$$

This completes the proof of (4.6). □

Remark 4.1. If the initial approximation $\boldsymbol{B}_h(0)$ is divergence free, the splitting of $\boldsymbol{B}_h(t)$ into $\boldsymbol{B}_h^+(t) + \boldsymbol{B}_h^\perp(t)$ in the proof is not necessary because of $\boldsymbol{B}_h^\perp(t) \equiv \mathbf{0}$. But the above theorem shows that the initial approximation $\boldsymbol{B}_h(0)$ does not need to be divergence free for the semidiscrete scheme (4.3) and (4.4) to result in good approximations of $\boldsymbol{E}$ and $\boldsymbol{B}$.

Acknowledgment

The author wants to thank Prof. J. Pasciak for his many discussions and suggestions which lead to a great improvement of the paper.

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