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REDUCING THE CONSTRUCTION COST OF THE COMPONENT-BY-COMPONENT CONSTRUCTION OF GOOD LATTICE RULES

J. DICK AND F. Y. KUO

Abstract. The construction of randomly shifted rank-1 lattice rules, where the number of points n is a prime number, has recently been developed by Sloan, Kuo and Joe for integration of functions in weighted Sobolev spaces and was extended by Kuo and Joe and by Dick to composite numbers. To construct d-dimensional rules, the shifts were generated randomly and the generating vectors were constructed component-by-component at a cost of $O(n^2d^2)$ operations. Here we consider the situation where n is the product of two distinct prime numbers p and q. We still generate the shifts randomly but we modify the algorithm so that the cost of constructing the, now two, generating vectors component-by-component is only $O(n(p+q)d^2)$ operations. This reduction in cost allows, in practice, construction of rules with millions of points. The rules constructed again achieve a worst-case strong tractability error bound, with a rate of convergence $O(p^{-1+\delta}q^{-1/2})$ for $\delta > 0$.

1. Introduction

In the recent Sloan, Kuo, and Joe paper [10], the d-dimensional integral

$$I_d(f) = \int_{[0,1]^d} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

for functions f belonging to certain weighted Sobolev spaces was approximated by a certain class of equal-weight quasi–Monte Carlo (QMC) quadrature rules, namely "randomly shifted rank-1 lattice rules":

$$R_{n,d}(f,\boldsymbol{\Delta}_1,\ldots,\boldsymbol{\Delta}_t) = \frac{1}{tn}\sum_{m=1}^{t}\sum_{i=1}^{n} f\left(\left\{\frac{i\boldsymbol{z}}{n}+\boldsymbol{\Delta}_m\right\}\right),$$

where $\boldsymbol{\Delta}_1,\ldots,\boldsymbol{\Delta}_t$ are t independent random "shifts" drawn from a uniform distribution on $[0,1]^d$. Here $\boldsymbol{z}$ is a d-dimensional integer vector called the "generating vector", and the braces around the vector indicate that we take the fractional part of each component of the vector.

The d-dimensional weighted Sobolev spaces considered in [10] are in fact tensor product reproducing kernel Hilbert spaces. (See [1] or [12] for properties of such spaces.) These weighted Sobolev spaces are parameterized by two sequences $\boldsymbol{\beta} =$

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$(\beta_1, \beta_2, \ldots)$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots)$ of "weights" satisfying

$$\frac{\gamma_1}{\beta_1} \geq \frac{\gamma_2}{\beta_2} \geq \cdots .$$

Many recent papers include analysis of integration in these spaces. (For example, see [12] and [13].) In particular, Sloan, Kuo, and Joe developed an algorithm in [9] for constructing shifted rank-1 lattice rules, which are rules of the form

$$Q_{n,d}(f) = \frac{1}{n} \sum_{i=1}^{n} f\left(\left\{\frac{i\boldsymbol{z}}{n} + \boldsymbol{\Delta}\right\}\right).$$

Both the shift $\boldsymbol{\Delta}$ and the generating vector $\boldsymbol{z}$ are constructed component-by-component, and the cost to construct such a rule up to d dimensions is $O(n^3 d^2)$ operations, with the number of points n taken to be a prime number. (The idea for such an algorithm originates from [11].) The construction was later generalized by Kuo and Joe in [8] to rules with a composite number of points. In [10], Sloan, Kuo and Joe used a number of random shifts to replace the construction of the single shift in [9] and as a result the cost of construction was reduced to $O(n^2 d^2)$ operations, with the additional advantage of a possible probabilistic error estimation. (See [4] for the key underlying concepts behind such randomization ideas.) In detail they showed that the generating vector $\boldsymbol{z}$ can be constructed one component at a time by minimizing the quantity

$$e_{n,d}^2(\boldsymbol{z}) = -\prod_{j=1}^{d} \left(\beta_j + \tfrac{\gamma_j}{3}\right) + \frac{1}{n} \sum_{i=1}^{n} \prod_{j=1}^{d} \left(\beta_j + \gamma_j \left[B_2\left(\left\{\tfrac{iz_j}{n}\right\}\right) + \tfrac{1}{3}\right]\right),$$

over the set $\mathbb{Z}_n^d = \{1, 2, \ldots, n-1\}^d$ with n being a prime number. Here $B_2(x) = x^2 - x + 1/6$ is the Bernoulli polynomial of degree 2. We shall call $e_{n,d}(\boldsymbol{z})$ the "worst-case error" of randomly shifted rank-1 lattice rules in weighted Sobolev spaces, as it is the worst-case error of rank-1 lattice rules in some related function spaces. The definition of worst-case error and the full details of why such quantity can be used as the selection criterion for $\boldsymbol{z}$ in a randomly shifted rank-1 lattice rule can be found in [10].

The motivation for this paper is to find an algorithm that reduces the cost of construction even further so it is possible to construct, in practice, rules with even a larger number of points. Instead of taking n to be a prime number, we choose n to be the product of two distinct prime numbers p and q. We are thus considering rank-1 lattice rules with points given by the set

$$\left\{\left\{\frac{i\boldsymbol{z}}{p} + \frac{k\boldsymbol{w}}{q}\right\} : 1 \leq i \leq p,\, 1 \leq k \leq q\right\},$$

where $\boldsymbol{z} \in \mathbb{Z}_p^d = \{1, 2, \ldots, p-1\}^d$ and $\boldsymbol{w} \in \mathbb{Z}_q^d = \{1, 2, \ldots, q-1\}^d$ are two generating vectors. This idea is not new. In 1960, Korobov pointed out in [6] the advantage of the decomposition $n = pq$ with $p \approx q^2$. This fact was later mentioned again in Hua and Wang's book [5] in 1981.

In Section 2 we derive the existence of a pair $(\boldsymbol{z}, \boldsymbol{w})$ such that $e_{p,q,d}^2(\boldsymbol{z}, \boldsymbol{w})$ satisfies a strong QMC tractability error bound, where strong QMC tractability means that the minimal number of function evaluations n in a quasi–Monte Carlo rule

$$Q_{n,d}(f) = \frac{1}{n} \sum_{i=1}^{n} f(\boldsymbol{x}_i)$$

needed to reduce the initial error $I_d(f)$ by a factor of $\varepsilon > 0$ is bounded by a polynomial in ε^{-1} independently of d. Moreover, we show that there exists a pair $(\boldsymbol{z}, \boldsymbol{w})$ such that the better rate of convergence $O(n^{-1+\delta})$ for any $\delta > 0$ can be achieved. The proofs in Section 2 are based on averaging arguments and are not constructive.

In Section 3 we show how the component-by-component algorithm of [10] can be modified to reduce the construction cost. The corresponding worst-case error for these modified rules with $n = pq$ satisfies

$$\begin{aligned} e^2_{p,q,d}(\boldsymbol{z}, \boldsymbol{w}) &= -\prod_{j=1}^{d} \left(\beta_j + \tfrac{\gamma_j}{3}\right) \\ &\quad + \frac{1}{pq} \sum_{i=1}^{p} \sum_{k=1}^{q} \prod_{j=1}^{d} \left(\beta_j + \gamma_j \left[B_2\left(\left\{\tfrac{iz_j}{p} + \tfrac{kw_j}{q}\right\}\right) + \tfrac{1}{3}\right]\right). \end{aligned} \tag{1.1}$$

The new algorithm constructs the vectors $\boldsymbol{z} = (z_1, \ldots, z_d)$ and $\boldsymbol{w} = (w_1, \ldots, w_d)$ component-by-component. We set z_1 and w_1 to 1, then for each s satisfying $2 \le s \le d$, z_s is found by minimizing the average of $e^2_{p,q,s}(\boldsymbol{z}, \boldsymbol{w})$ over all $w_s \in \{1, \ldots, q-1\}$; and then with this z_s fixed, w_s is obtained by minimizing $e^2_{p,q,s}(\boldsymbol{z}, \boldsymbol{w})$ (see Algorithm 3.1 for more details). The cost of construction for this algorithm is $O(n(p+q)d^2)$ operations. We show that with some minor restrictions on p and q, the square worst-case errors of rules constructed using Algorithm 3.1 are better than the QMC mean (see Lemma 8 of [12]):

$$E^2_{n,d} = \frac{1}{n}\left(\prod_{j=1}^{d}\left(\beta_j + \tfrac{\gamma_j}{2}\right) - \prod_{j=1}^{d}\left(\beta_j + \tfrac{\gamma_j}{3}\right)\right),$$

in weighted Sobolev spaces. We show also that the rules constructed using Algorithm 3.1 achieve the rate of convergence $O(p^{-1+\delta} q^{-1/2})$, for any $\delta > 0$. In the final section, Section 4, we outline numerical experiments and present the numerical results. We consider the performance of Algorithm 3.1 using different decompositions of n and various choices of weights.

We note that the reduction in cost of Algorithm 3.1 can be substantial: If $p \approx q \approx n^{1/2}$, then the cost is reduced from $O(n^2 d^2)$ to approximately $O(n^{3/2} d^2)$. If, for example, $n = 10^6$, then the cost is reduced by a factor of about a thousand. This reduction in cost allows the construction of rules with millions of points.

Throughout the paper, we will make use of the fact that for $x \in [0, 1]$,

$$B_2(x) = \frac{1}{2\pi^2} \sum_{h=-\infty}^{\infty}{}' \frac{e^{2\pi i h x}}{h^2},$$

where the prime on the sum indicates that we omit the $h = 0$ term. We will also make use of the Riemann zeta function

$$\zeta(k) = \sum_{h=1}^{\infty} \frac{1}{h^k}, \quad k > 1,$$

and in particular $\zeta(2) = \pi^2/6$.

2. The existence of a good randomly shifted rank-1 lattice rule

We will use the averaging argument which was used in various other papers. First we derive the expression for the mean square worst-case error over all generating vectors. We then argue the existence of good vectors with square worst-case error better than the mean.

We can separate the cases of $i = p$ and $k = q$ in the square worst-case error expression (1.1) and obtain

$$\begin{aligned} e^2_{p,q,d}(\boldsymbol{z},\boldsymbol{w}) &= -\prod_{j=1}^{d}\left(\beta_j + \tfrac{\gamma_j}{3}\right) + \frac{1}{pq}\prod_{j=1}^{d}\left(\beta_j + \tfrac{\gamma_j}{2}\right) \\ &+ \frac{1}{pq}\sum_{i=1}^{p-1}\prod_{j=1}^{d}\left(\beta_j + \gamma_j\left[B_2\left(\left\{\tfrac{iz_j}{p}\right\}\right) + \tfrac{1}{3}\right]\right) \\ &+ \frac{1}{pq}\sum_{k=1}^{q-1}\prod_{j=1}^{d}\left(\beta_j + \gamma_j\left[B_2\left(\left\{\tfrac{kw_j}{q}\right\}\right) + \tfrac{1}{3}\right]\right) \\ &+ \frac{1}{pq}\sum_{i=1}^{p-1}\sum_{k=1}^{q-1}\prod_{j=1}^{d}\left(\beta_j + \gamma_j\left[B_2\left(\left\{\tfrac{iz_j}{p} + \tfrac{kw_j}{q}\right\}\right) + \tfrac{1}{3}\right]\right). \end{aligned} \tag{2.1}$$

We define the mean of the square worst-case error:

$$M^2_{p,q,d} := \frac{1}{(p-1)^d(q-1)^d}\sum_{\boldsymbol{z}\in\mathbb{Z}_p^d}\sum_{\boldsymbol{w}\in\mathbb{Z}_q^d} e^2_{p,q,d}(\boldsymbol{z},\boldsymbol{w}).$$

Theorem 2.1. *Let p and q be two distinct prime numbers. We have*

$$\begin{aligned} M^2_{p,q,d} &= -\prod_{j=1}^{d}\left(\beta_j + \tfrac{\gamma_j}{3}\right) + \frac{1}{pq}\prod_{j=1}^{d}\left(\beta_j + \tfrac{\gamma_j}{2}\right) \\ &+ \frac{p-1}{pq}\prod_{j=1}^{d}\left(\beta_j + \gamma_j\left(\tfrac{1}{3} - \tfrac{1}{6p}\right)\right) \\ &+ \frac{q-1}{pq}\prod_{j=1}^{d}\left(\beta_j + \gamma_j\left(\tfrac{1}{3} - \tfrac{1}{6q}\right)\right) \\ &+ \frac{(p-1)(q-1)}{pq}\prod_{j=1}^{d}\left(\beta_j + \gamma_j\left(\tfrac{1}{3} + \tfrac{1}{6pq}\right)\right). \end{aligned}$$

This expression is an easy consequence of the following result:

Lemma 2.2. *Let p and q be two distinct prime numbers. For $i \neq p$ and $k \neq q$, we have*

$$\frac{1}{p-1}\sum_{z=1}^{p-1} B_2\left(\left\{\tfrac{iz}{p}\right\}\right) = -\frac{1}{6p},$$

$$\frac{1}{p-1}\sum_{z=1}^{p-1} B_2\left(\left\{\tfrac{iz}{p} + \tfrac{kw}{q}\right\}\right) = \frac{1}{p(p-1)}B_2\left(\left\{\tfrac{pkw}{q}\right\}\right) - \frac{1}{p-1}B_2\left(\left\{\tfrac{kw}{q}\right\}\right),$$

and

$$\frac{1}{(p-1)(q-1)}\sum_{z=1}^{p-1}\sum_{w=1}^{q-1} B_2\left(\left\{\frac{iz}{p}+\frac{kw}{q}\right\}\right)=\frac{1}{6pq}.$$

Proof. The first equation was used in several papers before. For completeness we include a proof here. For $i \neq p$, we have

$$\frac{1}{p-1}\sum_{z=1}^{p-1} B_2\left(\left\{\frac{iz}{p}\right\}\right)=\frac{1}{2\pi^2}\sum_{h=-\infty}^{\infty}{}' \left[\frac{1}{h^2}\left(\frac{1}{p-1}\sum_{z=1}^{p-1} e^{2\pi \mathrm{i} hiz/p}\right)\right].$$

Since

$$\frac{1}{p-1}\sum_{z=1}^{p-1} e^{2\pi \mathrm{i} hiz/p}=\begin{cases}1, & \text{if } h \text{ is a multiple of } p,\\ -\frac{1}{p-1}, & \text{otherwise,}\end{cases}$$

we have

$$\begin{aligned}
&\frac{1}{p-1}\sum_{z=1}^{p-1} B_2\left(\left\{\frac{iz}{p}\right\}\right)\\
&= \frac{1}{2\pi^2}\left[\sum_{h\equiv 0\,(\mathrm{mod}\ p)}{}' \frac{1}{h^2}-\frac{1}{p-1}\sum_{h\not\equiv 0\,(\mathrm{mod}\ p)}{}' \frac{1}{h^2}\right]\\
&= \frac{1}{2\pi^2}\left[\sum_{m=-\infty}^{\infty}{}' \frac{1}{m^2p^2}-\frac{1}{p-1}\left(\sum_{h=-\infty}^{\infty}{}' \frac{1}{h^2}-\sum_{m=-\infty}^{\infty}{}' \frac{1}{m^2p^2}\right)\right]\\
&= \frac{1}{2\pi^2}\left[\frac{2\zeta(2)}{p^2}-\frac{1}{p-1}\left(2\zeta(2)-\frac{2\zeta(2)}{p^2}\right)\right]=-\frac{1}{6p}.
\end{aligned}$$

Similarly we can obtain for $i \neq p$,

$$\begin{aligned}
&\frac{1}{p-1}\sum_{z=1}^{p-1} B_2\left(\left\{\frac{iz}{p}+\frac{kw}{q}\right\}\right)\\
&= \frac{1}{2\pi^2}\sum_{h=-\infty}^{\infty}{}' \left[\frac{e^{2\pi \mathrm{i} hkw/q}}{h^2}\left(\frac{1}{p-1}\sum_{z=1}^{p-1} e^{2\pi \mathrm{i} hiz/p}\right)\right]\\
&= \frac{1}{p(p-1)}B_2\left(\left\{\frac{pkw}{q}\right\}\right)-\frac{1}{p-1}B_2\left(\left\{\frac{kw}{q}\right\}\right).
\end{aligned}$$

Finally, it follows from the two results above that for $i \neq p$ and $k \neq q$,

$$\frac{1}{(p-1)(q-1)}\sum_{z=1}^{p-1}\sum_{w=1}^{q-1} B_2\left(\left\{\frac{iz}{p}+\frac{kw}{q}\right\}\right)=\frac{1}{6pq}.$$

This completes the proof. □

Now we find an upper bound for the mean $M_{p,q,d}^2$.

Theorem 2.3. *Let p and q be two distinct prime numbers. We have*

$$M_{p,q,d}^2 \;\leq\; \frac{1}{(p-1)(q-1)}\left(\prod_{j=1}^{d}\left(\beta_j+\frac{\gamma_j}{2}\right)-\prod_{j=1}^{d}\left(\beta_j+\frac{\gamma_j}{3}\right)\right).$$

Proof. Using the property that

$$\prod_{j=1}^{d}(b_j+a_j) = \sum_{\mathfrak{u}\subseteq\mathcal{D}}\left(\prod_{j\notin\mathfrak{u}} b_j \prod_{j\in\mathfrak{u}} a_j\right)$$

$$= \prod_{j=1}^{d} b_j + \sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{D}}\left(\prod_{j\notin\mathfrak{u}} b_j \prod_{j\in\mathfrak{u}} a_j\right), \tag{2.2}$$

where $\mathcal{D}=\{1,2,\ldots,d\}$, we can write $M_{p,q,d}^2$ from Theorem 2.1 as

$$M_{p,q,d}^2 = \frac{1}{pq}\sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{D}}\left(S(\mathfrak{u})\prod_{j\notin\mathfrak{u}}\left(\beta_j+\tfrac{\gamma_j}{3}\right)\prod_{j\in\mathfrak{u}}\left(\tfrac{\gamma_j}{6}\right)\right),$$

where

$$S(\mathfrak{u}) = 1+(p-1)\left(-\tfrac{1}{p}\right)^{|\mathfrak{u}|}+(q-1)\left(-\tfrac{1}{q}\right)^{|\mathfrak{u}|}+(p-1)(q-1)\left(\tfrac{1}{pq}\right)^{|\mathfrak{u}|}.$$

For $1\le|\mathfrak{u}|\le d$, if $|\mathfrak{u}|$ is even, we have

$$S(\mathfrak{u}) \le 1+\frac{p-1}{p^2}+\frac{q-1}{q^2}+\frac{(p-1)(q-1)}{p^2q^2} \le \frac{pq}{(p-1)(q-1)},$$

and if $|\mathfrak{u}|$ is odd, we have

$$S(\mathfrak{u}) = 1-\frac{p-1}{p^{|\mathfrak{u}|}}-\frac{q-1}{q^{|\mathfrak{u}|}}+\frac{(p-1)(q-1)}{p^{|\mathfrak{u}|}q^{|\mathfrak{u}|}} \le 1 \le \frac{pq}{(p-1)(q-1)}.$$

Thus

$$\begin{aligned} M_{p,q,d}^2 &\le \frac{1}{(p-1)(q-1)}\sum_{\emptyset\neq\mathfrak{u}\subseteq\mathcal{D}}\left(\prod_{j\notin\mathfrak{u}}\left(\beta_j+\tfrac{\gamma_j}{3}\right)\prod_{j\in\mathfrak{u}}\left(\tfrac{\gamma_j}{6}\right)\right) \\ &= \frac{1}{(p-1)(q-1)}\left(\prod_{j=1}^{d}\left(\beta_j+\tfrac{\gamma_j}{2}\right)-\prod_{j=1}^{d}\left(\beta_j+\tfrac{\gamma_j}{3}\right)\right). \end{aligned}$$

This completes the proof. □

We now write the square worst-case error as one sum. As we shall see later, this allows us to apply Jensen's inequality (see Theorem 19 of [3]), which states that for $\{a_i\}$ a sequence of positive numbers,

$$\sum a_i \le \left(\sum a_i^{\lambda}\right)^{\frac{1}{\lambda}} \quad \text{for } 0<\lambda\le 1.$$

Lemma 2.4. *We can write*

$$e_{p,q,d}^2(\boldsymbol{z},\boldsymbol{w}) = \sideset{}{'}\sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d \\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \,(\mathrm{mod}\, p) \\ \boldsymbol{h}\cdot\boldsymbol{w}\equiv 0 \,(\mathrm{mod}\, q)}} \prod_{j=1}^{d} r\left(2,\beta_j+\tfrac{\gamma_j}{3},\tfrac{\gamma_j}{2\pi^2},h_j\right),$$

where

$$r(\alpha,\beta,\gamma,h) = \begin{cases}\beta & \text{if } h=0,\\ \gamma|h|^{-\alpha} & \text{if } h\neq 0,\end{cases}$$

and the prime on the sum means that we omit $\boldsymbol{h}=(0,\ldots,0)$.

Proof. We can rewrite (1.1) as follows:

$$\begin{aligned} &e^2_{p,q,d}(\boldsymbol{z},\boldsymbol{w}) \\ &= -\prod_{j=1}^{d}\left(\beta_j+\tfrac{\gamma_j}{3}\right) \\ &\quad +\frac{1}{pq}\sum_{i=1}^{p}\sum_{k=1}^{q}\prod_{j=1}^{d}\left(\sum_{h=-\infty}^{\infty} e^{2\pi \mathrm{i} h(iz_j/p+kw_j/q)}\, r\left(2,\beta_j+\tfrac{\gamma_j}{3},\tfrac{\gamma_j}{2\pi^2},h\right)\right) \\ &= \sum_{\boldsymbol{h}\in\mathbb{Z}^d}{}' \left[\left(\frac{1}{p}\sum_{i=1}^{p} e^{2\pi \mathrm{i} i\boldsymbol{h}\cdot\boldsymbol{z}/p}\right)\left(\frac{1}{q}\sum_{k=1}^{q} e^{2\pi \mathrm{i} k\boldsymbol{h}\cdot\boldsymbol{w}/q}\right) \right. \\ &\qquad\qquad \left. \times\prod_{j=1}^{d} r\left(2,\beta_j+\tfrac{\gamma_j}{3},\tfrac{\gamma_j}{2\pi^2},h_j\right)\right]. \end{aligned}$$

The result now follows from the fact that $\sum_{i=1}^{p} e^{2\pi \mathrm{i} i\boldsymbol{h}\cdot\boldsymbol{z}/p}$ is p if $\boldsymbol{h}\cdot\boldsymbol{z}$ is a multiple of p and 0 otherwise, and similarly $\sum_{k=1}^{q} e^{2\pi \mathrm{i} k\boldsymbol{h}\cdot\boldsymbol{w}/q}$ is q if $\boldsymbol{h}\cdot\boldsymbol{w}$ is a multiple of q and 0 otherwise. □

The following theorem is a consequence of Theorem 2.3.

Theorem 2.5. *Let p and q be two distinct prime numbers.*

(a) *There exist $\boldsymbol{z}\in\mathbb{Z}_p^d$ and $\boldsymbol{w}\in\mathbb{Z}_q^d$ such that*

$$e^2_{p,q,d}(\boldsymbol{z},\boldsymbol{w}) \le \frac{1}{(p-1)(q-1)}\left(\prod_{j=1}^{d}\left(\beta_j+\tfrac{\gamma_j}{2}\right)-\prod_{j=1}^{d}\left(\beta_j+\tfrac{\gamma_j}{3}\right)\right).$$

(b) *There exist $\boldsymbol{z}\in\mathbb{Z}_p^d$ and $\boldsymbol{w}\in\mathbb{Z}_q^d$ such that*

$$e^2_{p,q,d}(\boldsymbol{z},\boldsymbol{w}) \le (p-1)^{-\frac{1}{\lambda}}(q-1)^{-\frac{1}{\lambda}}\prod_{j=1}^{d}\left(\left(\beta_j+\tfrac{\gamma_j}{3}\right)^{\lambda}+2\zeta(2\lambda)\left(\tfrac{\gamma_j}{2\pi^2}\right)^{\lambda}\right)^{\frac{1}{\lambda}}$$

for all λ satisfying $\frac{1}{2}<\lambda\le 1$.

Proof. Clearly there exist $\boldsymbol{z}\in\mathbb{Z}_p^d$ and $\boldsymbol{w}\in\mathbb{Z}_q^d$ such that $e^2_{p,q,d}(\boldsymbol{z},\boldsymbol{w})\le M_{p,q,d}$. Thus result (a) follows from Theorem 2.3.

To prove (b), let $\alpha=2$, $\bar{\boldsymbol{\beta}}=(\bar\beta_j)=\left(\beta_j+\frac{\gamma_j}{3}\right)$, and $\bar{\boldsymbol{\gamma}}=(\bar\gamma_j)=\left(\frac{\gamma_j}{2\pi^2}\right)$. First, it follows from Lemma 2.4 that

$$e^2_{p,q,d}(\boldsymbol{z},\boldsymbol{w}) = e^2_{p,q,d}(\alpha,\bar{\boldsymbol{\beta}},\bar{\boldsymbol{\gamma}};\boldsymbol{z},\boldsymbol{w}) = \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^d\\ \boldsymbol{h}\cdot\boldsymbol{z}\equiv 0\ (\mathrm{mod}\ p)\\ \boldsymbol{h}\cdot\boldsymbol{w}\equiv 0\ (\mathrm{mod}\ q)}}{}'\prod_{j=1}^{d} r\left(\alpha,\bar\beta_j,\bar\gamma_j,h_j\right).$$

Using Jensen's inequality and the property

$$[r(\alpha,\beta,\gamma,h)]^{\lambda} = r(\alpha\lambda,\beta^{\lambda},\gamma^{\lambda},h),$$

we can show that

$$e^2_{p,q,d}(\alpha,\bar{\boldsymbol{\beta}},\bar{\boldsymbol{\gamma}};\boldsymbol{z},\boldsymbol{w}) \le \left[e^2_{p,q,d}(\alpha\lambda,\bar{\boldsymbol{\beta}}^{\lambda},\bar{\boldsymbol{\gamma}}^{\lambda};\boldsymbol{z},\boldsymbol{w})\right]^{\frac{1}{\lambda}} \tag{2.3}$$

for all $\frac{1}{2} < \lambda \le 1$. Second, it follows from (a) that there exist $\boldsymbol{z} \in \mathbb{Z}_p^d$ and $\boldsymbol{w} \in \mathbb{Z}_q^d$ such that

$$e^2_{p,q,d}(\alpha, \bar{\boldsymbol{\beta}}, \bar{\boldsymbol{\gamma}}; \boldsymbol{z}, \boldsymbol{w}) \le \frac{1}{(p-1)(q-1)} \prod_{j=1}^{d} \left(\bar{\beta}_j + 2\zeta(\alpha)\bar{\gamma}_j\right).$$

From this with a change of parameters, we see that there exist $\boldsymbol{z} \in \mathbb{Z}_p^d$ and $\boldsymbol{w} \in \mathbb{Z}_q^d$ such that

$$e^2_{p,q,d}(\alpha\lambda, \bar{\boldsymbol{\beta}}^\lambda, \bar{\boldsymbol{\gamma}}^\lambda; \boldsymbol{z}, \boldsymbol{w}) \le \frac{1}{(p-1)(q-1)} \prod_{j=1}^{d} \left(\bar{\beta}_j^\lambda + 2\zeta(\alpha\lambda)\bar{\gamma}_j^\lambda\right). \tag{2.4}$$

The result (b) now follows from (2.3) and (2.4). □

The following theorem shows the existence of a pair $(\boldsymbol{z}, \boldsymbol{w})$ that achieves strong QMC tractability under a certain condition on the weights.

Theorem 2.6. *There exists a pair* $(\boldsymbol{z}, \boldsymbol{w})$ *such that*

$$e_{p,q,d}(\boldsymbol{z}, \boldsymbol{w}) \le C_d(\delta)\, n^{-1+\delta} e_{0,d}, \quad \textit{for all} \quad 0 < \delta \le \tfrac{\alpha-1}{2},$$

where $C_d(\delta)$ *is independent of* n*. Moreover, if*

$$\sum_{j=1}^{\infty} \left(\frac{\gamma_j}{\beta_j}\right)^{\frac{1}{2(1-\delta)}} < \infty,$$

then

$$C_d(\delta) \le C_\infty(\delta) < \infty;$$

that is, $e_{p,q,d}(\boldsymbol{z}, \boldsymbol{w})/e_{0,d}$ *is* $O(n^{-1+\delta})$ *for* $\delta > 0$*, independently of* d*.*

Proof. The initial error in the weighted Sobolev spaces is

$$e_{0,d} = \prod_{j=1}^{d} \left(\beta_j + \tfrac{\gamma_j}{3}\right)^{\frac{1}{2}}.$$

It follows from (b) in Theorem 2.5 that there exists a pair $(\boldsymbol{z}, \boldsymbol{w})$ such that

$$\begin{aligned}
e_{p,q,d}(\boldsymbol{z}, \boldsymbol{w}) &\le [(p-1)(q-1)]^{-\frac{1}{2\lambda}} \prod_{j=1}^{d} \left(\left(\beta_j + \tfrac{\gamma_j}{3}\right)^\lambda + 2\zeta(2\lambda)\left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda\right)^{\frac{1}{2\lambda}} \\
&\le 3^{\frac{1}{2\lambda}} n^{-\frac{1}{2\lambda}} \prod_{j=1}^{d} \left(1 + \frac{2\zeta(2\lambda)\left(\frac{\gamma_j}{2\pi^2}\right)^\lambda}{\left(\beta_j + \frac{\gamma_j}{3}\right)^\lambda}\right)^{\frac{1}{2\lambda}} \prod_{j=1}^{d} \left(\beta_j + \tfrac{\gamma_j}{3}\right)^{\frac{1}{2}} \\
&\le 3^{\frac{1}{2\lambda}} n^{-\frac{1}{2\lambda}} \prod_{j=1}^{d} \left(1 + 2(2\pi^2)^{-\lambda}\zeta(2\lambda)\left(\frac{\gamma_j}{\beta_j}\right)^\lambda\right)^{\frac{1}{2\lambda}} e_{0,d},
\end{aligned}$$

for all $\frac{1}{2} < \lambda \le 1$. Now with the substitution of

$$-1 + \delta = -\frac{1}{2\lambda},$$

the condition $\frac{1}{2} < \lambda \le 1$ becomes $0 < \delta \le \frac{1}{2}$ and we obtain

$$e_{p,q,d}(\boldsymbol{z}, \boldsymbol{w}) \le C_d(\delta)\, n^{-1+\delta} e_{0,d} \quad \text{for all } 0 < \delta \le \tfrac{1}{2},$$

where

$$C_d(\delta) = 3^{1-\delta} \prod_{j=1}^{d} \left[1 + 2(2\pi^2)^{-\frac{1}{2(1-\delta)}} \zeta\left(\tfrac{1}{1-\delta}\right) \left(\tfrac{\gamma_j}{\beta_j}\right)^{\frac{1}{2(1-\delta)}} \right]^{1-\delta} \le C_\infty(\delta),$$

and

$$\begin{aligned} &C_\infty(\delta) \\ &= 3^{1-\delta} \exp\left((1-\delta) \sum_{j=1}^{\infty} \log\left(1 + 2(2\pi^2)^{-\frac{1}{2(1-\delta)}} \zeta\left(\tfrac{1}{1-\delta}\right) \left(\tfrac{\gamma_j}{\beta_j}\right)^{\frac{1}{2(1-\delta)}} \right) \right) \\ &\le 3^{1-\delta} \exp\left((1-\delta) 2(2\pi^2)^{-\frac{1}{2(1-\delta)}} \zeta\left(\tfrac{1}{1-\delta}\right) \sum_{j=1}^{\infty} \left(\tfrac{\gamma_j}{\beta_j}\right)^{\frac{1}{2(1-\delta)}} \right), \end{aligned}$$

where we have used the fact that $\log(1+x) \le x$ for $x \ge 0$. It is clear from this expression that for $0 < \delta \le \frac{1}{2}$, $C_\infty(\delta) < \infty$ provided

$$\sum_{j=1}^{\infty} \left(\frac{\gamma_j}{\beta_j}\right)^{\frac{1}{2(1-\delta)}} < \infty.$$

This completes the proof. □

3. Component-by-component construction of good randomly shifted rank-1 lattice rules

We want to construct two d-dimensional vectors $\boldsymbol{z} \in \mathbb{Z}_p^d$ and $\boldsymbol{w} \in \mathbb{Z}_q^d$ component-by-component. For each s satisfying $2 \le s \le d$, we can write (see (2.1))

$$\begin{aligned} &e^2_{p,q,s}((z_1,\ldots,z_s),(w_1,\ldots,w_s)) \\ &= \left(\beta_s + \tfrac{\gamma_s}{3}\right) e^2_{p,q,s-1}((z_1,\ldots,z_{s-1}),(w_1,\ldots,w_{s-1})) + \frac{\gamma_s}{6pq} \prod_{j=1}^{s-1} \left(\beta_j + \tfrac{\gamma_j}{2}\right) \\ &\quad + \frac{\gamma_s}{pq} \sum_{i=1}^{p-1} \left[\prod_{j=1}^{s-1} \left(\beta_j + \gamma_j \left[B_2\left(\left\{\tfrac{iz_j}{p}\right\}\right) + \tfrac{1}{3} \right] \right) B_2\left(\left\{\tfrac{iz_s}{p}\right\}\right) \right] \\ &\quad + \frac{\gamma_s}{pq} \sum_{k=1}^{q-1} \left[\prod_{j=1}^{s-1} \left(\beta_j + \gamma_j \left[B_2\left(\left\{\tfrac{kw_j}{q}\right\}\right) + \tfrac{1}{3} \right] \right) B_2\left(\left\{\tfrac{kw_s}{q}\right\}\right) \right] \\ &\quad + \frac{\gamma_s}{pq} \sum_{i=1}^{p-1} \sum_{k=1}^{q-1} \left[\prod_{j=1}^{s-1} \left(\beta_j + \gamma_j \left[B_2\left(\left\{\tfrac{iz_j}{p} + \tfrac{kw_j}{q}\right\}\right) + \tfrac{1}{3} \right] \right) \right. \\ &\qquad\qquad \left. \times B_2\left(\left\{\tfrac{iz_s}{p} + \tfrac{kw_s}{q}\right\}\right) \right]. \end{aligned}$$

We define the mean of $e^2_{p,q,s}((z_1,\ldots,z_s),(w_1,\ldots,w_s))$ over all $w_s \in \mathbb{Z}_q$ by

$$\begin{aligned} &\theta^2_{p,q,s}((z_1,\ldots,z_{s-1}),(w_1,\ldots,w_{s-1}); z_s) \\ &:= \frac{1}{q-1} \sum_{w_s=1}^{q-1} e^2_{p,q,s}((z_1,\ldots,z_s),(w_1,\ldots,w_s)). \end{aligned}$$

The explicit expression for $\theta^2_{p,q,s}$ (see (3.1) below) can be derived using Lemma 2.2.

We present an algorithm for constructing $\boldsymbol{z}$ and $\boldsymbol{w}$ component-by-component. For each s satisfying $2 \le s \le d$, z_s is found by minimizing over the quantity $\theta^2_{p,q,s}$, and then with this z_s fixed, w_s is found by minimizing over the square worst-case error. Since $B_2(\{x\}) = B_2(1-\{x\})$, we have

$$\begin{aligned} &\theta^2_{p,q,s}((z_1,\ldots,z_{s-1}),(w_1,\ldots,w_{s-1});z_s)\\ = \quad &\theta^2_{p,q,s}((z_1,\ldots,z_{s-1}),(w_1,\ldots,w_{s-1});p-z_s), \end{aligned}$$

and thus the search of z_s can be reduced to the set $\{1,2,\ldots,\frac{p-1}{2}\}$.

Algorithm 3.1 (Partial Search)**.** *Given two distinct prime numbers p and q:*

1. *Set z_1 and w_1, the first components of $\boldsymbol{z}$ and $\boldsymbol{w}$, to* 1.
2. *For $s = 2,\ldots,d$, do the following:*
   (a) *Find $z_s \in \{1,2,\ldots,\frac{p-1}{2}\}$ to minimize*

$$\begin{aligned} &\theta^2_{p,q,s}((z_1,\ldots,z_{s-1}),(w_1,\ldots,w_{s-1});z_s)\\ &= \left(\beta_s + \tfrac{\gamma_s}{3}\right) e^2_{p,q,s-1}((z_1,\ldots,z_{s-1}),(w_1,\ldots,w_{s-1})) + \frac{\gamma_s}{6pq}\prod_{j=1}^{s-1}\left(\beta_j + \tfrac{\gamma_j}{2}\right)\\ &\quad + \frac{\gamma_s}{pq}\sum_{i=1}^{p-1}\left[\prod_{j=1}^{s-1}\left(\beta_j + \gamma_j\left[B_2\left(\left\{\tfrac{iz_j}{p}\right\}\right) + \tfrac{1}{3}\right]\right) B_2\left(\left\{\tfrac{iz_s}{p}\right\}\right)\right]\\ &\quad - \frac{\gamma_s}{6pq^2}\sum_{k=1}^{q-1}\prod_{j=1}^{s-1}\left(\beta_j + \gamma_j\left[B_2\left(\left\{\tfrac{kw_j}{q}\right\}\right) + \tfrac{1}{3}\right]\right)\\ &\quad + \frac{\gamma_s}{pq}\sum_{i=1}^{p-1}\sum_{k=1}^{q-1}\left[\prod_{j=1}^{s-1}\left(\beta_j + \gamma_j\left[B_2\left(\left\{\tfrac{iz_j}{p} + \tfrac{kw_j}{q}\right\}\right) + \tfrac{1}{3}\right]\right)\right.\\ &\quad\quad \left.\times\left(\tfrac{1}{q(q-1)}B_2\left(\left\{\tfrac{qiz_s}{p}\right\}\right) - \tfrac{1}{q-1}B_2\left(\left\{\tfrac{iz_s}{p}\right\}\right)\right)\right]. \end{aligned} \tag{3.1}$$

   (b) *Find $w_s \in \{1,2,\ldots,q-1\}$ to minimize*

$$\begin{aligned} &e^2_{p,q,s}((z_1,\ldots,z_s),(w_1,\ldots,w_s))\\ &= -\prod_{j=1}^{s}\left(\beta_j + \tfrac{\gamma_j}{3}\right) + \frac{1}{pq}\sum_{i=1}^{p}\sum_{k=1}^{q}\prod_{j=1}^{s}\left(\beta_j + \gamma_j\left[B_2\left(\left\{\tfrac{iz_j}{p} + \tfrac{kw_j}{q}\right\}\right) + \tfrac{1}{3}\right]\right). \end{aligned}$$

For each s satisfying $2 \le s \le d$, the search for z_s requires $O(p^2qs)$ operations and the search for w_s requires $O(pq^2s)$ operations, with a total of $O(n(p+q)s)$ operations. Thus the cost for constructing an n-point rule up to dimension d is $O(n(p+q)d^2)$ operations. Similar to other component-by-component construction algorithms, this cost can be reduced to $O(n(p+q)d)$ at the expense of $O(n)$ storage.

The following theorem shows that with some minor restrictions on p and q, the randomly shifted rank-1 lattice rules constructed by Algorithm 3.1 are in fact better than average QMC rules.

Theorem 3.2. *Let $n = pq$ where p and q are two distinct prime numbers, and let $\hat{\boldsymbol{z}} \in \mathbb{Z}_p^d$ and $\hat{\boldsymbol{w}} \in \mathbb{Z}_q^d$ be constructed using Algorithm 3.1. If*

$$p,\, q \geq 2\exp\left(\frac{1}{6}\sum_{j=1}^{\infty}\frac{\gamma_j}{\beta_j}\right),$$

then for each $s = 1, 2, \ldots, d$, we have

$$\begin{aligned} & e_{p,q,s}^2((1, \hat{z}_2, \ldots, \hat{z}_s), (1, \hat{w}_2, \ldots, \hat{w}_s)) \\ & \quad \leq \frac{1}{pq}\left(\prod_{j=1}^{s}\left(\beta_j + \tfrac{\gamma_j}{2}\right) - \prod_{j=1}^{s}\left(\beta_j + \tfrac{\gamma_j}{3}\right)\right) = E_{n,s}^2. \end{aligned}$$

Proof. For $s = 1$, there is only one n-point lattice rule: the n-point rectangle rule. Thus we may take $z_1 = 1$ and $w_1 = 1$ to obtain

$$e_{p,q,1}^2(1,1) = M_{p,q,1}^2 = \frac{\gamma_1}{6p^2q^2} \leq \frac{\gamma_1}{6n} = E_{n,1}^2,$$

where the third to last expression follows from Theorem 2.1. Hence the result is true for $s = 1$.

For each s satisfying $2 \leq s \leq d$, suppose that two $(s-1)$-dimensional vectors $(1, \hat{z}_2, \ldots, \hat{z}_{s-1})$ and $(1, \hat{w}_2, \ldots, \hat{w}_{s-1})$ have already been constructed using Algorithm 3.1 and they satisfy

$$\begin{aligned} & e_{p,q,s-1}^2((1, \hat{z}_2, \ldots, \hat{z}_{s-1}), (1, \hat{w}_2, \ldots, \hat{w}_{s-1})) \\ & \quad \leq \frac{1}{pq}\left(\prod_{j=1}^{s-1}\left(\beta_j + \tfrac{\gamma_j}{2}\right) - \prod_{j=1}^{s-1}\left(\beta_j + \tfrac{\gamma_j}{3}\right)\right) = E_{n,s-1}^2. \end{aligned} \tag{3.2}$$

Following step 2 of Algorithm 3.1, we choose $\hat{z}_s \in \mathbb{Z}_p$ to minimize $\theta_{p,q,s}^2$ (which is the average of $e_{p,q,s}^2$ over all w_s), and then with this $\hat{z}_s$ fixed, we choose $\hat{w}_s \in \mathbb{Z}_q$ to minimize $e_{p,q,s}^2$. Thus these choices of $\hat{z}_s$ and $\hat{w}_s$ satisfy

$$\begin{aligned} & e_{p,q,s}^2((1, z_2, \ldots, \hat{z}_s), (1, w_2, \ldots, \hat{w}_s)) \\ & \quad \leq \theta_{p,q,s}^2((1, z_2, \ldots, z_{s-1}), (1, w_2, \ldots, w_{s-1}); \hat{z}_s) \\ & \quad \leq \frac{1}{p-1}\sum_{z_s=1}^{p-1}\theta_{p,q,s}^2((1, z_2, \ldots, z_{s-1}), (1, w_2, \ldots, w_{s-1}); z_s). \end{aligned} \tag{3.3}$$

The result is proved if we can show that this last expression is bounded by $E_{n,s}^2$.

It follows from (3.1) and Lemma 2.2 that

$$\begin{aligned}
&\frac{1}{p-1}\sum_{z_s=1}^{p-1}\theta_{p,q,s}^2((1,z_2,\ldots,z_{s-1}),(1,w_2,\ldots,w_{s-1});z_s)\\
&= \left(\beta_s+\tfrac{\gamma_s}{3}\right)e_{p,q,s-1}^2((1,z_2,\ldots,z_{s-1}),(1,w_2,\ldots,w_{s-1}))\\
&\quad+\frac{\gamma_s}{6pq}\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{2}\right)-\frac{\gamma_s}{6p^2q}\sum_{i=1}^{p-1}\prod_{j=1}^{s-1}\left(\beta_j+\gamma_j\left[B_2\left(\left\{\tfrac{iz_j}{p}\right\}\right)+\tfrac{1}{3}\right]\right)\\
&\quad-\frac{\gamma_s}{6pq^2}\sum_{k=1}^{q-1}\prod_{j=1}^{s-1}\left(\beta_j+\gamma_j\left[B_2\left(\left\{\tfrac{kw_j}{q}\right\}\right)+\tfrac{1}{3}\right]\right)\\
&\quad+\frac{\gamma_s}{6p^2q^2}\sum_{i=1}^{p-1}\sum_{k=1}^{q-1}\prod_{j=1}^{s-1}\left(\beta_j+\gamma_j\left[B_2\left(\left\{\tfrac{iz_j}{p}+\tfrac{kw_j}{q}\right\}\right)+\tfrac{1}{3}\right]\right)\\
&\le \left(\beta_s+\tfrac{\gamma_s}{3}\right)e_{p,q,s-1}^2((1,z_2,\ldots,z_{s-1}),(1,w_2,\ldots,w_{s-1}))\\
&\quad+\frac{\gamma_s}{6pq}\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{2}\right)+\frac{\gamma_s}{6p^2q^2}\times G, \qquad (3.4)
\end{aligned}$$

where

$$\begin{aligned}
G = \sum_{i=1}^{p-1}\sum_{k=1}^{q-1}\Bigg[&\prod_{j=1}^{s-1}\left(\beta_j+\gamma_j\left[B_2\left(\left\{\tfrac{iz_j}{p}+\tfrac{kw_j}{q}\right\}\right)+\tfrac{1}{3}\right]\right)\\
&-\prod_{j=1}^{s-1}\left(\beta_j+\gamma_j\left[B_2\left(\left\{\tfrac{iz_j}{p}\right\}\right)+\tfrac{1}{3}\right]\right)\\
&-\prod_{j=1}^{s-1}\left(\beta_j+\gamma_j\left[B_2\left(\left\{\tfrac{kw_j}{q}\right\}\right)+\tfrac{1}{3}\right]\right)\Bigg].
\end{aligned}$$

Later we will show that $G \le 0$ under the assumption

$$p,\, q \ge 2\exp\left(\frac{1}{6}\sum_{j=1}^{\infty}\frac{\gamma_j}{\beta_j}\right),$$

after which upon combining (3.2), (3.3) and (3.4), it will follow that

$$\begin{aligned}
&e_{p,q,s}^2((1,\hat{z}_2,\ldots,\hat{z}_s),(1,\hat{w}_2,\ldots,\hat{w}_s))\\
&\le \left(\beta_s+\tfrac{\gamma_s}{3}\right)\frac{1}{pq}\left(\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{2}\right)-\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{3}\right)\right)+\frac{\gamma_s}{6pq}\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{2}\right)\\
&= \frac{1}{pq}\left(\prod_{j=1}^{s}\left(\beta_j+\tfrac{\gamma_j}{2}\right)-\prod_{j=1}^{s}\left(\beta_j+\tfrac{\gamma_j}{3}\right)\right)=E_{n,s}^2.
\end{aligned}$$

It will then follow inductively that the result is true for all $s = 1, 2, \ldots, d$.

To prove that $G \le 0$, we use property (2.2) to rewrite G as

$$G = -(p-1)(q-1)\prod_{j=1}^{s-1}\left(\beta_j + \tfrac{\gamma_j}{3}\right) + \sum_{\emptyset \neq \mathfrak{u} \subseteq \{1,2,\ldots,s-1\}} \left[\prod_{j \notin \mathfrak{u}}\left(\beta_j + \tfrac{\gamma_j}{3}\right)\prod_{j \in \mathfrak{u}} \gamma_j \times H_{\mathfrak{u}}\right],$$

where

$$\begin{aligned} H_{\mathfrak{u}} &= \sum_{i=1}^{p-1}\sum_{k=1}^{q-1}\left(\prod_{j\in\mathfrak{u}} B_2\left(\left\{\tfrac{iz_j}{p} + \tfrac{kw_j}{q}\right\}\right)\right. \\ &\qquad\qquad \left. - \prod_{j\in\mathfrak{u}} B_2\left(\left\{\tfrac{iz_j}{p}\right\}\right) - \prod_{j\in\mathfrak{u}} B_2\left(\left\{\tfrac{kw_j}{q}\right\}\right)\right) \\ &= \frac{1}{(2\pi^2)^{|\mathfrak{u}|}}\sum_{i=1}^{p-1}\sum_{k=1}^{q-1}\left[\prod_{j\in\mathfrak{u}}\sum_{h=-\infty}^{\infty}{}' \frac{e^{2\pi \mathrm{i} h(iz_j/p + kw_j/q)}}{h^2}\right. \\ &\qquad\qquad \left. - \prod_{j\in\mathfrak{u}}\sum_{h=-\infty}^{\infty}{}' \frac{e^{2\pi \mathrm{i} h i z_j/p}}{h^2} - \prod_{j\in\mathfrak{u}}\sum_{h=-\infty}^{\infty}{}' \frac{e^{2\pi \mathrm{i} h k w_j/q}}{h^2}\right]. \end{aligned}$$

Now let $\boldsymbol{z}_{\mathfrak{u}}$ denote the $|\mathfrak{u}|$-dimensional vector containing those components of $\boldsymbol{z}$ whose indices belong to $\mathfrak{u}$. Then we can rewrite $H_{\mathfrak{u}}$ as

$$\begin{aligned} H_{\mathfrak{u}} &= \frac{1}{(2\pi^2)^{|\mathfrak{u}|}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^{|\mathfrak{u}|} \\ h_j \neq 0\,\forall j}} \left[\frac{1}{h_1^2\cdots h_{|\mathfrak{u}|}^2}\sum_{i=1}^{p-1}\sum_{k=1}^{q-1}\left(e^{2\pi\mathrm{i}(i\boldsymbol{h}\cdot\boldsymbol{z}_{\mathfrak{u}}/p + k\boldsymbol{h}\cdot\boldsymbol{w}_{\mathfrak{u}}/q)}\right.\right. \\ &\qquad\qquad \left.\left. - e^{2\pi\mathrm{i}i\boldsymbol{h}\cdot\boldsymbol{z}_{\mathfrak{u}}/p} - e^{2\pi\mathrm{i}k\boldsymbol{h}\cdot\boldsymbol{w}_{\mathfrak{u}}/q}\right)\right]. \end{aligned}$$

It can be shown that

$$\begin{aligned} &\sum_{i=1}^{p-1}\sum_{k=1}^{q-1}\left(e^{2\pi\mathrm{i}(i\boldsymbol{h}\cdot\boldsymbol{z}_{\mathfrak{u}}/p + k\boldsymbol{h}\cdot\boldsymbol{w}_{\mathfrak{u}}/q)} - e^{2\pi\mathrm{i}i\boldsymbol{h}\cdot\boldsymbol{z}_{\mathfrak{u}}/p} - e^{2\pi\mathrm{i}k\boldsymbol{h}\cdot\boldsymbol{w}_{\mathfrak{u}}/q}\right) \\ &\quad = \begin{cases} p+q-1, & \text{if } \boldsymbol{h}\cdot\boldsymbol{z}_{\mathfrak{u}} \not\equiv 0 \pmod{p} \text{ and } \boldsymbol{h}\cdot\boldsymbol{w}_{\mathfrak{u}} \not\equiv 0 \pmod{q}, \\ -(p-1)(q-1) \le p+q-1, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus

$$H_{\mathfrak{u}} \le \frac{p+q-1}{(2\pi^2)^{|\mathfrak{u}|}} \sum_{\substack{\boldsymbol{h}\in\mathbb{Z}^{|\mathfrak{u}|} \\ h_j \neq 0\,\forall j}} \frac{1}{h_1^2\cdots h_{|\mathfrak{u}|}^2} = \frac{p+q-1}{(2\pi^2)^{|\mathfrak{u}|}}\left(2\zeta(2)\right)^{|\mathfrak{u}|} = \frac{p+q-1}{6^{|\mathfrak{u}|}},$$

which leads to

$$\begin{aligned} G &\leq -(p-1)(q-1)\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{3}\right) \\ &\quad +(p+q-1)\sum_{\emptyset\neq\mathfrak{u}\subseteq\{1,2,\ldots,s-1\}}\left[\prod_{j\notin\mathfrak{u}}\left(\beta_j+\tfrac{\gamma_j}{3}\right)\prod_{j\in\mathfrak{u}}\left(\tfrac{\gamma_j}{6}\right)\right] \\ &= -(p-1)(q-1)\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{3}\right) \\ &\quad +(p+q-1)\left(\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{2}\right)-\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{3}\right)\right) \\ &= -pq\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{3}\right)+(p+q-1)\prod_{j=1}^{s-1}\left(\beta_j+\tfrac{\gamma_j}{2}\right). \end{aligned}$$

Now consider

$$R=\frac{(p+q-1)\prod_{j=1}^{s-1}\left(\beta_j+\frac{\gamma_j}{2}\right)}{pq\prod_{j=1}^{s-1}\left(\beta_j+\frac{\gamma_j}{3}\right)}=\left(\tfrac{1}{p}+\tfrac{1}{q}-\tfrac{1}{pq}\right)\prod_{j=1}^{s-1}\left(1+\tfrac{\gamma_j}{6\beta_j+2\gamma_j}\right).$$

Since

$$p,\ q\geq 2\exp\left(\frac{1}{6}\sum_{j=1}^{\infty}\frac{\gamma_j}{\beta_j}\right),$$

we have

$$\begin{aligned} R &= \left(\tfrac{1}{p}+\tfrac{1}{q}-\tfrac{1}{pq}\right)\exp\left(\sum_{j=1}^{s-1}\log\left(1+\tfrac{\gamma_j}{6\beta_j+2\gamma_j}\right)\right) \\ &\leq \left(\tfrac{1}{p}+\tfrac{1}{q}\right)\exp\left(\tfrac{1}{6}\sum_{j=1}^{\infty}\tfrac{\gamma_j}{\beta_j}\right)\leq 1, \end{aligned}$$

which leads to $G\leq 0$. This completes the proof. □

The condition of

$$p,\ q\geq 2\exp\left(\frac{1}{6}\sum_{j=1}^{\infty}\frac{\gamma_j}{\beta_j}\right)$$

is not unreasonable at all. For example, for $\beta_j=1$ and $\gamma_j=0.5^j$ we need $p,q\geq 3$; for $\beta_j=1$ and $\gamma_j=0.9^j$ we need $p,q\geq 9$; for $\beta_j=1$ and $\gamma_j=1/j^2$ we need $p,q\geq 3$.

The next theorem shows that the randomly shifted lattice rules constructed by Algorithm 3.1 achieve the rate of convergence $O(p^{-1+\delta}q^{-1/2})$ for any $\delta>0$.

Theorem 3.3. *Let p and q be two distinct prime numbers, and let $\hat{\boldsymbol{z}} \in \mathbb{Z}_p^d$ and $\hat{\boldsymbol{w}} \in \mathbb{Z}_q^d$ be constructed using Algorithm 3.1. Then for each $s = 1, 2, \ldots, d$, we have*

$$\begin{aligned} & e_{p,q,s}^2((1, \hat{z}_2, \ldots, \hat{z}_s), (1, \hat{w}_2, \ldots, \hat{w}_s)) \\ \leq \quad & (p-1)^{-\frac{1}{\lambda}} (q-1)^{-1} \prod_{j=1}^{s} \left(\left(\beta_j + \tfrac{\gamma_j}{3}\right)^\lambda + 2\zeta(2\lambda) \left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda \right)^{\frac{1}{\lambda}}, \end{aligned}$$

for all $\frac{1}{2} < \lambda \leq 1$.

Proof. For $s = 1$, it follows from the proof of Theorem 3.2 that

$$e_{p,q,1}^2(1,1) = \frac{\gamma_1}{6p^2q^2}.$$

For any λ satisfying $\frac{1}{2} < \lambda \leq 1$, we have

$$\begin{aligned} \frac{\gamma_1}{6p^2q^2} \quad & \leq \quad p^{-2}q^{-2} \left(\left(\beta_1 + \tfrac{\gamma_1}{3}\right) + 2\zeta(2) \left(\tfrac{\gamma_1}{2\pi^2}\right) \right) \\ & \leq \quad p^{-2}q^{-2} \left(\left(\beta_1 + \tfrac{\gamma_1}{3}\right)^\lambda + 2^\lambda [\zeta(2)]^\lambda \left(\tfrac{\gamma_1}{2\pi^2}\right)^\lambda \right)^{\frac{1}{\lambda}}, \end{aligned}$$

where the second inequality follows by applying Jensen's inequality to the sum. It can be easily verified that $p^{-2} < (p-1)^{-\frac{1}{\lambda}}$, $q^{-2} < (q-1)^{-1}$, $2^\lambda \leq 2$, and by Jensen's inequality, $[\zeta(2)]^\lambda \leq \zeta(2\lambda)$. Thus we have

$$e_{p,q,1}^2(1,1) = \frac{\gamma_1}{6p^2q^2} \leq (p-1)^{-\frac{1}{\lambda}} (q-1)^{-1} \left(\left(\beta_1 + \tfrac{\gamma_1}{3}\right)^\lambda + 2\zeta(2\lambda) \left(\tfrac{\gamma_1}{2\pi^2}\right)^\lambda \right)^{\frac{1}{\lambda}}.$$

Hence the result is true for $s = 1$.

For each s satisfying $2 \leq s \leq d$, suppose that two $(s-1)$-dimensional vectors $(1, \hat{z}_2, \ldots, \hat{z}_{s-1})$ and $(1, \hat{w}_2, \ldots, \hat{w}_{s-1})$ have already been constructed using Algorithm 3.1 and that they satisfy

$$\begin{aligned} & e_{p,q,s-1}^2((1, \hat{z}_2, \ldots, \hat{z}_{s-1}), (1, \hat{w}_2, \ldots, \hat{w}_{s-1})) \\ \leq \quad & (p-1)^{-\frac{1}{\lambda}} (q-1)^{-1} \prod_{j=1}^{s-1} \left(\left(\beta_j + \tfrac{\gamma_j}{3}\right)^\lambda + 2\zeta(2\lambda) \left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda \right)^{\frac{1}{\lambda}} \end{aligned} \tag{3.5}$$

for all $\frac{1}{2} < \lambda \leq 1$.

Following step 2 of Algorithm 3.1, we first choose $\hat{z}_s \in \mathbb{Z}_p$ to minimize $\theta_{p,q,s}^2$, which is the average of $e_{p,q,s}^2$ over all w_s. From the expression of $e_{p,q,s}^2$ in Lemma 2.4, $\theta_{p,q,s}^2$ can be written as

$$\begin{aligned} & \theta_{p,q,s}^2((1, z_2, \ldots, z_{s-1}), (1, w_2, \ldots, w_{s-1}); z_s) \\ = \quad & \frac{1}{q-1} \sum_{w_s=1}^{q-1} \sideset{}{'}\sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^s \\ \boldsymbol{h}\cdot(1,z_2,\ldots,z_s) \equiv 0 \, (\mathrm{mod}\ p) \\ \boldsymbol{h}\cdot(1,w_2,\ldots,w_s) \equiv 0 \, (\mathrm{mod}\ q)}} \prod_{j=1}^{s} r\left(2, \beta_j + \tfrac{\gamma_j}{3}, \tfrac{\gamma_j}{2\pi^2}, h_j\right) \\ = \quad & \left(\beta_s + \tfrac{\gamma_s}{3}\right) e_{p,q,s-1}^2((1, z_2, \ldots, z_{s-1}), (1, w_2, \ldots, w_{s-1})) + \tfrac{1}{q-1} F(z_s), \end{aligned} \tag{3.6}$$

where we have separated out the $h_s = 0$ terms, and
(3.7)

$$F(z_s) = \sum_{w_s=1}^{q-1} \sum_{h_s \in \mathbb{Z}}{}' \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^{s-1} \\ \boldsymbol{h}\cdot(1,z_2,\ldots,z_{s-1}) \equiv -h_s z_s \,(\mathrm{mod}\ p) \\ \boldsymbol{h}\cdot(1,w_2,\ldots,w_{s-1}) \equiv -h_s w_s \,(\mathrm{mod}\ q)}} \left(\frac{\gamma_s}{2\pi^2} |h_s|^{-2} \prod_{j=1}^{s-1} r\left(2, \beta_j + \tfrac{\gamma_j}{3}, \tfrac{\gamma_j}{2\pi^2}, h_j\right) \right).$$

Since $F(z_s)$ is the only dependency of $\theta^2_{p,q,s}$ on z_s, our choice of $\hat{z}_s$ will satisfy $F(\hat{z}_s) \le F(z_s)$ for all $z_s \in \mathbb{Z}_p$. Thus for any $\frac{1}{2} < \lambda \le 1$, we have $[F(\hat{z}_s)]^\lambda \le [F(z_s)]^\lambda$ for all $z_s \in \mathbb{Z}_p$, which leads to

$$(3.8) \quad [F(\hat{z}_s)]^\lambda \le \frac{1}{p-1} \sum_{z_s=1}^{p-1} [F(z_s)]^\lambda, \ \text{ or } \ F(\hat{z}_s) \le (p-1)^{-\frac{1}{\lambda}} \left(\sum_{z_s=1}^{p-1} [F(z_s)]^\lambda \right)^{\frac{1}{\lambda}}.$$

Using Jensen's inequality and the property $r(\alpha,\beta,\gamma,h)^\lambda = r(\alpha\lambda, \beta^\lambda, \gamma^\lambda, h)$, we obtain from (3.7) that

$$(3.9) \qquad \sum_{z_s=1}^{p-1} [F(z_s)]^\lambda \le \left(\frac{\gamma_s}{2\pi^2}\right)^\lambda \sum_{h_s \in \mathbb{Z}}{}' \frac{T(h_s)}{|h_s|^{2\lambda}},$$

where

$$T(h_s) = \sum_{z_s=1}^{p-1} \sum_{w_s=1}^{q-1} \sum_{\substack{\boldsymbol{h} \in \mathbb{Z}^{s-1} \\ \boldsymbol{h}\cdot(1,z_2,\ldots,z_{s-1}) \equiv -h_s z_s \,(\mathrm{mod}\ p) \\ \boldsymbol{h}\cdot(1,w_2,\ldots,w_{s-1}) \equiv -h_s w_s \,(\mathrm{mod}\ q)}} \prod_{j=1}^{s-1} r\left(2\lambda, \left(\beta_j + \tfrac{\gamma_j}{3}\right)^\lambda, \left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda, h_j\right).$$

To simplify the notation, let us write this expression as

$$T(h) = \sum_{z=1}^{p-1} \sum_{w=1}^{q-1} \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z} \equiv -hz \,(\mathrm{mod}\ p) \\ \boldsymbol{h}\cdot\boldsymbol{w} \equiv -hw \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}),$$

with

$$R(\boldsymbol{h}) := \prod_{j=1}^{s-1} r\left(2\lambda, \left(\beta_j + \tfrac{\gamma_j}{3}\right)^\lambda, \left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda, h_j\right),$$

and we are interested in finding an upper bound to

$$\sum_h{}' \frac{T(h)}{|h|^{2\lambda}} = \sum_{\substack{h\equiv 0 \,(\mathrm{mod}\ p) \\ h \equiv 0 \,(\mathrm{mod}\ q)}}{}' \frac{T(h)}{|h|^{2\lambda}} + \sum_{\substack{h\equiv 0 \,(\mathrm{mod}\ p) \\ h \not\equiv 0 \,(\mathrm{mod}\ q)}}{}' \frac{T(h)}{|h|^{2\lambda}} + \sum_{\substack{h\not\equiv 0 \,(\mathrm{mod}\ p) \\ h \equiv 0 \,(\mathrm{mod}\ q)}}{}' \frac{T(h)}{|h|^{2\lambda}} + \sum_{\substack{h\not\equiv 0 \,(\mathrm{mod}\ p) \\ h \not\equiv 0 \,(\mathrm{mod}\ q)}}{}' \frac{T(h)}{|h|^{2\lambda}}.$$

If h is a multiple of both p and q, then

$$T(h) = (p-1)(q-1) \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z} \equiv 0 \,(\mathrm{mod}\ p) \\ \boldsymbol{h}\cdot\boldsymbol{w} \equiv 0 \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}) \ \text{ and } \ \sum_{\substack{h\equiv 0 \,(\mathrm{mod}\ p) \\ h \equiv 0 \,(\mathrm{mod}\ q)}}{}' \frac{1}{|h|^{2\lambda}} = \frac{2\zeta(2\lambda)}{p^{2\lambda} q^{2\lambda}}.$$

If h is a multiple of p but not q, then

$$T(h) = (p-1) \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \,(\mathrm{mod}\ p)\\ \boldsymbol{h}\cdot\boldsymbol{w}\not\equiv 0 \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}) \text{ and } \sum_{\substack{h\equiv 0 \,(\mathrm{mod}\ p)\\ h\not\equiv 0 \,(\mathrm{mod}\ q)}}{}' \frac{1}{|h|^{2\lambda}} = \frac{2\zeta(2\lambda)(q^{2\lambda}-1)}{p^{2\lambda}q^{2\lambda}}.$$

If h is a multiple of q but not p, then

$$T(h) = (q-1) \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z}\not\equiv 0 \,(\mathrm{mod}\ p)\\ \boldsymbol{h}\cdot\boldsymbol{w}\equiv 0 \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}) \text{ and } \sum_{\substack{h\not\equiv 0 \,(\mathrm{mod}\ p)\\ h\equiv 0 \,(\mathrm{mod}\ q)}}{}' \frac{1}{|h|^{2\lambda}} = \frac{2\zeta(2\lambda)(p^{2\lambda}-1)}{p^{2\lambda}q^{2\lambda}}.$$

If h is neither a multiple of p nor a multiple of q, then

$$\begin{aligned} T(h) &= \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z}\not\equiv 0 \,(\mathrm{mod}\ p)\\ \boldsymbol{h}\cdot\boldsymbol{w}\not\equiv 0 \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}) \\ &= \sum_{\boldsymbol{h}} R(\boldsymbol{h}) - \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \,(\mathrm{mod}\ p)\\ \boldsymbol{h}\cdot\boldsymbol{w}\equiv 0 \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}) - \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \,(\mathrm{mod}\ p)\\ \boldsymbol{h}\cdot\boldsymbol{w}\not\equiv 0 \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}) - \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z}\not\equiv 0 \,(\mathrm{mod}\ p)\\ \boldsymbol{h}\cdot\boldsymbol{w}\equiv 0 \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}) \end{aligned}$$

and

$$\sum_{\substack{h\not\equiv 0 \,(\mathrm{mod}\ p)\\ h\not\equiv 0 \,(\mathrm{mod}\ q)}}{}' \frac{1}{|h|^{2\lambda}} = \frac{2\zeta(2\lambda)(p^{2\lambda}-1)(q^{2\lambda}-1)}{p^{2\lambda}q^{2\lambda}}.$$

Putting all of the above together, we have

$$\begin{aligned} \sum_{h}{}' \frac{T(h)}{|h|^{2\lambda}} &= \frac{2\zeta(2\lambda)}{p^{2\lambda}q^{2\lambda}} \Bigg[(p^{2\lambda}-1)(q^{2\lambda}-1) \sum_{\boldsymbol{h}} R(\boldsymbol{h}) \\ &\quad - \big[(p^{2\lambda}-1)(q^{2\lambda}-1) - (p-1)(q-1)\big] \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \,(\mathrm{mod}\ p)\\ \boldsymbol{h}\cdot\boldsymbol{w}\equiv 0 \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}) \\ &\quad - \big[(p^{2\lambda}-1)(q^{2\lambda}-1) - (p-1)(q^{2\lambda}-1)\big] \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z}\equiv 0 \,(\mathrm{mod}\ p)\\ \boldsymbol{h}\cdot\boldsymbol{w}\not\equiv 0 \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}) \\ &\quad - \big[(p^{2\lambda}-1)(q^{2\lambda}-1) - (p^{2\lambda}-1)(q-1)\big] \sum_{\substack{\boldsymbol{h}\cdot\boldsymbol{z}\not\equiv 0 \,(\mathrm{mod}\ p)\\ \boldsymbol{h}\cdot\boldsymbol{w}\equiv 0 \,(\mathrm{mod}\ q)}} R(\boldsymbol{h}) \Bigg] \\ &\le \frac{2\zeta(2\lambda)(p^{2\lambda}-1)(q^{2\lambda}-1)}{p^{2\lambda}q^{2\lambda}} \sum_{\boldsymbol{h}} R(\boldsymbol{h}) \\ &\le 2\zeta(2\lambda) \prod_{j=1}^{s-1} \left(\left(\beta_j + \tfrac{\gamma_j}{3}\right)^{\lambda} + 2\zeta(2\lambda) \left(\tfrac{\gamma_s}{2\pi^2}\right)^{\lambda} \right). \end{aligned} \tag{3.10}$$

We now conclude from (3.8), (3.9), and (3.10) that

$$F(\hat{z}_s) \le 2^{\frac{1}{\lambda}} \left[\zeta(2\lambda)\right]^{\frac{1}{\lambda}} \tfrac{\gamma_s}{2\pi^2} (p-1)^{-\frac{1}{\lambda}} \prod_{j=1}^{s-1} \left(\left(\beta_j + \tfrac{\gamma_j}{3}\right)^{\lambda} + 2\zeta(2\lambda) \left(\tfrac{\gamma_s}{2\pi^2}\right)^{\lambda} \right)^{\frac{1}{\lambda}}. \tag{3.11}$$

With this $\hat{z}_s$ fixed, $\hat{w}_s \in \mathbb{Z}_q$ is chosen to minimize $e^2_{p,q,s}$. Thus it follows from (3.5), (3.6), and (3.11) that these choices of $\hat{z}_s$ and $\hat{w}_s$ satisfy

$$\begin{aligned}
&e^2_{p,q,s}((1,\hat{z}_2,\ldots,\hat{z}_s),(1,\hat{w}_2,\ldots,\hat{w}_s)) \\
&\quad\le \theta^2_{p,q,s}((1,\hat{z}_2,\ldots,\hat{z}_{s-1}),(1,\hat{w}_2,\ldots,\hat{w}_{s-1});\hat{z}_s) \\
&\quad\le \left(\left(\beta_s+\tfrac{\gamma_s}{3}\right)+2^{\frac{1}{\lambda}}\left[\zeta(2\lambda)\right]^{\frac{1}{\lambda}}\tfrac{\gamma_s}{2\pi^2}\right) \\
&\qquad\qquad \times (p-1)^{-\frac{1}{\lambda}}(q-1)^{-1}\prod_{j=1}^{s-1}\left(\left(\beta_j+\tfrac{\gamma_j}{3}\right)^\lambda+2\zeta(2\lambda)\left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda\right)^{\frac{1}{\lambda}} \\
&\quad\le \left(\left(\beta_s+\tfrac{\gamma_s}{3}\right)^\lambda+2\zeta(2\lambda)\left(\tfrac{\gamma_s}{2\pi^2}\right)^\lambda\right)^{\frac{1}{\lambda}} \\
&\qquad\qquad \times (p-1)^{-\frac{1}{\lambda}}(q-1)^{-1}\prod_{j=1}^{s-1}\left(\left(\beta_j+\tfrac{\gamma_j}{3}\right)^\lambda+2\zeta(2\lambda)\left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda\right)^{\frac{1}{\lambda}} \\
&\quad= (p-1)^{-\frac{1}{\lambda}}(q-1)^{-1}\prod_{j=1}^{s}\left(\left(\beta_j+\tfrac{\gamma_j}{3}\right)^\lambda+2\zeta(2\lambda)\left(\tfrac{\gamma_j}{2\pi^2}\right)^\lambda\right)^{\frac{1}{\lambda}},
\end{aligned}$$

where the third inequality follows from applying Jensen's inequality to the sum in the first factor. Hence it follows inductively that the result is true for all $s = 1, 2, \ldots, d$. □

Remark. We want to have an algorithm so that the choice of z_s is independent of w_s. A natural criterion for choosing z_s is then to minimize $\theta^2_{p,q,s}$ as it is the average of $e^2_{p,q,s}$ over all possible w_s and we know that we can always find a w_s such that

$$e^2_{p,q,s}((z_1,\ldots,z_s),(w_1,\ldots,w_s)) \le \theta^2_{p,q,s}((z_1,\ldots,z_{s-1}),(w_1,\ldots,w_{s-1});z_s)$$

for any z_s. In this way we separate the search for z_s and w_s, which yields the reduction of the construction cost.

As we do not have a concise formula for

$$\frac{1}{q-1}\sum_{w_s=1}^{q-1}\left[e^2_{p,q,s}((z_1,\ldots,z_s),(w_1,\ldots,w_s))\right]^\lambda$$

with $\frac{1}{2} < \lambda < 1$, which could be used instead of $\theta^2_{p,q,s}$ in Algorithm 3.1, the bound in Theorem 3.3 cannot be improved by the arguments used in the proof.

4. Numerical experiments

By means of numerical calculations we test how well the Partial Search algorithm (Algorithm 3.1) performs for various choices of decompositions $n = pq$. Since the theory suggests a rate of convergence $O(p^{-1+\delta}q^{-1/2})$ for $\delta > 0$, it would seem intuitive to choose p much larger than q to ensure a low worst-case error. On the other hand, the cost of the construction is $O(n(p+q)d^2)$ operations, which is minimized when choosing p and q to be roughly the same, that is, $p \approx q \approx n^{1/2}$. Table 1 shows a comparison of the theoretical rate of convergence against the cost of construction.

In particular, we test whether for fixed n larger values of p (which means higher costs for the construction) will indeed lead to better rates of convergence. Furthermore we compare numerical results from the Partial Search algorithm with those

TABLE 1. Partial search analysis

$n = pq$	cost of construction $O(n(p+q)d^2)$	rate of convergence $O(p^{-1+\delta}q^{-1/2})$, $\delta > 0$
$p = q$	$O(n^{1.5}d^2)$	$O(n^{-0.75+\delta})$
$p = q^2$	$O(n^{1.67}d^2)$	$O(n^{-0.83+\delta})$
$p = q^3$	$O(n^{1.75}d^2)$	$O(n^{-0.875+\delta})$

from the Full Search algorithm (Algorithm 4.1 below), which requires a cost of $O(n^2d^2)$ operations. Results from the Kuo and Joe paper [8] can be used to justify Algorithm 4.1. In [2], Dick showed that rules constructed by Algorithm 4.1 achieve a rate of convergence $O(n^{-1+\delta})$ for $\delta > 0$.

Algorithm 4.1 (Full Search)**.** *Given any composite number n:*

1. *Set z_1, the first component of $\boldsymbol{z}$, to* 1.
2. *For $s = 2, 3, \ldots, d-1, d$, find $z_s \in \{1 \le z \le \frac{n-1}{2} : \gcd(z,n) = 1\}$ such that*

$$e_{n,s}^2(z_1,\ldots,z_s) = -\prod_{j=1}^{s}\left(\beta_j + \tfrac{\gamma_j}{3}\right) + \frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{s}\left(\beta_j + \gamma_j\left[B_2\left(\left\{\tfrac{iz_j}{n}\right\}\right) + \tfrac{1}{3}\right]\right),$$

is minimized.

We take $\boldsymbol{\beta} = \mathbf{1}$ and six different sequences of $\boldsymbol{\gamma}$:

$$\gamma_j = 0.9^j,\ \gamma_j = 0.5^j,\ \gamma_j = 0.1^j,\ \gamma_j = 1/j^2,\ \gamma_j = 1/j^6,\ \text{and}\ \gamma_j = 1/j,$$

and we consider different decompositions with $p \approx q$, $p \approx q^2$, and $p \approx q^3$. For each choice of $n = pq$, we compare the following:

(1) worst-case error $e_{n,100}$ for the 100-dimensional rule constructed by the Full Search algorithm (Algorithm 4.1),
(2) worst-case error $e_{p,q,100}$ for the 100-dimensional rule constructed by the Partial Search algorithm (Algorithm 3.1),
(3) root QMC mean $E_{n,100}$.

The results of these comparisons are presented in Tables 2, 3, and 4. The observed rates of convergence $O(n^{-\omega})$ for successive worst-case errors are also included in these tables.

We observe from our numerical results that the worst-case errors for rules constructed by the Partial Search algorithm are only slightly worse than those constructed by the Full Search algorithm, and they are all significantly better than the root QMC mean, which is supported by Theorem 3.2. The observed rates of convergence for rules constructed by the Partial Search algorithm are very close to those constructed by the Full Search algorithm. This indicates that in practice, we can use the cheaper Partial Search algorithm as a replacement for the costly Full Search algorithm.

Contrary to what Theorem 3.3 may suggest, the observed rates of convergence show no correlation with the choice of decompositions $p \approx q$, $p \approx q^2$ or $p \approx q^3$. (It would appear that the observed rate of convergence depends on the rate of decay of the weights $\boldsymbol{\gamma}$.) From this we may conclude that there is no advantage in taking p to be much larger than q. The Partial Search algorithm performs just as well with the cheaper choice of $p \approx q$. In this case the algorithm has only a construction cost of $O(n^{1.5}d^2)$ operations, and we are now able to use 10^6 or even 10^7 points: see Table 5 for results with n roughly 2 million, 4 million, and 8 million.

Tables of numerical results:

TABLE 2. Full Search and Partial Search with $p \approx q$

	$n = pq$			Full		Partial		QMC Mean
	n	p	q	$e_{n,100}$	$\widetilde{\omega}$	$e_{p,q,100}$	$\widetilde{\omega}$	$E_{n,100}$
$\gamma_j = 0.9^j$	2021	47	43	5.0496e-02	0.669	5.2455e-02	0.658	1.4320e-01
	8633	97	89	1.9124e-02	0.660	2.0187e-02	0.664	6.9286e-02
	32399	181	179	7.9942e-03		8.3845e-03		3.5765e-02
$\gamma_j = 0.5^j$	2021	47	43	3.7133e-04	0.932	3.8948e-04	0.891	1.0370e-02
	8633	97	89	9.5914e-05	0.929	1.0685e-04	0.938	5.0172e-03
	32399	181	179	2.8070e-05		3.0918e-05		2.5899e-03
$\gamma_j = 0.1^j$	2021	47	43	6.8716e-05	0.998	6.8744e-05	0.998	3.0398e-03
	8633	97	89	1.6123e-05	0.999	1.6135e-05	0.997	1.4708e-03
	32399	181	179	4.3028e-06		4.3188e-06		7.5920e-04
$\gamma_j = 1/j^2$	2021	47	43	6.9041e-04	0.882	7.3900e-04	0.872	1.4074e-02
	8633	97	89	1.9196e-04	0.877	2.0846e-04	0.872	6.8097e-03
	32399	181	179	6.0179e-05		6.5794e-05		3.5151e-03
$\gamma_j = 1/j^6$	2021	47	43	2.1076e-04	0.997	2.1089e-04	0.997	9.2246e-03
	8633	97	89	4.9526e-05	0.998	4.9568e-05	0.994	4.4632e-03
	32399	181	179	1.3234e-05		1.3306e-05		2.3039e-03
$\gamma_j = 1/j$	2021	47	43	1.4932e-02	0.697	1.5220e-02	0.681	5.5014e-02
	8633	97	89	5.4266e-03	0.680	5.6640e-03	0.679	2.6618e-02
	32399	181	179	2.2089e-03		2.3072e-03		1.3740e-02

TABLE 3. Full Search and Partial Search with $p \approx q^2$

	$n = pq$			Full		Partial		QMC Mean
	n	p	q	$e_{n,100}$	$\widetilde{\omega}$	$e_{p,q,100}$	$\widetilde{\omega}$	$E_{n,100}$
$\gamma_j = 0.9^j$	2171	167	13	4.7989e-02	0.656	5.2102e-02	0.659	1.3817e-01
	6821	359	19	2.2650e-02	0.668	2.4493e-02	0.642	7.7948e-02
	24331	839	29	9.6878e-03		1.0832e-02		4.1271e-02
$\gamma_j = 0.5^j$	2171	167	13	3.4980e-04	0.934	4.1686e-04	0.932	1.0005e-02
	6821	359	19	1.2002e-04	0.934	1.4345e-04	0.882	5.6444e-03
	24331	839	29	3.6586e-05		4.6724e-05		2.9886e-03
$\gamma_j = 0.1^j$	2171	167	13	6.3964e-05	0.999	6.4096e-05	0.997	2.9329e-03
	6821	359	19	2.0389e-05	0.998	2.0477e-05	0.996	1.6546e-03
	24331	839	29	5.7304e-06		5.7696e-06		8.7608e-04
$\gamma_j = 1/j^2$	2171	167	13	6.5151e-04	0.886	7.4688e-04	0.860	1.3579e-02
	6821	359	19	2.3620e-04	0.884	2.7891e-04	0.844	7.6610e-03
	24331	839	29	7.6780e-05		9.5401e-05		4.0563e-03
$\gamma_j = 1/j^6$	2171	167	13	1.9614e-04	0.998	1.9683e-04	0.995	8.9002e-03
	6821	359	19	6.2592e-05	0.996	6.3035e-05	0.995	5.0212e-03
	24331	839	29	1.7632e-05		1.7790e-05		2.6586e-03
$\gamma_j = 1/j$	2171	167	13	1.4118e-02	0.690	1.5024e-02	0.681	5.3079e-02
	6821	359	19	6.4111e-03	0.688	6.8911e-03	0.667	2.9946e-02
	24331	839	29	2.6732e-03		2.9501e-03		1.5855e-02

TABLE 4. Full Search and Partial Search with $p \approx q^3$

	$n = pq$			Full		Partial		QMC Mean
	n	p	q	$e_{n,100}$	$\widetilde{\omega}$	$e_{p,q,100}$	$\widetilde{\omega}$	$E_{n,100}$
$\gamma_j = 0.9^j$	2429	347	7	4.4700e-02	0.660	4.9211e-02	0.646	1.3062e-01
	14597	1327	11	1.3675e-02	0.677	1.5449e-02	0.646	5.3284e-02
	28639	2203	13	8.6638e-03		9.9991e-03		3.8041e-02
$\gamma_j = 0.5^j$	2429	347	7	3.1333e-04	0.930	3.7871e-04	0.878	9.4587e-03
	14597	1327	11	5.9134e-05	0.920	7.8475e-05	1.035	3.8584e-03
	28639	2203	13	3.1809e-05		3.9061e-05		2.7546e-03
$\gamma_j = 0.1^j$	2429	347	7	5.7162e-05	0.998	5.7507e-05	0.998	2.7727e-03
	14597	1327	11	9.5467e-06	0.999	9.6067e-06	1.004	1.1311e-03
	28639	2203	13	4.8675e-06		4.8822e-06		8.0751e-04
$\gamma_j = 1/j^2$	2429	347	7	5.9242e-04	0.886	7.2990e-04	0.862	1.2838e-02
	14597	1327	11	1.2096e-04	0.900	1.5567e-04	0.879	5.2369e-03
	28639	2203	13	6.5966e-05		8.6099e-05		3.7388e-03
$\gamma_j = 1/j^6$	2429	347	7	1.7529e-04	0.997	1.7685e-04	0.996	8.4142e-03
	14597	1327	11	2.9345e-05	0.998	2.9631e-05	1.007	3.4324e-03
	28639	2203	13	1.4975e-05		1.5029e-05		2.4505e-03
$\gamma_j = 1/j$	2429	347	7	1.3075e-02	0.691	1.4314e-02	0.672	5.0181e-02
	14597	1327	11	3.7855e-03	0.687	4.2868e-03	0.656	2.0470e-02
	28639	2203	13	2.3830e-03		2.7541e-03		1.4614e-02

TABLE 5. Partial Search with $p \approx q$

	$n = pq$			Partial		QMC Mean
	n	p	q	$e_{p,q,100}$	$\widetilde{\omega}$	$E_{n,100}$
$\gamma_j = 0.9^j$	2005007	1423	1409	5.5119e-04	0.671	4.5464e-03
	4003997	2003	1999	3.4651e-04	0.656	3.2172e-03
	8037211	2837	2833	2.1932e-04		2.2708e-03
$\gamma_j = 0.5^j$	2005007	1423	1409	7.1750e-07	0.957	3.2922e-04
	4003997	2003	1999	3.7002e-07	0.945	2.3297e-04
	8037211	2837	2833	1.9148e-07		1.6443e-04
$\gamma_j = 0.1^j$	2005007	1423	1409	7.0272e-08	1.002	9.6509e-05
	4003997	2003	1999	3.5137e-08	1.254	6.8293e-05
	8037211	2837	2833	1.4670e-08		4.8203e-05
$\gamma_j = 1/j^2$	2005007	1423	1409	1.9173e-06	0.845	4.4684e-04
	4003997	2003	1999	1.0686e-06	0.833	3.1620e-04
	8037211	2837	2833	5.9812e-07		2.2318e-04
$\gamma_j = 1/j^6$	2005007	1423	1409	2.1845e-07	1.001	2.9287e-04
	4003997	2003	1999	1.0929e-07	1.029	2.0724e-04
	8037211	2837	2833	5.3371e-08		1.4628e-04
$\gamma_j = 1/j$	2005007	1423	1409	1.4475e-04	0.685	1.7466e-03
	4003997	2003	1999	9.0134e-05	0.669	1.2360e-03
	8037211	2837	2833	5.6561e-05		8.7238e-04

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School of Mathematics, The University of New South Wales, Sydney, New South Wales 2052, Australia

E-mail address: `josi@maths.unsw.edu.au`

Department of Mathematics, The University of Waikato, Private Bag 3105, Hamilton, New Zealand

Current address: School of Mathematics, The University of New South Wales, Sydney, New South Wales 2052, Australia

E-mail address: `fkuo@maths.unsw.edu.au`