CANONICAL VECTOR HEIGHTS
ON K3 SURFACES WITH PICARD NUMBER THREE—
AN ARGUMENT FOR NONEXISTENCE

ARTHUR BARAGAR

Abstract. In this paper, we investigate a K3 surface with Picard number three and present evidence that strongly suggests a canonical vector height cannot exist on this surface.

1. Introduction and background

Let $V$ be a K3 surface over a number field $K$. A vector height on $V/K$ is a function

$$ h : V/K \to \text{Pic}(V/K) \otimes \mathbb{R} $$

with the following two properties: (1) For any $\sigma \in \text{Aut}(V/K)$,

$$ h(\sigma P) = \sigma_* h(P) + O(1), $$

where $\sigma_* = (\sigma^{-1})^*$ is the pushforward of $\sigma$ and $O(1)$ is a vector function with bounded components; and (2) for any Weil height $h_D(P)$ associated to a divisor $D$, we have

$$ h_D(P) = h(P) \cdot D + O(1). $$

Vector heights exist and are unique up to bounded vector functions [Ba1].

Given a basis $D = \{D_1, \ldots, D_n\}$ for $\text{Pic}(V/K) \otimes \mathbb{R}$, let $D^* = \{D_1^*, \ldots, D_n^*\}$ be the dual basis defined by the property that $D_i \cdot D_j^* = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta symbol (that is, the matrix $[\delta_{ij}]$ is the identity matrix). Let $J = [D_i \cdot D_j]$ be the intersection matrix with respect to the basis $D$. Then $J^{-1}$ is the intersection matrix with respect to the basis $D^*$, and $J$ is the change of basis matrix from the basis $D^*$ to the basis $D$.

Given Weil heights $h_{D_i}$, the function

$$ h(P) = \sum_{i=1}^n h_{D_i}(P)D_i^* $$

is a vector height.

We call a vector height $\hat{h}$ a canonical vector height if $\hat{h}$ is a vector height and for every $\sigma \in \text{Aut}(V/K)$ and $P \in V/K$,

$$ \hat{h}(\sigma P) = \sigma_* \hat{h}(P). $$
The existence of a canonical vector height makes it possible or easier to answer certain arithmetic questions. If $V/K$ is an algebraic K3 surface over a number field $K$ that has Picard number two and an infinite group of automorphisms $\text{Aut}(V/K)$, then there exists a canonical vector height on $V/K$ [Ba1]. The goal of this paper is to numerically verify (though not rigorously prove) that a canonical vector height does not exist on a certain K3 surface with Picard number three and thereby to give convincing evidence that they do not exist in general.

Suppose $\sigma \in \text{Aut}(V/K)$ and that $\sigma^*$ has a maximal real eigenvalue $\omega > 1$ with associated eigenvector $E \in \text{Pic}(V) \otimes \mathbb{R}$. Silverman [S] defined the height

$$
\hat{h}_E(P) = \lim_{n \to \infty} \omega^{-n} h_E(\sigma^n P),
$$

where $h_E$ is a Weil height with respect to $E$. This height is canonical with respect to $\sigma$, since $\hat{h}_E(\sigma P) = \omega \hat{h}_E(P)$. Of particular use to us is the property that $\hat{h}_E(P)$ is independent of the choice one makes for Weil height $h_E$.

Suppose now that there exists a canonical vector height $\hat{h}$ on $V/K$. Then the function $\hat{h}(P) \cdot E$ is a Weil height with respect to the divisor $E$, so

$$
\hat{h}_E(P) = \lim_{n \to \infty} \omega^{-n} \hat{h}(\sigma^n P) \cdot E
$$

$$
= \lim_{n \to \infty} \omega^{-n} \sigma^* \hat{h}(P) \cdot E
$$

$$
= \lim_{n \to \infty} \omega^{-n} \hat{h}(P) \cdot (\sigma^*)^{-n} E
$$

$$
= \lim_{n \to \infty} \omega^{-n} \hat{h}(P) \cdot \omega^n E
$$

$$
= \hat{h}(P) \cdot E.
$$

Thus, if we can calculate $\hat{h}_E(P)$, then this will give us a linear equation for $\hat{h}(P)$.

Our idea for demonstrating that no such $\hat{h}$ can exist is to calculate $\hat{h}_E(P)$ for enough $\sigma$ so that we arrive at an inconsistent system of linear equations.

2. The example

We will look at a surface $V$ defined by a $(2, 2, 2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{Q}$. Such surfaces have been studied by Wheler [Wh], Wang [Wa], Billard [Bi], and the author [Ba2]. The proofs of some of the following statements can be found in these sources. A $(2, 2, 2)$ form can be written in the form

$$
F(X, Y, Z) = F_{00}(Y, Z) X_0^2 + F_{01}(Y, Z) X_0 X_1 + F_{11}(Y, Z) X_1^2,
$$

where $X = (X_0, X_1)$, etc., and the polynomials $F_{ij}$ are $(2, 2)$ forms in $\mathbb{P}^1 \times \mathbb{P}^1$. If the variety is nonsingular, then the surface $V$ defined by $F(X, Y, Z) = 0$ is a K3 surface. Let

$$
p_1 : V \to \mathbb{P}^1 \times \mathbb{P}^1
$$

$$(X, Y, Z) \mapsto (Y, Z)$$

be the projection onto the second two coordinates. Define $p_2$ and $p_3$ in a similar fashion. Generically, the projection $p_1$ defines a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ by the surface $V$. The exception is when there exists a point $(Y', Z') \in \mathbb{P}^1 \times \mathbb{P}^1$ such that we simultaneously have

$$
F_{00}(Y', Z') = F_{01}(Y', Z') = F_{11}(Y', Z') = 0.
$$
Then $V$ includes the line $(X, Y', Z')$, which is a $-2$ curve on $V$. In such a case, the Picard number for $V$ is at least 4. If each of $p_1$, $p_2$, and $p_3$ define a double cover everywhere, then $V$ has Picard number three. Let $\pi_1(X, Y, Z) = X$ be the projection onto the first component, and define $\pi_2$ and $\pi_3$ similarly. Let $H$ be a point in $\mathbb{P}^1$ and let $D_i$ be the divisor class defined by $\pi_i^*(H)$. Then $D = \{D_1, D_2, D_3\}$ is a basis for $\text{Pic}(V)$ and the intersection matrix with respect to this basis is

$$J = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$ 

If $(X, Y, Z)$ is a point on $V$, then $p_i^{-1}(Y, Z) = \{(X, Y, Z), (X', Y, Z)\}$ and the map

$$\sigma_1: V \rightarrow V$$

$$(X, Y, Z) \mapsto (X', Y, Z)$$

is an automorphism of $V$. Explicitly,

$$X' = \begin{cases} \left[ F_{01}(Y, Z)X_1 + F_{00}(Y, Z)X_0, -F_{00}(Y, Z)X_1 \right] & \text{if this is in } \mathbb{P}^1 \\
\left[ F_{01}(Y, Z)X_0 + F_{11}(Y, Z)X_1, -F_{11}(Y, Z)X_0 \right] & \text{otherwise.} \end{cases}$$

The maps $\sigma_2$ and $\sigma_3$ can be defined similarly. Let $T_i = [\sigma_i^*]_{D^*}$ be the the pullback of $\sigma_i$ in the basis $D^*$. Then

$$T_1 = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad T_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & -1 \end{bmatrix}.$$ 

One can easily verify that $T_i^2 = 1$ and $T_i^3 = J^{-1}T_i = J^{-1}$.

We investigate the surface $V/\mathbb{Q}$ that is defined by $F(X, Y, Z) = 0$, where (in affine coordinates)

$$F([x, 1], [y, 1], [z, 1]) = (y^2z^2 + y + 4)x^2 + (y^2 + z^2)x + (yz^2 + z + 2y^2).$$

It is fairly easy to check that $F(X, Y, Z) = 0$ is nonsingular over $\mathbb{Z}/2\mathbb{Z}$, so it is nonsingular over $\mathbb{Q}$, and only a little tedious to check that the projections $p_i$ are everywhere double covers. Thus, the Picard number for $V$ is 3, as desired.

The surface $V$ includes the point $P_0 = [[0, 1], [-1, 1], [-1, 1]]$. With the exception of these three properties (smooth, Picard number three, and with a rational point), there is nothing special about our choice for $V$, and we presume that it is a random representative of the class.

Let $h$ be the usual logarithmic height on $\mathbb{P}^1(\mathbb{Q})$. That is, if $X = [X_0, X_1] \in \mathbb{P}^1(\mathbb{Q})$, then we can choose $X_0, X_1 \in \mathbb{Z}$ with $\gcd(X_0, X_1) = 1$. We define $h(X) = \log(\max\{|X_0|, |X_1|\})$. This induces several Weil heights on $V$:

$$h_{D_i}(P) = h(\pi_i(P)),$$

and from this, we can define the vector height

$$h(P) = \sum_{i=1}^{3} h_{D_i}(P)D_i^*.$$ 

The matrices $T_1$ and $T_1T_2$ do not have any eigenvalues larger than one. (Interpreted as isomorphisms of the hyperbolic surface $x^2J^{-1}x = 1$, these are, respectively, a reflection and a parabolic translation.) Thus, we must look at a combination of three of these generating matrices before we will find one with
For each of these automorphisms, the largest eigenvalue is again associated eigenvector \( E \). The associated eigenvectors are, respectively, 
\[
\begin{bmatrix}
E_1 \\
E_2 \\
E_3 \\
E_4 \\
E_5 \\
\end{bmatrix}
\]
Values of \( \rho \) are as follows:
\[
\begin{array}{cccc}
n & \rho^{-n} h_{E_132} (\sigma_{321}^n P_0) & \rho^{-n} h_{E_{231}} (\sigma_{123}^n P_0) & \rho^{-n} h_{E_{312}} (\sigma_{213}^n P_0) & \rho^{-n} h_{E_{213}} (\sigma_{132}^n P_0) \\
1 & .672800 & .316293 & .711937 & 1.007649 \\
2 & .664771 & .325946 & .772532 & .975334 \\
3 & .665283 & .325731 & .770970 & .972127 \\
4 & .665303 & .325732 & .771149 & .972045 \\
5 & .665311 & .325733 & .771154 & .972055 \\
\end{array}
\]

A positive eigenvalue. The matrix \( T_1 T_2 T_3 \) has an eigenvalue \( \omega = \alpha^3 \) and associated eigenvector \( \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} \alpha^2, \alpha, 1 \end{bmatrix} \) where \( \alpha = \frac{\sqrt[3]{5}}{2} \). As an element of Pic(\( V \)) \( \otimes \mathbb{R} \), \( E_{123} = \alpha^2 D_1^2 + \alpha D_2^2 + D_3^2 \). Since in the following all expressions will be in the basis \( D^* \), let us drop the explicit reference to \( D^* \), so by \( E_{123} \) mean \( E_{123}^* \). Note that \( T_1 T_2 T_3 = \begin{bmatrix} \sigma_1^2 \sigma_2^2 \sigma_3^2 \end{bmatrix} = \left( \begin{bmatrix} \sigma_3 \sigma_2 \sigma_1 \end{bmatrix} \right)^n \). We therefore choose \( \sigma = \sigma_{321} = \sigma_3 \sigma_2 \sigma_1 \). We set
\[
h_{E_{123}}(P) = h(P) \cdot E_{123}
\]
and calculate
\[
\rho^{-n} h_{E_{123}} (\sigma_{321}^n (P_0))
\]
These calculations are shown in Table I. The table is rather short, since calculating this value for \( n = 6 \) involves integer arithmetic with million digit integers. Similar calculations are made for \( \sigma_{123} = \sigma_1 \sigma_2 \sigma_3 \), \( \sigma_{213} = \sigma_2 \sigma_1 \sigma_3 \), and \( \sigma_{312} = \sigma_3 \sigma_1 \sigma_2 \). For each of these automorphisms, the largest eigenvalue is again \( \omega = \alpha^3 \), and the associated eigenvectors are, respectively, \( E_{231} = \begin{bmatrix} 1, \alpha^2, \alpha \end{bmatrix} \), \( E_{312} = \begin{bmatrix} \alpha, 1, \alpha^2 \end{bmatrix} \), and \( E_{213} = \begin{bmatrix} \alpha, \alpha^2, 1 \end{bmatrix} \).

Let \( A \) be the matrix whose rows are \( E_{123} \), \( E_{231} \), and \( E_{312} \). Let \( B = (B_1, B_2, B_3) = \left( \hat{h}_{E_{123}}(P_0), \hat{h}_{E_{231}}(P_0), \hat{h}_{E_{312}}(P_0) \right) \). Suppose now that a canonical vector height \( \hat{h} \) exists for \( V \). Then,
\[
\hat{h}_{E}(P_0) = \hat{h}(P_0) \cdot E = E \cdot \hat{h}(P_0) = [E]^t J^{-1} [\hat{h}(P_0)],
\]
so
\[
AJ^{-1} [\hat{h}(P_0)] = B,
\]
\[
[\hat{h}(P_0)] = JA^{-1} B.
\]

The exact value of \( JA^{-1} \) is known; \( B \) is approximately \( [169405, .326915, .176779] \). This gives us the approximation \( [\hat{h}(P_0)] \approx [169405, .326915, .176779] \). We therefore have
\[
\hat{h}_{E_{213}}(P_0) = \hat{h}(P_0) \cdot E_{213} \approx 331699,
\]
which is not very close to the value .972055 shown in Table I.

From Table I it would appear that our estimate for the entries of \( B \) are accurate to \( \pm .00002 \). Since
\[
\hat{h}_{E_{213}}(P_0) = E_{213} \cdot \hat{h}(P_0)
\]
\[
= [E_{213}]^t J^{-1} (JA^{-1} B)
\]
\[
= [E_{213}]^t A^{-1} B
\]
\[
= .345492 B_1 + .904508 B_2 - .25 B_3,
\]
we expect an error in our calculation of no more than ±0.00003. Since our two
calculations differ significantly, we conclude that no such canonical vector height
could exist on this K3 surface and therefore that it is unlikely that a canonical
vector height should exist on any K3 surface with Picard number greater than 2,
except for perhaps in some very special cases.

Remark. The referee noted that the eigenvalue \( \omega \) is a quadratic unit, and wondered
whether this is always the case. For \( n = 2 \) and \( 3 \), it is. More generally, \( \omega \) is either
a quadratic unit or a Salem number. Suppose \( J \) is the intersection matrix for some
surface \( V \). Then \( J \) has one positive eigenvalue and \( n - 1 \) negative eigenvalues (by
the Hodge index theorem). Thus the surface \( x^T J x = x \cdot x = 0 \) is a cone, and in
particular, does not contain any planes. Suppose \( T = \sigma^* \) for some \( \sigma \in \text{Aut}(V) \).
Then \( T^4 J T = J \) and \( T \) has integer entries. If \( v \) is an eigenvector for \( T \) and \( T v = \lambda v \),
then \( v \cdot v = T v \cdot T v = \lambda^2 v \cdot v \). Hence, either \( \lambda^2 = 1 \) or \( v \cdot v = 0 \). Suppose \( T \) has
three (possibly equal) real eigenvalues \( \alpha, \beta, \gamma \), none equal to either \( \pm 1 \). Let
their associated eigenvectors be \( u, v, \) and \( w \), respectively (where these vectors are
linearly independent). Then \( u \cdot u = v \cdot v = w \cdot w = 0 \). Note that \( u \cdot v \neq 0 \), since if
it did, then the surface \( x \cdot x = 0 \) would contain the plane spanned by \( u \) and \( v \), and
as noted earlier, this surface contains no planes. Let \( y = u + v \). Then \( y \cdot y = 2 u \cdot v \).
But \( y \cdot y = T y \cdot T y = 2 \alpha \beta u \cdot v \). Hence \( \alpha \beta = 1 \). Similarly, \( \alpha \gamma = 1 \) and \( \beta \gamma = 1 \),
which implies \( \alpha = \beta = \gamma = 1 \), a contradiction. Thus, there can be at most two
real eigenvalues not equal to \( \pm 1 \). Suppose now that \( T \) has an eigenvalue \( \lambda \) that
is not real. Let its associated eigenvector be \( v = v_R + iv_I \) where \( v_R \) and \( v_I \) are
real vectors. Then \( v_I \) are linearly independent. Since \( v \cdot v = \lambda^2 v \cdot v \), and
\( \lambda^2 \neq 1 \), we get \( v \cdot v = 0 \), which implies \( v_R \cdot v_R + v_I \cdot v_I = 0 \) and \( v_R \cdot v_I = 0 \).
Since \( v \cdot v = \lambda \lambda^* v \cdot v \), we get \( |\lambda| = 1 \) or \( v_R \cdot v_R - v_I \cdot v_I = 0 \). If we have the latter, then
the plane spanned by \( v_R \) and \( v_I \) is in the cone \( x \cdot x = 0 \), which is a contradiction.
Thus \( |\lambda| = 1 \). Finally, since \( T \) has integer entries, the minimal polynomial for \( \omega \)
divides the characteristic polynomial for \( T \). Hence, \( \omega \) is an algebraic integer, it has
only one other real conjugate \( \pm \omega^{-1} \) (since \( \det T = \pm 1 \)), and all its other conjugates
are complex with magnitude one. Such a number is either a quadratic unit or a
Salem number. When \( n = 2 \) or \( 3 \), there can be no complex eigenvalues, so \( \omega \) must
be a quadratic unit. Geometrically (for \( n = 3 \)), \( T \) is a translation or glide reflection
that translates along the line with endpoints the associated eigenvectors.

3. Further analysis of the error in \( B \)

In the previous section, we stated that the estimates for the entries of \( B \) look
to be accurate to ±0.00002. This was based on the observation that the sequence
that converges to the canonical height \( h_E \) converges geometrically and that the
difference of the fourth and fifth iteration for our various calculations is no more
than ±.00002. Let us now present a more sophisticated argument.

Let \( \sigma \) be an automorphism of \( V \) whose pull back \( \sigma^* \) has a maximal real eigenvalue
\( \omega > 1 \) with associated eigenvector \( E \). Let \( h_E \) be a Weil height with respect to the
divisor \( E \). Then

\[
(2) \quad h_E(\sigma P) = h_{\sigma^* E}(P) + O(1) = h_{\omega E}(P) + O(1) = \omega h_E(P) + O(1).
\]

The function implied by the \( O(1) \) is bounded independent of \( P \). To make our argu-
ment completely rigorous, we would have to find explicit bounds for this error term.
This can possibly be done, following the ideas presented by Call and Silverman in
Table 2. The quantity described in Eq. 3 for $\sigma = \sigma_{321}$ and various values of $n$ and $m$.

<table>
<thead>
<tr>
<th>$n \backslash m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>11.3</td>
<td>-2.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>11.3</td>
<td>-2.3</td>
<td>3.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>11.3</td>
<td>-2.3</td>
<td>2.9</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>11.3</td>
<td>-2.3</td>
<td>2.9</td>
<td>2.7</td>
<td>15.5</td>
</tr>
</tbody>
</table>

[C.S.], but the extra effort is probably not worth it. Instead, we will use our data to get an idea of the range of the error term. Note that

$$h_E(\sigma^n(P)) = \omega h_E(\sigma^{n-1}P) + O(1)$$

$$= \omega^2 h_E(\sigma^{n-2}P) + \omega O(1) + O(1)$$

$$= \omega^n h_E(P) + \left(\frac{\omega^n - 1}{\omega - 1}\right) O(1),$$

where the bound on the function implied by $O(1)$ in (3) is the same as the bound for the function implied by the $O(1)$ in (2). Thus, for $n > m$, we get

$$\omega^{-n} h_E(\sigma^n P) - \omega^{-m} h_E(\sigma^m P) = \omega^{-n} \left(\omega^{n-m} h_E(\sigma^m P) + \left(\frac{\omega^{n-m} - 1}{\omega - 1}\right) O(1)\right) - \omega^{-m} h_E(\sigma^m P)$$

$$= \left(\frac{\omega^{-m} - \omega^{-n}}{\omega - 1}\right) O(1),$$

where the bound on the function implied by the $O(1)$ is again the same as in (2).

Using the values in Table 1 and turning (4) around, we get data on the function implied by the $O(1)$ in (2). More precisely, if we set

$$x_n = x_n(\sigma) = \omega^{-n} h_E(\sigma^n P),$$

then for $n > m$,

$$\left(\frac{\omega - 1}{\omega^{-m} - \omega^{-n}}\right) (x_n - x_m) = O(1),$$

so the left-hand side in the above equation gives us some information about the function. We tabulate these values of $\sigma = \sigma_{321}$ in Table 2. Note that $x_0 = 0$, since $h(P_0) = [0, 0, 0]$.

Similar calculations were made for $\sigma_{321}$, $\sigma_{213}$, and $\sigma_{312}$. The absolute values ranged from .09 to 19.5. Unfortunately, this gives us only a lower bound on the desired bound. Still, assuming an upper bound of 100, we get an error of

$$B_1 - x_5(\sigma_{321}) = \lim_{n \to \infty} x_n(\sigma_{321}) - x_5(\sigma_{321}) = \pm \left(\frac{\omega^{-5}}{\omega - 1}\right) 100 \approx \pm .000003,$$

which is a factor of ten better than the error we arrived at in the previous section.
4. Further evidence

The point $P_0$ has an interesting unexpected feature:

$$\sigma_2 \sigma_3 \sigma_2(P_0) = P_0,$$
$$\sigma_3 \sigma_2 \sigma_3(P_0) = P_0.$$

Thus, if $\widehat{\mathbf{h}}$ exists, then $\widehat{\mathbf{h}}(P_0)$ must be in the eigenspace associated to the eigenvalue 1 for $\sigma_2^* \sigma_3^* \sigma_2^*$ and in the eigenspace associated to the eigenvalue 1 for $\sigma_3^* \sigma_2^* \sigma_3^*$. The intersection is one dimensional, and we find

$$\widehat{\mathbf{h}}(P_0) = kD_2^* + kD_3^*$$
for some $k$. Since $\widehat{\mathbf{h}}(P_0) \cdot E = \widehat{\mathbf{h}}_E(P_0)$, we find

$$k = 1.94135 \pm 0.00001$$

when we set $E = E_{123}$ and

$$k = 0.651466 \pm 0.00006$$

when we set $E = E_{321}$, again leading to a contradiction.

References

[Ba2] A. Baragar, Rational points on K3 surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, Math. Ann. 305 (1996), 541–558. MR 97g:14020
[Wa] L. Wang, Rational points and canonical heights on K3-surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, Contemporary Math. 186 (1995), 273–289. MR 97h:14024

Department of Mathematical Sciences, University of Nevada, Las Vegas, Las Vegas, Nevada 89154-4020
E-mail address: baragar@unlv.nevada.edu