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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the American Mathematical Society classification scheme. The 2000 Mathematics Subject Classification can be found in print starting with the 1999 annual index of *Mathematical Reviews*. The classifications are also accessible from `www.ams.org/msc/`.

9[65N30, 65N50, 65N15]—*Finite element methods and B-splines*, by Klaus Höllig, Frontiers in applied mathematics, SIAM, Philadelphia, PA, 2003, viii+145 pp., ISBN 0-89871-533-4, $65.00

With the standard h-version finite element method available to overcome complicated boundaries, with discontinuous Galerkin methods available so that $h-p$ versions can also overcome complicated boundaries, and with various mesh-less methods constructed to conquer complicated boundaries, why would one even want to consider a method based on straight boxes, to boot uniform, where the mesh-geometry decidedly cannot match a complicated geometry?

Professor Höllig, together with twelve collaborators whom he mentions by name in the Preface, begs to differ. After all, uniform boxes are easy to keep track of while unstructured meshes are not. He makes his case very eloquently without resorting to hyperbole. He fairly and honestly presents the difficulties and how one may overcome them. His answer is *web*-splines, where "w" stands for weighted, "e" for extended, and "b" for B-splines (in the classical sense of smoothest B-splines in this account).

OK, what can you do if your mesh-domain is a serious mismatch with the actual domain? There will be small slivers that will certainly cause grief in the condition number of the stiffness matrix. (At least one may think so. The corresponding right hand side may be correspondingly small, but it is beyond current theoretical numerical analysis to unravel this effect with the abrupt changes in element geometry and size going on here.)

The answer in this book is "e", for extended. Ingeniously using Marsden's identity to go outside the slivers, thus creating a uniformly sizeable domain for the support of this spline extended from the sliver while keeping the actual book-keeping parameters describing the spline "inside" in a uniform manner (due to a "polynomial" relation in Marsden's identity for these book-keeping parameters), the condition number problem for the stiffness matrix seems overcome. To me, that extentional "e" in web seems to be the most important key idea in this development of web-splines.

(Curiously, while Professor Höllig is usually very conscientious about historical references, there is no reference to Martin J. Marsden, *An identity for spline functions with application to variation-diminishing spline approximation*, J. Approximation Theory 3, 1970, 7–49, where the short proof given of the identity is credited to T.N.E. Greville.)

Well, so much for the "e" in web. Now for the "w", for weighted. This has to do with essential boundary conditions, say homogeneous Dirichlet ones. Again, the geometry of the boundary is represented in a lousy fashion by those uniform boxes.

Aha—weigh down those B-splines by multiplying them by a cut-off function that vanishes on the boundary. This is an idea that, as Professor Höllig points out, can be found in the classical numerical analysis text of Kantorovich and Krylov from 1941 (or 1936, depending on how one counts). In the English translation from 1958, L.V. Kantorovich and V.I. Krylov, *Approximate methods of higher analysis* (Interscience Publishers, New York, 1958), this idea is on top of page 276 (Chapter IV, Variational Methods, §2 Ritz's Method and the Method of B.G.Galerkin, subsection 3, The application of the methods of Ritz and B.G.Galerkin to the solution of partial differential equations of the second order, equations (43) and (44)). Well, the idea needs some careful thought if one is interested in rates of convergence! Professor Höllig credits Rvachev, 1995 and onwards, and coworkers, for recent work motivated by geometric design that has made this idea computationally feasible in practice in boundary value problems for partial differential equations. A main problem is that the weight function has to be tied both to how it itself decays to zero at the boundary and how the actual solution of the partial differential equation to be approximated so does. These two things interact in the approximation theory needed, and it seems that fairly strong a priori knowledge is needed. Professor Höllig gives an honest account of these tricky issues. (An alternative approach for Dirichlet boundary conditions may be penalty methods. From what I recall having seen, e.g., in the context of wavelets, this approach can lead to rather ugly "splashes" in the approximation at the boundary.)

The book also considers mesh refinements via hierarchical bases, e.g., toward corner singularities.

This account is aimed at novices and experts alike, theoreticians and practitioners of numerical analysis. Here is a rough overview of the organisation of it.

Chapters 1, 2, 3, and 4 up to Section 4.2, consists of rather well-known material. It does describe, however, the basic difficulties associated with a uniform B-spline method that do not match a complicated boundary. For a reader who just wishes to get a "feel for things", read:

Section 4.3, Weight Functions,

Section 4.4, Web-splines,

Section 4.5, Hierarchical Bases.

These sections give the central ideas.

The rest of this monograph, Chapters 5–8, consists of theoretical numerical analysis and implementation issues, together with examples of how web-splines may be applied to partial differential equations of interest in practice. It is a fine blend of the practical and theoretical, expertly done. It certainly helps to make the case for web-splines convincing.

This monograph is short, to the point, and delightfully written. Occasional "Germanisms" add to the charm of the exposition, at least to this reviewer: "Merely for existence theorems and error estimates some background in functional analysis is helpful". Gazing into my crystal ball, in about a decade and a half from now we shall know if the ideas so eloquently presented here will be part of the mainstream practical methods for approximating solutions of partial differential equations. Certainly, the ease of book-keeping of elements is a big point in favor of web-splines. But, what about internal discontinuities? Even if one could use more general B-splines with coalescing nodes to pick up less smooth solutions, what about if the discontinuities do not fit the uniform boxes? Tons of complicated hierarchical refinements to the rescue (perhaps combined with interior weight functions if enough a priori knowledge is available)?

Well, it is early days yet for those weighted extended B-splines, and it is not fair for a reviewer to ask that they should instantly solve all problems in numerical approximation of solutions of partial differential equations (but a fool tends to ask questions).

In conclusion, this timely, short, and elegant monograph makes the case for web-splines in a very honest fashion.

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10[65H05, 65H10, 65F10, 65F22]—*Solving nonlinear equations with Newton's method*, by C. T. Kelley, Fundamentals of algorithms, SIAM, Philadelphia, PA, 2003, xiii+104 pp., ISBN 0-89871-546-6, softcover, $39.00

In the new series *Fundamentals of algorithms*, SIAM wants to publish short monographs that give the state-of-the-art of numerical algorithms, including sufficient theoretical background to understand the methods, but also details on implementation, and their limitations, a guideline on how to choose a method for a particular problem, and how to deal with failures.

This first volume deals with Newton's method for the solution of nonlinear equations and systems of nonlinear equations. It perfectly fits the objectives of the series. A book like Kelley's *Iterative methods for solving linear and nonlinear equations* (SIAM, 1995) places its contents in a broader perspective.

The nice thing about this book is that it builds up the practical algorithms progressively just as a numerical analyst would design it. One starts with an elementary implementation, tests it and discovers some shortcomings, finds a remedy for them, which leads to an improved version, etc. All the aspects are dealt with: adequate selection of starting vectors; efficient computation of the update steps; stopping criterion; convergence analysis; memory and time efficiency; and, not to forget, the human time, i.e., the time needed to find an appropriate algorithm for your problem if you do not want to write the algorithm yourself. All these topics appear already in the first introductory chapter, mostly illustrated in the scalar case.

When one is dealing with systems of equations, then the update step involves the computation of the approximate Jacobian and the solution of a linear system. So, a considerable part of the text deals with efficient computation of approximate Jacobians and with the (iterative) solution of (large sparse) linear systems. While Chapter 2 deals with direct solvers for the linear system, Chapter 3 treats Newton-Krylov methods using the iterative solvers GMRES, BiCGSTAB, and TFQMR, and the final Chapter 4 discusses quasi-Newton methods represented by Broyden's update of the Jacobian. Besides the examples that are worked out with the different methods (Chendrasekhar's H-equation, two-point boundary value problem, Ornstein-Zernike equation, convection-diffusion equation), also some suggestions for stimulating projects are given at the end of the chapters.

The algorithms are given in pseudo-code and in MATLAB code which allows the reader to experiment with the methods on some simple, but also on some less trivial examples. Of course the code need not be typed because it can also be downloaded from www.siam.org/books/fa01.

The book contains a wealth of hints for improving efficiency and performance, and it gives many enlightening explanations of what can go wrong and how one should deal with it. However, such a short text can never be complete in all the aspects that one may encounter in solving these problems. For example, not all possibilities of dealing with sparsity, or not all quasi-Newton methods are dealt with. Parallel implementations or special systems, like polynomial systems in several variables, are not discussed. But all the topics that are included are covered in detail. A pleasure to read and very accessible for the nonspecialist.

Except for some elementary knowledge of numerical analysis and linear algebra, no other knowledge is assumed. So it is a text that will not only inspire numerical analysts and students, but it will also be a valuable reference text for researchers and engineers who are involved in a larger software project.

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