ON THE ABSOLUTE MAHLER MEASURE OF POLYNOMIALS HAVING ALL ZEROS IN A SECTOR. II

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ABSTRACT. Let $\alpha$ be an algebraic integer of degree $d$, not 0 or a root of unity, all of whose conjugates $\alpha_i$ are confined to a sector $|\arg z| \leq \theta$. In the paper On the absolute Mahler measure of polynomials having all zeros in a sector, G. Rhin and C. Smyth compute the greatest lower bound $c(\theta)$ of the absolute Mahler measure $\left( \prod_{i=1}^{d} \max(1, |\alpha_i|) \right)^{1/d}$ of $\alpha$, for $\theta$ belonging to nine subintervals of $[0, 2\pi/3]$. In this paper, we improve the result to thirteen subintervals of $[0, \pi]$ and extend some existing subintervals.

1. Introduction

Let $P(z) \neq z$ be a monic polynomial with integer coefficients, irreducible over the rationals, of degree $d \geq 1$, and having zeros $\alpha_1, \ldots, \alpha_d$. Its relative Mahler measure $M(P)$, given by

$$M(P) = \prod_{i=1}^{d} \max(1, |\alpha_i|),$$

is either 1 (if $P$ is cyclotomic) or thought to be bounded away from 1 by an absolute constant (if $P$ is not cyclotomic) [B1, B2]. When the zeros of $P$ are restricted to a closed set $V$ which does not contain the whole unit circle, however, one can say much more. Then, from a result of Langevin [LA] there is a constant $C_V > 1$ such that the absolute Mahler measure $\Omega(P) := M(P)^{1/d}$ for such $P$ is either 1 or else satisfies

$$\Omega(P) \geq C_V.$$

So we try to find the largest value for the constants $C_V$ when $V$ is the sector $\{ z : |\arg z| \leq \theta \}$, where $0 \leq \theta < \pi$. We denote this best value by $c(\theta)$. It is clear that $c(\theta)$ is a nonincreasing function of $\theta$ and, using the polynomials $z^{2k+1} - 2$ as $k \to \infty$, that $c(\theta) \to 1$ as $\theta \to \pi$.

In a previous paper [RS], G. Rhin and C. Smyth succeeded in finding $c(\theta)$ exactly for $\theta$ in nine intervals. They conjectured that $c(\theta)$ is a “staircase” function of $\theta$ which is constant except for finitely many left discontinuities in any interval $[0, \Theta)$ for $\Theta < \pi$. They used auxiliary functions of the type

$$f_i(\theta) = \max_{z \in W_\theta} \left| z^{a_i} \prod_j P_{ij}(z)^{e_{ij}} \right|^{-1/(2a_i + \sum_j e_{ij} \deg P_{ij})}$$

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in the sector $W_\theta = \{ \vert z \vert < 1, \vert \arg z \vert \leq \theta \}$. Then they find:

**Theorem.** There is a continuous, monotonically decreasing function $f(\theta) > 1$ for $0 \leq \theta \leq 2\pi/3$ and there is a staircase function $g(\theta) > 1$ such that

$$\min(f(\theta), g(\theta)) \leq c(\theta) \leq g(\theta) \quad (0 \leq \theta < \pi).$$

The function $f(\theta)$ is given by $f(\theta) := \max_{i=1}^{9} f_i(\theta)$. The function $g(\theta)$ is a decreasing staircase function having left discontinuities at the angles given (in degrees) in Table 4 of [RS]. The corresponding absolute measure is the new smaller value of $g(\theta)$ which is the smallest value of $\Omega(P)$ that could be found, for $P$ having all its zeros in $\vert \arg z \vert \leq \theta$.

In the proof of the Theorem, Rhin and Smyth referred to Langevin’s proof [LA], which has three basic ingredients:

(i) the observation that the set $V_1 = V \cap \{z \in \mathbb{C} : \vert z \vert \leq 1 \}$ has transfinite diameter less than 1,

(ii) a result of Kakeya to the effect that for any set $W$ of transfinite diameter less than 1 and symmetric about the real axis there is a nonzero polynomial $A$ with integer coefficients such that $\sup_{z \in W} \vert A(z) \vert < 1$,

(iii) deduction of $\Omega(P) \geq C_\theta$ from (i) and (ii) using $W := \{ z : z \in V \text{ and } \bar{z} \in V \}$.

For the computation of $f(\theta) = \max_{i=1}^{9} f_i(\theta)$, they use, for each $f_i$, an auxiliary polynomial $A$ as in (ii), and they choose such $A$ of the form $z^a R(z)$, where $a$ is a positive integer and $R$ is a reciprocal polynomial of degree $r$ with integer coefficients, i.e.,

$$A(z) = z^a \prod_j P_j(z)^{c_j}.$$  

The function

$$m(\theta) = \sup_{z \in W_\theta} \vert A(z) \vert^{1/(2a+r)}$$  

is then associated with $A$. Then Langevin’s argument of (iii) above gives

$$\Omega(P) \geq \frac{1}{m(\theta)} \quad \text{if } \gcd(P, A) = 1$$

for $P$ irreducible, of degree $d$, with integer coefficients. For, if $\alpha_1, \ldots, \alpha_d$ are the zeros of $P$, then, since $R(z) = z^r R(z^{-1})$, one has

$$1 \leq \prod_{i=1}^{d} \vert a_i^a R(\alpha_i) \vert = \prod_{\vert \alpha_i \vert \leq 1} \vert a_i^a R(\alpha_i) \vert \times \prod_{\vert \alpha_i \vert > 1} \vert a_i^{a+r} R(\alpha_i^{-1}) \vert$$

$$= \prod_{\vert \alpha_i \vert \leq 1} \vert a_i^a R(\alpha_i) \vert \times \prod_{\vert \alpha_i \vert > 1} \vert (\alpha_i^{-1})^a R(\alpha_i^{-1}) \vert \times \prod_{\vert \alpha_i \vert > 1} \alpha_i^{2a+r}$$

$$\leq m(\theta)^{(2a+r)\delta M(P)^{2a+r}}$$

whence $\Omega(P) \geq 1/m(\theta)$.

Then each $f_i(\theta)$ was defined, as in equation (1), to be the function $1/m(\theta)$ corresponding to a polynomial $A$ chosen so that $f(\theta_i) > g(\theta_i)$ and so that the length of the interval $[\theta_i, \theta_i']$ over which $f(\theta) > g(\theta)$ was as long as possible. Thus, if $g(\theta_i) = \Omega(P_*)$ (Table 4 in [RS]), then $\Omega(P_*) < f_i(\theta_i)$. From (2) it follows that $P$ is a factor of $A$ and that, among polynomials with all conjugates in $\vert \arg z \vert \leq \theta_i$, only factors of $A$ can have absolute measure less than $f_i(\theta_i)$. Now $P_*$ does indeed divide
A, and in fact it has the smallest absolute measure among factors $A$ of measure $> 1$. It follows that $\Omega(P_j)$ is the smallest value of the absolute measure for polynomials having all zeros in $|\arg z| \leq \theta$ for $\theta \in [\theta_1, \theta']$. Hence, $c(\theta) = \Omega(P_j)$ for these $\theta$.

One of the main problems in the previous paper was to find for each interval suitable polynomials to use to obtain a good auxiliary function. In fact they only used a heuristic process and produced a table of good polynomials to use to obtain a good auxiliary function. In fact they only used a heuristic process and produced a table of good polynomials for almost all polynomials of one of the following six types:

$$
\begin{align*}
&z^nQ(z + z^{-1} - k) \quad (k = 3, 2, 1, 0) \quad \text{(types 1, 2, 3, 4)}, \\
&z^nS(z + z^{-1} - 2) \quad \text{where } S(x) = Q(1)x^nQ(1 + 1/x) \quad \text{(type 5),} \\
&z^n(Q(z) + Q(1/z)) \quad \text{(type 6)}.
\end{align*}
$$

Here $Q$ is a degree $n$ monic polynomial with small coefficients, also with $Q(1) = \pm 1$ for the fifth type. As pointed out in [RS] p. 301 “The reason for polynomials of these types giving good polynomials appears mysterious, however.”

The second author gave in [WU] an algorithm improving the ones given by P. Borwein and T. Erdelyi [BE] and L. Habsieger and B. Salvy [HS] to find polynomials which have to be involved in such auxiliary functions $f_i(\theta)$. This method gives better lower bounds for $c(\theta)$ for four new intervals of $\theta$ between 0 and $7\pi/9$.

Table 1 shows the 13 intervals $[\theta_i, \theta'_i]$ where $f(\theta) > g(\theta)$, so that $c(\theta) = g(\theta) = g(\theta_i)$ for $\theta$ in those intervals; i.e., $c(\theta)$ is known exactly. Here $c(\theta) = c(\theta_i) = \Omega(p)$ for $\theta \in [\theta_i, \theta'_i]$. The fifth column presents the results from [RS]. The polynomial $P$ is read off from Table 3. The function $f(\theta)$ is given by $f(\theta) := \max_{i=1}^{13} f_i(\theta)$ where the $f_i(\theta)$ are defined as in (1) and the $a_i$, $P_j$ and the $c_{ij}$ are given by Table 2 using the polynomials of Table 3. The function $g(\theta)$ employs the polynomials listed in Table 4 of [RS], where we add the polynomials $P_{25}$ and $P_{31}$ of our Table 6.

Table 2 gives the auxiliary functions

$$
A_i(z) = z^{a_i} \prod_j P_{ij}(z)^{e_{ij}}
$$

used to compute $f_i(\theta)$ for $i = 1, \cdots, 13$.

Table 3 shows the reciprocal polynomials used in Tables 1 and 2 where $d = \deg P$ and $\varphi(P) = \max\{|\arg z| : P(z) = 0\}$.

**Table 1.**

<table>
<thead>
<tr>
<th>$i$</th>
<th>$c(\theta)$</th>
<th>$\theta_i$</th>
<th>$\theta'_i$ in [RS]</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.618034</td>
<td>0.000000</td>
<td>17.40</td>
<td>$P_2$</td>
</tr>
<tr>
<td>2</td>
<td>1.539222</td>
<td>26.408740</td>
<td>26.65</td>
<td>$P_2$</td>
</tr>
<tr>
<td>3</td>
<td>1.493633</td>
<td>30.440145</td>
<td>30.74</td>
<td>$P_8$</td>
</tr>
<tr>
<td>4</td>
<td>1.303055</td>
<td>47.941432</td>
<td>49.46</td>
<td>$P_9$</td>
</tr>
<tr>
<td>5</td>
<td>1.300734</td>
<td>50.830684</td>
<td>50.96</td>
<td>$P_{12}$</td>
</tr>
<tr>
<td>6</td>
<td>1.259269</td>
<td>60.890196</td>
<td>63.87</td>
<td>$P_{15}$</td>
</tr>
<tr>
<td>7</td>
<td>1.210608</td>
<td>73.631615</td>
<td>74.04</td>
<td>$P_{19}$</td>
</tr>
<tr>
<td>8</td>
<td>1.154618</td>
<td>80.241034</td>
<td>82.43</td>
<td>$P_{21}$</td>
</tr>
<tr>
<td>9</td>
<td>1.129338</td>
<td>86.708519</td>
<td>91.40</td>
<td>$P_{23}$</td>
</tr>
<tr>
<td>10</td>
<td>1.096504</td>
<td>101.353607</td>
<td>101.99</td>
<td>$P_{25}$</td>
</tr>
<tr>
<td>11</td>
<td>1.055423</td>
<td>112.647119</td>
<td>115.32</td>
<td>$P_{29}$</td>
</tr>
<tr>
<td>12</td>
<td>1.033097</td>
<td>127.355699</td>
<td>129.47</td>
<td>$P_{31}$</td>
</tr>
<tr>
<td>13</td>
<td>1.020306</td>
<td>137.102805</td>
<td>137.15</td>
<td>$P_{37}$</td>
</tr>
</tbody>
</table>
As we have seen in the introduction, the search for a good auxiliary function $A$ Let $\theta$ be a fixed angle. For a nonzero polynomial $A \in \mathbb{Z}[z]$ we define $a = a(A)$ to be the multiplicity of the root 0 of $A$ and $\|A\| = \sup_{z \in W_0} |A(z)|$. As we have seen in the introduction, the search for a good auxiliary function $f$ for $W_0$ is equivalent to seeking a polynomial $A$ (such that $z^{-a}A$ is reciprocal) in $\mathbb{Z}[z]$ and such that $\|A\|^{\frac{1}{\deg a(A) + 1}}$ is as small as possible.

Let $A_n$ be the polynomial of degree $n$ such that

$$\|A_n\|^{\frac{1}{\deg a(A) + 1}} = \min_{A \in \mathbb{Z}[z]} \|A\|^{\frac{1}{\deg a(A) + 1}}.$$

### Table 2.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$P_i$</th>
<th>$P_0$</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
<th>$P_4$</th>
<th>$P_5$</th>
<th>$|P_i|$</th>
<th>$|P_0|$</th>
<th>$|P_1|$</th>
<th>$|P_2|$</th>
<th>$|P_3|$</th>
<th>$|P_4|$</th>
<th>$|P_5|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.050000</td>
<td>0.000000</td>
<td>0.500000</td>
<td>1.000000</td>
<td>1.500000</td>
<td>2.000000</td>
<td>2.500000</td>
<td>3.000000</td>
<td>3.500000</td>
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<td>4.500000</td>
<td>5.000000</td>
<td>5.500000</td>
<td>6.000000</td>
</tr>
</tbody>
</table>

### Table 3.

<table>
<thead>
<tr>
<th>$P_i$</th>
<th>$x(P_i)$</th>
<th>$d$</th>
<th>Highest half of coefficients of $P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>1.000000</td>
<td>0.000000</td>
<td>2</td>
</tr>
<tr>
<td>$P_2$</td>
<td>1.610000</td>
<td>0.000000</td>
<td>2</td>
</tr>
<tr>
<td>$P_3$</td>
<td>1.630000</td>
<td>0.000000</td>
<td>3</td>
</tr>
<tr>
<td>$P_4$</td>
<td>1.640000</td>
<td>0.000000</td>
<td>4</td>
</tr>
</tbody>
</table>

### Table 4.

<table>
<thead>
<tr>
<th>$P_i$</th>
<th>$x(P_i)$</th>
<th>$d$</th>
<th>Highest half of coefficients of $P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>1.000000</td>
<td>0.000000</td>
<td>2</td>
</tr>
<tr>
<td>$P_2$</td>
<td>1.610000</td>
<td>0.000000</td>
<td>2</td>
</tr>
<tr>
<td>$P_3$</td>
<td>1.630000</td>
<td>0.000000</td>
<td>3</td>
</tr>
<tr>
<td>$P_4$</td>
<td>1.640000</td>
<td>0.000000</td>
<td>4</td>
</tr>
</tbody>
</table>

### Table 5.

<table>
<thead>
<tr>
<th>$P_i$</th>
<th>$x(P_i)$</th>
<th>$d$</th>
<th>Highest half of coefficients of $P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$</td>
<td>1.000000</td>
<td>0.000000</td>
<td>2</td>
</tr>
<tr>
<td>$P_2$</td>
<td>1.610000</td>
<td>0.000000</td>
<td>2</td>
</tr>
<tr>
<td>$P_3$</td>
<td>1.630000</td>
<td>0.000000</td>
<td>3</td>
</tr>
<tr>
<td>$P_4$</td>
<td>1.640000</td>
<td>0.000000</td>
<td>4</td>
</tr>
</tbody>
</table>

2. Search for good polynomials for the auxiliary functions

Let $\theta < \pi$ be a fixed angle. For a nonzero polynomial $A \in \mathbb{Z}[z]$ we define $a = a(A)$ to be the multiplicity of the root 0 of $A$ and $\|A\| = \sup_{z \in W_0} |A(z)|$. As we have seen in the introduction, the search for a good auxiliary function $f$ for $W_0$ is equivalent to seeking a polynomial $A$ (such that $z^{-a}A$ is reciprocal) in $\mathbb{Z}[z]$ and such that $\|A\|^{\frac{1}{\deg a(A) + 1}}$ is as small as possible.

Let $A_n$ be the polynomial of degree $n$ such that

$$\|A_n\|^{\frac{1}{\deg a(A) + 1}} = \min_{A \in \mathbb{Z}[z]} \|A\|^{\frac{1}{\deg a(A) + 1}}.$$
We can define
\[
\tau_{\theta} = \lim_{n \to \infty} \| A_n \|^n \frac{1}{n+1}
\]
as a generalization of \( t_{\zeta}(W_\theta) \) which is the integer transfinite diameter of \( W_\theta \) (in this case the exponent of \( \| A_n \| \) is \( 1/n \)).

Then the factors of the polynomials \( A_n \) lead to good auxiliary functions as follows. It is difficult to compute the polynomials \( A_n \) for large \( n \), so we will compute some polynomials \( A'_n \) of sufficiently large degree (say 40) where the norm \( \| A'_n \| \) is sufficiently small and use their factors \( Q_j \) inside the function \( f \). For this we use the following algorithm, which was already described in [WU].

Step 1. We use the LLL algorithm to find a polynomial \( Q(x) \) of degree \( m \) (say 30) in \( \mathbb{Z}[x] \) which has a small sup norm in the interval \([2 \cos \theta, 2]\). Then we choose the integer \( a \) such that the polynomial \( A = z^{a+m}Q(z+1/z) \) has a norm \( \| A \|^{1/(2a+2m)} \) as small as possible.

It is well known that the LLL algorithm gives better results when used in low dimension. So, in Step 2 we will show that \( A'_n \) has an explicit factor of large degree.

Step 2. We use the previous bound and a generalization of the orthogonal Müntz-Legendre polynomials to find polynomials that must divide \( A'_n = A \) (where \( n = a + 2m \)) in \( \mathbb{Z}[z] \).

Step 3. We use now the LLL algorithm to find new polynomial factors of \( A'_n \).

By this algorithm, we find the polynomials \( P_{25} \) and \( P_{31} \), which not only improve the function \( g(\theta) \), but also give better bounds for \( c(\theta) \) in the intervals \([101.35, 101.99]\) and \([127.35, 129.47]\). We also find polynomials (for example \( P_{13} \)) that do not improve the function \( g(\theta) \) but give us a new interval in which \( c(\theta) \) is known exactly. Furthermore, we find some other polynomials for the auxiliary function which enable us to extend existing intervals.

3. Computation of the auxiliary functions

We use (for a fixed \( \theta \)) the auxiliary function
\[
f(z) = |z|^a \prod_{j=1}^{J} |Q_j(z)|^{e_j}
\]
where the polynomials \( Q_j \) are those which have been computed in Section 2 and such that the positive rationals \( a \) and \( e_j \) satisfy the linear condition
\[2a + \sum_{j=1}^{J} e_j \deg Q_j = 1.\]

The optimal function \( f \) is obtained by semi-infinite linear programming [WU, RS]. This gives four new intervals for \( c(\theta) \). Moreover, technical improvements allow us to enlarge some intervals found earlier.

References


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