FIVE CONSECUTIVE POSITIVE ODD NUMBERS, 
NONE OF WHICH CAN BE EXPRESSED 
AS A SUM OF TWO PRIME POWERS

YONG-GAO CHEN

Abstract. In this paper, we prove that there is an arithmetic progression of 
positive odd numbers for each term \( M \) of which none of five consecutive odd 
numbers \( M, M - 2, M - 4, M - 6 \) and \( M - 8 \) can be expressed in the form 
\( 2^n \pm p^a \), where \( p \) is a prime and \( n, a \) are nonnegative integers.

Introduction

By calculation, we find that almost all positive odd numbers can be expressed in the form \( 2^n + p \), where \( n \) is a positive integer and \( p \) is prime. For example, 
\( 5 = 2 + 3, 7 = 2 + 5, 9 = 2 + 7, 11 = 2^2 + 7, 13 = 2 + 11, 15 = 2 + 13, 17 = 2^2 + 13 \), etc. The first counterexample is 127. In 1934, Romanoff [11] proved 
that the set of positive odd numbers which can be expressed in the form \( 2^n + p \) 
has positive asymptotic density in the set of all positive odd numbers, where \( n \) is 
a nonnegative integer and \( p \) is prime. For a positive integer \( n \) and an integer \( a \), 
let \( a \equiv a_i \pmod{m_i} \) be called a covering system if every integer \( b \) satisfies 
\( b \equiv a_i \pmod{m_i} \) for at least one value of \( i \). By employing a 
covering system, P. Erdős [8] proved that there is an infinite arithmetic progression 
of positive odd numbers each of which has no representation of the form \( 2^n + p \). 
Cohn and Selfridge [7] proved that there exist infinitely many odd numbers which 
are neither the sum nor the difference of two prime powers. In [3] Chen proved the 
following result: the set of positive integers which have no representation of the form 
\( 2^n \pm p^a q^b \), where \( p, q \) are distinct odd primes and \( n, a, b \) are nonnegative integers, 
has positive lower asymptotic density in the set of all positive odd integers. That is, 
the lower asymptotic density of the set of positive odd integers \( k \) such that \( k - 2^n \) 
has at least three distinct prime factors for all positive integers \( n \) is positive. In [5] 
Chen showed that the set of positive odd integers \( k \) such that \( k - 2^n \) has at least 
three distinct prime factors for all positive integers \( n \) contains an infinite arithmetic 
progression. For further related information see Chen [4], [1], Guy [9], A19, B21, 
F13], Jaeschke [10], and Stanton and Williams [12]. The following question is a 
natural one: Are there two consecutive positive odd numbers neither of which can 
be expressed as a sum of two prime powers?
In this paper, we show that the answer to the question is affirmative. In fact, we go much further.

**Theorem 1.** Let \( k_1, \ldots, k_s \) be integers, let \( \{a_{ij} \pmod{m_{ij}}\}_{j=1}^t \) be \( s \) covering systems with \( 0 \leq a_{ij} < m_{ij} \), and let \( p_{ij} \) be primes with \( m_{ij} \) the order of \( 2 \pmod{p_{ij}} \) \((1 \leq j \leq t, 1 \leq i \leq s)\) such that if \( p_{ij} = p_{uv} \), then
\[
2^{a_{ij}} - k_i \equiv 2^{a_{uv}} - k_u \pmod{p_{ij}}.
\]
Then there exists an arithmetic progression of positive odd numbers for each term \( M \) of which none of \( M + k_i \) \((1 \leq i \leq s)\) can be expressed in the form \( 2^n \pm p^\alpha \), where \( p \) is a prime and \( n, \alpha \) are nonnegative integers.

**Theorem 2.** There exists an arithmetic progression of positive odd numbers for each term \( M \) of which none of five consecutive odd numbers \( M, M - 2, M - 4, M - 6 \) and \( M - 8 \) can be expressed in the form \( 2^n \pm p^\alpha \), where \( p \) is a prime and \( n, \alpha \) are nonnegative integers.

**Remark.** By the proofs of Theorems 1 and 2, there is an integer \( M \leq 2^{253000} \) such that none of five consecutive odd numbers \( M, M - 2, M - 4, M - 6 \) and \( M - 8 \) can be expressed in the form \( 2^n \pm p^\alpha \). Currently, we cannot give an explicit value of \( M \).

2. Proofs

**Lemma 1.** Let \( p \) be an odd prime and let \( T \) be a positive integer. Then \( 2^{p^T} - 1 \) has at least \( T \) distinct prime factors.

**Proof.** Let \( q_i \) \((i = 1, 2, \ldots, T)\) be primes with
\[
q_i \mid \frac{2^{p^i} - 1}{2^{p^{i-1}} - 1}.
\]
Then \( q_1, q_2, \ldots, q_T \) are distinct primes. This completes the proof of Lemma 1. \( \square \)

**Lemma 2.** Let \( p \) be an odd prime and let \( m \) be the order of \( 2 \pmod{p} \). If
\[
2^m = 1 + p^d, \quad p \nmid d,
\]
and \( p^n|2^n - 1 \) for two integers \( n \geq 0 \) and \( u > 0 \), then \( n = mp^{u-l}v \) for some integer \( v \).

**Proof.** By using induction on \( r \), we can prove that
\[
2^{mp^r} = 1 + p^{l+r}d_r, \quad p \nmid d_r, \quad r = 0, 1, \ldots.
\]
By \( p|2^n - 1 \) and \( m \) being the order of \( 2 \pmod{p} \), we have \( m|n \). Let \( n = mp^hv' \), \( p \nmid v' \). Then
\[
2^n = 2^{mp^hv'} = 1 + p^{l+h}d_h', \quad p \nmid d_h'.
\]
Since \( p^u|2^n - 1 \), we have \( u \leq l + h \). Hence \( h \geq n - l \). Let \( v = v'h^{u-l} \). This completes the proof of Lemma 2. \( \square \)

**Lemma 3.** Let \( p_1, \ldots, p_t \) be primes such that each prime repeats at most \( s \) times. Then there exist \( t \) distinct primes \( q_1, \ldots, q_t \) such that
\[
q_i|2^{p_i^{s+i}} - 1, \quad q_i \neq p_j, \quad \text{for all } i, j.
\]
Proof. For each prime \( p \), by Lemma 1 we may take a set \( S(p) \) of primes with \(|S(p)| = t + s\) such that
\[
q_i | 2^{p^{t+s}} - 1.
\]
Since there are at most \( s \) indexes \( i \) with \( p_i = p \), we may appoint a prime \( q_i \in S(p) \setminus \{p_1, \ldots, p_t\} \) for each \( i \) with \( p_i = p \) such that if \( p_i = p_j = p \), then \( q_i \neq q_j \). If \( p_i \neq p_j \), then, by \( q_i \in S(p_i) \) and \( q_j \in S(p_j) \) we have
\[
q_i | 2^{p^{t+s}} - 1, \quad q_j | 2^{p^{t+s}} - 1.
\]
Hence \( q_i \neq q_j \). Thus, these \( q_i \) are distinct such that
\[
q_i | 2^{p^{t+s}} - 1, \quad q_i \neq p_j, \quad \text{for all } i, j.
\]
This completes the proof of Lemma 3. \( \square \)

Proof of Theorem 1. If \( p_{iu} = p_{iv} \), then, by \( m_{iu} \) and \( m_{iv} \) being the orders of \( 2 \pmod{p_{iu}} \) and \( 2 \pmod{p_{iv}} \), respectively, we have \( m_{iu} = m_{iv} \). By
\[
2^{a_{iu}} - k_i \equiv 2^{a_{iv}} - k_i \pmod{p_{iu}}
\]
and \( m_{iu} \) of the order of \( 2 \pmod{p_{iu}} \), we have
\[
a_{iv} \equiv a_{iu} \pmod{m_{iu}}.
\]
Hence \( a_{iu} \pmod{m_{iu}} = a_{iv} \pmod{m_{iv}} \). Thus, without loss of generality, we may assume that for each \( i \), primes \( p_{i1}, \ldots, p_{iu} \) are distinct. Let \( T = s + t_1 + \cdots + t_s \).

By Lemma 3, for each \( p_{ij} \), we may appoint a prime \( q_{ij} \) such that all primes \( q_{ij} \) \((1 \leq j \leq t_i, 1 \leq i \leq s)\) are distinct,
\[
q_{ij} | 2^{p_{ij}^s} - 1, \quad 1 \leq j \leq t_i, 1 \leq i \leq s,
\]
and \( q_{ij} \neq p_{uv} \) for all \( 1 \leq j \leq t_i, 1 \leq i \leq s, 1 \leq v \leq t_u, 1 \leq u \leq s \). Let \( r_{ij} \) be integers such that \( 0 \leq r_{ij} < p_{ij} \) and
\[
(1) \quad r_{ij} \equiv 2^{a_{ij}} - k_i \pmod{p_{ij}}, \quad 1 \leq j \leq t_i, 1 \leq i \leq s.
\]
Let
\[
2^{m_{ij}} = 1 + p_{ij}^{l_{ij}} t_{ij}, \quad p / t_{ij}, \quad 1 \leq j \leq t_i, 1 \leq i \leq s,
\]
and \( l = \max_{i,j} l_{ij} \). If there exists a nonnegative integer \( b \equiv a_{ij} \pmod{m_{ij}} \) with
\[
(2) \quad p_{ij}^{l_{ij}+T} k_i^b \equiv - r_{ij},
\]
then let \( b_{ij} \) be the least one of such \( b \). If there are no such \( b \), then let \( b_{ij} = a_{ij} \). Let \( m \) be a positive integer with
\[
2^m \geq \max_{i,j} p_{ij}^{l_{ij}+T} + \max_i |k_i| + 1.
\]
Take an integer \( M \) with
\[
(3) \quad M \equiv r_{ij} \pmod{p_{ij}^{l_{ij}+T}},
\]
\[
M \equiv 2^{b_{ij}} - k_i \pmod{q_{ij}}, \quad 1 \leq j \leq t_j, 1 \leq i \leq s,
\]
\[
M \equiv 1 + 2^m + 2^{m+1} \pmod{2^{m+2}}.
\]
If \( p_{ij} = p_{uv} \), then \( r_{ij} = r_{uv} \) by the condition. Again, \( q_{ij} \) are distinct and each \( q_{ij} \) is different from any \( p_{uv} \). So such an \( M \) exists by the Chinese Remainder Theorem. Now we prove that none of \( M + k_i (1 \leq i \leq s) \) can be expressed in the form \( 2^n \pm p^n \), where \( p \) is a prime and \( n, \alpha \) are nonnegative integers. In order to prove this, it is enough to show that for each \( i \) and any nonnegative integer \( n \), \( M + k_i - 2^n \) has at least two distinct positive prime factors. Since \( \{a_{ij} (\mod m_{ij})\}_j^{1} \) is a covering system, there exists a \( j \) with

\[
 n \equiv a_{ij} (\mod m_{ij}).
\]

By (1), (3) and \( 2^{m_{ij}} \equiv 1 (\mod p_{ij}) \), we have

\[
 M + k_i - 2^n \equiv r_{ij} + k_i - 2^{a_{ij}} \equiv 0 (\mod p_{ij}).
\]

Let

\[
 M + k_i - 2^n = p^{a_{ij}} K_{ij}, \quad p_{ij} \not\mid K_{ij}, \quad \alpha_{ij} \geq 1.
\]

If \( \alpha_{ij} < l + T \), then by

\[
 |M + k_i - 2^n| = |1 + 2^m + 2^{m+1} + 2^{m+2} + 2^{m+2}u + k_i - 2^n| \\
 \geq |1 + 2^m + 2^{m+1} + 2^{m+2} - 2^n| - |k_i| \\
 \geq 2^m - 1 - |k_i| \geq p^{l+T}_{ij},
\]

we have \( |K_{ij}| > 1 \). In this case, \( M + k_i - 2^n \) has at least two distinct prime factors.

If \( \alpha_{ij} \geq l + T \), then \( n \equiv a_{ij} (\mod m_{ij}) \) and

\[
 r_{ij} + k_i - 2^n \equiv M + k_i - 2^n \equiv 0 (\mod p^{l+T}_{ij}).
\]

Hence \( n \equiv b_{ij} (\mod m_{ij}) \) and by (2),

\[
 2^{b_{ij}}(1 - 2^{n-b_{ij}}) \equiv 2^{b_{ij}} - k_i + k_i - 2^n \equiv r_{ij} + k_i - 2^n \equiv 0 (\mod p^{l+T}_{ij}).
\]

Thus

\[
 p^{l+T}_{ij} \not\mid 2^{n-b_{ij}} - 1.
\]

By Lemma 2 we have \( n - b_{ij} = m_{ij}p^{l+T}_{ij}v_{ij} \) for some integer \( v_{ij} \). By

\[
 q_{ij} \not\mid 2^{p^{l+T}_{ij}} - 1,
\]

we have

\[
 q_{ij} \not\mid 2^{n-b_{ij}} - 1.
\]

That is,

\[
 q_{ij} \not\mid 2^n - 2^{b_{ij}}.
\]

Hence

\[
 M + k_i - 2^n \equiv 2^{b_{ij}} - k_i + k_i - 2^n \equiv 2^{b_{ij}} - 2^n \equiv 0 (\mod q_{ij}).
\]

Thus \( q_{ij} \mid K_{ij} \) and then \( M + k_i - 2^n \) has at least two distinct prime factors. This completes the proof of Theorem 1.
Proof of Theorem 2. Let $k_1 = 0$, $k_2 = -2$, $k_3 = -4$, $k_4 = -6$ and $k_5 = -8$. Take

$$\{a_{1j} \mod m_{1j}\}^8_{j=1} = \{0 \mod 2, 3 \mod 4, 5 \mod 8, 9 \mod 16, 17 \mod 32, 33 \mod 64, 1 \mod 128\},$$

$$\{a_{2j} \mod m_{2j}\}^7_{j=1} = \{1 \mod 2, 0 \mod 4, 6 \mod 8, 10 \mod 16, 18 \mod 32, 34 \mod 64, 2 \mod 64\},$$

$$\{a_{3j} \mod m_{3j}\}^{26}_{j=1} = \{0 \mod 3, 2 \mod 4, 3 \mod 5, 1 \mod 10, 4 \mod 12, 2 \mod 15, 1 \mod 18, 7 \mod 20, 8 \mod 24, 19 \mod 25, 24 \mod 25, 11 \mod 36, 23 \mod 36, 25 \mod 40, 25 \mod 45, 40 \mod 45, 20 \mod 48, 44 \mod 48, 9 \mod 50, 39 \mod 50, 37 \mod 60, 35 \mod 72, 4 \mod 75, 5 \mod 120, 29 \mod 150, 215 \mod 360\},$$

$$\{a_{4j} \mod m_{4j}\}^9_{j=1} = \{0 \mod 2, 1 \mod 4, 7 \mod 8, 11 \mod 16, 19 \mod 32, 35 \mod 64, 67 \mod 128, 3 \mod 256, 131 \mod 256\}$$

$$\{a_{5j} \mod m_{5j}\}^{13}_{j=1} = \{1 \mod 2, 2 \mod 3, 2 \mod 5, 4 \mod 9, 6 \mod 10, 6 \mod 12, 10 \mod 18, 0 \mod 20, 24 \mod 30, 34 \mod 36, 48 \mod 60, 34 \mod 90, 88 \mod 180\}. $$

Noting that $\{a_j \mod m_j\}^k_{j=1}$ is a covering system if and only if for every integer $n$ with $0 \leq n < \text{l.c.m.} \{m_1, \ldots, m_k\}$ there exists a $j$ with $n \equiv a_j \mod m_j$, we can verify that $\{a_{1j} \mod m_{1j}\}^8_{j=1}$, $\{a_{2j} \mod m_{2j}\}^7_{j=1}$, $\{a_{3j} \mod m_{3j}\}^{26}_{j=1}$, $\{a_{4j} \mod m_{4j}\}^9_{j=1}$ and $\{a_{5j} \mod m_{5j}\}^{13}_{j=1}$ are all covering systems. Now, for every $a_{ij} \mod m_{ij}$ we appoint a prime $p_{ij}$ such that $m_{ij}$ is the order of 2 (mod $p_{ij}$) and if $p_{ij} = p_{uv}$, then

$$2^{a_{ij}} - k_i \equiv 2^{a_{uv}} - k_u \mod p_{ij}.$$ 

(4)

Case 1. Let $p_{11} = p_{21} = p_{41} = p_{51} = 3$. Then

$$2^0 - 0 \equiv 2^1 - (-2) \equiv 2^0 - (-6) \equiv 2^1 - (-8) \mod 3.$$ 

Case 2. Let $p_{12} = p_{22} = p_{32} = p_{42} = 5$. Then

$$2^3 - 0 \equiv 2^0 - (-2) \equiv 2^2 - (-4) \equiv 2^1 - (-6) \mod 5.$$
If \( m > \frac{1}{2} \), these primes are 601, 1801; 631, 23311; 97, 673; 251, 4051; 61, 1321, respectively.

Note that both Fermat numbers \( 2^{128}+1 \) and \( 2^{64}+1 \) are composite, let \( p_{18} = p_{47}, p_{17} \) be two distinct prime divisors of \( 2^{64}+1 \), and let \( p_{48}, p_{49} \) be two distinct prime divisors of \( 2^{128}+1 \). Then (4) follows from the following fact:

\[
2^{2k} + 1 - 0 \equiv 2^{2k} + 2 - (-2) \equiv 2^{2k} + 3 - (-6) \pmod{2^{2k} + 1}.
\]

**Case 3.** Let

\[
p_{13} = p_{23} = p_{43} = 17, \quad p_{14} = p_{24} = p_{44} = 257,
\]

\[
p_{15} = p_{25} = p_{45} = 65537, \quad p_{16} = p_{26} = p_{46} = 641, \quad p_{27} = 6700417.
\]

These primes are 601, 1801; 631, 23311; 97, 673; 251, 4051; 61, 1321, respectively.

**Case 4.** Let

\[
p_{31} = p_{32} = 7, \quad p_{33} = p_{34} = 31,
\]

\[
p_{34} = p_{35} = 11, \quad p_{35} = p_{36} = 13,
\]

\[
p_{37} = p_{38} = 19, \quad p_{38} = p_{39} = 41,
\]

\[
p_{3(12)} = p_{5(10)} = 109, \quad p_{3(13)} = 37.
\]

Then

\[
2^0 - (-4) \equiv 2^2 - (-8) \pmod{7}, \quad 2^3 - (-4) \equiv 2^2 - (-8) \pmod{31},
\]

\[
2^1 - (-4) \equiv 2^6 - (-8) \pmod{11}, \quad 2^4 - (-4) \equiv 2^6 - (-8) \pmod{13},
\]

\[
2^1 - (-4) \equiv 2^{10} - (-8) \pmod{19}, \quad 2^7 - (-4) \equiv 2^9 - (-8) \pmod{41},
\]

\[
2^{11} - (-4) \equiv 2^{34} - (-8) \pmod{109}.
\]

**Case 5.** Each of 25, 45, 48, 50, 60 is the order of 2 modulus two distinct primes. These primes are 601, 1801; 631, 23311; 97, 673; 251, 4051; 61, 1321, respectively. If \( m > 1 \) and \( m \neq 6 \), then there exists at least one prime \( p \) with \( m \) the order of 2 \( \pmod{p} \) (see [1], [2], [13]). Thus we may appoint a prime \( p_{ij} \) for each of the remaining \( a_{ij} \pmod{m_{ij}} \). Now, Theorem 2 follows from Theorem 1. \( \square \)

**Acknowledgment**

I am grateful to the referee for his/her suggestion.

**References**


[6] Y. G. Chen, On integers of the forms \( k^{-r} - 2^n \) and \( k^{r}2^n + 1 \), *J. Number Theory* 98(2003), 310-319. MR [2003m:11004]


Department of Mathematics, Nanjing Normal University, Nanjing 210097, Peoples Republic of China

E-mail address: ygchen@pine.njnu.edu.cn