FIVE CONSECUTIVE POSITIVE ODD NUMBERS, 
NONE OF WHICH CAN BE EXPRESSED 
AS A SUM OF TWO PRIME POWERS

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Abstract. In this paper, we prove that there is an arithmetic progression of 
positive odd numbers for each term $M$ of which none of five consecutive odd 
numbers $M, M - 2, M - 4, M - 6$ and $M - 8$ can be expressed in the form 
$2^n \pm p^a$, where $p$ is a prime and $n, a$ are nonnegative integers.

Introduction

By calculation, we find that almost all positive odd numbers can be expressed 
in the form $2^n + p$, where $n$ is a positive integer and $p$ is prime. For example, 
$5 = 2 + 3, 7 = 2 + 5, 9 = 2 + 7, 11 = 2^2 + 7, 13 = 2 + 11, 15 = 2 + 13, 
17 = 2^2 + 13$, etc. The first counterexample is 127. In 1934, Romanoff \[11\] proved 
that the set of positive odd numbers which can be expressed in the form $2^n + p$ 
has positive asymptotic density in the set of all positive odd numbers, where $n$ is 
a nonnegative integer and $p$ is prime. For a positive integer $n$ and an integer $a$, 
let $a \pmod{n} = \{a + nk : k \in \mathbb{Z}\}$. \{\{a_i \pmod{m_i}\}_{i=1}^k\} is called a covering system if 
every integer $b$ satisfies $b \equiv a_i \pmod{m_i}$ for at least one value of $i$. By employing a 
covering system, P. Erdős \[8\] proved that there is an infinite arithmetic progression 
of positive odd numbers each of which has no representation of the form $2^n + p$. 
Cohn and Selfridge \[7\] proved that there exist infinitely many odd numbers which 
are neither the sum nor the difference of two prime powers. In \[3\] Chen proved the 
following result: the set of positive integers which have no representation of the form 
$2^n \pm p^aq^b$, where $p, q$ are distinct odd primes and $n, a, b$ are nonnegative integers, 
has positive lower asymptotic density in the set of all positive odd integers. That is, 
the lower asymptotic density of the set of positive odd integers $k$ such that $k - 2^n$ 
has at least three distinct prime factors for all positive integers $n$ is positive. In \[5\] 
Chen showed that the set of positive odd integers $k$ such that $k - 2^n$ has at least 
three distinct prime factors for all positive integers $n$ contains an infinite arithmetic 
progression. For further related information see Chen \[3, 4, 6\], Guy \[9\], A19, B21, 
F13], Jaeschke \[10\], and Stanton and Williams \[12\]. The following question is a 
natural one: Are there two consecutive positive odd numbers neither of which can 
be expressed as a sum of two prime powers?
In this paper, we show that the answer to the question is affirmative. In fact, we go much further.

**Theorem 1.** Let \( k_1, \ldots, k_s \) be integers, let \( \{a_{ij} \mod m_{ij}\}_{j=1}^t \) be \( s \) covering systems with \( 0 \leq a_{ij} < m_{ij} \), and let \( p_{ij} \) be primes with \( m_{ij} \) the order of \( 2 \mod p_{ij} \) \((1 \leq j \leq t, 1 \leq i \leq s)\) such that if \( p_{ij} = p_{uv} \), then

\[
2^{a_{ij}} - k_i \equiv 2^{a_{uv}} - k_u \mod p_{ij}.
\]

Then there exists an arithmetic progression of positive odd numbers for each term \( M \) of which none of \( M + k_i \) \((1 \leq i \leq s)\) can be expressed in the form \( 2^n \pm p^\alpha \), where \( p \) is a prime and \( n, \alpha \) are nonnegative integers.

**Theorem 2.** There exists an arithmetic progression of positive odd numbers for each term \( M \) of which none of five consecutive odd numbers \( M, M-2, M-4, M-6 \) and \( M-8 \) can be expressed in the form \( 2^n \pm p^\alpha \), where \( p \) is a prime and \( n, \alpha \) are nonnegative integers.

**Remark.** By the proofs of Theorems 1 and 2, there is an integer \( M \leq 2^t * 2^{3000} \) such that none of five consecutive odd numbers \( M, M-2, M-4, M-6 \) and \( M-8 \) can be expressed in the form \( 2^n \pm p^\alpha \). Currently, we cannot give an explicit value of \( M \).

2. Proofs

**Lemma 1.** Let \( p \) be an odd prime and let \( T \) be a positive integer. Then \( 2^{p^T} - 1 \) has at least \( T \) distinct prime factors.

**Proof.** Let \( q_i \ (i = 1, 2, \ldots, T) \) be primes with

\[
q_i \mid \frac{2^{p^i} - 1}{2^{p^{i-1}} - 1}.
\]

Then \( q_1, q_2, \ldots, q_T \) are distinct primes. This completes the proof of Lemma 1. \( \square \)

**Lemma 2.** Let \( p \) be an odd prime and let \( m \) be the order of \( 2 \mod p \). If

\[
2^m = 1 + p^l d, \quad p \nmid d,
\]

and \( p^n | 2^n - 1 \) for two integers \( n \geq 0 \) and \( u > 0 \), then \( n = mp^{u-l}v \) for some integer \( v \).

**Proof.** By using induction on \( r \), we can prove that

\[
2^{mp^r} = 1 + p^{l+r}d_r, \quad p \nmid d_r, \quad r = 0, 1, \ldots.
\]

By \( p | 2^n - 1 \) and \( m \) being the order of \( 2 \mod p \), we have \( m | n \). Let \( n = mp^hv' \), \( p \nmid v' \). Then

\[
2^n = 2^{mp^hv'} = 1 + p^{l-h}d'_h, \quad p \nmid d'_h.
\]

Since \( p^u | 2^n - 1 \), we have \( u \leq l + h \). Hence \( h \geq u - l \). Let \( v = v'p^{h-u+l} \). This completes the proof of Lemma 2. \( \square \)

**Lemma 3.** Let \( p_1, \ldots, p_t \) be primes such that each prime repeats at most \( s \) times. Then there exist \( t \) distinct primes \( q_1, \ldots, q_t \) such that

\[
q_i | 2^{p_i^{s+t}} - 1, \quad q_i \neq p_j, \quad \text{for all } i, j.
\]
Proof. For each prime \( p \), by Lemma 1 we may take a set \( S(p) \) of primes with 
\[ |S(p)| = t + s \]

such that
\[ q | 2^{p^{t+s}} - 1. \]

Since there are at most \( s \) indexes \( i \) with \( p_i = p \), we may appoint a prime \( q_i \in S(p) \setminus \{p_1, \ldots, p_t\} \) for each \( i \) with \( p_i = p \) such that if \( p_i = p_j = p \), then \( q_i \neq q_j \). If \( p_i \neq p_j \), then, by \( q_i \in S(p_i) \) and \( q_j \in S(p_j) \) we have
\[ q_i | 2^{p^{t+s}} - 1, \quad q_j | 2^{p^{t+s}} - 1. \]

Hence \( q_i \neq q_j \). Thus, these \( q_i \) are distinct such that
\[ q_i | 2^{p^{t+s}} - 1, \quad q_j \neq p_j, \quad \text{for all } i, j. \]

This completes the proof of Lemma 3. \( \square \)

Proof of Theorem 1. If \( p_{iu} = p_{iv} \), then, by \( m_{iu} \) and \( m_{iv} \) being the orders of \( 2 \pmod{p_{iu}} \) and \( 2 \pmod{p_{iv}} \), respectively, we have \( m_{iu} = m_{iv} \). By
\[ 2^{a_{iu}} - k_i \equiv 2^{a_{iv}} - k_i \pmod{p_{iu}} \]

and \( m_{iu} \) the order of \( 2 \pmod{p_{iu}} \), we have
\[ a_{iv} \equiv a_{iu} \pmod{m_{iu}}. \]

Hence \( a_{iu} \pmod{m_{iu}} = a_{iv} \pmod{m_{iv}} \). Thus, without loss of generality, we may assume that for each \( i \), primes \( p_{i1}, \ldots, p_{iu} \) are distinct. Let \( T = s + t_1 + \cdots + t_s \).

By Lemma 3, for each \( p_{ij} \), we may appoint a prime \( q_{ij} \) such that all primes \( q_{ij} \) \((1 < j < t_i, 1 < i < s)\) are distinct,
\[ q_{ij} | 2^{p^{ij}} - 1, \quad 1 \leq j \leq t_i, 1 \leq i \leq s, \]
and \( q_{ij} \neq p_{uv} \) for all \( 1 < j < t_i, 1 \leq i \leq s, 1 \leq v \leq t_u, 1 \leq u \leq s \). Let \( r_{ij} \) be integers such that \( 0 \leq r_{ij} < p_{ij} \) and
\[ r_{ij} \equiv 2^{a_{ij}} - k_i \pmod{p_{ij}}, \quad 1 \leq j \leq t_i, 1 \leq i \leq s. \]

Let
\[ 2^{m_{ij}} = 1 + p_{ij}^{l_{ij}} t_{ij}, \quad p \not| t_{ij}, \quad 1 \leq j \leq t_i, 1 \leq i \leq s, \]

and \( l = \max_{i,j} l_{ij} \). If there exists a nonnegative integer \( b \equiv a_{ij} \pmod{m_{ij}} \) with
\[ p_{ij}^{l+T} | 2^{b} - k_i - r_{ij}, \]
then let \( b_{ij} \) be the least one of such \( b \). If there are no such \( b \), then let \( b_{ij} = a_{ij} \). Let \( m \) be a positive integer with
\[ 2^{m} \geq \max_{i,j} p_{ij}^{l+T} + \max_{i} |k_i| + 1. \]

Take an integer \( M \) with
\[ M \equiv r_{ij} \pmod{p_{ij}^{l+T}}, \]
\[ M \equiv 2^{b_{ij}} - k_i \pmod{q_{ij}}, \quad 1 \leq j \leq t_j, 1 \leq i \leq s, \]
\[ M \equiv 1 + 2^{m} + 2^{m+1} \pmod{2^{m+2}}. \]
If \( p_{ij} = p_{uv} \), then \( r_{ij} = r_{uv} \) by the condition. Again, \( q_{ij} \) are distinct and each \( q_{ij} \) is different from any \( p_{uv} \). So such an \( M \) exists by the Chinese Remainder Theorem. Now we prove that none of \( M + k_i \) (1 \( \leq i \leq s \)) can be expressed in the form \( 2^n \pm p^\alpha \), where \( p \) is a prime and \( n, \alpha \) are nonnegative integers. In order to prove this, it is enough to show that for each \( i \) and any nonnegative integer \( n \), \( M + k_i - 2^n \) has at least two distinct positive prime factors. Since \( \{ a_{ij} \pmod{m_{ij}} \}_{j=1}^{t_i} \) is a covering system, there exists a \( j \) with

\[ n \equiv a_{ij} \pmod{m_{ij}}. \]

By (1), (3) and \( 2^{m_{ij}} \equiv 1 \pmod{p_{ij}} \), we have

\[ M + k_i - 2^n \equiv r_{ij} + k_i - 2^{a_{ij}} \equiv 0 \pmod{p_{ij}}. \]

Let

\[ M + k_i - 2^n = p_{ij}^{\alpha_{ij}} K_{ij}, \quad p_{ij} \nmid K_{ij}, \quad \alpha_{ij} \geq 1. \]

If \( \alpha_{ij} < l + T \), then by

\[
|M + k_i - 2^n| = |1 + 2^m + 2^{m+1} + 2^{m+2}u + k_i - 2^n| \\
\geq |1 + 2^m + 2^{m+1} + 2^{m+2}u - 2^n| - |k_i| \\
\geq 2^m - 1 - |k_i| \geq p_{ij}^{l+T},
\]

we have \( |K_{ij}| > 1 \). In this case, \( M + k_i - 2^n \) has at least two distinct prime factors.

If \( \alpha_{ij} \geq l + T \), then \( n \equiv a_{ij} \pmod{m_{ij}} \) and

\[ r_{ij} + k_i - 2^n \equiv M + k_i - 2^n \equiv 0 \pmod{p_{ij}^{l+T}}. \]

Hence \( n \equiv b_{ij} \pmod{m_{ij}} \) and by (2),

\[
2^{b_{ij}}(1 - 2^{n-b_{ij}}) \equiv 2^{b_{ij}} - k_i + k_i - 2^n \equiv r_{ij} + k_i - 2^n \equiv 0 \pmod{p_{ij}^{l+T}}.
\]

Thus

\[ p_{ij}^{l+T} | 2^{n-b_{ij}} - 1. \]

By Lemma 2 we have \( n - b_{ij} = m_{ij} p_{ij}^T v_{ij} \) for some integer \( v_{ij} \). By

\[ q_{ij} | 2^{p_{ij}^T} - 1, \]

we have

\[ q_{ij} | 2^{n-b_{ij}} - 1. \]

That is,

\[ q_{ij} | 2^n - 2^{b_{ij}}. \]

Hence

\[ M + k_i - 2^n \equiv 2^{b_{ij}} - k_i + k_i - 2^n \equiv 2^{b_{ij}} - 2^n \equiv 0 \pmod{q_{ij}}. \]

Thus \( q_{ij} | K_{ij} \) and then \( M + k_i - 2^n \) has at least two distinct prime factors. This completes the proof of Theorem 1. \( \square \)
Proof of Theorem 2. Let \( k_1 = 0, k_2 = -2, k_3 = -4, k_4 = -6 \) and \( k_5 = -8 \). Take

\[
\{ a_{ij} \pmod{m_{1j}} \}_{j=1}^{8} = \{ 0 \pmod{2}, 3 \pmod{4}, 5 \pmod{8}, \\
9 \pmod{16}, 17 \pmod{32}, 33 \pmod{64}, \\
1 \pmod{128}, 65 \pmod{128} \},
\]
\[
\{ a_{2j} \pmod{m_{2j}} \}_{j=1}^{7} = \{ 1 \pmod{2}, 0 \pmod{4}, 6 \pmod{8}, \\
10 \pmod{16}, 18 \pmod{32}, 34 \pmod{64}, \\
2 \pmod{64} \},
\]
\[
\{ a_{3j} \pmod{m_{3j}} \}_{j=1}^{26} = \{ 0 \pmod{3}, 2 \pmod{4}, 3 \pmod{5}, \\
1 \pmod{10}, 4 \pmod{12}, 2 \pmod{15}, \\
1 \pmod{18}, 7 \pmod{20}, 8 \pmod{24}, \\
19 \pmod{25}, 24 \pmod{25}, 11 \pmod{36}, \\
23 \pmod{36}, 25 \pmod{40}, 25 \pmod{45}, \\
40 \pmod{45}, 20 \pmod{48}, 44 \pmod{48}, \\
9 \pmod{50}, 39 \pmod{50}, 37 \pmod{60}, \\
35 \pmod{72}, 4 \pmod{75}, 5 \pmod{120}, \\
29 \pmod{150}, 215 \pmod{360} \},
\]
\[
\{ a_{4j} \pmod{m_{4j}} \}_{j=1}^{9} = \{ 0 \pmod{2}, 1 \pmod{4}, 7 \pmod{8}, \\
11 \pmod{16}, 19 \pmod{32}, 35 \pmod{64}, \\
67 \pmod{128}, 3 \pmod{256}, 131 \pmod{256} \}
\]
\[
\{ a_{5j} \pmod{m_{5j}} \}_{j=1}^{13} = \{ 1 \pmod{2}, 2 \pmod{3}, 2 \pmod{5}, \\
4 \pmod{9}, 6 \pmod{10}, 6 \pmod{12}, \\
10 \pmod{18}, 0 \pmod{20}, 24 \pmod{30}, \\
34 \pmod{36}, 48 \pmod{60}, 34 \pmod{90}, \\
88 \pmod{180} \}.
\]

Noting that \( \{ a_j \pmod{m_j} \}_{j=1}^{k} \) is a covering system if and only if for every integer \( n \) with \( 0 \leq n < \text{l.c.m.} \{ m_1, \ldots, m_k \} \) there exists a \( j \) with \( n = a_j \pmod{m_j} \), we can verify that \( \{ a_{1j} \pmod{m_{1j}} \}_{j=1}^{8}, \{ a_{2j} \pmod{m_{2j}} \}_{j=1}^{7}, \{ a_{3j} \pmod{m_{3j}} \}_{j=1}^{26}, \{ a_{4j} \pmod{m_{4j}} \}_{j=1}^{9}, \{ a_{5j} \pmod{m_{5j}} \}_{j=1}^{13} \) are all covering systems. Now, for every \( a_{ij} \pmod{m_{ij}} \) we appoint a prime \( p_{ij} \) such that \( m_{ij} \) is the order of \( 2 \pmod{p_{ij}} \) and if \( p_{ij} = p_{uv} \), then

\[
2^{a_{ij}} - k_i \equiv 2^{a_{uv}} - k_u \pmod{p_{ij}}.
\]

**Case 1.** Let \( p_{11} = p_{21} = p_{41} = p_{51} = 3 \). Then

\[
2^0 - 0 \equiv 2^1 - (2) \equiv 2^0 - (-6) \equiv 2^1 - (-8) \pmod{3}.
\]

**Case 2.** Let \( p_{12} = p_{22} = p_{32} = p_{42} = 5 \). Then

\[
2^3 - 0 \equiv 2^0 - (-2) \equiv 2^2 - (-4) \equiv 2^1 - (-6) \pmod{5}.
\]
Case 3. Let
\[ p_{13} = p_{23} = p_{43} = 17, \quad p_{14} = p_{24} = p_{44} = 257, \]
\[ p_{15} = p_{25} = p_{45} = 65537, \quad p_{16} = p_{26} = p_{46} = 641, \quad p_{27} = 6700417. \]

Note that both Fermat numbers \( F_6 \) and \( F_7 \) are composite, let \( p_{18} = p_{47}, p_{17} \) be two distinct prime divisors of \( 2^{64} + 1 \), and let \( p_{48}, p_{49} \) be two distinct prime divisors of \( 2^{128} + 1 \). Then (4) follows from the following fact:
\[ 2^{2k+1} - 0 \equiv 2^{2k+2} - (-2) \equiv 2^{2k+3} - (-6) \pmod{2^{2k} + 1}. \]

Case 4. Let
\[ p_{31} = p_{52} = 7, \quad p_{33} = p_{53} = 31, \]
\[ p_{34} = p_{55} = 11, \quad p_{35} = p_{56} = 13, \]
\[ p_{37} = p_{57} = 19, \quad p_{38} = p_{58} = 41, \]
\[ p_{3(12)} = p_{5(10)} = 109, \quad p_{3(13)} = 37. \]

Then
\[ 2^0 - (-4) \equiv 2^2 - (-8) \pmod{7}, \quad 2^3 - (-4) \equiv 2^2 - (-8) \pmod{31}, \]
\[ 2^1 - (-4) \equiv 2^6 - (-8) \pmod{11}, \quad 2^4 - (-4) \equiv 2^6 - (-8) \pmod{13}, \]
\[ 2^1 - (-4) \equiv 2^{10} - (-8) \pmod{19}, \quad 2^7 - (-4) \equiv 2^9 - (-8) \pmod{41}, \]
\[ 2^{11} - (-4) \equiv 2^{34} - (-8) \pmod{109}. \]

Case 5. Each of 25, 45, 48, 50, 60 is the order of 2 modulus two distinct primes. These primes are 601, 1801; 631, 2331; 97, 673; 251, 4051; 61, 1321, respectively. If \( m > 1 \) and \( m \neq 6 \), then there exists at least one prime \( p \) with \( m \) the order of \( 2 \pmod{p} \) (see [4, 5, 13]). Thus we may appoint a prime \( p_{ij} \) for each of the remaining \( a_{ij} \pmod{m_{ij}} \). Now, Theorem 2 follows from Theorem 1.

\[ \Box \]

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References

[3] Y. G. Chen, On integers of the form \( 2n \pm p_{1}^{a_1} \cdots p_{r}^{a_r} \), Proc. Amer. Math. Soc. 128(2000), 1613-1616. MR [2000i:11006]


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