SOME PROPERTIES OF THE GAMMA AND PSI FUNCTIONS,
WITH APPLICATIONS

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Abstract. In this paper, some monotoneity and concavity properties of the gamma, beta and psi functions are obtained, from which several asymptotically sharp inequalities follow. Applying these properties, the authors improve some well-known results for the volume $\Omega_n$ of the unit ball $B^n \subset \mathbb{R}^n$, the surface area $\omega_{n-1}$ of the unit sphere $S^{n-1}$, and some related constants.

1. Introduction

For real and positive values $x$ and $y$, the Euler gamma function, the beta and psi (or polygamma) functions are defined as

\begin{equation}
\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},
\end{equation}

respectively. For the extensions to complex variables and for the basic properties of these functions, see [AS], [AAR], [Mi], [T], and [WW]. Over the past half century, many authors have obtained various properties and inequalities for these very important functions (see [A1], [A2], [A5], [G], [Ke], [K2], [L], [MSC] and bibliographies therein).

Formulas for the volumes of geometric bodies sometimes involve the gamma function. This topic and related inequalities have been studied recently in [AQ], [A1]–[A4], [BP], [EL]. Let $B^n$ and $S^{n-1}$ be the unit ball and unit sphere in $\mathbb{R}^n$, respectively, $\Omega_n$ be the volume of $B^n$, and $\omega_{n-1}$ denote the surface area of $S^{n-1}$.

Set $\Omega_0 = 1$. It is well known that $\Omega_n$ is increasing for $2 \leq n \leq 5$ and decreasing for $n \geq 5$, while $\omega_{n-1}$ is increasing for $2 \leq n \leq 7$ and decreasing for $n \geq 7$ (see [BH] pp. 263–264 and [AVV1] p. 38):

\begin{equation}
\Omega_n = \frac{2\pi}{n} \Omega_{n-2} = \frac{\pi^{n/2}}{\Gamma(1 + n/2)}, \quad \omega_{n-1} = n \Omega_n = 2 \frac{\pi^{n/2}}{\Gamma(n/2)}.
\end{equation}

Throughout this paper, we let $\gamma = 0.57721\cdots = -\psi(1)$ denote the Euler-Mascheroni constant, $(2n-1)!! = (2n-1)(2n-3)\cdots1$ for $n \in \mathbb{N} = \{k; k$ is a natural number$\}$,

\begin{equation}
I_n = \int_0^{\pi/2} \sin^{n-2} t \, dt, \quad J_n = \int_0^{\pi/2} \sin^{(2-n)/(n-1)} t \, dt,
\end{equation}

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Theorem C. \( (1.9) \)

\[ A_n = \left( \frac{\omega_n-1}{2^nc_n} \right)^{1/(n-1)}, \quad B_n = b_n^{-1}, \quad D_n = \frac{1}{n-1} \left( \frac{\omega_n-1}{c_n} \right)^{1/(n-1)}, \]

\( (1.10) \)

\[ b_n = J_n/(n-1) \quad \text{and} \quad c_n = (2J_n)^{1-n}\omega_{n-2}, \]

for \( n \in \mathbb{N} \) with \( n \geq 2 \). These constants have applications in some fields of mathematics such as geometry of Grassmannian subspaces of \( \mathbb{R}^n \) \( [KR] \), optimization theory \( [B] \), geometric function theory \( [AVV1] \) pp. 234–246], \( [V] \), \( [Vu] \), as well as geometry of spaces of constant curvature \( [BH] \).

These are some examples of the fields where the gamma function is frequently used. The accumulated literature on the gamma function is so vast that it is difficult for someone working chiefly in these specialized areas of research to find the property of the gamma function which, in this particular application, would lead to the desired conclusion. This is exactly how we were led to explore the gamma function, our main research area being geometric function theory; see \( [V] \), \( [Vu] \), \( [AVV1] \), \( [AQ] \). Some of the latest developments on special functions that have been studied in this context are reviewed in \( [AVV2] \). The statement of some of our main results, which we think have independent interest as such, now follows.

**Theorem A.** For \( n = 1, 2, 3, \ldots \)

\[ 2^{1/2} \sqrt{\frac{\pi}{2n+1}} < \frac{\Omega_n}{\Omega_{n-1}} < 2^{1/2} \sqrt{\frac{\pi}{2n+1}}. \]

**Theorem B.** For \( x \geq 1 \), let \( P(x) = \max\{\sqrt{x}, \sqrt{4x-1}\} \). Then for \( x \geq 1 \)

\[ x^{(1-\gamma)x^{-1}} P(x) \leq \Gamma(x)P(x) \leq 2\Gamma(x + 1/2). \]

**Theorem C.** For \( x > 1 \)

\[ \log x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \log x - \frac{1}{2x} - \frac{2\gamma - 1}{2x^2}. \]

Theorems A, B, and C are special cases of the more technical Theorems 1.17, 1.12, and 2.1, respectively. We next proceed to review some earlier results and to discuss our preliminary and main results.

Various properties of \( \Omega_n \) and \( \omega_{n-1} \) have been proved in these contexts. For example, it was shown in \( [B] \) that for \( n \geq 3 \),

\( (1.6) \)

\[ \sqrt{\frac{2\pi}{n+1}} < \frac{\Omega_n}{\Omega_{n-1}} < \sqrt{\frac{2\pi}{n}} \]

and

\( (1.7) \)

\[ \sqrt{\frac{2\pi}{n-1}} < \frac{\omega_{n-1}}{\omega_{n-2}} < \sqrt{\frac{2\pi}{n-2}}, \]

and a new property of \( \Omega_n \) was obtained in \( [AQ] \). Several inequalities for \( A_n, B_n, c_n, D_n, I_n \) and \( J_n \) were shown or stated in \( [AVV1] \). For example, it was proved that for \( n \geq 3 \),

\( (1.8) \)

\[ \sqrt{\frac{2\pi}{2n-2}} < I_n < \sqrt{\frac{2\pi}{2n-4}}, \]

\( (1.9) \)

\[ -1 < (n-1)(J_n - n + 1 - \log 2) < 1 - (\pi/2) \log 2 < 0, \]

\( (1.10) \)

\[ \lim_{n \to \infty} A_n/n = 1, \quad \lim_{n \to \infty} B_n = \lim_{n \to \infty} D_n = 2, \]
and

(1.11) \[ 1 < B_n < 2 \quad \text{for} \quad n \geq 2. \]

(See [AVVII, pp. 42-43].)

It is the purpose of the present paper to show some properties of \( \Gamma(x) \), \( B(x, y) \), and \( \psi(x) \) and then, based on these properties, to improve some well-known results for \( \Omega_n \), \( \omega_{n-1} \), \( A_n \), \( B_n \), \( b_n \), \( I_n \), and \( J_n \), such as (1.6)–(1.11). We now state some of the main results of this paper below.

1.12. **Theorem.** (1) The function \( f_1(x) \equiv \Gamma(x + 1/2)/[\sqrt{x} \Gamma(x)] \) is strictly increasing and log-concave from \((0, \infty)\) onto \((0, 1)\), \( f_2(x) \equiv f_1(1/x) \) is log-convex on \((0, \infty)\), and \( f_3(x) \equiv f_1(1/x) \) is convex on \((0, \infty)\).

(2) The function \( f_4(x) \equiv x - [\Gamma(x + 1/2)/\Gamma(x)]^2 = x[1 - f_1(x)^2] \) is strictly increasing and concave from \((0, \infty)\) onto \((0, 1/4)\). Moreover, for \( x \geq 1 \),

\begin{equation}
(1.13) \quad x^{(1-\gamma)x-1} P(x) \leq \Gamma(x) P(x) \leq 2\Gamma(x + 1/2)
\end{equation}

\[ \leq \sqrt{\pi x} \Gamma(x) (2/\sqrt{\pi})^{1-1/x} \leq \sqrt{\pi x} (2/\sqrt{\pi})^{1-1/x} x^{x-1/2}, \]

with equality in each instance iff \( x = 1 \), where \( P(x) = \max\{\sqrt{\pi x}, \sqrt{4x - 1}\} \). The third and the fourth inequalities in (1.13) are asymptotically sharp as \( x \) tends to \( \infty \).

(3) For each fixed \( x > 0 \), define functions \( f_5 \) and \( f_7 \) on \((0, 1)\) by

\[ f_5(s) = \frac{1}{1 - s} \left[ \log x - \frac{1}{s} \frac{\Gamma(x + s)}{\Gamma(x)} \right], \quad f_6(s) = \frac{\beta(x) - f_5(s)}{x}, \quad f_7(s) = \frac{s\psi(x) - \alpha(x)}{1 - s}, \]

respectively, where \( \alpha(x) = \psi(x + 1) - \log x \) and \( \beta(x) = \log x - \psi(x) \). Then \( f_5 \) and \( f_6 \) are both strictly decreasing on \((0, 1)\) with ranges \((\alpha(x), \beta(x))\) and \((\xi(x), \eta(x))\), respectively, while \( f_7 \) is strictly increasing from \((0, 1)\) onto \((0, \theta(x))\), where

\[ \xi(x) = \beta(x) - \alpha(x), \quad \eta(x) = \frac{1}{2} \psi'(x) - \beta(x) \quad \text{and} \quad \theta(x) = \alpha(x) - \frac{1}{2} \psi'(x + 1). \]

In particular, for \( x > 0 \) and \( s \in (0, 1) \),

\begin{equation}
(1.14) \quad \exp\{s(s - 1)\beta(x)\} < \exp\{s(1-s)\rho(s,x)\} < \frac{\Gamma(x + s)}{x^s \Gamma(x)} < \exp\{s(s - 1)\omega(s,x)\} \leq \exp\{s(s - 1)\alpha(x)\},
\end{equation}

where

\[ \rho(s,x) = \max\{s\xi(x) - \beta(x), -\alpha(x) + (1 - s)\theta(x)/s\} \]

and

\[ \omega(s,x) = \min\{s\eta(x) - \beta(x), -\alpha(x)\}. \]

The above theorem yields some inequalities of \( \Gamma(x) \). Some similar or related results have been proved recently in [AQ], [BP], [EL], [AS], [K2], [McI], [Mc3].

1.15. **Remarks.** (1) For \( n \in \mathbb{N} \), let \( G(n, k) \) be the so-called Grassmannian, and let \( \tau_n \) denote the invariant measure on \( G(n, 1) \), that is, on the set of all straight lines through the origin (cf. [KR] p. 206). Set \( g(n) = \tau_n(G(n, 1)) \). Then the functions \( f_1(x) \), \( f_4(x) \) and \( f_5(s) \) in Theorem 1.12 are related to \( g(n) \). Therefore, Theorem 1.12 is related to some results in [KR]. For example, it was proved in [KR] p. 214] that \( g(n) \) is increasing for \( n \in \mathbb{N} \), while Theorem 1.12(1) says that \( g(n)/\sqrt{n} \) is also strictly increasing for \( n \in \mathbb{N} \) since \( g(n) = \sqrt{n\pi/2f_1(n/2)} \) by the expression of \( g(n) \) (see [KR] p. 214]).
The double inequality (1.14) and its Corollary 3.2 are related to a result in [LL], which gives the estimates
\[(1 - s) \log(x + s/2) < D_s(x) < (1 - s) \log\left(x + \left(\Gamma(s)\right)^{1/(s-1)}\right)\]
for \(s \in (0, 1)\) and \(x > 0\), where \(D_s(x) = \log \Gamma(x + 1) - \log \Gamma(x + s)\). For some other related results, see [A3] p. 365, [Me1], [MSC], etc.

1.16. Theorem. (1) The function \(g_1(x) \equiv [xB(x,1/2)]^{1/x}\) is strictly decreasing and convex from \((0,\infty)\) onto \((1,4)\).

(2) There exists a unique \(x_1 \in (1.5, 2)\) such that the function
\[g_2(x) = \frac{1}{2x + 1} \left[\frac{1}{2} B\left(x, \frac{1}{2}\right)\right]^{1/(2x)} B\left(\frac{1}{4x}, \frac{1}{2}\right),\]
for \(x \in (0,\infty)\), is strictly decreasing on \((0,x_1]\) and increasing on \([x_1,\infty)\), with \(g_2(1/2) = \pi^2/4\) and \(\lim_{x \to \infty} g_2(x) = 2\).

(3) There exists a unique \(x_2 \in (33,34)\) such that the function
\[g_3(x) \equiv 2^{1/(2x)} \left(1 + \frac{1}{2x}\right) g_2(x),\]
for \(x \in (0,\infty)\), is strictly decreasing on \((0,x_2]\) and increasing on \([x_2,\infty)\), with \(g_3(1/2) = \pi^2\) and \(\lim_{x \to \infty} g_3(x) = 2\).

As an example of the applications of the above two theorems, we shall prove the following improvements upon some results for \(\Omega_n, \omega_{n-1}, A_n, B_n, b_n, I_n\) and \(J_n\).

1.17. Theorem. (1) For \(n \in \mathbb{N}\setminus\{1,2\}\), the function \(F_1(n) \equiv 2n - \pi I_n^{-2}\) is strictly increasing with \(F_1(3) = 6 - \pi\) and \(\lim_{n \to \infty} F_1(n) = 3\). In particular, for \(n \geq 3\),
\[\sqrt{\frac{\pi}{2n - (6 - \pi)}} \leq I_n < \sqrt{\frac{\pi}{2n - 3}},\]
with equality iff \(n = 3\).

(2) The function \(F_2(n) \equiv n - 2\pi[\omega_{n-2}/\omega_{n-1}]^2\) is strictly increasing for \(n \in \mathbb{N}\setminus\{1,2\}\), with \(F_2(3) = 3 - (\pi/2)\) and \(\lim_{n \to \infty} F_2(n) = 3/2\). In particular, for \(n \geq 3\),
\[2\sqrt{\frac{\pi}{2n - 6 + \pi}} \leq \frac{\omega_{n-1}}{\omega_{n-2}} < 2\sqrt{\frac{\pi}{2n - 3}},\]
with equality iff \(n = 3\).

(3) The function \(F_3(n) \equiv n - 2\pi[\Omega_{n-1}/\Omega_n]^2\) is strictly increasing for \(n \in \mathbb{N}\), with \(F_3(1) = 1 - \pi/2\) and \(\lim_{n \to \infty} F_3(n) = -1/2\). In particular, for \(n \in \mathbb{N}\),
\[2\sqrt{\frac{\pi}{2n + \pi - 2}} \leq \frac{\Omega_n}{\Omega_{n-1}} < 2\sqrt{\frac{\pi}{2n + 1}},\]
with equality iff \(n = 1\).

(4) Let \(n \in \mathbb{N}\). Then \(b_n\) is strictly decreasing for \(n \geq 2\), and \(B_n\) is strictly increasing for \(n \geq 2\). In particular, for \(n \geq 2\),
\[\frac{\pi}{2} \leq B_n < 2, \quad (n - 1) \left(\frac{\pi}{2}\right)^{1/(n-1)} \leq J_n \leq (n - 1) \min\left\{2^{1/(n-1)}, \frac{\pi}{2}\right\},\]
with equality in each instance iff \(n = 2\).
(5) Let $a = 1 - \log 2$, $b = 3 - 0.5B(1/4, 1/2) = 0.37794 \ldots$, $c = [\pi^2 - 3(\log 4)^2]/24 = 0.171007 \ldots$, and $n \in \mathbb{N}$. Then the function $F_4(n) \equiv (n-1)(n-J_n-a)$ is strictly increasing for $n \geq 3$, with $F_4(3) = d \equiv 2(b-a) = 0.14217 \ldots$ and $\lim_{n \to \infty} F_4(n) = c$. In particular, for $n \geq 3$,

\begin{equation}
\frac{1}{a_n} \leq (n-1)(n-J_n-a) < c,
\end{equation}

with equality iff $n = 3$.

(6) Let $n \in \mathbb{N}$. Then the function $F_5(n) \equiv A_n/n$ is strictly decreasing for $2 \leq n \leq 5$ and increasing for $n \geq 5$. In particular,

\begin{equation}
\lim_{n \to \infty} \frac{1}{A_n} = 1, \quad C \leq \frac{1}{n} A_n \begin{cases} \leq A_2/2 = \pi^2/8, & \text{for } 2 \leq n \leq 5, \\ < 1, & \text{for } n \geq 5, \end{cases}
\end{equation}

where

\[ C = \frac{1}{5} A_5 = \sqrt{\frac{\pi}{10}} \left( \frac{2}{3} \right)^{1/2} \frac{\Gamma(1/8)}{\Gamma(5/8)} = 0.8411396629 \ldots. \]

(7) Let $n \in \mathbb{N}$. Then $D_n$ is strictly decreasing for $2 \leq n \leq 69$ and increasing for $n \geq 69$. In particular,

\begin{equation}
\lim_{n \to \infty} D_n = 2, \quad D \leq D_n \begin{cases} \leq D_2 = \pi^2, & \text{for } 2 \leq n \leq 69, \\ < 2, & \text{for } n \geq 69, \end{cases}
\end{equation}

where

\[ D = D_{69} = \sqrt{\frac{2\pi}{68}} \left( \frac{33!}{67!!} \right)^{1/68} \frac{\Gamma(1/136)}{\Gamma(69/136)} = 1.9853487779 \ldots. \]

2. Some properties of $\psi(x)$

In this section, we obtain some properties of $\psi(x)$, which are also needed in the proofs of the theorems stated in Section 1.

2.1. Theorem. (1) The function $h_1(x) \equiv \psi(x + 1/2) - \psi(x) - [1/(2x)]$ is strictly decreasing and convex from $(0, \infty)$ onto $(0, \infty)$. Moreover, $h_2(x) \equiv h_1(1/x)$ is convex on $(0, \infty)$.

(2) The function $h_3(x) \equiv xh_1(x)$ is strictly decreasing from $(0, \infty)$ onto $(0, 1/2)$.

(3) The function $h_4(x) \equiv x^2h_1(x)$ is strictly increasing from $(0, \infty)$ onto $(0, 1/8)$.

In particular, for $x > 1$,

\begin{equation}
\psi(x) + \frac{1}{2x} + \frac{1.5 - \log 4}{x^2} \leq \psi \left( x + \frac{1}{2} \right) < \psi(x) + \frac{1}{2x} + \frac{1}{8x^2}.
\end{equation}

(4) The function $h_5(x) \equiv x^2[\psi(x) - \log x] + (x/2)$ is strictly decreasing and convex from $(0, \infty)$ onto $(-1/12, 0)$. In particular, for $x > 1$,

\begin{equation}
\log x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \log x - \frac{1}{2x} - \frac{2\gamma - 1}{2x^2}.
\end{equation}

(5) Let $\eta$ and $\theta$ be functions defined in Theorem 1.12(3). Then $\eta$ and $\theta$ are both strictly decreasing from $(0, \infty)$ onto $(0, \infty)$.
Proof. (1) Differentiation gives

\[ h_1'(x) = \frac{h_6(x)}{x^2} = \psi'\left(x + \frac{1}{2}\right) - \psi'(x) + \frac{1}{2x^2} = \frac{1}{2x^2} - \int_0^\infty \frac{te^{-xt}}{1 + e^{-t/2}} dt, \]

where

\[ h_6(x) = \int_0^\infty \frac{ue^{-u}}{1 + e^{-u/2}} - \frac{1}{2}, \]

since [AS, Section 6.4.1]

(2.4) \[ \psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty \frac{t^n e^{-xt}}{1 - e^{-t}} dt. \]

Clearly, \( h_6 \) is strictly decreasing from \((0, \infty)\) onto \((0, 1/2)\). Hence \( h_1' \) is strictly increasing from \((0, \infty)\) onto \((-\infty, 0)\), and the monotoneity and convexity of \( h_1 \) follow.

By [AS, Section 6.3.5], \( h_1(x) = \psi(x + 1/2) - \psi(x + 1) + [1/(2x)] \) so that \( h_1(0^+) = \infty \).

The limiting value \( \lim_{x \to \infty} h_1(x) = 0 \) follows from [AS, Section 6.3.18].

The convexity of \( h_2 \) follows from the monotoneity of \( h_6 \), since

\[ h_2'(x) = -h_1'(1/x)/x^2 = h_6(1/x). \]

(2) Put \( y = 1/x \). Then

\[ h_3(x) = \frac{h_2(y)}{y} \quad \text{and} \quad \frac{h_3'(y)}{h_3(y)} = h_2'(y), \]

and hence, the monotoneity of \( h_3 \) follows from the convexity of \( h_2 \) and the Monotone

l’Hôpital’s Rule [AVV, Theorem 1.25].

By [AS, Section 6.3.5], \( h_3(x) \) can be written as

\[ h_3(x) = \frac{x}{2} \left[ 2\psi\left(x + \frac{1}{2}\right) - \psi(x + 1) - \psi(x) \right], \]

and hence, \( h_3(0^+) = 1/2 \). It follows from [AS, Section 6.3.18] that \( \lim_{x \to \infty} h_3(x) = 0 \).

(3) Write \( h_4(x) = h_2(y)/y^2 \), where \( y = 1/x \), and let

\[ h_7(y) = h_6\left(\frac{1}{y}\right) = \int_0^\infty \frac{ue^{-u}}{1 + e^{-u/2}} = \frac{1}{2}, \quad h_8(y) = y. \]

Then \( h_7(0) = h_8(0) = 0 \), and by (2.4),

(2.5) \[ \frac{h_7'(y)}{h_7(y)} = \frac{h_7(y)}{2h_8(y)}. \]

(2.6) \[ \frac{h_7'(y)}{h_7(y)} = h_9(y) = \frac{1}{2} \int_0^\infty \frac{u^2 e^{-u(1+y/2)}}{(1 + e^{-u/2})^2} du. \]

Clearly, \( h_9 \) is strictly decreasing in \( y \) on \((0, \infty)\). Hence the monotoneity of \( h_4 \) follows from the Monotone

l’Hôpital’s Rule [AVV, Theorem 1.25].

Clearly, \( h_4(0^+) = \lim_{x \to 0} xh_3(x) = 0 \). By l’Hôpital’s Rule, (2.5) and (2.6),

\[ \lim_{x \to \infty} h_4(x) = \lim_{y \to 0} \frac{h_2(y)}{y^2} = h_9(0) = \frac{1}{16} \int_0^\infty u^2 e^{-u} du = \frac{1}{8}. \]

Inequality (2.2) is clear.
(4) Using [AS 6.3.21], we obtain by differentiation
\[
\begin{align*}
h'_5(x) &= h_{10}(x) = 2x[\psi(x) - \log x] + x^2 \left[ \psi'(x) - \frac{1}{x} \right] + \frac{1}{2} \\
&= -2x \left[ \frac{1}{2x} + \int_0^\infty \frac{2tdt}{(t^2 + x^2)(e^{2\pi t} - 1)} \right] + \frac{1}{2} \\
&\quad + x^2 \left[ \frac{1}{2x^2} + \int_0^\infty \frac{4xtdt}{(t^2 + x^2)(e^{2\pi t} - 1)} \right] \\
&= 4x^3 \int_0^\infty \frac{tdt}{(t^2 + x^2)(e^{2\pi t} - 1)} - 4x \int_0^\infty \frac{tdt}{(t^2 + x^2)(e^{2\pi t} - 1)} \\
&= -4x \int_0^\infty \frac{t^3dt}{(t^2 + x^2)(e^{2\pi t} - 1)} = -4x \int_0^\infty \frac{u^3du}{(1 + u^2)^2(e^{2\pi ux} - 1)}.
\end{align*}
\]
where \( u = t/x \). Since \( x \mapsto x/(e^{2\pi ux} - 1) \) is strictly decreasing on \((0, \infty)\), \( h_{10} \) is strictly increasing on \((0, \infty)\). Hence the monotoneity and convexity of \( h_5 \) follow. Clearly, \( h_5(0^+) = 0, h_5(1) = (1/2) - \gamma \). By [AS 6.3.18], \( h_5(\infty) = -1/12 \).

(5) By differentiation and (2.4), we get
\[
2x\eta'(x) = 2x \left[ \frac{1}{2} \psi''(x) - \frac{1}{x} + \psi'(x) \right] = \int_0^\infty \frac{xt(2 - t)e^{-xt}}{1 - e^{-t}}dt - 2
\]
which is strictly increasing from \((0, \infty)\) onto \((-\infty, 0)\) since the function \( v \mapsto v(2 - v)/(1 - e^{-v}) \) is strictly decreasing from \((0, \infty)\) onto \((-\infty, 2)\). This yields the monotoneity of \( \eta \).

By [AS 6.3.5], we can rewrite \( \eta(x) \) as
\[
\eta(x) = \frac{1}{2} \psi'(x) + 1 + \frac{1}{2x^2} - \log x + \psi(x) + 1 - \frac{1}{x},
\]
from which we see that \( \eta(0^+) = \infty \). The limiting value \( \lim_{x \to \infty} \eta(x) = 0 \) follows from [AS 6.3.18 and 6.4.12].

Next, similarly, we have
\[
2x\theta'(x) = \int_0^\infty \frac{xt(2 + t)e^{-(x+1)t}}{1 - e^{-t}}dt - 2 = \frac{1}{x} \int_0^\infty \frac{u(2 + u/x)e^{-u}}{e^{u/x} - 1}du - 2,
\]
which is strictly increasing from \((0, \infty)\) onto \((-2, 0)\) since \( v \mapsto v(2 + v)/(e^v - 1) \) is strictly decreasing from \((0, \infty)\) onto \((0, 2)\). Hence the monotoneity of \( \theta \) follows.

Clearly, \( \theta(0^+) = \infty \). By [AS 6.3.18 and 6.4.12], we obtain \( \lim_{x \to \infty} \theta(x) = 0 \). □

2.7. Remarks. (1) The double inequality (2.3) improves the inequality obtained in [ABRVV, Theorem 3.1].

(2) It is well known that [AS 6.1.3]
\[
(2.8) \quad \gamma = \lim_{n \to \infty} d_n, \quad d_n = \sum_{k=1}^n \frac{1}{k} - \log n,
\]
and by [AS 6.3.2],
\[
d_n - \gamma = \psi(n + 1) - \log n.
\]
S. R. Tims and J. A. Tyrrell [TT] obtained the bounds
\[ \frac{1}{2(n+1)} < d_n - \gamma < \frac{1}{2(n-1)}, \quad n \geq 2, \]
for \( d_n - \gamma \), which was improved by R. M. Young [Y] as
\[ \frac{1}{2(n+1)} < d_n - \gamma < \frac{1}{2n}, \quad \text{for} \quad n \in \mathbb{N}, \]
while G. D. Anderson et al. [ABRVV] proved
\[ \frac{1-\gamma}{n} < d_n - \gamma < \frac{1}{2n}, \quad \text{for} \quad n \in \mathbb{N}, \]
Recently, H. Alzer [A3, Theorem 3] established the double inequality
\[ \frac{1}{2(n+\alpha)} \leq d_n - \gamma < \frac{1}{2(n+\beta)}, \quad \text{for} \quad n \in \mathbb{N}, \]
where \( \alpha = \{1/[2(1-\gamma)]\} - 1 = 0.1826 \cdots \) and \( \beta = 1/6 \). Let \( h_5 \) be as in Theorem 2.1(4). Then
\[ h_5(n) = n^2 \left\{ \psi(n+1) - \log n - \frac{1}{n} \right\} + \frac{n}{2} = n^2(d_n - \gamma) - \frac{n}{2}, \]
and hence our Theorem 2.1(4) implies the following estimates for \( d_n - \gamma \):

\[ 2.13. \textbf{Corollary.} \quad \text{For} \quad n \in \mathbb{N}, \]
\[ \frac{1}{2n} - \frac{\alpha}{n^2} < d_n - \gamma \leq \frac{1}{2n} - \frac{\beta}{n^2}, \]
with equality iff \( n = 1 \), where the constants \( \alpha = 1/12 = 0.08333 \cdots \) and \( \beta = \gamma - 1/2 = 0.07721 \cdots \) are best possible.

Some further results on the approximation of \( \gamma \) are given in [K1].

3. Proofs of the main theorems

In this section we prove the theorems stated in Section 1.

3.1. Proof of Theorem 1.12. (1) Logarithmic differentiation gives
\[ f_1'(x)/f_1(x) = h_1(x), \]
where \( h_1 \) is as in Theorem 2.1(1), and hence the monotoneity and log-concavity of \( f_1 \) follow from Theorem 2.1(1). By [AS 6.1.15 and 6.1.37], we have
\[ f_1(0^+) = \lim_{x \to 0} \sqrt{x} \frac{\Gamma(x+1/2)}{\Gamma(x+1)} = 0 \]
and
\[ \lim_{x \to \infty} f_1(x) = \lim_{x \to \infty} \frac{e^{-x}(x+1/2)(x+1/2)}{\sqrt{2e^{-x}x^{x-1/2}}} = 1. \]
Since \( f_2'(x)/f_2(x) = -h_4(1/x) \), where \( h_4 \) is as in Theorem 2.1(3), the log-convexity of \( f_2 \) follows from Theorem 2.1(3).

Next, by differentiation,
\[ f_3'(x) = -f_1'(x)/f_1(x)^2 = -h_1(x)/f_1(x), \]
which is strictly increasing on \((0, \infty)\) by Theorem 2.1(1), and hence the convexity of \( f_3 \) follows.
(2) Set \( y = \Gamma(x + 1/2)/\Gamma(x) \). Then
\[
f_1'(x) = 1 - 2y^2 \left[ \psi \left( x + \frac{1}{2} \right) - \psi(x) \right]
\]
and
\[
\frac{1}{2y^2} f_1''(x) = - \left[ \psi \left( x + \frac{1}{2} \right) - \psi(x) \right]^2 f_8(x),
\]
where
\[
f_8(x) = f_9'(x) = 2 + \frac{\psi'(x + 1/2) - \psi'(x)}{[\psi(x + 1/2) - \psi(x)]^2},
\]
\[
f_9(x) = 2x - \left[ \psi \left( x + \frac{1}{2} \right) - \psi(x) \right]^{-1} = \frac{2h_4(x)}{h_3(x) + 1/2}
\]
and \( h_3 \) and \( h_4 \) are as in Theorem 2.1. By Theorem 2.1(2)-(3), \( f_9 \) is strictly increasing from \((0, \infty)\) onto \((0, 1/2)\), so that \( f_8(x) > 0 \) for \( x \in (0, \infty) \), and hence \( f_1''(x) < 0 \) for \( x \in (0, \infty) \). This yields the concavity of \( f_4 \). Furthermore, by part (1) and Theorem 2.1(2),
\[
f_4'(x) > \lim_{x \to \infty} f_4'(x)
\]
\[
= 1 - 2 \lim_{x \to \infty} f_1(x)^2 \cdot x \left[ \psi \left( x + \frac{1}{2} \right) - \psi(x) \right]
\]
\[
= 1 - 2 \left[ \lim_{x \to \infty} f_1(x) \right]^2 \left[ \lim_{x \to \infty} h_3(x) + \frac{1}{2} \right] = 0
\]
for \( x \in (0, \infty) \), and hence the monotonicity of \( f_4 \) follows.

By part (1), \( f_4(0^+) = \lim_{x \to 0^+} [x - x f_1(x)^2] = 0 \). Applying l'Hôpital's Rule, we obtain
\[
\lim_{x \to \infty} f_4(x) = \lim_{x \to \infty} \frac{1 - f_1(x)^2}{1/x} = \lim_{t \to 0} \frac{1 - f_2(t)^2}{t} = -2 \lim_{t \to 0} f_2(t) f_2'(t) = 2 \lim_{t \to 0} f_2(t)^2 h_4(1/t) = 2 \lim_{x \to \infty} f_1(x)^2 h_4(x) = 1/4.
\]

The second and third inequalities in (1.13) follow from the monotonicity of \( f_4 \) on \((0, \infty)\) and the log-convexity of \( f_2 \) on \((0, 1)\), while the first and fourth inequalities in (1.13) hold by \([AQ] \) Theorem 1.5. The equality case is clear. The asymptotic sharpness of the third and fourth inequalities in (1.13) follows from the monotonicity of \( f_1 \) and \([AQ] \) Theorem 1.5.

(3) For \( x > 0 \), let
\[
f_{10}(s) = \log x - \frac{1}{s} \log \frac{\Gamma(x + s)}{\Gamma(x)}, \quad f_{11}(s) = 1 - s,
\]
\[
f_{12}(s) = s \psi(x + s) - \log \Gamma(x + s) + \log \Gamma(x) \quad \text{and} \quad f_{13}(s) = s^2.
\]
Then \( f_{5}(s) = f_{10}(s) / f_{11}(s), f_{10}(1) = f_{11}(1) = f_{12}(0) = f_{13}(0) = 0, \) and
\[
f'_{10}(s) = f'_{12}(s), \quad f'_{11}(s) = f'_{13}(s), \quad f'_{12}(s) = \frac{1}{2} \psi'(x + s).
\]
Since \( \psi'(t) \) is strictly decreasing on \((0, \infty)\), the monotonicity of \( f_5 \) follows from the Monotone l'Hôpital's Rule \([AVV] \) Theorem 1.25].
By l'Hôpital's Rule,
\[ f_5(1^-) = \lim_{s \to 1^-} \frac{f_{12}(s)}{f_{13}(s)} = \psi(x + 1) - \log x = \alpha(x), \]
and
\[ f_5(0^+) = f_{10}(0^+) = \log x - \lim_{s \to 0^+} \frac{\log \Gamma(x + s) - \log \Gamma(x)}{s}. \]

Let \( f_{14}(s) = s\beta(x) - sf_5(s), \) \( f_{15}(s) = s^2, \)
\[ f_{16}(s) = \psi'(x + s)(1 - s)^2 - 2\{[\log x - \psi(x + s)](1 - s) + s\log x - \log \Gamma(x + s) + \log \Gamma(x)\}, \]
and \( f_{17}(s) = (1 - s)^3. \) Then \( f_6(s) = f_{14}(s)/f_{15}(s), f_{14}(0) = f_{15}(0) = f_{14}'(0) = f_{15}'(0) = f_{16}(1) = f_{17}(1) = 0 \)
and
\[ \frac{f_{14}''(s)}{f_{15}''(s)} = \frac{f_{16}(s)}{2f_{17}(s)}, \quad \frac{f_{14}'(s)}{f_{15}'(s)} = -\frac{1}{3}\psi''(x + s). \]

It is well known that \( \psi'' \) is strictly increasing on \((0, \infty). \) Hence the monotonicity of \( f_6 \) follows from the Monotone l'Hôpital's Rule \([AVV1, \text{ Theorem} \ 1.25].\)

By l'Hôpital's Rule, we get
\[ f_6(0^+) = \lim_{s \to 0^+} \frac{f_{14}'(s)}{f_{15}'(s)} = \lim_{s \to 0} \frac{f_{14}''(s)}{f_{15}''(s)} = \frac{f_{16}(0)}{2f_{17}(0)} = \frac{1}{2}\psi'(x) - \beta(x) = \eta(x). \]

The limiting value \( f_6(1^-) \) follows from the result for \( f_5. \)

Next, let
\[ f_{18}(s) = s\log x - \log \frac{\Gamma(x + s)}{\Gamma(x)} - s(1 - s)\alpha(x) \quad \text{and} \quad f_{19}(s) = (1 - s)^2. \]

Then \( f_7(s) = f_{18}(s)/f_{19}(s), f_{18}(1) = f_{19}(1) = f_{18}'(1) = f_{19}'(1) = 0 \)
and
\[ \frac{f_{18}''(s)}{f_{19}''(s)} = \alpha(x) - \frac{1}{2}\psi'(x + s), \]

which is strictly increasing in \( s \) on \((0, 1)\) by the well-known monotonicity of \( \psi'. \)

Hence the monotonicity of \( f_7 \) follows from the Monotone l'Hôpital's Rule \([AVV1, \text{ Theorem} \ 1.25]\).

Clearly, \( f_7(0^+) = 0. \) By l'Hôpital's Rule, \( f_7(1^-) = \theta(x). \)

The inequalities in (1.14) are clear.

3.2. Corollary. (1) For \( x > 1 \) and \( s \in (0, 1), \)
\[ x^s \exp \left\{ \frac{s(s - 1)}{2x} \left( 1 + \frac{1}{6x} \right) \right\} < \frac{\Gamma(x + s)}{\Gamma(x)} < x^s \exp \left\{ \frac{s(s - 1)}{2x} \left( 1 - \frac{1}{6x} \right) \right\}. \]

(2) For \( s \in [0, 1), \) let \( \lambda(s) = 1 + 1/[3(1 - s)] \) and let \( \beta \) be as in Theorem 1.12(3).

Then for \( x > (1 - s)/2 \) and \( s \in (0, 1), \)
\[ \Gamma(x + s) \Gamma(x) > x^s \exp \left\{ -s(1 - s)\beta \left( \frac{1 - s}{2} \right) \right\} > x^s \exp \{-s\lambda(s)\}. \]
Proof. By (2.3) and AS 6.3.5, we have
\[ \beta(x) < \frac{1}{2x} \left( 1 + \frac{1}{6x} \right) \] and \[ \alpha(x) > \frac{1}{2x} \left( 1 - \frac{1}{6x} \right), \]
and hence part (1) follows from (1.14).

Next, by (2.4),
\[ x \beta'(x) = 1 - x \psi'(x) = 1 - \int_0^\infty \frac{xte^{-xt}}{1 - e^{-t}} \, dt = 1 - \frac{1}{x} \int_0^\infty \frac{ue^{-u}}{1 - e^{-u/x}} \, du, \]
which is strictly increasing from \((0, \infty)\) onto \((-\infty, 0)\) since \(u \mapsto v/(1-e^{-v})\) is strictly increasing from \((0, \infty)\) onto \((1, \infty)\). Hence \(\beta\) is strictly decreasing and convex from \((0, \infty)\) onto itself. This, together with the first inequality in (2.3), yields (3.3). \(\square\)

3.4. Proof of Theorem 1.16. (1) By (1.1), we can rewrite \(g_1(x)\) as
\[ g_1(x) = \left[ \frac{\sqrt{\pi}x\Gamma(x)}{\Gamma(x+1/2)} \right]^{1/x} = \left[ \frac{\sqrt{\pi}(x+1)}{\Gamma(x+1/2)} \right]^{1/x}. \]

Logarithmic differentiation gives
\[ g_1'(x)/g_1(x) = g_4(x)/g_6(x), \]
where \(g_6(x) = x^2\) and
\[ g_5(x) = x \left[ \psi(x+1) - \psi \left( x + \frac{1}{2} \right) \right] + \log \frac{\Gamma(x+1)}{\sqrt{\pi} \Gamma(x+1/2)}, \]
with \(g_5(0) = g_6(0) = 0\), \(\lim_{x \to \infty} g_5(x) = -\infty\), and
\[ \frac{g_5'(x)}{g_6(x)} = g_7(x) = \frac{1}{2} \left[ \psi'(x+1) - \psi' \left( x + \frac{1}{2} \right) \right]. \]

Since \(\psi''(x)\) is strictly increasing, \(g_7'(x) > 0\). Hence \(g_4\) is strictly increasing on \((0, \infty)\) by the Monotone l'Hôpital’s Rule [AVV Theorem 1.25]. By l'Hôpital’s Rule, (3.5), (3.6), and AS 6.4.2, 6.4.4, 6.4.10 and 23.2.24,
\[ g_4(0^+) = g_7(0^+) = \frac{1}{2} \left[ \psi'(1) - \psi' \left( \frac{1}{2} \right) \right] = -\zeta(2) = -\frac{\pi^2}{6} \]
and
\[ \lim_{x \to \infty} g_4(x) = \lim_{x \to \infty} g_7(x) = 0. \]

Here \(\zeta(x)\) is the Riemann zeta function. Hence \(g_4\) is strictly increasing from \((0, \infty)\) onto \((-\pi^2/6, 0)\), so that the monotonicity of \(g_1\) follows from (3.5). Since \(g_1'(x) = -g_1(x) \cdot [-g_4(x)]\), \(g_1'\) is strictly increasing on \((0, \infty)\) and the convexity of \(g_1\) follows.

By l'Hôpital’s Rule and AS 6.3.2, 6.3.3 and 6.3.18,
\[ g_1(0^+) = \exp \left\{ \lim_{x \to 0^+} \log \frac{\Gamma(x+1)}{\log \Gamma(x+1/2)} + (\log \pi)/2 \right\} = \exp(\psi(1) - \psi(1/2)) = 4 \]
and
\[ \lim_{x \to \infty} g_1(x) = \exp \left\{ \lim_{x \to \infty} \left[ \psi(x+1) - \psi \left( x + \frac{1}{2} \right) \right] \right\} = 1. \]
(2) By (1.1) and \([\text{AS}, 6.1.15]\), \(g_2(x)\) can be rewritten as
\[
g_2(x) = \frac{\sqrt{\pi} \Gamma(1 + 1/(4x))}{\Gamma(3/2 + 1/(4x))} \left[ \frac{\sqrt{\pi}}{2^{1/2}} \Gamma(x) \right]^{1/(2x)}.
\]

Logarithmic differentiation gives
\[
g_2'(x) = 2x \left[ \psi(x) - \psi \left( x + \frac{1}{2} \right) \right] - 2 \log \frac{\sqrt{\pi} \Gamma(x)}{2 \Gamma(x + 1/2)} + \psi \left( \frac{1}{4x} + \frac{3}{2} \right) - \psi \left( \frac{1}{4x} + 1 \right).
\]

Since
\[
g_2'(x) = 2x \left[ \psi'(x) - \psi'(x + \frac{1}{2}) \right] + \frac{1}{4x^2} \left[ \psi' \left( \frac{1}{4x} + 1 \right) - \psi' \left( \frac{1}{4x} + \frac{3}{2} \right) \right],
\]
which is positive for \(x \in (0, \infty)\) by the monotoneity of \(\psi'\) on \((0, \infty)\), \(g_2\) is strictly increasing on \((0, \infty)\).

Let \(h_3\) be as in Theorem 2.1. Then \(g_8\) can be rewritten as
\[
g_8(x) = -2h_3(x) - 1 - 2 \log \frac{\sqrt{\pi} \Gamma(x)}{2 \Gamma(x + 1/2)} + \psi \left( \frac{1}{4x} + \frac{3}{2} \right) - \psi \left( \frac{1}{4x} + 1 \right).
\]

Therefore, by \([\text{AS}, 6.3.18\text{ and } 6.1.37]\),
\[
g_8(0^+) = -2 + \log 4 + \lim_{x \to 0} \left\{ 2 \log x + \log \left[ \frac{1}{4x} + \frac{3}{2} \right] \right\} = -\infty
\]
and
\[
\lim_{x \to \infty} g_8(x) = -1 + \psi \left( \frac{3}{2} \right) - \psi(1) - 2 \log \frac{\sqrt{\pi}}{2} - 2 \log \frac{\Gamma(x)}{\Gamma(x + 1/2)}
\]
\[
= 1 - \log \pi - 2 \lim_{x \to \infty} \log \left[ \sqrt{\frac{e}{x}} \left( 1 + \frac{1}{2x} \right)^{-x} \right] = \infty.
\]
Hence \(g_8\) has a unique zero \(x_1 \in (0, \infty)\) such that \(g_8(x) < 0\) for \(x \in (0, x_1)\) and \(g_8(x) > 0\) for \(x > x_1\), so that the piecewise monotoneity of \(g_2\) follows from (3.7).

By \([\text{AS}, 6.3.7\text{ and } 6.3.8]\), we have
\[
\psi \left( \frac{2}{3} \right) - \psi \left( \frac{1}{6} \right) = \psi \left( 1 - \frac{1}{3} \right) - \psi \left( \frac{1}{6} \right)
\]
\[
= \pi \cot \frac{\pi}{3} + \psi \left( \frac{1}{3} \right) - \psi \left( \frac{1}{6} \right)
\]
\[
= \pi \cot \frac{\pi}{3} + \frac{1}{2} \psi \left( \frac{1}{6} \right) + \frac{1}{2} \psi \left( \frac{1}{6} + \frac{1}{2} \right) + \log 2 - \psi \left( \frac{1}{6} \right)
\]
\[
= \pi \cot \frac{\pi}{3} + \frac{1}{2} \psi \left( \frac{2}{3} \right) - \psi \left( \frac{1}{6} \right) + \log 2,
\]
so that
\[
\psi \left( \frac{2}{3} \right) - \psi \left( \frac{1}{6} \right) = 2 \pi \cot \frac{\pi}{3} + \log 4.
\]
Consequently, by [AS, 6.3.2, 6.3.4 and 6.3.5],
\[
g_s\left(\frac{3}{2}\right) = 3 \left[ \psi\left(\frac{3}{2}\right) - \psi(2) \right] - 2 \log \frac{\sqrt{\pi} \Gamma(3/2)}{2 \Gamma(2)} + \psi\left(\frac{10}{6}\right) - \psi\left(\frac{7}{6}\right)
\]
\[
= 3(1 - \log 4) - 2 \log \frac{\pi}{4} + \left(\frac{2}{3} + 1\right) - \psi\left(\frac{1}{6} + 1\right)
\]
\[
= -\frac{3}{2} - \log 4 - 2 \log \pi + \psi\left(\frac{2}{3}\right) - \psi\left(\frac{1}{6}\right)
\]
\[
= -\frac{3}{2} - 2 \log(2\pi) + 2\pi \cot \frac{\pi}{3} + \log 4 = -0.16186 \ldots < 0.
\]
On the other hand, by [AS, 6.3.2, 6.3.4, 6.1.12, 6.3.5 and 6.3.8],
\[
g_s(2) = 4 \left\{ \log 4 - \frac{5}{3} \right\} - 2 \log \frac{2}{3} + \left[ \psi\left(\frac{5}{8}\right) + \frac{8}{5} \right] - \left[ 8 + \psi\left(\frac{1}{8}\right) \right]
\]
\[
= 3 \log 4 + 2 \log 3 - \frac{196}{15} + \psi\left(\frac{5}{8}\right) - \psi\left(\frac{1}{8}\right)
\]
\[
= 3 \log 4 + 2 \log 3 - \frac{196}{15} + \psi\left(\frac{1}{8} + \frac{1}{2}\right) - \psi\left(\frac{1}{8}\right)
\]
\[
= 2 \log 12 - \frac{196}{15} + 2 \left[ \psi\left(\frac{1}{4}\right) - \psi\left(\frac{1}{8}\right) \right].
\]
It follows from [GR, 8.366.4 and 8.363.6, pp. 944–945] that
\[
\psi\left(\frac{1}{4}\right) = -\pi - \frac{\pi}{2} - 3 \log 2
\]
and
\[
\psi\left(\frac{1}{8}\right) = -\gamma - 3 \log 2 - \frac{\pi}{2} \cot \frac{\pi}{8} + \sum_{k=1}^{7} \left(\cos \frac{k\pi}{4}\right) \log \left(2 \sin \frac{k\pi}{8}\right)
\]
\[
= -\gamma - 4 \log 2 - \frac{\pi}{2} \cot \frac{\pi}{8} + \sqrt{2} \log \frac{\sin(\pi/8)}{\sin(3\pi/8)}
\]
\[
= -\gamma - 4 \log 2 - \frac{\pi}{2} \cot \frac{\pi}{8} + \sqrt{2} \log \tan \frac{\pi}{8},
\]
so that
\[
\psi\left(\frac{1}{4}\right) - \psi\left(\frac{1}{8}\right) = -\frac{\pi}{2} + \log 2 + \frac{\pi}{2} \cot \frac{\pi}{8} + \sqrt{2} \log \cot \frac{\pi}{8},
\]
and hence,
\[
g_s(2) = 3 \log 4 + 2 \log 3 - \frac{196}{15} - \pi + \pi \cot \frac{\pi}{8} + 2 \sqrt{2} \log \cot \frac{\pi}{8} = 0.225224 \ldots > 0.
\]
Consequently, \(x_1 \in (3/2, 2)\).

Clearly,
\[
g_2\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \cdot \frac{\pi}{2} \cdot \frac{\Gamma(1/2)}{\Gamma(1)} = \frac{\pi^2}{4}.
\]
It follows from [AS, 6.1.37] that \(\lim_{x \to \infty} g_2(x) = 2\).

(3) Logarithmic differentiation gives

\[ \frac{4x^2 g_3(x)}{g_3(x)} = g_9(x) \equiv g_8(x) + \frac{2}{2x + 1} - 2 - \log 4 \]

\[ = 2x \left[ \psi(x) - \psi \left( x + \frac{1}{2} \right) \right] + 2 \log \frac{\Gamma \left( x + 1/2 \right)}{\sqrt{\pi} (x)} + \psi \left( \frac{1}{4x} + \frac{1}{2} \right) - \psi \left( \frac{1}{4x} \right) - 4x. \]

By [AS] 6.4.1, for each \( x > 0, \)

\[ g_9(x) = 2x \int_0^\infty \frac{te^{-xt} dt}{1 + e^{-t/2}} + \frac{1}{4x^2} \int_0^\infty \frac{te^{-\left[1+(1/4x)\right]t} dt}{1 + e^{-t/2}} \]

\[ > 2x \int_0^\infty \frac{te^{-xt} dt}{1 + e^{-t/2}} = \frac{2}{x} \int_0^\infty \frac{(xt)e^{-xt}d(xt)}{1 + e^{-t/2}} \]

\[ = \frac{2}{x} \int_0^\infty \frac{ue^{-u} du}{1 + e^{-u/(2x)}} > \frac{1}{x} \int_0^\infty ue^{-u} du = \frac{1}{x}, \]

and hence,

\[ g_9'(x) = g_8'(x) - \frac{4}{(2x + 1)^2} > \frac{1}{x} - \frac{4}{(2x + 1)^2} = \frac{4x^2 + 1}{x(2x + 1)^2} > 0, \]

so that \( g_9 \) is strictly increasing on \((0, \infty).\) Since \( g_9(0^+) = -\infty \) and \( \lim_{x \to \infty} g_9(x) = \infty, \)

\( g_9 \) has a unique zero \( x_2 \) such that \( g_9(x) < 0 \) for \( x \in (0, x_2) \) and \( g_9(x) > 0 \) for \( x \in (x_2, \infty). \) This, together with (3.9), yields the piecewise monotonicity of \( g_3. \)

Applying the formulas [AS] 6.3.2, 6.3.4, 6.3.5 and 6.1.12

(3.10) \[ \psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \psi \left( n + \frac{1}{2} \right) = -\log 4 + 2 \sum_{k=1}^{n} \frac{1}{2k - 1}. \]

(3.11) \[ \Gamma \left( n + \frac{1}{2} \right) = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n} \sqrt{\pi}, \]

and [GR] 8.370 and 8.372

(3.12) \[ \psi \left( \frac{x + 1}{2} \right) - \psi \left( \frac{x}{2} \right) = 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{x + k}. \]

we get

\[ \frac{1}{2n} g_9(n) = g_{10}(n) \equiv 2 \log 2 + \log (2n - 1)! - \frac{1}{n} \log (n - 1)! + 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{2nk+1} - \sum_{k=1}^{n} \frac{1}{k(2k - 1)} - \frac{1}{n} \]

for \( n \in \mathbb{N}, \) and hence, by the properties of the Leibniz series and by computation,

\[ g_{10}(33) = 2 \log 2 + \log \frac{65!}{32!} + 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{66k + 1} - \sum_{k=1}^{33} \frac{1}{k(2k - 1)} - \frac{1}{33} \]

\[ < 2 \log 2 + \log \frac{65!}{32!} + 2 \sum_{k=0}^{30000} \frac{(-1)^k}{66k + 1} - \sum_{k=1}^{33} \frac{1}{k(2k - 1)} - \frac{1}{33} \]

\[ = -0.000381 \cdots < 0 \]
and
\[
g_{10}(34) = \log 2 - 2 + \frac{1}{34} \log \frac{67!!}{33!} + 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{68k+1} - \sum_{k=1}^{34} \frac{1}{k(2k-1)} - \frac{1}{34} > \log 2 - 2 + \frac{1}{34} \log \frac{67!!}{33!} + 2 \sum_{k=0}^{\infty} \frac{(-1)^k}{68k+1} - \sum_{k=1}^{34} \frac{1}{k(2k-1)} - \frac{1}{34} = \quad 0.000065 \ldots > 0.
\]
Hence \(x_2 \in (33, 34)\).

Finally, it is clear that \(g_3(1/2) = 4g_2(1/2) = \pi^2\) and \(\lim_{x \to \infty} g_3(x) = \lim_{x \to \infty} g_2(x) = 2\). \(\square\)

3.13. **Proof of Theorem 1.17.** (1) By the expression
\[
I_n = \frac{1}{2} B \left( \frac{n-1}{2}, \frac{1}{2} \right),
\]
\(F_1(n)\) can be rewritten as
\[
F_1(n) = 2n - 4 \left[ \frac{\Gamma \left( \frac{9}{2} \right)}{\Gamma \left( \frac{n-1}{2} \right)} \right]^2 = 4f_4 \left( \frac{n-1}{2} \right) + 2
\]
by (1.1), where \(f_4\) is as in Theorem 1.12. Hence the monotonicity of \(F_1\) and the limiting value \(\lim_{n \to \infty} F_1(n) = 3\) follow from Theorem 1.12(2). Clearly, \(F_1(1) = 6 - \pi\).

Thus, for \(n \geq 3, 6 - \pi \leq F_1(n) < 3\), so that (1.18) holds.

(2) By (1.2),
\[
F_2(n) = 2 \left\{ \frac{n-1}{2} - \left[ \frac{\Gamma \left( \frac{n-1}{2} + \frac{1}{2} \right)}{\Gamma \left( \frac{n-1}{2} \right)} \right]^2 + \frac{1}{2} \right\} = 2f_4 \left( \frac{n-1}{2} \right) + 1,
\]
which is strictly increasing for \(n \geq 3 \) with \(F_2(3) = 3 - \pi/2\) and \(\lim_{n \to \infty} F_2(n) = 3/2\) by Theorem 1.12(2). Hence the result follows.

(3) It follows from (1.2) that
\[
\frac{\Omega_{n-1}}{\Omega_n} = \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \frac{9}{2} + 1 \right)}{\Gamma \left( \frac{n+1}{2} \right)} = \frac{\omega_n}{\omega_{n+1}},
\]
so that
\[
F_3(n) = n - 2\pi \left( \frac{\omega_n}{\omega_{n+1}} \right)^2 = F_2(n + 2) - 2.
\]
Hence part (3) follows from part (2).

(4) It follows from (1.2), the expression [AVVI (2.32), p. 41]
\[
J_n = \frac{1}{2} B \left( \frac{1}{2(n-1)}, \frac{1}{2} \right)
\]
and [KS] 6.1.15] that
\[
b_n = \frac{\sqrt{\pi}}{2(n-1)} \frac{\Gamma \left( \frac{n-1}{2n-1} \right)}{\Gamma \left( \frac{1}{2(n-1)} + \frac{1}{2} \right)} = \sqrt{\pi} F_6 \left( \frac{1}{2(n-1)} \right),
\]
where $F_6(x) = \Gamma(x + 1)/\Gamma(x + 1/2)$. Since

$$F_6'(x) = F_6(x)[\psi(x + 1) - \psi(x + 1/2)] > 0$$

by the monotoneity of $\psi(x)$, $F_6$ is strictly increasing on $(0, \infty)$. Hence the monotoneity of $b_n$ follows from (3.15), and

\begin{equation}
1 = \sqrt{\pi} F_6(0) < b_n \leq b_2 = \pi/2.
\end{equation}

(3.16)

Next, let $g_1$ be as in Theorem 1.16(1). Then by (1.4) and (3.14),

$$B_n = \left[ \frac{1}{2(n-1)} B\left(\frac{1}{2(n-1)}, \frac{1}{2}\right) \right]^{n-1} = \sqrt{g_1\left(\frac{1}{2(n-1)}\right)},$$

and hence the monotoneity of $B_n$ follows from Theorem 1.16(1). Moreover, for $n \geq 2$,

\begin{equation}
\frac{\pi}{2} = B_2 \leq B_n < \lim_{n \to \infty} B_n = \sqrt{g_1(0^+)} = 2.
\end{equation}

(3.17)

The second double inequality in (1.21) follows from (3.16) and (3.17).

(5) We know that $n - J_n - a$ has the expression

$$n - J_n - a = \int_0^{\pi/2} \left[1 - (\sin t)^{1/(n-1)}\right] \tan t/2 \, dt.$$

(Cf. [AVV1, p. 43].) Hence

\begin{equation}
F_4(n) = \int_0^{\pi/2} F_7\left(\frac{1}{n-1}, t\right) \tan t/2 \, dt,
\end{equation}

(3.18)

where $F_7(x, t) = [1 - (\sin t)^{x}] / x$. Since the function

$$F_8(x, t) = \frac{\partial}{\partial x} \left[1 - (\sin t)^x\right] = (\sin t)^x \log \frac{1}{\sin t}$$

is strictly decreasing in $x$ on $(0, \infty)$ for each fixed $t \in (0, \pi/2)$, it follows from the Monotone L'Hôpital’s Rule [AVV1, Theorem 1.25] that $F_7$ is strictly decreasing in $x$ on $(0, \infty)$. Hence $F_4$ is strictly increasing for $n \geq 3$.

Clearly, $F_4(3) = 2(3 - J_3 - a) = 2(b - a)$. Set $x = 1/[2(n-1)]$. Then, by (3.14),

\begin{align*}
F_4(n) &= \frac{1}{2x} \left[ \frac{1}{2x} + 1 - \frac{1}{2} B\left(x, \frac{1}{2}\right) - a \right] \\
&= \frac{1}{4x^2} \left[ x \log 4 - \sqrt{\pi} \frac{\Gamma(x+1)}{\Gamma\left(x + \frac{1}{2}\right)} + 1 \right].
\end{align*}
Hence, by l'Hôpital's Rule,
\[
F_4(\infty) = \lim_{n \to \infty} F_4(n) = \frac{1}{4} \lim_{x \to 0} \frac{x \log 4 + 1 - [\sqrt{\pi}(x+1)/\Gamma(x+1/2)]}{x^2} = \frac{1}{8} \lim_{x \to 0} \frac{1}{x} \left\{ \log 4 - \sqrt{\pi} \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \left[ \psi(x+1) - \psi \left( x + \frac{1}{2} \right) \right] \right\} = -\frac{\sqrt{\pi}}{8} \lim_{x \to 0} \frac{\Gamma(x+1)}{\Gamma(x+1/2)} \times \left\{ \left[ \psi(x+1) - \psi \left( x + \frac{1}{2} \right) \right]^2 + \left[ \psi(x+1) - \psi \left( x + \frac{1}{2} \right) \right] \right\} = -\frac{1}{8} \left[ (\log 4)^2 + \frac{\pi^2}{6} - \psi \left( \frac{1}{2} \right) \right] = \frac{1}{8} \left[ \pi^2 - (\log 4)^2 \right] = c.
\]

The inequality (1.22) is clear.

(6) By (1.2), (1.4), (1.5) and [AS] 6.1.15, we have
\[
A_n = \left[ \frac{\pi^{n/2} \Gamma \left( \frac{n+1}{2} \right)}{2^n \Gamma \left( \frac{n}{2} + 1 \right)} \right]^{1/(n-1)} = \frac{\Gamma \left( \frac{1}{n+1} \right)}{\Gamma \left( \frac{1}{n+1} + \frac{1}{2} \right)}.
\]

Putting \( x = (n-1)/2 \) and applying [AS] 6.1.15, we get

(3.19)
\[
F_5(n) = \frac{\sqrt{\pi}}{2} \frac{1}{2x+1} \left[ \frac{\sqrt{\pi}}{2} \frac{\Gamma(x)}{\Gamma \left( x + \frac{1}{2} \right)} \right]^{1/2x} = \frac{1}{2} g_2(x),
\]

where \( g_2 \) is as in Theorem 1.16(2).

Next, computation gives

(3.20)
\[
F_5(4) = \frac{\sqrt{\pi}}{2} \left[ \frac{\sqrt{\pi}}{2} \frac{\Gamma(3/2)}{\Gamma(2)} \right]^{1/3} = \frac{\Gamma(1+1/16)}{\Gamma(1+9/16)} = 0.8890941475 \cdots
\]

and

(3.21)
\[
F_5(5) = \frac{\sqrt{\pi}}{2} \left[ \frac{\sqrt{\pi}}{2} \frac{\Gamma(2)}{\Gamma(2+1/2)} \right]^{1/4} = \frac{\Gamma(1+1/8)}{\Gamma(1+5/8)} = 0.8411396629 \cdots.
\]

The result for \( F_5 \) now follows from (3.19), (3.20), (3.21) and Theorem 1.16(2).

The limiting value \( \lim_{n \to \infty} (1/n)A_n = 1 \) follows from Theorem 1.16(2), and the inequality in (1.23) is clear.

(7) Put \( x = (n-1)/2 \). Then, by (3.19),

(3.22)
\[
D_n = 2^{n/(n-1)} \frac{1}{n-1} A_n = 2^{n/(n-1)} \frac{n}{n-1} \cdot \frac{A_n}{n} = g_3(x),
\]
where \( g_3 \) is as in Theorem 1.16(3). Hence, by Theorem 1.16(3), \( D_n \) is strictly decreasing for \( 2 \leq n \leq 67 \) and strictly increasing for \( n \geq 69 \).

By computation, we obtain

\[
D_{67} = \frac{2^{67/66}}{66} A_{67} = 1 \frac{\pi^{67/2}}{66} \left( \frac{\Gamma(33)}{\Gamma(33 + 1/2)} \right)^{1/66} \frac{\Gamma(1/132)}{\Gamma(67/132)}
\]

\[
= \frac{\sqrt{2\pi}}{66} \left( \frac{32!}{65!!} \right)^{1/66} \frac{\Gamma(1/132)}{\Gamma(67/132)} = 1.9853539227 \cdots ,
\]

\[
D_{68} = \frac{2^{68/67}}{67} A_{68} = 1 \frac{\pi^{68/2}}{67} \left( \frac{\Gamma(33 + 1/2)}{\Gamma(34)} \right)^{1/67} \frac{\Gamma(1/134)}{\Gamma(68/134)}
\]

\[
= \frac{\sqrt{2\pi}}{67} \left( \frac{32!}{65!!} \right)^{1/67} \frac{\Gamma(1/134)}{\Gamma(68/134)} = 1.9853491481 \cdots ,
\]

\[
D_{69} = \frac{2^{69/68}}{68} A_{69} = 1 \frac{\pi^{69/2}}{68} \left( \frac{\Gamma(34)}{\Gamma(34 + 1/2)} \right)^{1/68} \frac{\Gamma(1/136)}{\Gamma(69/136)}
\]

\[
= \frac{\sqrt{2\pi}}{68} \left( \frac{33!}{67!!} \right)^{1/68} \frac{\Gamma(1/136)}{\Gamma(69/136)} = 1.9853487792 \cdots ,
\]

\[
D_{70} = \frac{2^{70/69}}{69} A_{70} = 1 \frac{\pi^{70/2}}{69} \left( \frac{\Gamma(35)}{\Gamma(34 + 1/2)} \right)^{1/69} \frac{\Gamma(1/138)}{\Gamma(70/138)}
\]

\[
= \frac{\sqrt{2\pi}}{69} \left( \frac{34!}{67!!} \right)^{1/69} \frac{\Gamma(1/138)}{\Gamma(70/138)} = 1.98535124396 \cdots .
\]

These values, together with the conclusion just proved above, yield the piecewise monotonicity of \( D_n \).

Clearly, \( D_2 = \pi^2 \). By (3.22) and (1.23), \( \lim_{n \to \infty} D_n = 2 \). Hence (1.24) holds. \( \square \)

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REFERENCES


THE GAMMA AND PSI FUNCTIONS, WITH APPLICATIONS


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