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MORE ON THE TOTAL NUMBER OF PRIME FACTORS OF AN ODD PERFECT NUMBER

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ABSTRACT. Let $\sigma(n)$ denote the sum of the positive divisors of n. We say that n is perfect if $\sigma(n) = 2n$. Currently there are no known odd perfect numbers. It is known that if an odd perfect number exists, then it must be of the form $N = p^\alpha \prod_{j=1}^k q_j^{2\beta_j}$, where $p, q_1, \ldots, q_k$ are distinct primes and $p \equiv \alpha \equiv 1 \pmod 4$. Define the total number of prime factors of N as $\Omega(N) := \alpha + 2\sum_{j=1}^k \beta_j$. Sayers showed that $\Omega(N) \geq 29$. This was later extended by Iannucci and Sorli to show that $\Omega(N) \geq 37$. This paper extends these results to show that $\Omega(N) \geq 47$.

1. INTRODUCTION

Here and throughout, n is any natural number, and N is a hypothetical odd perfect number. Let $\sigma(n)$ denote the sum of the positive divisors of n. We say that n is perfect if $\sigma(n) = 2n$. It is known that if $\sigma(n) = 2n$ and n is even, then $n = 2^{k-1}(2^k - 1)$ where $2^k - 1$ is a Mersenne prime. Currently there are no known odd perfect numbers. First shown by Euler, it is well known that if an odd perfect number exists, then it must be of the form

$$N = p^\alpha \prod_{j=1}^k q_j^{2\beta_j}, \tag{1.1}$$

where $p, q_1, \ldots, q_k$ are distinct primes and $p \equiv \alpha \equiv 1 \pmod 4$.

Based on (1.1), we define the total number of prime factors of an odd perfect number as

$$\Omega(N) := \alpha + 2\sum_{j=1}^k \beta_j, \tag{1.2}$$

and we define the total number of distinct prime factors of N as

$$\omega(N) := 1 + k. \tag{1.3}$$

A number of bounds have been derived for $\Omega(N)$ and $\omega(N)$. Cohen showed that $\Omega(N) \geq 23$ [2]. Sayers showed that $\Omega(N) \geq 29$ [11]. This was later extended by Iannucci and Sorli to show that $\Omega(N) \geq 37$ [7]. This paper extends these results to give

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Theorem 1.1. *If N is an odd perfect number, then $\Omega(N) \geq 47$.*

As a result of the calculations made to prove Theorem 1.1, we get

Theorem 1.2. *If $\omega(N) > 8$, then $\Omega(N) \geq 49$. If $\omega(N) > 9$, then $\Omega(N) \geq 51$.*

2. Definitions and notation

We define the function $\sigma_{-1}(n)$ as

$$\sigma_{-1}(n) := \sum_{d|n} d^{-1} = \frac{\sigma(n)}{n}. \tag{2.1}$$

A number of simple results concerning $\sigma_{-1}(n)$ are summarized below.

Lemma 2.1. *Let n be any natural number. Then*

- *$\sigma_{-1}(n)$ is a multiplicative function, i.e., if $(n, m) = 1$, then $\sigma_{-1}(n \cdot m) = \sigma_{-1}(n)\sigma_{-1}(m)$.*
- *$\sigma_{-1}(n) > 1$ for all $n > 1$.*
- *$\sigma_{-1}(n) = 2$ if and only if n is perfect.*
- *$\frac{p+1}{p} \leq \sigma_{-1}(p^a) < \sigma_{-1}(p^{a+1}) < \frac{p}{p-1}$ for all primes p and integers $a > 1$.*

Lemma 2.2. *Let N be an odd perfect number. Then*

- *$\omega(N) \geq 8$ [1, 4].*
- *if $3 \nmid N$, then $\omega(N) \geq 11$ [5, 8].*
- *if $3 \nmid N$ and $5 \nmid N$, then $\omega(N) \geq 15$ [10].*
- *if $3 \nmid N$, $5 \nmid N$ and $7 \nmid N$, then $\omega(N) \geq 27$ [10].*

We adopt and modify the notation of [7]

Definition 2.3. Let N be an odd perfect number of the form $N = p^\alpha \prod q_i^{2\beta_i}$, where N has at most a_j of the β_i equal to b_j. The statement $[x : \alpha : b_1(a_1), \ldots, b_m(a_m)]$ means

- if $x = 1$, then N cannot be of the form;
- if $x \geq 3$ and N is of this form, then $P \nmid N$ for all primes $3 \leq P \leq x$.

If $a_j = *$, then we can have an arbitrary number of the β_i being b_j. If the α is not explicitly mentioned, then this statement is assumed to hold for all $\alpha \equiv 1 \pmod 4$.

For example, the statement $[17 : 5 : 1(2), 2(3)]$ would say that if

$$N = p^5 q_1^2 \cdots q_k^2 r_1^4 \cdots r_l^4,$$

is an odd perfect number where $p, q_1, \ldots, r_l$ are distinct primes, with $k \leq 2$ and $l \leq 3$, then $P \nmid n$ for $P = 3, 5, 7, 11, 13, 17$. Of course this is vacuously true as this N has at most 6 distinct prime factors, and it is known that $\omega(N) \geq 8$ for all odd perfect numbers. The statement $[17 : 1(2), 2(3)]$ instead would say that if $N = p^\alpha q_1^2 \cdots q_k^2 r_1^4 \cdots r_l^4$, is an odd perfect number, then the same conclusion holds, regardless of the value of α.

Lemma 2.4. *Using the notation of Definition 2.3,*

- *$[1 : 5(1), 1(*)]$,*
- *$[1 : 6(1), 1(*)]$,*
- *$[1 : 3(1), 2(1), 1(*)]$.*

Proof. These are exactly as stated in [7]. □

3. The algorithm and proof of Theorem 1.1

Suppose $N = p^\alpha \prod q_i^{2\beta_i}$, as before. To prove that $\Omega(N) \geq 47$, we assume that $\Omega(N) = \alpha + \sum 2\beta_i \leq 45$ and obtain a contradiction for every combination of α and β_i.

First select every partition

$$B = \{\alpha, \underbrace{2\beta_1, \ldots, 2\beta_1}_{a_1 \text{ times}}, \ldots, \underbrace{2\beta_m, \ldots, 2\beta_m}_{a_m \text{ times}}\}$$

of 45 such that

(1) $\alpha \equiv 1 \pmod 4$,
(2) $\omega(N) \geq 8$. (i.e., $1 + \sum_{i=1}^n a_i \geq 8$), and
(3) the existence of such a partition does not violate any result in Lemma 2.4.

The statement $[x : B]$ is to be taken as the same as $[x : \alpha : b_1(a_1), \ldots, b_m(a_m)]$.

It should be noted that we do not need to consider corresponding partitions $37, 39, 41$ or 43, because if $\alpha + \sum 2a_ib_i < 45$, then this case is proven by appending $45 - \alpha - \sum 2a_ib_i$ to this partition, which does sum to 45. For example, the partition

$$[1, \underbrace{2, \ldots, 2}_{19 \text{ times}}]$$

of 39 is shown to give a contradiction if the partition

$$[1, \underbrace{2, \ldots, 2}_{19 \text{ times}}, 6]$$

is shown to give a contradiction. There are 2539 partitions that satisfy condition (1). We have only 1310 of these which satisfy condition (2). Of these, 1268 satisfy condition (3). Using the results of [7], we could have reduced this list even more.

Based on Lemma 2.2,

- by proving $[3 : B]$ for all B with $8 \leq 1 + \sum a_i \leq 10$, we will have shown that $\omega(N) \geq 11$, which is a contradiction;
- by proving $[5 : B]$ for all B with $11 \leq 1 + \sum a_i \leq 14$, we will have shown that $\omega(N) \geq 15$, which is a contradiction;
- by proving $[7 : B]$ for all B with $15 \leq 1 + \sum a_i \leq 26$, we will have shown that $\omega(N) \geq 27$, which is a contradiction.

By (1.1) it is then clear that $\Omega(N) \geq 47$.

For example, as a small subset of these 1268 cases, we prove the following.

$[3 : 1 : 1(4), 6(3)]$	$[7 : 1 : 1(13), 2(1), 7(1)]$	$[3 : 13 : 1(3), 2(1), 3(1), 4(2)]$
$[5 : 1 : 1(9), 2(3), 7(1)]$	$[3 : 13 : 1(5), 3(1), 4(2)]$	$[5 : 1 : 1(10), 2(1), 3(1), 7(1)]$
$[7 : 1 : 1(15), 7(1)]$	$[3 : 5 : 1(5), 2(1), 13(1)]$	$[5 : 1 : 1(6), 2(2), 3(1), 4(1), 5(1)]$

We prove any given result by contradiction. For example, to prove the statement $[3 : 13 : 1(5), 3(1), 4(2)]$, we assume $3|N$. The case $3^{13}||N$ yields an immediate contradiction as $3 \not\equiv 1 \pmod 4$. Here $p^a||N$ means $p^a|N$ and $p^{a+1} \nmid N$. The next case would be $3^2||N$ (after which we try 3^6 and finally 3^8). Assuming that $3^2||N$ implies $\sigma(3^2)|N$, which implies $13|N$ (see Table 1). Next we assume that $13|N$ and consider the cases in order $13^{13}||N$, $13^2||N$, $13^6||N$ and finally $13^8||N$. We keep descending in this manner until such time as we derive a contradiction. As in [7], we consider the primes in the order from smallest to largest.

TABLE 1. Beginning of proof of $[3:13:1(5),3(1),4(2)]$

```
3^13  => 2^2 547 1093  xs=2
3^2  => 13
  13^13  => 2 7^2 29 5229043 22079
    7^2  => 3 19
      19^2  => 3 127  S=2.005554070
      19^6  => 701 70841
        29^2  => 13 67  xs=prime
        29^8  => 13 67 14437 41203  xs=prime
      19^8  => 3^2 127 523 29989  xs=3
    7^6  => 29 4733
      29^2  => 13 67
        67^2  => 3 7^2 31
          31^2  => 3 331  xs=prime
          31^8  => 3^2 331 81343 3637  xs=3
        67^8  => 3^2 7^2 31 30152894311  xs=prime
      29^8  => 13 67 14437 41203  xs=prime
    7^8  => 3^2 19 37 1063  S=2.052904805
  13^2  => 3 61
    61^13  => 2 31 52379047267 50689400581
      31^2  => 3 331
        331^2  => 3 7 5233  xs=3
        331^6  => 2180921 604842197
          2180921^2  => 1478526139 3217  xs=prime
          2180921^8  => 19 653977 12583 3217 9883 c_33 xs=prime
          ...
```

As in [7], we only partially factor large numbers, unless it becomes necessary to completely factor them. In the output these are denoted as "c_n" where n is the number of digits of the unfactored number.

There are four particular contradictions that we test for.

(1) Excess of a given prime:

By assuming $p^k||N$ we derive the contradiction that $p^{k+1}|N$. This is denoted in the output by "xs=p" where p is the prime in question.

(2) Excess of the number of primes:

We have more primes than we are allowed, given the restrictions on $\omega(N)$ for this case. This is denoted in the output by "xs=prime". Incompletely factored numbers are counted as contributing two primes, even though this may be too low. (Incompletely factored numbers are known not to be perfect powers.)

(3) Partition cannot be satisfied:

The factors that must divide N, along with their powers, cannot satisfy the partition. For example, if we find two primes, p and q, that must divide N at least 3 times, ($p^3|N$ and $q^3|N$), but the partition allows only one prime to divide N with a power greater than 2, we would have this contradiction. This is denoted in the output by "exponent bounds exceeded".

This contradiction is extremely rare, and was only used 6 times for the 1268 tests.

(4) Excess of σ_{-1}:
A floating point lower bound for $\sigma_{-1}(N)$ using known factors gives $\sigma_{-1}(N) > 2$. This is denoted in the code by "S=$\sigma_{-1}(N)$", giving a floating point approximation for $\sigma_{-1}(N)$.

It should be noted that when we start with a prime p other than 3, and we have already proven a contradiction for all primes between 3 and p, then we may assume that $P \nmid N$ for all $3 \leq P < p$. This is taken into account in contradiction (1).

This procedure is done on all of the 1268 tests to prove the results. The tests and the code are available at [6].

When an incompletely factored number needed to be factored, the following methods were used:

- A search was done of the online tables of factorizations of $\sigma(p^a)$ [12].
- If this failed, ecm was used to find a factor, using the code of T. Granlund, found at [13]. After ecm found a factor, further factorization was not always needed.

4. Comments on Theorem 1.2

This paper proved that $\Omega(N) \geq 47$. There is only one test that blocked a proof that $\Omega(N) \neq 47$, but this requires the factorization of a 301-digit number. In particular, in attempting to prove $\Omega(N) \neq 47$ we need to prove:

- $[3 : 1 : 1(4), 2(1), 8(1), 9(1)]$, which requires a factor of $\sigma(\sigma(11^{18})^{16})$, a 301-digit number.

There are four tests that blocked the proof that $\Omega(N) \neq 49$. They are:

- $[3 : 1 : 1(5), 2(1), 8(1), 9(1)]$ and $[3 : 1 : 1(3), 2(2), 8(1), 9(1)]$, which both require a factor of $\sigma(\sigma(11^{18})^{16})$, a 301-digit number.
- $[3 : 1 : 1(4), 3(1), 8(1), 9(1)]$, which requires a factor of $\sigma(\sigma(547^{18})^{16})$, a 789-digit number.
- $[3 : 1 : 1(3), 2(2), 6(1), 11(1)]$, which requires a factor of $\sigma(\sigma(3221^{12})^{22})$, a 927-digit number.

It is worth noting that these are the only tests which we could not prove computationally, and for each either $\omega(N) = 8$ or 9. This proves Theorem 1.2.

5. Conclusions and comments

Each time that we descend a level of the algorithm, we must choose a prime to work with. Currently the algorithm will take the smallest available prime. This is not always the best choice. If a better choice could be made, some calculations may become feasible which currently are not. In particular, a number of results of the form $[739 : 1(*), 2(*)]$ (see [3, 9]), could be shown which are currently infeasible, due to time considerations.

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