COMPUTING ISOGENY COVARIANT DIFFERENTIAL MODULAR FORMS

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Abstract. We present the computation modulo $p^2$ and explicit formulas for the unique isogeny covariant differential modular form of order one and weight $\chi_{-p-1,-1}$ called $f_{\text{jet}}$, an isogeny covariant differential modular form of order two and weight $\chi_{-p^2-p-1,-1}$ denoted by $f_{\text{jet}}h_{\text{jet}}$, and an isogeny covariant differential modular form $h_{\text{jet}}$ of order two and weight $\chi_{1-p^2,0,-1}$.

1. Introduction

In this paper we introduce explicit formulas modulo $p^2$ for various differential modular forms discussed by Buium in [3], [2], [4]. The central modular form discussed is the unique, up to multiplication by an element in $\mathbb{Z}_p^*$, isogeny covariant differential modular form of order one and weight $\chi_{-p-1,-1}$ called $f_{\text{jet}}$, $f_{\text{jet}}^1$, and $f_{\text{jet}}^{1,2}$, respectively, in [3], [2], [4]. For the rest of this paper we mean “unique up to multiplication by an element in $\mathbb{Z}_p^*$” when we say “unique”, and we will refer to the unique isogeny covariant differential modular form of order one and weight $\chi_{-p-1,-1}$ by $f_{\text{jet}}$. This modular form has many interesting connections detailed in [1], [3], [2], and [4]. We compute $f_{\text{jet}}$ in a $p$-adic fashion following the construction of $f_{\text{jet}}$ detailed in [3] which allows us to compute modulo $p^n$ or specifically modulo $p^2$. Then we use the explicit formula from this computation to provide modulo $p^2$ formulas for order two differential modular forms. The specific order two isogeny covariant differential modular forms we describe are $f_{\text{jet}}h_{\text{jet}}$ from [3] also referred to as $k^2$ in [2] or $f_{\text{jet}}^{1,2}$ in [4] and $h_{\text{jet}}$ from [3] also referred to as $k^2_p/f^1_p$ in [2]. We note that modulo $p$ neither of these order two modular forms contain any second order terms.

The strategy is simple. We know that the isogeny covariant differential modular forms $f_{\text{jet}}h_{\text{jet}}$ and $h_{\text{jet}}$ of order two and weights $\chi_{-p^2-p-1,-1}$ and $\chi_{1-p^2,0,-1}$, respectively, are $f_{\text{jet}}h_{\text{jet}} = \phi(f_{\text{jet}})$, where $\phi$ is the lifting of the Frobenius morphism, and outside the locus, where $f_{\text{jet}}$ modulo $p$ is zero $h_{\text{jet}} = \frac{\phi(f_{\text{jet}})}{f_{\text{jet}}}$. We should note that $h_{\text{jet}}$ is defined only outside this zero locus of $f_{\text{jet}}$ modulo $p$. In [5] we have the explicit computation of $f_{\text{def}}$ (the $p$-derivation analog of the Kodaira-Spencer class) which is the reduction modulo $p$ of the unique isogeny covariant differential modular form of weight $\chi_{-p-1,-1}$. By uniqueness, $f_{\text{jet}} \equiv c f_{\text{def}}$ modulo $p$ for some $c \in \mathbb{Z}_p^*$. Here we compute $f_{\text{jet}}$ directly allowing us to give a formula for the unique
isogeny covariant differential modular form modulo $p^2$ and not just modulo $p$. We also then are able to describe the order two terms that occur in $f_{\text{jet}} h_{\text{jet}}$ and $h_{\text{jet}}$ modulo $p^2$ but not modulo $p$.

For both context and notation we give the relevant definitions of differential modular forms. Let $p > 3$ be a prime number. Let $M^0 = \mathbb{Z}_p[a_4, a_6, \Delta^{-1}]$, $M^1 = \mathbb{Z}_p[a_4, a_6, \delta a_4, \delta a_6, \Delta^{-1}]$, and $M^2 = \mathbb{Z}_p[a_4, a_6, \delta a_4, \delta a_6, \delta^2 a_4, \delta^2 a_6, \Delta^{-1}]$, where $\Delta = -2^4(4a_4^3 + 27a_6^2)$ and $\mathbb{Z}_p$ is the $p$-adic integer. We note that $a_4, a_6, \delta a_4, \delta a_6, \delta^2 a_4, \delta^2 a_6$ are variables over $\mathbb{Z}_p$ and that $\hat{\delta}$ represents the $p$-adic completion. Then the elements of $M^2$ are called $\delta$ modular forms of order one and elements of $M^2$ are called $\delta$ modular forms of order two as defined by Buium in [3].

Recall now that a $p$-derivation is a set theoretic map, $\delta : A \rightarrow B$, from a ring $A$ to an $A$-algebra $B$ such that

\begin{align*}
\delta(x + y) &= \delta x + \delta y + C_p(x, y), \\
\delta(xy) &= y^p \delta x + x^p \delta y + p \delta x \delta y
\end{align*}

for all $x, y \in A$, where $C_p(X, Y) = \frac{X^p + Y^p - (X + Y)^p}{p}$. In Section 2 we will expand these axioms into a more complete list of properties of $p$-derivations. For now, if $A$ is a complete discrete valuation ring $R$, where $R$ has maximal ideal generated by $p$ and an algebraically closed residue field $k$, and if $\phi$ is the unique lifting of the Frobenius morphism to $A$, then the $p$-derivation given by $\delta(x) = (\phi(x) - x^p)/p$ is unique on $R$.

Now we use the $R$ and $\delta$ from our example and set

$M(R) = \{(a, b) \in R^2 | 4a^3 + 27b^2 \in R^*\}.$

Then the set $M(R)$ is in one-to-one correspondence with the set of pairs consisting of an elliptic curve over $R$ and an invertible 1-form; namely, each $(\bar{a}_4, \bar{a}_6) \in M(R)$ corresponds to $(E, dx/2y)$, where $E$ is the projective closure of the affine plane curve $y^2 = x^3 + \bar{a}_4 x + \bar{a}_6$. For any $f \in M^1$, if we substitute $a, b, \delta a, \delta b$ in for $a_4, a_6, \delta a_4, \delta a_6$, then $f$ defines a map (still denoted by $f$) from $M(R)$ to $R$. This element in $M^1$ is in fact uniquely determined by the map from $M(R)$ to $R$. Similar statements are true for $f \in M^2$.

We define a $\delta$-character of order $\leq 1$ to be a group homomorphism $\chi : R^* \rightarrow R^*$ of the form $\chi = \chi_{m,n}$, where

$\chi_{m,n}(\lambda) = \lambda^m \left( \frac{\phi(\lambda)}{\lambda^p} \right)^n.$

Then a $\delta$-modular function of order one has weight $\chi$ if for any $\lambda \in R^*$

$f(\lambda^4 a, \lambda^6 b) = \chi(\lambda) f(a, b)$

for all $(a, b) \in M(R)$. We can easily extend the definition of $\delta$-characters to higher orders. Namely, a $\delta$-character of order $\leq 2$ is a group homomorphism $\chi : R^* \rightarrow R^*$ of the form $\chi = \chi_{m,n,r}$ where

$\chi_{m,n,r}(\lambda) = \lambda^m \left( \frac{\phi(\lambda)}{\lambda^p} \right)^n \left( \frac{\phi^2(\lambda)}{\lambda^p} \right)^r.$

The criterion for a $\delta$-modular function of order two to have weight $\chi$ is exactly the same as the criterion for a $\delta$-modular function of order one to have weight $\chi$. A $\delta$-modular form is a $\delta$-modular function with a weight.
A \delta\text{-modular form is isogeny covariant} if for any two pairs \((a, b)\) and \((\bar{a}, \bar{b})\) with an etale isogeny of degree \(N\) between the corresponding elliptic curves that pulls back \(\frac{dx}{y}\) to \(\frac{dx}{y}\)

\[ f(a, b) = N^{-k/2}f(\bar{a}, \bar{b}), \]

where \(k\) is a constant that depends solely on the weight. Note that for \(\chi = \chi_{m,n}\) the constant is \(k = m + n(1 - p)\) and for \(\chi = \chi_{m,n,r}\) the constant is \(k = m + n(1 - p) + r(1 - p^2)\).

**Theorem 1.1.** The isogeny covariant differential modular form of order one and weight \(\chi_{-p-1, -1}\) modulo \(p^2\) is

\[ f_{\text{jet}} = \left[ -\frac{72a_6^2\delta a_4 + 48a_4^p\delta a_6}{\Delta^p} \right] \gamma_{2p,p} + h + pH, \]

where \(\gamma_{2p,p}\) and \(h\) are polynomials in \(M^0_1 := M^0 \otimes \mathbb{Z}_p/(p^2)\), \(H\) is a polynomial in \(M^1_1 := M^1 \otimes \mathbb{Z}_p/(p)\), and \(H\) is a nonhomogeneous quadratic in \(\delta a_4\) and \(\delta a_6\).

Explicit formulas for \(h\) and \(H\) are given in Theorem 6.11 and an explicit formula for \(\gamma_{2p,p}\) is given in Proposition 6.2.

**Theorem 1.2.** The isogeny covariant differential modular form \(f_{\text{jet}}h_{\text{jet}}\) of order two and weight \(\chi_{-p^2-p, -1-1}\) modulo \(p^2\) is

\[ f_{\text{jet}}h_{\text{jet}} = \left[ -\frac{72a_6^2\delta a_4 + 48a_4^p\delta a_6}{\Delta^p} \right] \gamma_{2p,p}(a_4^p, a_6^p) + h^* + p\left[ -\frac{72a_6^2\delta^2 a_4 + 48a_4^p\delta^2 a_6}{\Delta^p} \right] \gamma_{2p,p}(a_4^p, a_6^p) + J, \]

where \(h^*\) is a polynomial in \(M^1_1 := M^1 \otimes \mathbb{Z}_p/(p^2)\) and \(J\) is a polynomial in \(M^1_1\).

**Corollary 1.3.** The isogeny covariant differential modular form \(h_{\text{jet}}\) of order two and weight \(\chi_{1-p^2, 0-1}\) modulo \(p^2\) is

\[ h_{\text{jet}} = \left[ -\frac{72a_6^2\delta a_4 + 48a_4^p\delta a_6}{\Delta^p} \right] \gamma_{2p,p}(a_4^p, a_6^p) + \Delta^p h^* \frac{\Delta^p - 2p([-72a_6^2\delta a_4 + 48a_4^p\delta a_6] \gamma_{2p,p} + \Delta^p h)}{\Delta^p - 2p([-72a_6^2\delta a_4 + 48a_4^p\delta a_6] \gamma_{2p,p} + \Delta^p h + J)} + \frac{[72a_6^2\delta^2 a_4 + 48a_4^p\delta^2 a_6] \gamma_{2p,p}(a_4^p, a_6^p) + \Delta^p J}{\Delta^p - 2p([-72a_6^2\delta a_4 + 48a_4^p\delta a_6] \gamma_{2p,p} + \Delta^p h)}, \]

where \(h, h^*, H, J\) are the same as in Theorems 1.1 and 1.2.

What follows is preliminary information to the calculation of \(f_{\text{jet}}\). Let \(E\) be the elliptic curve in Weierstrass form over \(M^0\) defined by the homogeneous equation

\[ f(X,Y,W) = WY^2 - X^3 - a_4XW^2 - a_6W^3. \]

Let \(U\) and \(V\) be the affine open subsets of \(E\) given by the equations \(f(x, y, 1)\) and \(f(z, 1, w)\), respectively. So

\[ U = \text{Spec } M^0[X, Y]/(f(X, Y, 1)) = \text{Spec } M^0[x, y], \]
\[ V = \text{Spec } M^0[Z, W]/(f(Z, 1, W)) = \text{Spec } M^0[z, w], \]
and on $U \cap V$

\[
z = -x/y, \\
w = -1/y,
\]
whence $E = U \cup V$. Next we define the first jets of $U$ and $V$ to be the sets

\[
U^1 = \text{Spec}\ M^1[1, Y, \delta X, \delta Y]/(f(X, Y, 1), \delta f(X, Y, 1)) = \text{Spec}\ M^1[x, y, \delta x, \delta y], \\
V^1 = \text{Spec}\ M^1[Z, W, \delta Z, \delta W]/(f(1, Z, W), \delta f(1, Z, W)) = \text{Spec}\ M^1[z, w, \delta z, \delta w].
\]

Then $E^1$, the first jet space of $E$, is the gluing of $U^1$ and $V^1$ by the maps

\[
z = -x/y, \\
w = -1/y, \\
\delta z = \frac{x^p \delta y - y^p \delta x}{y^p(y^p + p\delta y)}, \\
\delta w = \frac{\delta y}{y^p(y^p + p\delta y)}.
\]

We can extend the group law on $E$ to a group law on $E^1$. The group law arises naturally by construction from the group law on $E$ just as $E^1$ arises naturally by construction from $E$. This will be detailed explicitly in Section 3.

From now on we will also use the following notation. First by $M^1_n$ we mean $M^1 \otimes \mathbb{Z}_p/(p^{n+1})$. For example $M^1_0 = \mathbb{F}_p[a_4, a_6, \delta a_4, \delta a_6, \Delta^{-1}]$, where $\mathbb{F}_p$ is the finite field of $p$ elements. Second by $E^1_m$ we mean $E^1 \otimes M^1_m$, and by $E_m = E^0_m$ we mean $E \otimes M^0_m$. Also we will use $\delta(a_i)$ interchangeably for $\delta a_4$, $\delta(a_0)$ interchangeably for $\delta a_6$, etc.

To compute $f_{\text{jet}}$, the isogeny covariant $\delta$ modular form of weight $\chi_{-p-1,-1}$, we work from [2] Construction 4.1. The same construction is also described in [4] and [3]. First we find two sections $s_U$ and $s_V$ of the morphisms $U^1 \to U \otimes M^1$ and $V^1 \to V \otimes M^1$, respectively, such that $s_U$ defines a morphism from $U \otimes M^1$ to $U^1$ and $s_V$ defines a morphism from $V \otimes M^1$ to $V^1$. Then the difference of the sections under the group law induces a morphism $s_U - s_V : U \cap V \otimes M^1 \to E^1$. Let $\zeta$ be the $\delta z$ coordinate in the difference $s_U - s_V$. By the $\delta z$ coordinate, we mean the image of $\delta z \in U^1 \cap V^1$ under the ring homomorphism induced by the morphism $s_U - s_V : U \cap V \otimes M^1 \to E^1$. Let $\log_{f_{\text{jet}}^1}(\zeta)$ be the formal logarithm of the Frobenius twist of the formal group of the elliptic curve, namely

\[
\log_{f_{\text{jet}}^1}(\zeta) = \xi + \frac{p \phi(c_1)}{2} \xi^2 + \frac{p^2 \phi(c_2)}{3} \xi^3 + \cdots,
\]

where the $c_i$ are the coefficients of the power series expansion of the invariant differential [3]. Then $\log_{f_{\text{jet}}^1}(\zeta)$ is a cohomology class in $H^1(E \otimes M^1, \mathcal{O}) \simeq H^1(E, \mathcal{O}) \otimes M^1$, and this resulting class has a representative of the form $\sum a_n y^n + x \sum b_n y^n + x^2 \sum c_n y^n$. The modular form $f_{\text{jet}}$ is the coefficient $e_{-1}$ of $x^2/y$ in this representative which is the residue of the cohomology class, namely the image of the cohomology class under the Serre duality pairing.

We will actually work modulo $p^2$ which means that our end result will be $f_{\text{jet}}$ modulo $p^2$. In fact $f_{\text{jet}} \in M^1$ is a restricted power series whose coefficients expand exponentially in the number of terms in each coefficient of a power of $p$. Therefore, the formulas necessary to express $f_{\text{jet}}$ modulo $p^n$ for $n > 2$ are prohibitive in length. At this point we note that the formal logarithm of the Frobenius twist of the formal
group of the elliptic curve modulo $p^2$ is in fact the identity. This certainly simplifies one step of the computation modulo $p^2$; however, for $n > 4$ this formal logarithm is no longer trivial modulo $p^n$, meaning this step is not trivial for large $n$. As a preliminary step to computing $f_{jet}$ we detail some computation guidelines for $p$-derivations and the group law for $E^1$ modulo $p^2$.

2. Properties of $p$-derivations

Recall that a $p$-derivation is a set theoretic map, $\delta : A \rightarrow B$, from a ring $A$ to an $A$-algebra $B$ with $\delta(1) = 0$ such that

$$\delta(x + y) = \delta x + \delta y + C_p(x, y), \quad \delta(xy) = y^p\delta x + x^p\delta y + p\delta x\delta y$$

for all $x, y \in A$, where $C_p(X, Y) = \frac{X^{p} + Y^{p} - (X + Y)^{p}}{p}$. In the case when $A = B = R$, where $R$ is a complete discrete valuation ring with maximal ideal generated by $p$ and has an algebraically closed residue field, there is a unique $p$-derivation given by $\delta(x) = (\varphi(x) - x^p)/p$, where $\varphi$ is the unique lifting of the Frobenius morphism to $R$.

This definition implies that if $\varphi : A \rightarrow B$ is the ring homomorphism associated to $B$ being an $A$-algebra, then

$$(\varphi, \delta) : A \rightarrow W_2(B)$$

is a ring homomorphism, where $W_2(B)$ is the ring of Witt vectors of length two on $B$. With $(\varphi, \delta)$ as above, $\phi : A \rightarrow B$ defined by $\phi(x) = \varphi(x)^p + p\delta(x)$ is a ring homomorphism. In case $B = A$, this is a lifting of the Frobenius endomorphism $F(x) = x^p$ of $A/pA$.

While no further axioms for $p$-derivations beyond those in the definition are necessary for computation, the following $p$-derivation rules are very convenient for computation. Before introducing these rules we must define an extension of $C_p(X, Y)$.

Definition 2.1. For any $\sum q$, let

$$C_p^{\text{ext}}(\sum q) = \sum q^p - (\sum q)^p.$$  

Note that $C_p^{\text{ext}}(X + Y) = \frac{X^{p} + Y^{p} - (X + Y)^{p}}{p} = C_p(X, Y)$; thus, this is a very natural definition.

Lemma 2.2. Let $\delta : A \rightarrow B$ be a $p$-derivation, let $g = \sum q, x, y \in A$, and let $n > 0$ be an integer. Then the following are true.

1. $\delta(\sum q) = \sum \delta q + C_p^{\text{ext}}(\sum q)$.
2. $\delta(-1) = 0$.
3. $\delta(-x) = -\delta x$.
4. $\delta(x^n) = \sum_{k=1}^{n} \left( \begin{array}{c} n \\ k \\ \end{array} \right) p^{k-1} x^{(n-k)p}(\delta x)^k = \frac{-x^{np} + (x^p + p\delta x)^n}{p}$.
5. $\delta \left( \frac{1}{x} \right) = \frac{-\delta x}{x^p(x^p + p\delta x)}$.
6. $\delta \left( \frac{y}{x} \right) = \frac{x^p\delta y - y^p\delta x}{x^p(x^p + p\delta x)}$. 


3. The Group Law for the First p-Jet Space of \( E \)

We now want to make the group law on \( E^1 \) explicit. This is necessary since the main result requires us to subtract two sections using the group law. The group law on the first \( p \)-jet is induced by the group law on the elliptic curve \( E \), so we start by giving the group law on \( E \). Let \( \rho \) and \( \psi \) be the equations that define the group law on \( E \). So if \( (z_1, w_1) \oplus (z_2, w_2) = (z_3, w_3) \), then

\[
\begin{align*}
z_3 &= \rho(z_1, w_1, z_2, w_2), \\
w_3 &= \psi(z_1, w_1, z_2, w_2).
\end{align*}
\]

Then the group law on \( E^1 \) is an extension of the group law on \( E \) such that if \( (z_1, w_1, \delta z_1, \delta w_1) \oplus (z_2, w_2, \delta z_2, \delta w_2) = (z_3, w_3, \delta z_3, \delta w_3) \), then

\[
\begin{align*}
z_3 &= \rho(z_1, w_1, z_2, w_2), \\
w_3 &= \psi(z_1, w_1, z_2, w_2), \\
\delta z_3 &= \delta(\rho(z_1, w_1, z_2, w_2)), \\
\delta w_3 &= \delta(\psi(z_1, w_1, z_2, w_2)).
\end{align*}
\]

To find appropriate \( \rho \) and \( \psi \), we must consider actual formulas for the group law. On the elliptic curve \( E \), the group law can be explicitly formulated using the chord-tangent approach. In this approach we consider that every line intersects the elliptic curve at exactly three points counting multiplicity. We choose a specific point, \( O \), to be the origin; in this case the point we choose to be the origin is the point at infinity, \((0,1,0)\). We then define the inverse of a point \( P \) to be the third point on the line that intersects \( P \) and the origin. We denote this point by \(-P\). So if we want to add \( P \oplus Q \), we take the line through \( P \) and \( Q \) and let \( R \) be the third point on the line. Then we define \( P \oplus Q = -R \). This definition arises naturally from the theory of Weil divisors. We refer to the case when \( P = Q \) as the tangent case and \( P \neq Q \) as the tangent case. From now on we will focus on the chord case of the chord-tangent approach since that is the most general case and the case used when computing the group law for a \( p \)-jet space.

We use the standard procedure for finding explicit formulas for group law in the \( z \) and \( w \) coordinates. In these coordinates our origin is \((0,0)\). To start with, we recall data on \( V \); namely, that \( f(z, 1, w) = w - z^3 - a_4zw^2 - a_6w^3 \) is the curve we will be using and that any line through \( P = (z_0, w_0) \) and the origin \((0,0)\) will intersect \( f(z, 1, w) \) at the third point \((-z_0, -w_0)\). Whence \(-P = (-z_0, -w_0)\).

Now consider two points \( P_1 \) and \( P_2 \) denoted by \((z_i, w_i)\) for \( i = 1, 2 \), respectively. If we assume that \( z_1 \neq z_2 \), the line connecting these two points is

\[
w = \frac{w_2 - w_1}{z_2 - z_1}(z - z_1) + w_1.
\]

To find the sum \( P_1 \oplus P_2 \), we must find the three points counting multiplicity of the intersection of this line with the curve \( f(z, 1, w) \). If we substitute the line into the curve \( f(z, 1, w) \) we get a cubic equation in terms of \( z \). Finding these three points becomes a matter of finding the roots of the resulting cubic equation. On the other hand, we already know two of the roots, namely \( z = z_1 \) and \( z = z_2 \). The third root
is

\[ z = \frac{-2w_2z_1 - w_1z_1 + 2w_1z_2 - 3a_6w_2w_1^2z_2 - a_4w_1^2z_2^3 + w_2z_2 + 3w_2^2z_1a_6w_1 + w_2^2z_1^2a_4}{3a_6w_2w_1(w_2 - w_1) + 3z_2z_1(z_2 - z_1) + a_4(w_2^2z_1 - w_1^2z_2) + w_1 - w_2 + 2a_4w_2w_1(z_2 - z_1)}. \]

So for \( z_1 \neq z_2 \), the \( P_3 = P_1 \oplus P_2 \) has coordinates

\[ z_3 = \frac{-2w_2z_1 - w_1z_1 + 2w_1z_2 - 3a_6w_2w_1^2z_2 - a_4w_1^2z_2^3 + w_2z_2 + 3w_2^2z_1a_6w_1 + w_2^2z_1^2a_4}{3a_6w_2w_1(w_2 - w_1) + 3z_2z_1(z_2 - z_1) + a_4(w_2^2z_1 - w_1^2z_2) + w_1 - w_2 + 2a_4w_2w_1(z_2 - z_1)}, \]

\[ w_3 = \frac{3w_2z_2^2 + z_1w_2^2w_1a_4 - 3z_1w_1z_2^2 + w_1^2 - w_2w_1^2z_2a_4 - w_2^2}{3a_6w_2w_1(w_2 - w_1) + 3z_2z_1(z_2 - z_1) + a_4(w_2^2z_1 - w_1^2z_2) + w_1 - w_2 + 2a_4w_2w_1(z_2 - z_1)}. \]

From this information, if we want the formulation of group law on \( E_1 \) we must simply take the \( p \)-derivation of these equations. The resulting group law for \( P_1 = (z_1, w_1, \delta z_1, \delta w_1) \in V^1 \) is

\[ z_3 = -\frac{\alpha}{\mu}, \]

\[ w_3 = -\frac{\beta}{\mu}, \]

\[ \delta z_3 = -\frac{\mu^p\delta \alpha - \alpha^p\delta \mu}{\mu^p(\mu^p + p^2 \mu)}, \]

\[ \delta w_3 = -\frac{\mu^p\delta \beta - \beta^p\delta \mu}{\mu^p(\mu^p + p^2 \mu)}, \]

where \( P_3 = P_1 \oplus P_2 \),

\[ \alpha = -2w_2z_1 - w_1z_1 + 2w_1z_2 - 3a_6w_2w_1^2z_2 \]

\[ - a_4w_1^2z_2^3 + w_2z_2 + 3w_2^2z_1a_6w_1 + w_2^2z_1^2a_4, \]

\[ \beta = 3w_2z_2^2 + z_1w_2^2w_1a_4 - 3z_1w_1z_2^2 + w_1^2 - w_2w_1^2z_2a_4 - w_2^2, \]

\[ \mu = 3a_6w_2w_1(w_2 - w_1) + 3z_2z_1(z_2 - z_1) \]

\[ + a_4(w_2^2z_1 - w_1^2z_2) + w_1 - w_2 + 2a_4w_2w_1(z_2 - z_1), \]

and \( \delta \alpha, \delta \beta, \delta \mu \) are the respective \( p \)-derivatives which are not included here because of their lengthy nature. On the other hand, this group law also describes the group law on \( E_1^m \). For example, if \( m = 1 \), then we consider this same group law modulo \( p^2 \). This does shorten the expressions of \( \delta \alpha, \delta \beta, \) and \( \delta \mu \) for \( m \leq 5 \) to lengths that are possible to work with in computer algebra systems.

Besides shortening the expressions for \( \alpha, \beta, \) and \( \mu \) one other advantage of explicitly detailing the group law on \( E_1^1 \) rather than \( E^1 \) is that we may write \( \delta z_3 \) and \( \delta w_3 \) in terms of polynomials in \( \delta \alpha, \delta \beta, \) and \( \delta \mu \) by using their series expansions. Hence
we have the following description for the group law on $E^1_1$:

$$z_3 = -\frac{\alpha}{\mu},$$
$$w_3 = -\frac{\beta}{\mu},$$
$$\delta z_3 = \frac{1}{\mu^{2p}}(-\mu^p \delta \alpha + \alpha^p \delta \mu)(\mu^p - p\delta \mu),$$
$$\delta w_3 = \frac{1}{\mu^{2p}}(-\mu^p \delta \beta + \beta^p \delta \mu)(\mu^p - p\delta \mu),$$

where $\delta \alpha$, $\delta \beta$, and $\delta \mu$ are now expressions modulo $p^2$.

4. The section on $U$ that defines a map from $U \otimes M^1$ to $U^1$ and the section on $V$ that defines a map from $V \otimes M^1$ to $V^1$

We in fact want a specific map from $U$ to $U^1$; namely, the morphism which takes $\delta x$ and $\delta y$ to elements such that $\delta f(x, y, 1)$ is mapped to 0. To do this we use a variant of Hensel’s Lemma involving two variables which will be illuminated as we go along. To find the appropriate $\delta x$ and $\delta y$, we first consider the explicit expression of $\delta f(x, y, 1)$,

$$(4.1) \quad \delta f(x, y, 1) = -p^2 \delta x^3 + (-3x^p \delta x^2 - \delta a_4 \delta x + \delta y^2)p + (-3x^{2p} - a_4^p)\delta x + 2y^p \delta y - \delta a_{14} x^{p} - \delta a_{6} + C_p^{\text{ext}}(y^2 - x^3 - a_4 x - a_6),$$

and from this polynomial define $P_\gamma = -\delta a_{14} x^{p} - \delta a_{6} + C_p^{\text{ext}}(y^2 - x^3 - a_4 x - a_6)$. From now on for convenience of notation we will denote $f(x, y, 1)$ simply by $f$. We will also denote $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ by $f_x$ and $f_y$, respectively.

Now we let $A$ and $B$ be polynomials in $M^0[x, y]$ such that $Af_x + Bf_y = 1$. Specifically

$$A = \frac{2^4 (4a_4^2 + 6x^2 a_4 - 9xa_6)}{\Delta},$$
$$B = \frac{2^3 (9y)(2xa_4 - 3a_6)}{\Delta}.$$

By simple arithmetic, $A^p f_x^p + B^p f_y^p = 1 + A^p f_x^p + B^p f_y^p - (Af_x + Bf_y)^p = 1 + pC_p^{\text{ext}}(Af_x + Bf_y)$. Now we consider the relationship between $f_x^p$, $f_y^p$ and the coefficients of $\delta x$ and $\delta y$, respectively. First recall that $n = n^p + p\delta(n)$ for any positive integer $n$. So we can write the coefficients of $\delta x$ and $\delta y$ from equation (4.1) as

Coefficient of $\delta x = -3x^{2p} - a_4^p = -(3^p + p\delta(3))x^{2p} - a_4^p$

$$= -p\delta(3)x^{2p} + f_x^p + pC_p^{\text{ext}}(-3x^2 - a_4)$$

$$= f_x^p + p(-\delta(3)x^{2p} + C_p^{\text{ext}}(-3x^2 - a_4)),$$

Coefficient of $\delta y = 2y^p = (2^p + p\delta(2))y^p$

$$= f_y^p + p\delta(2)y^p.$$

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Then combining these with the equation $A^p f^p_x + B^p f^p_y = 1 + pC^\text{ext}_p(Af_x + Bf_y)$,

$$A^p(-3x^{2p} - a_4^p) + B^p(2y^p)$$

$$= A^p(f^p_x + p(-\delta(3)x^{2p} + C^\text{ext}_p(-3x^2 - a_4))) + B^p(f^p_y + p\delta(2)y^p)$$

$$= 1 + p(C^\text{ext}_p(Af_x + Bf_y) + A^p(-\delta(3)x^{2p} + C^\text{ext}_p(-3x^2 - a_4)) + B^p\delta(2)y^p).$$

Now if we let $R_{U,0} = C^\text{ext}_p Af_x + Bf_y + A^p(\delta(3)x^{2p} + C^\text{ext}_p(-3x^2 - a_4)) + B^p\delta(2)y^p$, then $A^p(-3x^{2p} - a_4^p) + B^p(2y^p) = 1 + pR_{U,0}.$

With the computations in the previous paragraph we now have enough tools to perform the iteration step in Hensel’s Lemma. We assume that

$$\delta x = -P_{U,0} A^p + p\eta,$$

$$\delta y = -P_{U,0} B^p + p\sigma,$$

and plug these assumptions into equation (4.1). Then we solve the resulting equation for $\eta$ and $\sigma$, keeping in mind that we are working modulo $p^2$. (Note: The procedure is the same working modulo $p^3$ etc., but in that case one must assume a $p^2$ term for the $\delta x$ and $\delta y$ and then perform the iteration twice.)

$$(-3x^p\delta x^2 - \delta a_4\delta x + \delta y^2)p + (-3x^{2p} - a_4^p)\delta x + 2y^p\delta y + P_{U,0}$$

$$= p(-3x^pP^2_{U,0}A^{2p} + \delta a_4P_{U,0}A^p + P^2_{U,0}B^{2p}$$

$$- P_{U,0}R_{U,0} + (-3x^{2p} - a_4^p)\eta + 2y^p\sigma).$$

Now if we let $P_{U,1} = -3x^pP^2_{U,0}A^{2p} + \delta a_4P_{U,0}A^p + P^2_{U,0}B^{2p} - P_{U,0}R_{U,0}$, then

$$\eta = -P_{U,1}A^p,$$

$$\sigma = -P_{U,1}B^p.$$

So the morphism that takes $x$ to $x$, $y$ to $y$, $\delta x$ to $-P_{U,0}A^p - pP_{U,1}A^p$, and $\delta y$ to $-P_{U,0}B^p - pP_{U,1}B^p$ will map $\delta f$ to 0. Our corresponding section, $s_U$, is

$$(x, y, -A^p(P_{U,0} + pP_{U,1}), -B^p(P_{U,0} + pP_{U,1})).$$

Next we find the section $s_V$ that defines a specific map from $V$ to $V^1$ such that under this map $\delta f(z, 1, w)$ is taken to 0. Since the techniques used are identical to those used to find $s_U$, we will omit most of the details. From now on for convenience of notation we will refer to $f(z, 1, w)$ as $g$, and $\frac{\partial g}{\partial z}$, $\frac{\partial g}{\partial w}$ will be referred to as $g_z$ and $g_w$, respectively.

Let $P_{V,0} = -\delta a_6 w^3 - \delta a_4 z^2 w^2 + C^\text{ext}_p(w - z^3 - a_4 z w^2 - a_6 w^3)$ and let $C$ and $D$ be polynomials in $M/[z, w]$ such that $Cg_z + Dg_w = 1$. Specifically

$$C = z(-\frac{3}{2}a_6 w - a_4 z), \quad D = -\frac{3}{2}a_6 w^2 - w a_4 z + 1.$$
Next let
\[ R_{V,0} = C_p^{\text{ext}}(Cg_z + Dg_w) + C_p(-\delta(3)z^{2p} + C_p^{\text{ext}}(-3z^2 - a_4w^2)) + D_p(-\delta(3)a_6^2w^{2p} - \delta(2)a_4^2z^p + C_p^{\text{ext}}(1-3a_6w^2 - 2a_4zw)) \]
and let
\[ P_{V,1} = -(a_4^2z^p + 3a_6^2w^{2p})(P_{V,0}D_p^3)^2 - (3\delta a_6w^{2p} + 2\delta a_4z^p)(-P_{V,0}D_p) - P_{V,0}R_{V,0}. \]

Then the section \( s_V \) defining a map from \( V \) to \( V^1 \) is
\[ (z, w, -CP(P_{V,0} + pP_{V,1}), -D_p(P_{V,0} + pP_{V,1})). \]

5. \( s_U - s_V \) UNDER THE GROUP LAW

We now need the \( \delta z \) coordinate also referred to as \( \zeta \) in the difference, \( s_U - s_V \), of these two sections under the group law. We will work with the element \( (z, w, z', w') \) where \( z' = \delta z \) and \( w' = \delta w \). Recall from the Introduction that our gluing maps on the intersection \( U^1 \cap V^1 \) are
\[ z = -x/y, \]
\[ w = -1/y, \]
\[ \delta z = \frac{x^p\delta y - y^p\delta x}{y^p(y^p + p\delta y)}, \]
\[ \delta w = \frac{\delta y}{y^p(y^p + p\delta y)}. \]

So if we let \( x' = \delta x \) and \( y' = \delta y \), then in terms of the coordinates on \( U^1 \), our element is \( (-x/y, -1/y, \frac{x^p y' - y^p x'}{y^p(y^p + py')}) \), which modulo \( p^2 \) is the same as \( (-x/y, -1/y, \frac{1}{y^p}(x^p y' - y^p x'))(y^p - py'), \frac{1}{y^p}(y' - py')) \). Then under the map \( s_U \), this element is mapped to
\[ \left( -x/y, -1/y, \frac{-x^p B_p(P_{U,0} + pP_{U,1}) + y^p A_p(P_{U,0} + pP_{U,1})}{y^p(y^p - pB_p(P_{U,0} + pP_{U,1}))}, \frac{-B_p(P_{U,0} + pP_{U,1})}{y^p(y^p - pB_p(P_{U,0} + pP_{U,1}))} \right) \]
which simplifies modulo \( p^2 \) to
\[ \left( -x/y, -1/y, \frac{-x^p B_p + y^p A_p)(y^p P_{U,0} + p(B_p P_{U,0}^2 + y^p P_{U,1})}{y^p}, \frac{-B_p(y^p P_{U,0} + p(B_p P_{U,0}^2 + y^p P_{U,1}))}{y^p} \right). \]

Under the map \( s_V \) the element \( (z, w, z', w') \) is mapped to
\[ (z, w, -CP(P_{V,0} + pP_{V,1}), -D_p(P_{V,0} + pP_{V,1})). \]

The image of the element \( (z, w, z', w') \) under the difference map \( s_U - s_V \) is the difference under the group law on \( E^1 \) of the image of \( (z, w, z', w') \) under \( s_U \) and the image of \( (z, w, z', w') \) under \( s_V \). In order to take the difference we must first take the inverse under the group law of the image of \( (z, w, z', w') \) under \( s_V \), which is
\[ (-z, -w, CP(P_{V,0} + pP_{V,1}), D_p(P_{V,0} + pP_{V,1})). \]
and then add this to the image of \((z, w, z', w')\) under \(s_U\). Specifically we will let
\[
\begin{align*}
z_1 &= -x/y = z, \\
w_1 &= -1/y = w, \\
\delta z_1 &= \frac{-x^p B^p + y^p A^p(y^p P_{U,0} + p(B^p P_{U,0}^2 + y^p P_{U,1}))}{y^{3p}} \\
&= (w^p z^p B^p + w^p A^p)(-P_{U,0} + p(w^p B^p P_{U,0}^2 - P_{U,1})), \\
\delta w_1 &= \frac{-B^p(y^p P_{U,0} + p(B^p P_{U,0}^2 + y^p P_{U,1}))}{y^{3p}} \\
&= w^{2p} B^p(-P_{U,0} + p(w^p B^p P_{U,0}^2 - P_{U,1})), \\
z_2 &= -z, \\
w_2 &= -w, \\
\delta z_2 &= C^p(P_{V,0} + pP_{V,1}), \\
\delta w_2 &= D^p(P_{V,0} + pP_{V,1}),
\end{align*}
\]
and apply the explicit formulation of the group law detailed in Section 3. Also since for the purpose of our computation we only need the \(\delta z_3\) term, this is the only one we will formulate in detail.

We are going to be analyzing \(\frac{1}{p^3}(-\mu^p \delta \alpha + \alpha^p \delta \mu) (\mu^p - p \delta \mu)\) with the above terms substituted in for \(z_1, w_1,\) etc. When we do this,
\[
\begin{align*}
\alpha &= 0, \\
\mu &= 2w + 6a_6w^3 + 6a_4zw^2 + 6z^3,
\end{align*}
\]
which, if we then add \(6f(z, 1, w)\), we have \(\mu = 8w = -8/y\).

**Proposition 5.1.** The \(\zeta = \delta z\) coordinate of \(s_U - s_V\) is
\[
\zeta = \frac{-\delta \alpha}{(8w)^p} + \frac{\alpha \delta \mu}{(8w)^{2p}},
\]
where the above expressions are used for \(z_1, z_2, \delta z_1, \delta z_2, \delta w_1,\) etc. in \(\delta \alpha\) and \(\delta \mu\).

The next step in the computation of \(f_{\text{def}}\) is to apply the formal logarithm of the Frobenius twist of the formal group of the elliptic curve to \(\zeta\). This is a triviality as mentioned in the Introduction by the following proposition.

**Proposition 5.2.** Let \(\log_{\mathcal{F}^{(p)}}(\xi)\) be the formal logarithm of the Frobenius twist of the formal group of the elliptic curve. Then
\[
\log_{\mathcal{F}^{(p)}}(\xi) = \zeta \mod p^2.
\]

**Proof.** Recall
\[
\log_{\mathcal{F}^{(p)}}(\xi) := \xi + \frac{p \phi(c_1)}{2} \xi^2 + \frac{p^2 \phi(c_2)}{3} \xi^3 + \cdots,
\]
where the \(c_i\) are the coefficients of the power series expansion of the invariant differential \([3, p. 127]\). From \([3, p. 113]\) we know that the invariant differential
\[
\omega(z) = (1 + 2a_4 z^4 + \cdots) dz
\]
and so \(c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 2a_4 \cdots\). However, the power of \(p\) in \(p^n/n\) is at least 2 for all \(n \geq 4\). Hence, modulo \(p^2\) the power series \(\log_{\mathcal{F}^{(p)}}(\xi)\) is the identity. \(\square\)
6. Residue of the cohomology class

Recall that any cohomology class in \( H^1(E \otimes M^1, \mathcal{O}) \simeq H^1(E, \mathcal{O}) \otimes M^1 \) has a representative of the form \( \sum a_n y^n + x \sum b_n y^n + x^2 \sum e_n y^n \). Let us refer to the coefficient \( e_{-1} \) in a sum \( \sum a_n y^n + x \sum b_n y^n + x^2 \sum e_n y^n \) as the residue of the sum. The final step in the computation of \( f_{\text{jet}} \) modulo \( p^2 \) is to take the residue of \( \zeta = \delta z \) coordinate of \( s_U - s_V \), which is a cohomology class as a result of the above proposition.

While the idea behind taking the residue is simple, namely write \( \zeta \) as \( \sum a_n y^n + x \sum b_n y^n + x^2 \sum e_n y^n \) and take the coefficient of \( x^2/y \) in this sum which is \( e_{-1} \), the practice is computationally unfeasible. Instead we break the process of finding the residue of \( \zeta \) into parts. The residue map has some useful properties; namely, it is linear and the residue of any function that is regular on \( U \) or any function that is regular on \( V \) is zero. So we can take the residue of the terms in \( \zeta \) and then add together the result to get the residue of \( \zeta \).

As a preliminary step to the task of analyzing the residue of the terms of \( \zeta = \delta z \), we write the following expressions in both coordinates of \( U \) and coordinates of \( V \):

\[
A = \frac{2^4(4a_4^2 + 6x^2a_4 - 9xa_6)}{w^2\Delta}, \quad B = \frac{2^3(9y)(2xa_4 - 3a_6)}{w^2\Delta}, \quad C = \frac{z(\frac{3}{2}a_6w - a_4z)}{y^2}\left(\frac{3}{2}a_6 - a_4x\right), \quad D = -\frac{3}{2}a_6w^2 - wa_4z + 1 = \frac{1}{y^2}\left(x^3 - \frac{1}{2}a_6\right),
\]

\[
P_{U,0} = \frac{P_{V,0}}{w^{3p}}.
\]

We also note that \( P_{U,0} \) and \( P_{U,1} \) are regular on \( U \), and that \( P_{V,0}, P_{V,1} \), and \( C_p^{\text{ext}}(3a_6w^3 + 3a_6w^3 + 3z^3 + 3z^3 + a_4w^2z + a_4w^2z + w + w + a_4w^2z + 2a_4w^2z) \) are regular on \( V \). So as an example the following combinations of taken from \( \frac{\delta a_6}{(8w)^{2p}} \) are regular on \( V \):

\[
-3p^3D^pC_p a_6^p w^p P_{V,0}, \quad p2w^p a_4^p w^p C_p P_{V,1}, \quad -C_p P_{V,0},
\]

More examples of regular combinations on \( V \), this time taken from \( \frac{\delta a_6}{(8w)^{2p}} \), are

\[
\frac{(3p^3w^3C_p a_6^p P_{V,0})}{(8w)^{2p}} \quad \text{and} \quad \frac{(-3p^3w^3a_6^p a_4^p P_{U,0})(2(3p^3)\delta(a_6)w^{3p})}{(8w)^{2p}},
\]

Since the residue of terms that are regular on either \( U \) or \( V \) is zero, we can exclude these terms from consideration in computing the residue class of \( \zeta \). This leads to the following proposition in which for brevity’s sake we let

\[
\Upsilon = C_p^{\text{ext}}(y^2 - x^3 - a_4x - a_6).
\]
Proposition 6.1. The residue of $\zeta$ is equal to the residue of
\[
\left(\frac{(1 - 2p)A^p}{y^p} + \frac{-2x^pB^p}{y^{2p}} + \frac{(-3pA_6 - 2a_4^pA^p)}{y^{3p}} + \frac{(-1 - 2p)x^pD^p}{y^{4p}}\right)P_{U,0} + \frac{p}{8p} \left(F_1C_p^\text{ext}(3a_6w^3 + 3a_6w^3 + 3z^3 + 3z^3 + a_4w^2z) + a_4w^2z + w + w + 2a_4w^2z + 2a_4w^2z) + F_2C_p^\text{ext}(Afx + Bfy) + F_3C_p^\text{ext}(-3x^2 - a_4) + F_4\Upsilon^2 + (F_5 + F_6 + F_7)\Upsilon + F_8\right),
\]
where $F_i$ are polynomials in $M_4^x[x^p, y^p, \Upsilon]$.

Proof. This is proved by the very precise removal of almost all regular terms using a computer algebra system. \(\square\)

It is now necessary to compute residues of terms whose residue may be nontrivial. Namely, we provide a formula for the residue of $\zeta$ which we will call $\gamma_{a,b}$. We let $\binom{n}{k}$ denote the binomial coefficient with the convention that $\binom{n}{k} = 0$ if $k > n$. Then from [5] we know

Proposition 6.2. Let $a$ and $b$ be positive integers. Let $m$ and $n \in \{0, 1, 2\}$ be integers such that $a = 3m + n$. Then the residue of $\zeta$ is

\[
\gamma_{a,b} = \begin{cases} 0 & \text{if } b \text{ is even}, \\ \sum_{k=0}^{\infty} \binom{m + k}{3k + 2 - n} \binom{m - 2k - 2 + n}{b \frac{b+1}{2}} (-1)^{m+k-\frac{b}{2}} (a_4)^{3k+2-n}(a_6)^{m-2k-2+n-\frac{b+1}{2}} & \text{if } b \text{ is odd}. \end{cases}
\]

Obviously, because of the convention for binomial coefficients, there will be integers $a$ and $b$ with $b$ odd for which $\gamma_{a,b}$ is 0. In fact, if $\frac{2b}{3} > a$, $\gamma_{a,b} = 0$ because of the binomial coefficient $\binom{m-2k-2+n}{b \frac{b+2}{3}}$. We now introduce a series of propositions that are just expanded formulas for expressions found in Proposition 6.1.

Proposition 6.3.
\[
\Upsilon = C_p^\text{ext}(y^2 - x^3 - a_4x - a_6)
= \frac{1}{p} \left(\sum_{k=1}^{p-1} \binom{p}{k} (-1)^{k} y^{2^p} - \sum_{k=1}^{p-1} \binom{p}{k} x^{3^k (a_4x + a_6)^{p-k}} - \sum_{k=1}^{p-1} \binom{p}{k} a_4^{p-k} x^{k}\right).
\]

Proposition 6.4.
\[
A^p = \frac{2^{4p}(4a_4^2 + 6x^2a_4 - 9xa_6)^p}{\Delta p}
= \frac{2^{4p}}{\Delta p} \left[4^p a_4^{2p} + 6p x^{2p} a_4^p - 9p x^p a_6^p + \sum_{k=1}^{p-1} \binom{p}{k} (4a_4^2)^{p-k} (6x^2a_4 - 9xa_6)^k + \sum_{k=1}^{p-1} \binom{p}{k} (6x^2a_4)^k (-9xa_6)^{p-k}\right].
\]
Proposition 6.5. Define 
\[ B^p = \frac{2^{3p}(9y)^p(2xa_4 - 3a_6)^p}{\Delta^p} \]
\[ = \frac{2^{3p}(9y)^p}{\Delta^p} \left[ 2^p x^p a_4^p - 3^p a_6^p + \sum_{k=1}^{p-1} \left( \frac{p}{k} \right) (2xa_4)^k (-3a_6)^{p-k} \right]. \]

Proposition 6.6. \[ C_p^{\text{ext}}(3a_6w^3 + 3a_6w^3 + 3z^3 + 3z^3 + a_4w^2z + a_4w^2z + w + w + 2a_4w^2z + 2a_4w^2z) \]
\[ = \frac{1}{p} \left[ 2(3a_6)^p + 2(3x^3)p + 2(1 + 2p)(a_4x)^p + (2 - 8p)y^{2p} \right] \left( -\frac{1}{y^{3p}} \right). \]

Proposition 6.7. \[ C_p^{\text{ext}}(Af_x + Bf_y) \]
\[ = \frac{1}{p} \sum_{k=1}^{p-1} \sum_{i=0}^{k} \sum_{j=0}^{p-i} \left( \frac{p}{k} \right) \left( \frac{k}{i} \right) \left( \frac{p-i}{j} \right) (-1)^{k-i} \]
\[ \times \left( \frac{9(2^4)}{\Delta} \right)^{p-i} (2a_4)^j (-3a_6)^{p-i-j} x^j y^{2(p-i)}. \]

Proposition 6.8. \[ C_p^{\text{ext}}(-3x^2 - a_4) = -\frac{1}{p} \sum_{k=1}^{p-1} \left( \frac{p}{k} \right) 3^k a_4^{p-k} x^{2k}. \]

Using this series of propositions, we can now explicitly write down the residue of \( \zeta \) by computing the residue of the formula in Proposition 6.1. First we introduce some more notation.

Definition 6.9. Define \( \mu_{a,b} \) to be the residue of \( \frac{x^a y^b}{y^p} \) where
\[ \Upsilon = C_p^{\text{ext}}(y^2 - x^3 - a_4x - a_6). \]

Definition 6.10. Define \( \tau_{a,b} \) to be the residue of \( \frac{x^a y^b}{y^p} \) where
\[ \Upsilon = C_p^{\text{ext}}(y^2 - x^3 - a_4x - a_6). \]

Using the formulas above, it is easy to check that for some, but not all, values, \( \mu_{a,b} \) will be zero. Similarly there are values of \( a \) and \( b \) for which \( \tau_{a,b} \) is nonzero and for which it is zero. Some examples of \( \mu_{a,b} \) are
\[ \mu_{p,3p} = 0, \]
\[ \mu_{p,p} = \frac{1}{p} \left[ -\sum_{k=1}^{p-1} \left( \frac{p}{k} \right) \sum_{i=0}^{p-k} \left( \frac{p-k}{i} \right) a_4^i a_6^{p-k-i} \gamma_{p+3i+k,p+3i} - \sum_{k=1}^{p-1} \left( \frac{p}{k} \right) a_4^k a_6^{p-k} \gamma_{p+k,p} \right]. \]

Note that both \( \gamma_{a,b} \) and \( \mu_{a,b} \) are in \( M_1^0 \). We can now prove the following theorem.
Theorem 6.11. The reduction modulo $p^2$ of $f_{j0}$ is

$$
\begin{align*}
&\left[ \frac{9p(2p - 4p - 2(3^p))a_0^p \delta(a_4) + 2p(- 6p^2 + 12p + 2(9p))a_0^p \delta(a_6)}{\Delta^p} \right] \gamma_{2p,p} \\
&+ \frac{1}{\Delta^p} \left[ 2p(1 - 2p^2)A_4 a_4^p \mu_{0,p} + (-18p(1 - 2p^2) + 2(27p))a_0^p \mu_{p,p} \\
&+ (21p(1 - 2p^2) - 2(18p^2))a_4^p \mu_{2p,p} + (2(18p) - 36p) a_0^p a_0^p \mu_{2p,3p} - 2(12p) a_4^p \mu_{3p,3p} \right] \\
&+ p \left( H_0 + H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + H_7 + H_8 \right)
\end{align*}
$$

where $H_0$ is

$$
\begin{align*}
&\frac{1}{p} \left( (1 - 2p^2) \Delta^p \right) \left[ - \delta(a_4) 6p^2 a_4^p \gamma_{3p,p} + \sum_{k=1}^{p-1} \binom{p}{k} (4a_4^p)^{p-k} \sum_{i=0}^{k} \binom{k}{i} (6a_4^p)^i (-9a_6^p)^{k-i} \\
&\times \left( - \delta(a_4) \gamma_{p+k+i,p} - \delta(a_6) \gamma_{p+k,i} + \mu_{k,i,p} \right) \\
&+ \sum_{k=1}^{p-1} \binom{p}{k} (6a_4^p)^k (-9a_6^p)^{p-k} \left( - \delta(a_4) \gamma_{2p+k,p} - \delta(a_6) \gamma_{p+k,p} + \mu_{p+k,p} \right) \right] \\
&- 2 \frac{2p}{\Delta^p} \left[ - \delta(a_4) 2p^2 a_4^p \gamma_{3p,p} \\
&+ \sum_{k=1}^{p-1} \binom{p}{k} (2a_4^p)^k (-3a_6^p)^{p-k} \left( - \delta(a_4) \gamma_{2p+k,p} - \delta(a_6) \gamma_{p+k,p} + \mu_{p+k,p} \right) \right] \\
&- (3a_6^p) \frac{2p}{\Delta^p} \left[ \sum_{k=1}^{p-1} \binom{p}{k} (4a_4^p)^{p-k} \sum_{i=0}^{k} \binom{k}{i} (6a_4^p)^i (-9a_6^p)^{k-i} \mu_{k+i,3p} \\
&+ \sum_{k=1}^{p-1} \binom{p}{k} (6a_4^p)^k (-9a_6^p)^{p-k} \mu_{p+k,3p} \right] \\
&- (2a_4^p) \frac{2p}{\Delta^p} \left[ \sum_{k=1}^{p-1} \binom{p}{k} (4a_4^p)^{p-k} \sum_{i=0}^{k} \binom{k}{i} (6a_4^p)^i (-9a_6^p)^{k-i} \mu_{p+k+i,3p} \\
&+ \sum_{k=1}^{p-1} \binom{p}{k} (6a_4^p)^k (-9a_6^p)^{p-k} \mu_{2p+k,3p} \right],
\end{align*}
$$

$H_1$ is

$$
\begin{align*}
&\frac{-1}{p} \left( 9\delta(a_4) a_4^p (3^p)^{\gamma_{3p,p}} \\
&- \frac{9}{2} a_4^p \left( 2(3a_6^p) \mu_{2p,3p} + 2(3^p) \mu_{2p,3p} + 2(1 + 2p^2) a_4^p \mu_{3p,3p} + (2 - 8p^2) \mu_{2p,p} \right) \\
&+ (9a_6^p - \frac{3}{2} a_4^p) \left( 2(3^p) \mu_{2p,3p} + 2(1 + 2p^2) a_4^p \mu_{2p,3p} + (2 - 8p^2) \mu_{2p,p} \right) \\
&- a_4^p \left( 2(3^p) \mu_{3p,3p} + (2 - 8p^2) \mu_{0,p} \right) \\
&+ \left( \frac{3}{2} a_4^p - 9 a_6^p \right) \delta(a_4) + \frac{9}{2} a_4^p \delta(a_6) \right) \left( 2(3^p) \gamma_{5p,3p} + (2 - 8p^2) \gamma_{2p,p} \right) \\
&+ \left( -3a_4^p + \frac{9}{2} a_0^p a_4^p \right) \left( 2(3^p) \mu_{5p,5p} + (2 - 8p^2) \mu_{2p,3p} \right) / \Delta^p,
\end{align*}
$$
\[ H_2 \text{ is} \]
\[
\frac{1}{p} \sum_{k=1}^{p-1} \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{p}{k} \binom{k}{i} (p-i) \left( -1 \right)^{k-i} \frac{\left( \frac{9(24)}{\Delta} \right)^{p-i} \left( 2a_4 \right)^{i} \left( -3a_0 \right)^{p-i-j}}{\Delta^p}
\times \left[ \left( 24a_4^{2p} - 36a_6^{3p} a_4^2 \right) \mu_{2p+j,3p+2(p-i)} + \left( 36a_6^{p} a_4^3 + 16a_4^{3p} - 54a_6^{2p} \right) \mu_{p+j,3p+2(p-i)} \\
+ 24a_6^{p} a_4^2 \mu_{j,3p+2(p-i)} + 36a_4^{p} \mu_{2p+j,p+2(p-i)} \\
+ 12a_4^{p} - 72a_6^{p} \mu_{p+j,p+2(p-i)} + 8a_4^{p} \mu_{j,p+2(p-i)} \right] / \Delta^p
\]

\[ H_3 \text{ is} \]
\[
-\frac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k} a_4^{p-k}
\times \left[ \left( 2304 a_4^{3p} + 10368 a_6^{2p} - 8640 a_6^{2p} a_4^2 + 1152 a_6^{2p} \right) \mu_{2p+2k,p} \\
+ (-1920 a_4^{3p} - 576 a_6^{2p} a_4^2) \mu_{2p+2k,p} \\
+ (512 a_4^{3p} + 10368 a_6^{2p} a_4^2 - 10368 a_6^{2p} a_4^2 + 2304 a_4^{3p}) \mu_{2k,p} \\
+ (-10368 a_6^{2p} a_4^2 + 3456 a_6^{2p} a_4^2 \\
+ 3072 a_6^{2p} + 7776 a_6^{2p} - 4608 a_6^{2p} a_4^2) \mu_{p+2k,3p} \\
+ (11520 a_6^{2p} a_4^2 + 1024 a_6^{3p} - 12096 a_6^{3p} a_4^2 - 2304 a_4^{3p}) \mu_{p+2k,3p} \\
+ (-5184 a_6^{3p} a_4^2 - 2304 a_6^{3p} a_4^2 + 1536 a_6^{3p} a_4^2 + 6912 a_6^{3p} a_4^2) \mu_{2k,3p} \right] / \Delta^{2p}
\]
\[
+ \left( \left( 1920 a_4^{3p} + 576 a_6^{2p} a_4^2 \right) \delta(a_4) \\
+ (8640 a_6^{2p} a_4^2 - 1152 a_4^{3p} - 2304 a_4^{3p} - 10368 a_6^{2p}) \delta(a_6) \right) \gamma_{2p+2k,p} \\
+ \left( (1792 a_4^{3p} - 1152 a_4^{3p} + 1728 a_6^{2p} a_4^2) \delta(a_4) \\
+ (1920 a_4^{3p} + 576 a_6^{2p} a_4^2) \delta(a_6) \right) \gamma_{p+2k,p} \\
+ \left( (6912 a_6^{3p} a_4^2 + 1728 a_6^{2p} a_4^2 \\
+ 2592 a_6^{3p} - 3072 a_4^{3p} - 2304 a_6^{2p} a_4^2) \delta(a_4) \\
+ (-512 a_4^{3p} - 10368 a_6^{2p} a_4^2 + 10368 a_6^{2p} a_4^2 - 2304 a_4^{3p}) \delta(a_6) \right) \gamma_{2k,p} \right] / \Delta^{2p}
\]
$H_4$ is

\[ -\frac{1}{2} \left( -3732480 \alpha_4^{2p} \alpha_4^{2p} - 829440 \alpha_6 \alpha_4^{4p} \\
+ 3359232 \alpha_6^{3p} \alpha_4^{p} + 3456 \alpha_4^{2p} \Delta^p + 774144 \alpha_4^{5p} \\
+ 1990656 \alpha_6^{3p} \alpha_4^{p} - 12960 \alpha_4^{3p} \alpha_6^{p} \Delta^p - 663552 \alpha_4^{4p} \right) \tau_{2p,p} \\
- \frac{1}{2} \left( 2592 \alpha_4^{2p} \Delta^p - 3317760 \alpha_4^{5p} - 7713792 \alpha_6^{2p} \alpha_4^{3p} \\
+ 19440 \alpha_6^{2p} \Delta^p + 9123840 \alpha_6^{4p} \alpha_4^{p} - 7278336 \alpha_6^{4p} \\
+ 2985984 \alpha_6^{2p} \alpha_4^{2p} - 1327104 \alpha_6^{4p} \alpha_4^{3p} + 3359232 \alpha_3^{3p} \alpha_4^{p} \\
+ 24576 \alpha_4^{3p} - 12960 \alpha_4^{3p} \alpha_6^{p} \Delta^p - 288 \alpha_4^{3p} \Delta^p \right) \tau_{p,p} \\
- \frac{1}{2} \left( -1769472 \alpha_4^{6p} - 8957952 \alpha_6^{3p} \alpha_4^{2p} - 11232 \alpha_4^{2p} \alpha_6^{p} \Delta^p \\
+ 2875392 \alpha_4^{5p} \alpha_6^{p} + 3456 \alpha_4^{4p} \Delta^p + 1327104 \alpha_4^{5p} \\
+ 20404224 \alpha_6^{3p} \alpha_4^{4p} - 10616832 \alpha_6^{4p} \alpha_4^{4p} \right) / (\Delta^3p) \\
+ \left( -\frac{1}{2} \right) \left( -393216 \alpha_4^{4p} + 22394880 \alpha_6^{3p} \alpha_4^{2p} - 3456 \alpha_4^{2p} \alpha_6^{p} \Delta^p \\
- 11197440 \alpha_6^{5p} \alpha_4^{p} - 10616832 \alpha_6^{5p} \alpha_4^{p} + 2592 \alpha_6^{3p} \alpha_4^{5p} \Delta^p \\
+ 3317760 \alpha_6^{4p} \alpha_4^{p} + 1152 \alpha_4^{4p} \Delta^p + 2654208 \alpha_6^{4p} \\
- 14929992 \alpha_4^{2p} \alpha_6^{3p} + 9953280 \alpha_6^{2p} \alpha_4^{4p} + 1152 \alpha_4^{3p} \Delta^p \right) \tau_{2p,3p} \\
- \frac{1}{2} \left( 12607488 \alpha_4^{4p} \alpha_6^{p} - 8957952 \alpha_6^{3p} \alpha_4^{2p} - 3244032 \alpha_6^{3p} \alpha_4^{4p} \\
+ 19408896 \alpha_6^{3p} \alpha_4^{3p} + 1536 \alpha_4^{4p} \Delta^p + 13436928 \alpha_6^{4p} \alpha_4^{p} \\
- 28864512 \alpha_6^{2p} \alpha_4^{4p} + 5184 \alpha_4^{2p} \alpha_6^{p} \Delta^p + 11664 \alpha_6^{3p} \Delta^p \\
- 1327104 \alpha_4^{4p} - 15552 \alpha_4^{3p} \alpha_6^{p} \Delta^p + 1769472 \alpha_4^{7p} \\
- 2304 \alpha_4^{3p} \alpha_6^{p} \Delta^p + 1990656 \alpha_6^{2p} \alpha_4^{3p} - 6718464 \alpha_6^{5p} \right) \tau_{p,3p} \\
- \frac{1}{2} \left( -20901888 \alpha_6^{3p} \alpha_4^{4p} - 18144 \alpha_4^{2p} \alpha_6^{p} \Delta^p - 3456 \alpha_4^{4p} \Delta^p \\
+ 9953280 \alpha_6^{3p} \alpha_4^{4p} + 13436928 \alpha_6^{4p} \alpha_4^{p} + 17280 \alpha_4^{3p} \alpha_6^{p} \Delta^p \\
+ 1769472 \alpha_6^{4p} \alpha_4^{4p} + 1664 \alpha_4^{4p} \Delta^p \\
- 2654208 \alpha_6^{3p} \alpha_4^{4p} - 1327104 \alpha_4^{5p} \alpha_6^{p} \right) / (\Delta^3p) \\
+ \left( -\frac{1}{2} \right) \left( -10368 \alpha_4^{3p} \alpha_6^{p} \Delta^p + 31104 \alpha_4^{2p} \alpha_6^{p} \Delta^p - 54 \Delta^2p \alpha_4^{p} \\
+ 10368 \Delta^p \alpha_6^{p} \alpha_4^{4p} - 23328 \Delta^p \alpha_6^{3p} \alpha_4^{p} - 4608 \alpha_4^{5p} \Delta^p \right) \tau_{2p,3p} / (\Delta^3p), \right]
\(H_5\) is
\[
\left(\frac{1}{2} \left( -14556672 a_6^{4p} - 6635520 a_6^{5p} + 6718464 a_6^{3p} a_4^{p} + 5971968 a_6^{2p} a_4^{2p} + 2880 a_4^{2p} \Delta^p + 18144 a_6^{2p} \Delta^p - 5184 a_4^{3p} \Delta^p - 15427584 a_4^{2p} a_4^{3p} - 2654208 a_6^{3p} - 8640 a_6^{4p} \Delta^p + 49152 a_4^{6p} + 18247680 a_6^{4p} a_4^{4p} \right) \delta(a_4) + \frac{1}{2} \left( 1548288 a_5^{5p} + 6912 a_4^{2p} \Delta^p - 25920 a_4^{2p} a_6^{2p} \Delta^p - 1658880 a_6^{3p} a_4^{2p} + 6718464 a_4^{3p} a_4^{2p} + 7464960 a_6^{2p} a_4^{2p} + 3981312 a_6^{2p} a_4^{4p} - 1327104 a_4^{4p} \right) \delta(a_6) + \frac{1}{2} \left( 3584 a_4^{4p} \delta(3) \Delta^p + 3456 \delta(3) a_4^{2p} a_6^{2p} \Delta^p + 4608 \delta(2) a_4^{4p} \Delta^p + 18 \delta(3) a_4^{3p} a_4^{2p} + 2304 a_4^{3p} a_4^{2p} \delta(3) \Delta^p) \right) \mu_{2p,p} \right) / (\Delta^{3p}),
\]
\(H_6\) is
\[
\left(\frac{1}{2} \left( 3840 a_4^{3p} \Delta^p - 25214976 a_6^{2p} a_4^{4p} + 3981312 a_6^{5p} + 7409664 a_4^{2p} a_6^{2p} + 4608 a_4^{2p} a_6^{4p} - 5087232 a_4^{3p} + 48273408 a_6^{2p} a_4^{4p} - 24634368 a_6^{2p} a_4^{4p} \delta(a_4) + \frac{1}{2} \left( 14556672 a_6^{4p} - 15427584 a_6^{2p} a_4^{3p} - 25920 a_6^{4p} a_4^{2p} - 2654208 a_6^{3p} - 6635520 a_4^{5p} + 38880 a_6^{2p} \Delta^p + 18247680 a_6^{4p} + 5184 a_4^{2p} \Delta^p + 6718464 a_6^{2p} a_4^{2p} + 5971968 a_6^{2p} a_4^{2p} - 576 a_4^{3p} \Delta^p + 49152 a_4^{2p} \right) \delta(a_6) + \frac{1}{2} \left( 5184 \delta(3) a_6^{3p} \Delta^p - 24 \delta(2) a_6^{2p} a_4^{3p} - 36 \delta(3) a_6^{2p} a_4^{3p} - 23328 \delta(2) a_6^{3p} \Delta^p + 36 \delta(2) a_6^{3p} \Delta^p - 6912 \delta(2) a_4^{4p} \Delta^p + 12672 a_4^{2p} \delta(3) a_4^{2p} \Delta^p - 15552 \delta(2) a_4^{2p} a_4^{2p} \Delta^p + 3456 \delta(3) a_4^{2p} a_4^{2p} \Delta^p + 10368 \delta(2) a_4^{2p} a_4^{2p} \Delta^p - 4608 \delta(3) a_4^{2p} a_4^{2p} \Delta^p + 9984 a_4^{2p} a_4^{3p} \delta(3) \Delta^p + 6 \delta(3) a_4^{2p} a_4^{3p} \right) \mu_{2p,p} \right) / (\Delta^{3p}),
\]
\[
\left(\frac{1}{2} \left( -1962624 a_6^{4p} - 33841152 a_6^{2p} a_4^{2p} - 22781952 a_4^{2p} a_6^{2p} + 5308116 a_4^{2p} a_4^{3p} + 52254720 a_6^{3p} a_4^{2p} - 29113344 a_4^{4p} a_4^{2p} + 54 \Delta^p a_4^{5p} - 786432 a_4^{3p} + 60192 a_6^{4p} a_4^{2p} + 10368 a_4^{2p} a_6^{4p} \Delta^p - 2304 a_4^{3p} \Delta^p + 1280 a_4^{2p} a_4^{2p} + 21565440 a_6^{2p} a_4^{4p} \right) \delta(a_4) + \frac{1}{2} \left( 40808448 a_6^{2p} a_4^{3p} - 22464 a_4^{2p} a_4^{2p} + 17915904 a_6^{3p} a_4^{2p} + 5750784 a_4^{3p} a_4^{2p} + 2654208 a_4^{5p} a_4^{2p} - 21233664 a_4^{3p} a_4^{2p} - 3538944 a_6^{4p} a_4^{2p} + 6912 a_4^{3p} \Delta^p \right) \delta(a_6) + \frac{1}{2} \left( -6912 \delta(2) a_6^{4p} a_4^{2p} \Delta^p - 2048 a_6^{4p} \delta(3) \Delta^p + 4608 a_4^{2p} \delta(3) \Delta^p - 14 \delta(3) a_4^{2p} a_4^{2p} \Delta^p + 23040 \delta(3) a_4^{2p} a_4^{2p} \Delta^p - 26880 a_4^{2p} \delta(3) a_4^{2p} a_4^{2p} + 2 \delta(2) a_6^{2p} a_4^{2p} \Delta^p \right) \mu_{2p,p} \right) / (\Delta^{3p}),
\]
$H_7$ is

\[
\left( \frac{1}{2} \right) \left( -17915904 a_6^{3p} a_4^{2p} + 25214976 a_4^{5p} a_6^p + 7776 a_6^{3p} \Delta^p \\
+ 38817792 a_6^{3p} a_4^{3p} + 4608 a_4^{3p} a_6^p \Delta^p + 3456 a_4^{2p} a_6^p \Delta^p \\
+ 3981312 a_6^{2p} a_4^{3p} + 26873856 a_6^{4p} a_4^p - 2654208 a_4^{4p} \\
- 13436928 a_6^{5p} - 30 \Delta^2 a_4^p a_4^p - 6488064 a_4^p a_6 a_4^{6p} + 3538944 a_4^{7p} \\
- 36 \Delta^2 a_6^p - 10368a_6^{3p} a_4^{2p} \Delta^p - 3072a_4^p a_4^{4p} \Delta^p - 57729024a_6^{2p} a_4^{4p} \delta(a_4) \right) \\
+ \frac{1}{2} \left( 54 \Delta^2 a_4^p a_4^p + 5184 a_4^{4p} a_6^{2p} \Delta^p - 21233664 a_4^{5p} a_6^p \\
+ 19906560 a_6^{2p} a_4^{4p} + 2304 a_4^{3p} \Delta^p + 6635520 a_6^{2p} a_4^{4p} \\
- 29859840 a_6^{2p} a_4^{3p} + 2304 a_4^{4p} \Delta^p - 6912 a_4^{2p} a_6^p \Delta^p \\
+ 44789760 a_6^{3p} a_4^{4p} + 5308416 a_4^p - 22394880 a_6^{2p} a_4^{4p} - 786432 a_4^{7p} \delta(a_6) \right) \\
+ \frac{1}{2} \left( 25920 a_6^{3p} \delta(3) a_4^p \Delta^p + 6144 a_4^{5p} \delta(3) \Delta^p - 36 \delta(2) a_4^p a_6^p \Delta^p \\
+ 11520 a_4^{3p} \delta(3) a_6^p \Delta^p + 6 \delta(2) a_4^p \Delta^2 a_4^p + 6 \delta(3) a_4^p \Delta^2 a_4^p \\
+ 18 \delta(3) a_6^p a_4^p \Delta^p - 35460 \delta(3)(3) a_6^p a_4^{3p} \Delta^p - 12288 a_4^{4p} \delta(3) a_6^p \Delta^p \right) \mu_{2p,3p} / (\Delta^3),
\]

and $H_8$ is

\[
\left( - (-1271808 a_4^{6p} + 995328 a_4^{5p} + 12068352 a_6^{3p} a_4^{3p} \\
- 6158592 a_6^{3p} a_4^{4p} + 1852416 a_4^{7p} a_6^p \\
- 6303744 a_6^{6p} a_4^{4p} + 1920 a_4^{3p} \Delta^p + 1440 a_4^{2p} a_6^p \Delta^p) \delta(a_4)^2 \\
- (9123840 a_6^{5p} a_4^{4p} - 7278336 a_6^{4p} + 24576 a_4^{6p} \\
- 3317760 a_4^{5p} - 7713792 a_6^{6p} a_4^{3p} - 2592 a_4^{3p} \Delta^p \\
+ 9072 a_6^{2p} \Delta^p - 4320 a_4^{2p} a_6^p \Delta^p - 1327104 a_6^{2p} a_4^{3p} \\
+ 1440 a_4^{2p} \Delta^p + 2985984 a_6^{2p} a_4^{4p} + 3359232 a_3^{3p} a_4^{7p} \delta(a_4) \delta(a_6) \\
- (387072 a_4^{5p} + 1728 a_4^{2p} \Delta^p - 1866240 a_6^{2p} a_4^{4p} \\
+ 995328 a_6^{4p} a_4^{4p} - 331776 a_4^{3p} + 1679616 a_6^{3p} a_4^{5p} \\
- 6480 a_6^{3p} a_6^p \Delta^p - 414720 a_6^{2p} a_4^{4p}) \delta(a_6)^2 \right) \\
- (18 \delta(2) a_4^{5p} \Delta^2 a_4^p - 18 \delta(3) a_6^{3p} \Delta^2 a_4^p + 1728 \delta(3) a_4^{2p} a_4^p \Delta^p \\
- 7776 \delta(2) a_6^{2p} a_4^p \Delta^p + 6336 a_4^{3p} \delta(3) a_6^p \Delta^p \\
- 2304 \delta(3) a_4^{2p} a_6^{3p} \Delta^p + 5184 \delta(2) a_4^{4p} a_6^p \Delta^p \\
+ 3 \delta(3) a_4^{2p} \Delta^2 a_4^p - 12 \delta(2) a_4^{3p} \Delta^2 a_4^p - 3456 \delta(2) a_4^{4p} \Delta^p \\
+ 11664 \delta(2) a_4^{3p} \Delta^p + 2592 \delta(3) a_6^{3p} \Delta^p - 4992 a_4^{4p} \delta(3) \Delta^p \delta(a_4) \\
- (1792 a_4^{5p} \delta(3) \Delta^p \Delta^p - 1152 a_4^{4p} \delta(3) \Delta^p + 1728 \delta(3) a_4^{3p} a_6^p \Delta^p \\
+ 9 \delta(3) a_6^p a_4^p \Delta^2 a_4^p + 2304 \delta(2) a_4^{4p} \Delta^p) \gamma_{2p,3p} / \Delta^{3p}.\]
Proof. We simply apply the most recent propositions to the actual formulas from Proposition 6.1. It should be noted that the $H_i$ correspond to the $F_i$ in Proposition 6.1 and that upon further analysis certain terms like $\frac{(1-2^n)G_i}{y^{2p}}$ have zero residue even though it is not immediately obvious that the term is regular. \hfill \square

From now on, when we refer to $H_0$, $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$, $H_7$, and $H_8$ we will mean the polynomials in this theorem. We note that $H_0$, $H_1$, $H_2$, $H_3$, $H_5$, $H_6$, $H_7$ are in $M_1^4$ and are linear in $\delta(a_4)$ and $\delta(a_6)$, $H_4 \in M_1^4$, and $H_8 \in M_1^4$ is quadratic in $\delta(a_4)$ and $\delta(a_6)$.

7. Order Two Modular Forms

We remind ourselves that $\phi$, the unique lifting of the Frobenius morphism to $R$, extends to a homomorphism from $M_1^4 \to M_2^4$ by taking, e.g., $a_4 \mapsto a_4^p + p\delta(a_4)$ and $\delta(a_6) \mapsto \delta(a_6)^p + p\delta(a_6)$. Hence if we start with a polynomial in $M_0^4$ like $\gamma_{a,b}$, then $\phi(\gamma_{a,b}) = \gamma_{a,b}(a_4^p + p\delta(a_4), a_6^p + p\delta(a_6)) \in M_1^4$, where by this notation we mean substitute $a_4^p + p\delta(a_4)$ in for $a_4$ and $a_6^p + p\delta(a_6)$ in for $a_6$.

**Definition 7.1.** Let $\gamma_{a,b}$ be the polynomial in $M_1^4$ such that $\phi(\gamma_{a,b}) = \gamma_{a,b}(a_4^p, a_6^p) + p\gamma_{a,b}$.

An explicit formula for $\gamma_{a,b}$ is simple to compute by expanding the formula

$$
\phi(\gamma_{a,b}) = \begin{cases} 
0 & \text{if } b \text{ is even}, \\
\sum_{k=0}^{\infty} \left( \frac{m+k}{3k+2-n} \right) \left( m - 2k - 2 + n \right)^{\frac{m+1}{2}} \\
\quad (-1)^{m+k-\frac{m+1}{2}} (a_4^p + p\delta(a_4))^{2k+2-n} (a_6^p + p\delta(a_6))^{m-2k-2+n-\frac{m+1}{2}} & \text{if } b \text{ is odd}
\end{cases}
$$

and modulo $p^2$, $\gamma_{a,b}$ is linear in $\delta(a_4), \delta(a_6)$. In addition $\phi(\gamma_{a,b}) = \gamma_{a,b}^p + p\delta(\gamma_{a,b})$; however, note that $\gamma_{a,b}$ does not equal $p\delta(\gamma_{a,b})$ because the latter is missing the terms from $\gamma_{a,b}^p$ whose coefficients are divisible by $p$.

**Definition 7.2.** Let $\mu_{a,b}$ be the polynomial in $M_1^4$ such that $\phi(\mu_{a,b}) = \mu_{a,b}(a_4^p, a_6^p) + p\mu_{a,b}$.

We recall that the isogeny covariant differential modular form $f_{\text{jet}}h_{\text{jet}}$ is $\phi(f_{\text{jet}})$.

**Theorem 7.3.** The reduction modulo $p^2$ of $f_{\text{jet}}h_{\text{jet}}$ is

$$
\begin{align*}
&\left[ -72a_4^{p^2}\delta(a_4)^p + 48a_6^{p^2}\delta(a_6)^p \right] \gamma_{2p,p}(a_4^p, a_6^p) \\
&\quad + \frac{1}{\Delta^{p^2}} \left[ -8a_4^{2p^2}\mu_{0,p}(a_4^p, a_6^p) + 72a_6^{p^2}\mu_{p,p}(a_4^p, a_6^p) \\
&\quad - 48a_4^{p^2}\mu_{2p,p}(a_4^p, a_6^p) - 24a_6^{2p^2}\mu_{3p,3p}(a_4^p, a_6^p) \right] \\
&\quad + p \left[ -72a_4^{p^2}\delta^2(a_4) + 48a_6^{p^2}\delta^2(a_6) \right] \gamma_{2p,p}(a_4^p, a_6^p) + p J_0,
\end{align*}
$$
where \( J_0 \) is

\[
\left( \frac{1}{\Delta p^2} \right) \left( -72a_6^{p^2} \delta(a_4)^p + 48a_4^{p^2} \delta(a_6)^p \right) + 72a_6^p \mu_{p,p} - 48a_4^p \mu_{2p,p} - 24a_4^{2p^2} \mu_{3p,3p} \\
+ \left( (27\delta(2) + 66\delta(3))a_6^p \delta(a_4)^p \right) + (-28\delta(3) - 42\delta(2))a_6^p \delta(a_6)^p) \gamma_{2p,p} + 20\delta(2)a_4^{2p^2} \mu_{0,p} \\
+ (-27\delta(2) - 66\delta(3))a_6^p \mu_{p,p} + (42\delta(2) + 28\delta(3))a_4^p \mu_{2p,p} \\
+ 18\delta(2)a_6^{2p} a_6^p \mu_{2p,3p} + (8\delta(3) + 24\delta(2))a_4^{2p^2} \mu_{3p,3p} \\
+ H_{10}^p + H_{10}^p + H_4^p + H_8^p + H_5^p + H_8^p + H_4^p + H_8^p + H_8^p \right).
\]

**Proof.** Let \( H_i \) be the polynomials from Theorem 6.11. 

The formula follows immediately from the fact that \( f_{jet}h_{jet} = \phi(f_{jet}). \)

Next working modulo \( p^2 \), \( h_{jet} = (f_{jet}h_{jet})/f_{jet} \) is \( h_0 + p \left( \frac{h_0 f_{jet}}{f_{jet}} + \frac{h_1 f_{jet}}{f_{jet}} \right) \), where \( f_0 \) is the coefficient of \( p^0 \) in \( f_{jet} \), \( f_1 \) is the coefficient of \( p \), \( h_0 \) is the coefficient of \( p^0 \) in \( f_{jet}h_{jet} \), and \( h_1 \) is the coefficient of \( p \). In particular

\[
f_0 = \left[ \frac{-72a_6^{p^2} \delta(a_4)^p + 48a_4^{p^2} \delta(a_6)^p}{\Delta p^2} \right] \gamma_{2p,p} \\
+ \frac{1}{\Delta p^2} \left[ -8a_4^{2p^2} \mu_{0,p} + 72a_6^p \mu_{p,p} - 48a_4^p \mu_{2p,p} - 24a_4^{2p^2} \mu_{3p,3p} \right],
\]

\[
f_1 = \frac{1}{\Delta p^2} \left[ (27\delta(2) + 66\delta(3))a_6^p \delta(a_4)^p \right] \\
+ (-28\delta(3) - 42\delta(2))a_6^p \delta(a_6)^p) \gamma_{2p,p} + 20\delta(2)a_4^{2p^2} \mu_{0,p} \\
+ (-27\delta(2) - 66\delta(3))a_6^p \mu_{p,p} + (42\delta(2) + 28\delta(3))a_4^p \mu_{2p,p} \\
+ 18\delta(2)a_6^{2p} a_6^p \mu_{2p,3p} + (8\delta(3) + 24\delta(2))a_4^{2p^2} \mu_{3p,3p} \\
+ \left( H_0 + H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + H_7 + H_8 \right),
\]

\[
h_0 = \left[ \frac{-72a_6^{p^2} \delta(a_4)^p + 48a_4^{p^2} \delta(a_6)^p}{\Delta p^2} \right] \gamma_{2p,p}(a_4^p, a_6^p) \\
+ \frac{1}{\Delta p^2} \left[ -8a_4^{2p^2} \mu_{0,p}(a_4^p, a_6^p) + 72a_6^p \mu_{p,p}(a_4^p, a_6^p) \\
- 48a_4^p \mu_{2p,p}(a_4^p, a_6^p) - 24a_4^{2p^2} \mu_{3p,3p}(a_4^p, a_6^p) \right],
\]

\[
h_1 = \left[ \frac{-72a_6^{p^2} \delta^2(a_4)^p + 48a_4^{p^2} \delta^2(a_6)^p}{\Delta p^2} \right] \gamma_{2p,p}(a_4^p, a_6^p) + J_0.
\]

Therefore we have the following explicit formulation for \( h_{jet} \) where \( J_0 \) is the polynomial from Theorem 7.8.
Theorem 7.4. The reduction modulo $p^2$ of $h_{\text{jet}}$ is
\[
\left( -72a_6^2 \delta(a_4) + 48a_4^2 \delta(a_6) \right) \gamma_{2p, p}(a_4, a_6) - 8a_4^2 \mu_0, p(a_4, a_6) \\
+ 72a_6^2 \mu_0, p(a_4, a_6) - 48a_4^2 \mu_2, p(a_4, a_6) - 24a_4^2 \mu_3, p(a_4, a_6) \right) \bigg/ \\
\left( \Delta^{2 - p} \left( -72a_6^2 \delta(a_4) + 48a_4^2 \delta(a_6) \right) \gamma_{2p, p} \\
- 8a_4^2 \mu_0, p + 72a_6^2 \mu_2, p - 48a_4^2 \mu_3, p \right) + pK_0 \\
+ \Delta^{2 - p} \left( -72a_6^2 \delta(a_4) + 48a_4^2 \delta(a_6) \right) \gamma_{2p, p} - 8a_4^2 \mu_0, p + 72a_6^2 \mu_2, p - 48a_4^2 \mu_3, p \right) \right),
\]
where $K_0$ is
\[
\Delta^{p} J_0 \left( -72a_6^2 \delta(a_4) + 48a_4^2 \delta(a_6) \right) \gamma_{2p, p} - 8a_4^2 \mu_0, p + 72a_6^2 \mu_2, p - 48a_4^2 \mu_3, p \right)
\]

\[= \left( \frac{1}{\Delta^2} \left( \left( 27\delta(2) + 66\delta(3) \right) a_6^p \delta(a_4) + \left( -28\delta(3) - 42\delta(2) \right) a_4^p \delta(a_6) \right) \gamma_{2p, p} \\
+ 20\delta(2) a_6^p \mu_0, p + \left( -27\delta(2) - 66\delta(3) \right) a_6^p \mu_2, p + \left( 42\delta(2) + 28\delta(3) \right) a_4^p \mu_3, p + 18\delta(2) a_6^p \mu_2, p, 3p + (8\delta(3) + 24\delta(2)) a_4^p \mu_3, p, 3p \right) + \sum_{i=0}^{8} H_i \bigg) \bigg/ \\
\left( \left( -72a_6^2 \delta(a_4) + 48a_4^2 \delta(a_6) \right) \gamma_{2p, p} \\
- 8a_4^2 \mu_0, p + 72a_6^2 \mu_2, p - 48a_4^2 \mu_3, p \right)^2 \Delta^{2 - 2p} \right).
\]

References


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