MATHEMATICS OF COMPUTATION
Volume 74, Number 251, Pages 1565–1572
S 0025-5718(04)01728-4
Article electronically published on December 8, 2004

ON THE NONEXISTENCE OF 2-CYCLES FOR THE $3x + 1$ PROBLEM

JOHN L. SIMONS

ABSTRACT. This article generalizes a proof of Steiner for the nonexistence of 1-cycles for the $3x + 1$ problem to a proof for the nonexistence of 2-cycles. A lower bound for the cycle length is derived by approximating the ratio between numbers in a cycle. An upper bound is found by applying a result of Laurent, Mignotte, and Nesterenko on linear forms in logarithms. Finally numerical calculation of convergents of $\log_2 3$ shows that 2-cycles cannot exist.

1. INTRODUCTION

The $3x+1$ problem is a notorious problem of elementary number theory. Let x_n be a natural number and consider a sequence, generated conditionally by $x_{n+1} = \frac{1}{2}x_n$ if x_n is even and by $x_{n+1} = \frac{1}{2}(3x_n + 1)$ if x_n is odd. Numerical verification indicates that for "all" natural numbers x_n the cycle $(1, 2)$ finally appears. A formal proof is lacking so far in spite of various approaches to the problem; see [10].

We call a cyclic solution an m-cycle if the numbers x_n appear in m sequences, each consisting of a subsequence of odd numbers followed by a subsequence of even numbers. Steiner [7] assumes the existence of a 1-cycle with k odd numbers and ℓ even numbers and proves four partial results:

(1) an inequality for the ratio $(k + \ell)/k$;
(2) a numerical lower bound for k, from which it follows that $(k + \ell)/k$ must be a convergent in the continued fraction expression of $\log_2 3$;
(3) an upper bound for k by applying a theorem of Baker [1, p. 45] on linear forms in two logarithms;
(4) a (very effective) lower bound for the partial quotient of the convergent of a possible solution.

Numerical calculation of partial quotients shows that the only 1-cycle that satisfies these conditions is $(1, 2)$.

As has been remarked by Lagarias [4], the result of that proof is rather weak considering the underlying number theory. We modify and generalize Steiner's approach to prove the nonexistence of 2-cycles (consisting of k_1 odd numbers, ℓ_1 even numbers, k_2 odd numbers and ℓ_2 even numbers).

Received by the editor February 13, 2003 and in revised form, May 4, 2004.
2000 *Mathematics Subject Classification.* Primary 11J86, 11K60; Secondary 11K31.
Key words and phrases. 3x+1 problem, cycles, linear form in logarithms, continued fractions.

Let $K = \sum_{i=1}^{2} k_i$, $L = \sum_{i=1}^{2} \ell_i$. We then derive

(1) a generalized inequality for the ratio $(K+L)/K$;
(2) a numerical lower bound for K, from which it follows that $(K+L)/K$ must be a convergent in the continued fraction expression of $\log_2 3$;
(3) an upper bound for K by applying a theorem of Laurent, Mignotte and Nesterenko [5] on linear forms in two logarithms;
(4) a lower bound for the partial quotient of the convergent of a possible solution.

Steiner's numerical calculation finally shows that no other 2-cycle satisfies these conditions. We show that the approach fails to prove the nonexistence of m-cycles for $m > 2$.

2. The nonexistence of 2-cycles

We call the twofold 1-cycle $(1, 2, 1, 2)$ a trivial 2-cycle and any other 2-cycle nontrivial. We will computationally exclude small values for x_n and K. The nonexistence of 2-cycles is proved by a series of lemmas along the line of Steiner's original proof, with a crucial lemma to satisfy the conditions for the continued fraction approximation part of the proof.

Lemma 1. *A necessary and sufficient condition for the existence of a 2-cycle is the existence of a solution (a_i, k_i, ℓ_i) of the diophantine system of equations*

$$\left\{ \begin{array}{rcrcl} -3^{k_1} a_1 & + & 2^{k_2+\ell_1} a_2 & = & 2^{\ell_1} - 1, \\ 2^{k_1+\ell_2} a_1 & - & 3^{k_2} a_2 & = & 2^{\ell_2} - 1. \end{array} \right. \tag{1}$$

Proof. Assume that such a solution exists. Then $a_i \not\equiv 0 \pmod 2$. By taking

$$x_0 = a_1 2^{k_1} - 1$$

which is an odd number, it is easily verified that

$$x_{k_1} = a_1 3^{k_1} - 1$$

is the first even number after k_1 odd numbers.

The first row equation of (1) then generates $\ell_1 - 1$ additional even numbers and shows that

$$x_{k_1+\ell_1} = a_2 2^{k_2} - 1$$

is the first appearing odd number. By induction a 2-cycle exists, which proves the necessity of the condition in the lemma.

Now assume that a 2-cycle exists. The first odd number in the subsequence of k_1 odd numbers can be written in the form

$$a_1 2^{k_1} - 1$$

with $a_1, k_1 > 0$, $a_1 \not\equiv 0 \pmod 2$. Hence

$$x_{k_1} = a_1 3^{k_1} - 1$$

is an even number and the beginning of a subsequence of ℓ_1 even numbers. The first odd number is then

$$x_{k_1+\ell_1} = \left(a_1 3^{k_1} - 1\right) / 2^{\ell_1}$$

which can be written in the form

$$a_2 2^{k_2} - 1$$

with $a_2, k_2 > 0$, $a_2 \not\equiv 0 \pmod 2$. By induction a solution of the diophantine system of equations (1) exists, which proves the sufficiency of the condition in the lemma. □

Note that $a_i = k_i = l_i = 1$ is a solution of the system (1) corresponding with the trivial 2-cycle $(1, 2, 1, 2)$.

Lemma 2. *If a solution of the diophantine system* (1) *of Lemma* 1 *exists, then* a_i, k_i *and* ℓ_i *satisfy the relation*

$$1 < 2^{K+L}/3^K = \prod_{i=1}^{2} \frac{a_i - 3^{-k_i}}{a_i - 2^{-k_i}}. \tag{2}$$

Proof. The first row equation of the system (1) can be rewritten in the form

$$2^{\ell_1} = \left(a_1 3^{k_1} - 1\right) / \left(a_2 2^{k_2} - 1\right).$$

Hence

$$2^{k_2+\ell_1}/3^{k_1} = \frac{a_1 - 3^{-k_1}}{a_2 - 2^{-k_2}},$$

and similarly from the second row equation

$$2^{k_1+\ell_2}/3^{k_2} = \frac{a_2 - 3^{-k_2}}{a_1 - 2^{-k_1}}.$$

Multiplication leads to the equal sign part of the lemma. Since $3^{-k_i} < 2^{-k_i}$, the lemma is proved. □

Lemma 3. *If* a_i, k_i *and* ℓ_i *satisfy the relation* (2) *of Lemma* 2*, then* a_i, k_i *and* ℓ_i *also satisfy the inequality*

$$0 < (K+L)\log 2 - K \log 3 < \sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1}. \tag{3}$$

Proof. Since

$$1 < \frac{a_i - 3^{-k_i}}{a_i - 2^{-k_i}} < \frac{a_i}{a_i - 2^{-k_i}} = \frac{a_i 2^{k_i}}{a_i 2^{k_i} - 1},$$

it follows from relation (2) that

$$1 < 2^{K+L}/3^K < \prod_{i=1}^{2} \frac{a_i 2^{k_i}}{a_i 2^{k_i} - 1}.$$

Taking logs and using $\log(1+x) < x$ if $x < 1$ leads to

$$0 < (K+L)\log 2 - K \log 3 < \sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1},$$

which proves this lemma. □

Note that this is a generalization of the result $0 < (k+\ell)\log 2 - k \log 3 < 1/(2^k - 1)$ in Steiner's proof. From there on Steiner derives a lower bound $k_{\min}$ with the property that if $k > k_{\min}$, then $(k+\ell)/k$ must be a convergent of the continued fraction expression of $\log_2 3$. A generalization is not straightforward, since if $K = \sum_{i=1}^{2} k_i$ is large, a single k_i can still take a small value. However for the expression

$$\sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1}$$

an effective upper bound can be derived by exploiting the average values of k_i and ℓ_i.

Lemma 4. *If a nontrivial* 2*-cycle exists, then*

$$\sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1} < 1.19 \cdot 2^{(L-K)/2}.$$

Proof. Let $k = K/2$. Let $\bar{a} > 0$ be defined by

$$\bar{a}^2 = \prod_{i=1}^{2} \frac{a_i 2^{k_i} - 1}{2^{k_i}}.$$

Let ρ_i be defined by

$$\rho_i \bar{a} 2^k = a_i 2^{k_i} - 1.$$

Hence

$$\frac{\rho_1}{\rho_2} = \frac{a_1 2^{k_1} - 1}{a_2 2^{k_2} - 1} < \left(\frac{2}{3}\right)^{k_1} \frac{a_1 3^{k_1} - 1}{a_2 2^{k_2} - 1} = \left(\frac{2}{3}\right)^{k_1} 2^{\ell_1}.$$

Since $\rho_1 \rho_2 = 1$, we have

$$\rho_1^2 = \frac{\rho_1}{\rho_2} < 2^{k_1 + \ell_1 - k_1 \log_2 3} < 2^{\ell_1 - \frac{1}{2} k_1}.$$

Let $\ell = L/2$. Then we have for ρ_1 (since $\frac{1}{4} k_1 + \frac{1}{2} \ell_2 \geq \frac{3}{4}$)

$$\rho_1 < 2^{\frac{1}{2}\ell_1 - \frac{1}{4} k_1} \leq 2^{\ell - \frac{3}{4}}.$$

In a similar way we can prove this inequality holds for ρ_2 and consequently we have

$$\sum_{i=1}^{2} \frac{1}{\rho_i} = \sum_{i=1}^{2} \rho_i < 2^{\ell + \frac{1}{4}}.$$

For a nontrivial 2-cycle with $a_1 a_2 \geq 3$ we have

$$\bar{a}^2 = \prod_{i=1}^{2} \frac{a_i 2^{k_i} - 1}{2^{k_i}} > \prod_{i=1}^{2} \frac{a_i (2^{k_i} - 1)}{2^{k_i}} > \frac{a_1 \frac{1}{2} 2^{k_i}}{2^{k_1}} \frac{a_2 \frac{2}{3} 2^{k_2}}{2^{k_2}} \geq 1.$$

It follows that

$$\sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1} = \sum_{i=1}^{2} \frac{1}{\rho_i \bar{a} 2^k} < \frac{1}{\bar{a}} 2^{\ell - k + \frac{1}{4}} < 1.19 \cdot 2^{(L-K)/2},$$

which proves this lemma. □

Lemma 5. *If a nontrivial* 2*-cycle exists, then* $(K+L)/K$ *must be a convergent in the continued fraction expansion of* $\log_2 3$.

Proof. From Lemma 3 we have $0 < (K+L)\log 2 - K\log 3$; hence

$$K + L > K\frac{\log 3}{\log 2} > 1.58K.$$

Suppose $(K+L) > 1.6K$. Then we have

$$(K+L)\log 2 - K\log 3 > (1.6\log 2 - \log 3)K > 0.009K.$$

We computationally checked that for all starting values $x_0 \leq 100$ the trivial cycle $(1,2)$ appears and that for all values k_1 and k_2 with $k_1 + k_2 \leq 100$ no integer solutions of the system (1) of Lemma 1 exist other than $a_i = k_i = \ell_i = 1$. So we will now explicitly assume that $K > 100$ and $x_i = a_i 2^{k_i} - 1 > 100$. From Lemma 3 we have

$$(K+L)\log 2 - K\log 3 < \sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1},$$

and thus $(K+L)\log 2 - K\log 3 < 0.02$, which contradicts the lower bound $0.009K > 0.9$, and hence $K + L < 1.6K$. Consequently

$$1.19 \cdot 2^{(L-K)/2} < 1.19 \cdot 2^{-0.2K} < \frac{\frac{1}{2}\log 2}{K} \text{ if } K > 100.$$

Substitution of this result in Lemma 3 and Lemma 4 leads to

$$0 < (K+L)\log 2 - K\log 3 < \frac{\frac{1}{2}\log 2}{K}$$

or equivalently

$$0 < \frac{K+L}{K} - \frac{\log 3}{\log 2} < \frac{1}{2K^2},$$

which proves this lemma. □

Lemma 6. *If a nontrivial* 2*-cycle exists, then* $K < 86\,000$.

Proof. Let $\Lambda = (K+L)\log 2 - K\log 3$. Then $\Lambda > 0$ from Lemma 3. According to a theorem on linear forms in two logarithms of Laurent, Mignotte and Nesterenko [5], if $\Lambda > 0$, then

$$\log\Lambda \geq -24.34 D^4 \log A_1 \log A_2 \max\left\{\log\left(\frac{K+L}{\log A_2} + \frac{K}{\log A_1}\right) + 0.14, \frac{21}{D}, \frac{1}{2}\right\}^2.$$

Here $D = 1$ is the degree of the extension field of $\mathbb{Q}$, $A_1 = 3$ and $A_2 = e$.

We now distinguish two cases for $T = \log\left(\frac{K+L}{\log 3} + K\right)$.

(a) If $T \leq 20.86$, since $K + L > 1.58K$ from Lemma 5, we have $K < 4.8 \cdot 10^8$. Also if $T \leq 20.86$, then $-\log\Lambda \leq 24.34(\log 3)21^2 < 11\,800$. From Lemma 5 we have $-\log\Lambda \geq 0.2K\log 2 - \log 1.19$. Thus $0.2K\log 2 - \log 1.19 < 11\,800$; hence $K < 86\,000$.

(b) If $T > 20.86$, since $K + L < 1.6K$ from Lemma 5, we have $K > 4.6 \cdot 10^8$. Also if $T > 20.86$, then $-\log\Lambda \leq 24.34(\log 3)(T + 0.14)^2 < 26.75(\log K + 1.04)^2$. From Lemma 5 we have $-\log\Lambda \geq 0.2K\log 2 - \log 1.19$. Thus, $0.2K\log 2 - \log 1.19 < 26.75(\log K + 1.04)^2$; hence $K < 24\,000$, which contradicts the lower bound $K > 4.6 \cdot 10^8$.

So $T \leq 20.86$ and $K < 86\,000$, which proves this lemma. □

Lemma 7. *If a nontrivial* 2*-cycle exists, then the partial quotient* a_{n+1} *in the continued fraction expansion of* $\log_2 3$*, corresponding with the solution* $(K+L)/K$*, is greater than* $3\,500$.

Proof. According to a theorem of Legendre [3, p. 153] we have for the partial quotients a_n of a possible solution $(K+L)/K$ of Lemma 5 the inequality

$$\left|\log_2 3 - \frac{K+L}{K}\right| > \frac{1}{(a_{n+1}+2)K^2}.$$

From Lemmas 3 and 4 we have $0 < (K+L)\log 2 - K\log 3 < 2^{(L-K)/2+\frac{1}{4}}$ or equivalently

$$\left|\log_2 3 - \frac{K+L}{K}\right| < \frac{2^{(L-K)/2+\frac{1}{4}}}{(\log 2)K}$$

From Lemma 5 we have $L - K < -0.4K$. Thus we have for K the inequality

$$\frac{1}{(a_{n+1}+2)K^2} < \frac{2^{-0.2K+\frac{1}{4}}}{(\log 2)K}$$

or equivalently

$$a_{n+1} > \frac{(\log 2)2^{0.2K-\frac{1}{4}}}{K} - 2 > 3\,500$$

if $K > 100$, which proves this lemma. □

Main Theorem 1. *There are no nontrivial* 2*-cycles for the* $3x+1$ *problem.*

Proof. Suppose such a 2-cycle exists. Then according to Lemma 5 the ratio $(K+L)/K$ must be a convergent in the continued fraction expansion of $\log_2 3$. According to Lemmas 4 and 6 we only need Steiner's calculations for the range $100 < K < 86\,000$. The only values of K and $K+L$ in this range for which $\Lambda > 0$ are $(306, 485)$ and $(15601, 24727)$. The corresponding partial quotients in the continued fraction expansion of $\log_2 3$ (taken from Steiner) satisfy $a_{n+1} < 25$. This upper bound contradicts the lower bound $3\,500$ of Lemma 7, which proves the theorem. □

3. Remarks

Remark 1. It is known from exterior calculations [2, pp. 215–218], [10, p. 23] that the cycle length $K+L$ of a possible cycle satisfies $K+L > 357\,638\,239$. Together with the upper bound $K < 86\,000$ of Lemma 6 this proves the nonexistence of 2-cycles. If for $m > 2$ a generalization of Lemma 4 can be found, the upper bound $K < 86\,000$ of Lemma 6 will increase, so this ad hoc line of proof should vanish for some $m > 2$.

Remark 2. The nonexistence of nontrivial 2-cycles can alternatively be proved by applying a result of de Weger [9, p. 108]. He uses a result of Waldschmidt [8] to derive upper bounds for linear forms of the type $a\log 2 - b\log 3$. In particular he (implicitly) proves that the equation $1 < 2^{k+\ell}/3^k < 1 + 3^{-0.1k}$ has for $k \geq 32$ no solutions. This can shorten the proof for the nonexistence of 2-cycles (and also Steiner's proof). The result of de Weger can be reformulated as $0 < (k+\ell)\log 2 - k\log 3 < 2^{-0.158k}$ has no solutions for $k \geq 32$. From Lemmas 4 and 5 it follows that $0 < (K+L)\log 2 - K\log 3 < 2^{-0.2K+\frac{1}{4}} (< 2^{-0.158K})$, so nontrivial 2-cycles cannot exist. De Weger's method can be applied for any coefficient $0 < \alpha < 1$ in

$1+3^{-\alpha k}$. If for $m>2$ a generalization of Lemma 4 can be found, the coefficient 0.2 in the exponent $-0.2K+\frac{1}{4}$ will decrease, so this line of proof can in principle be generalized for $m>2$.

Remark 3. There is no straightforward generalization to prove the nonexistence of m-cycles ($m>2$) for the $3x+1$ problem. We will sketch a trial proof for $m=3$ to demonstrate this. It is easily verified that Lemmas 1, 2 and 3 can be generalized to

$$0<(K+L)\log 2-K\log 3<\sum_{i=1}^{3}\frac{1}{a_i 2^{k_i}-1}.$$

We now have to find an upper bound for the right-hand part of this inequality as is done in Lemma 4 for 2-cycles. Let $k=K/3$ and let $\bar{a}$ and ρ_i, respectively, be defined by

$$\bar{a}^3=\prod_{i=1}^{3}\frac{a_i 2^{k_i}-1}{2^{k_i}},$$
$$\rho_i\bar{a}2^k=a_i 2^{k_i}-1.$$

Then $\bar{a}^3>0.375$. Substitution into the generalized inequality leads to

$$0<(K+L)\log 2-K\log 3<\sum_{i=1}^{3}\frac{1}{a_i 2^{k_i}-1}<\sum_{i=1}^{3}\frac{1}{\rho_i\bar{a}2^k}.$$

In a similar way as is done in Lemma 4 we can derive upper bounds for ρ_i^{-3}:

$$\rho_1^{-3}<2^{-\frac{1}{2}k_2-k_3+\ell_2+2\ell_3},$$
$$\rho_2^{-3}<2^{-\frac{1}{2}k_3-k_1+\ell_3+2\ell_1},$$
$$\rho_3^{-3}<2^{-\frac{1}{2}k_1-k_2+\ell_1+2\ell_2}.$$

If we assume that k_i and l_i satisfy

$$-\tfrac{1}{2}k_2-k_3+l_2+2l_3\le\tfrac{3}{2}(l_1+l_2+l_3),$$
$$-\tfrac{1}{2}k_3-k_1+l_3+2l_1\le\tfrac{3}{2}(l_1+l_2+l_3),$$
$$-\tfrac{1}{2}k_1-k_2+l_1+2l_2\le\tfrac{3}{2}(l_1+l_2+l_3).$$

then we have $\rho_i^{-1}<2^{\frac{3}{2}L}$ and consequently

$$0<(K+L)\log 2-K\log 3<1.39\cdot 2^{\frac{3}{2}L-K}.$$

This is a result similar to the result of Lemma 4 for 2-cycles and the proof could continue. The exception class of k_i and ℓ_i values which do not satisfy all these relations is, however, large. Let $k_1=7M+N_1$ and $\ell_3=7M+N_2$ ($N_2>N_1$) and all other $k_i=\ell_i=M$. Then only the first inequality is not satisfied. These arbitrarily chosen values are not necessarily a solution of the original system of diophantine equations, but all possible solutions must be checked separately. An alternative approach to generalize the results for m-cycles ($m>2$) is discussed in a forthcoming paper of Simons and de Weger [6].

Remark 4. This line of reasoning can also be used to (dis)prove the nonexistence of 2-cycles (and 1- cycles) for the $px+(p-2)r$ problem with $p\ge 3$. A similar reasoning as used in Lemmas 1, 2 and 3 leads to the generalized inequality

$$0<(K+L)\log 2-K\log 3<\sum_{i=1}^{2}\frac{r}{a_i 2^{k_i}-r}.$$

It can be proved that only "small" m-cycles can exist; however, the class of small cycles for the $px + (p-2)r$ problem does contain several m-cycles. The $3x + 5$ problem has for instance the 1-cycles $(1, 4, 2)$ and $(19, 31, 49, 76, 38)$, the 2-cycle $(23, 37, 58, 29, 46)$ and the 6-cycle $(187, \ldots, 427, \ldots, 1091, \ldots, 1847, \ldots, 781, \ldots, 883, \ldots, 374)$ with period 27. From such calculations a lower bound can be derived for the continued fraction approximation, and Laurent's theorem gives an upper bound for the period length of any possible m-cycle, m fixed.

Remark 5. The remark of Lagarias about the weakness of the result remains valid.

Acknowledgments

The author wishes to thank Dr. B.M.M. de Weger (University of Eindhoven), Prof. Dr. R. Tijdeman (University of Leiden), and the referee for valuable comments on an earlier version.

References

1. A. Baker, *Transcendental Number Theory*, Cambridge University Press, Cambridge, 1975. MR0422171 (54:10163)
2. R.K. Guy, *Unsolved Problems in Number Theory*, 2nd ed., Springer Verlag, Berlin, 1994. MR1299330 (96e:11002)
3. G.H. Hardy and E.M. Wright, *The theory of Numbers*, 4th edition, Oxford University Press, Oxford, 1975. MR0568909 (81i:10002)
4. J.C. Lagarias, *The $3x + 1$ problem and its generalizations.* American Mathematical Monthly **92** (1985), 3–23. MR0777565 (86i:11043)
5. M. Laurent, M. Mignotte and Yu. Nesterenko, *Formes linéaires en deux logarithmes et déterminants d'interpolation.* Journal of Number Theory **55** (1995), 285–321. MR1366574 (96h:11073)
6. J.L. Simons and B.M.M. de Weger, *Theoretical and computational bounds for m-cycles of the $3n + 1$ problem.* Accepted by Acta Arithmetica, 2004.
7. R.P. Steiner, *A theorem on the Syracuse problem*, Proceedings of the 7th Manitoba Conference on Numerical Mathematics and Computation, 1977, pp. 553–559. MR0535032 (80g:10003)
8. M. Waldschmidt, *A lower bound for linear forms in logarithms.* Acta Arithmetica **37** (1980), 257–283. MR0598881 (82h:10049)
9. B.M.M. de Weger, *Algorithms for diophantine equations*, CWI Tract 65, Centre for Mathematics and Computer Science, Amsterdam, 1990.
10. G.J. Wirsching, *The Dynamical System Generated by the $3n + 1$ Function*, Lecture Notes in Mathematics, Vol. 1681, Springer Verlag, Berlin, 1998. MR1612686 (99g:11027)

University of Groningen, PO Box 800, 9700 AV Groningen, The Netherlands
E-mail address: j.l.simons@bdk.rug.nl