ON THE NONEXISTENCE OF 2-CYCLES FOR THE 3x + 1 PROBLEM

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Abstract. This article generalizes a proof of Steiner for the nonexistence of 1-cycles for the 3x + 1 problem to a proof for the nonexistence of 2-cycles. A lower bound for the cycle length is derived by approximating the ratio between numbers in a cycle. An upper bound is found by applying a result of Laurent, Mignotte, and Nesterenko on linear forms in logarithms. Finally numerical calculation of convergents of \( \log_2 3 \) shows that 2-cycles cannot exist.

1. Introduction

The 3x + 1 problem is a notorious problem of elementary number theory. Let \( x_n \) be a natural number and consider a sequence, generated conditionally by \( x_{n+1} = \frac{1}{2} x_n \) if \( x_n \) is even and by \( x_{n+1} = \frac{1}{2}(3x_n + 1) \) if \( x_n \) is odd. Numerical verification indicates that for “all” natural numbers \( x_n \), the cycle \((1, 2)\) finally appears. A formal proof is lacking so far in spite of various approaches to the problem; see [10].

We call a cyclic solution an \( m \)-cycle if the numbers \( x_n \) appear in \( m \) sequences, each consisting of a subsequence of odd numbers followed by a subsequence of even numbers. Steiner [7] assumes the existence of a 1-cycle with \( k \) odd numbers and \( \ell \) even numbers and proves four partial results:

1. an inequality for the ratio \((k + \ell)/k\);
2. a numerical lower bound for \( k \), from which it follows that \((k + \ell)/k\) must be a convergent in the continued fraction expression of \( \log_2 3 \);
3. an upper bound for \( k \) by applying a theorem of Baker [1, p. 45] on linear forms in two logarithms;
4. a (very effective) lower bound for the partial quotient of the convergent of a possible solution.

Numerical calculation of partial quotients shows that the only 1-cycle that satisfies these conditions is \((1, 2)\).

As has been remarked by Lagarias [4], the result of that proof is rather weak considering the underlying number theory. We modify and generalize Steiner’s approach to prove the nonexistence of 2-cycles (consisting of \( k_1 \) odd numbers, \( \ell_1 \) even numbers, \( k_2 \) odd numbers and \( \ell_2 \) even numbers).

Received by the editor February 13, 2003 and in revised form, May 4, 2004.
2000 Mathematics Subject Classification. Primary 11J86, 11K60; Secondary 11K31.
Key words and phrases. 3x+1 problem, cycles, linear form in logarithms, continued fractions.

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Let \( K = \sum_{i=1}^{2} k_i \), \( L = \sum_{i=1}^{2} \ell_i \). We then derive

1. a generalized inequality for the ratio \((K + L)/K\);
2. a numerical lower bound for \( K \), from which it follows that \((K + L)/K\) must be a convergent in the continued fraction expression of \( \log 3 \);
3. an upper bound for \( K \) by applying a theorem of Laurent, Mignotte and Nesterenko \( \text{[5]} \) on linear forms in two logarithms;
4. a lower bound for the partial quotient of the convergent of a possible solution.

Steiner’s numerical calculation finally shows that no other 2-cycle satisfies these conditions. We show that the approach fails to prove the nonexistence of \( m \)-cycles for \( m > 2 \).

2. The nonexistence of 2-cycles

We call the twofold 1-cycle \((1, 2, 1, 2)\) a trivial 2-cycle and any other 2-cycle nontrivial. We will computationally exclude small values for \( x_n \) and \( K \). The nonexistence of 2-cycles is proved by a series of lemmas along the line of Steiner’s original proof, with a crucial lemma to satisfy the conditions for the continued fraction approximation part of the proof.

**Lemma 1.** A necessary and sufficient condition for the existence of a 2-cycle is the existence of a solution \((a_i, k_i, \ell_i)\) of the diophantine system of equations

\[
\begin{align*}
-3^{k_1}a_1 &+ 2^{k_2+\ell_1}a_2 = 2^{\ell_1} - 1, \\
2^{k_1+\ell_2}a_1 &- 3^{\ell_2}a_2 = 2^{\ell_2} - 1.
\end{align*}
\]

**Proof.** Assume that such a solution exists. Then \( a_i \neq 0 \pmod{2} \). By taking

\[ x_0 = a_1 2^{k_1} - 1 \]

which is an odd number, it is easily verified that

\[ x_{k_1} = a_1 3^{k_1} - 1 \]

is the first even number after \( k_1 \) odd numbers.

The first row equation of (1) then generates \( \ell_1 - 1 \) additional even numbers and shows that

\[ x_{k_1+\ell_1} = a_2 2^{k_2} - 1 \]

is the first appearing odd number. By induction a 2-cycle exists, which proves the necessity of the condition in the lemma.

Now assume that a 2-cycle exists. The first odd number in the subsequence of \( k_1 \) odd numbers can be written in the form

\[ a_1 2^{k_1} - 1 \]

with \( a_1, k_1 > 0, a_1 \neq 0 \pmod{2} \). Hence

\[ x_{k_1} = a_1 3^{k_1} - 1 \]

is an even number and the beginning of a subsequence of \( \ell_1 \) even numbers. The first odd number is then

\[ x_{k_1+\ell_1} = (a_1 3^{k_1} - 1)/2^{\ell_1} \]

which can be written in the form

\[ a_2 2^{k_2} - 1 \]
with $a_2, k_2 > 0$, $a_2 \not\equiv 0 \pmod{2}$. By induction a solution of the diophantine system of equations (1) exists, which proves the sufficiency of the condition in the lemma.

Note that $a_i = k_i = l_i = 1$ is a solution of the system (1) corresponding with the trivial 2-cycle $(1, 2, 1, 2)$.

**Lemma 2.** If a solution of the diophantine system (1) of Lemma 1 exists, then $a_i, k_i$ and $\ell_i$ satisfy the relation

$$1 < 2^{K+L}/3^K = \prod_{i=1}^{2} \frac{a_i - 3^{-k_i}}{a_i - 2^{-k_i}}.$$  

**Proof.** The first row equation of the system (1) can be rewritten in the form

$$2^{\ell_1} = (a_13^{k_1} - 1) / (a_22^{k_2} - 1).$$

Hence

$$2^{k_2+\ell_1}/3^{k_1} = \frac{a_1 - 3^{-k_1}}{a_2 - 2^{-k_2}},$$

and similarly from the second row equation

$$2^{k_1+\ell_2}/3^{k_2} = \frac{a_2 - 3^{-k_2}}{a_1 - 2^{-k_1}}.$$  

Multiplication leads to the equal sign part of the lemma. Since $3^{-k_i} < 2^{-k_i}$, the lemma is proved.

**Lemma 3.** If $a_i, k_i$ and $\ell_i$ satisfy the relation (2) of Lemma 2, then $a_i, k_i$ and $\ell_i$ also satisfy the inequality

$$0 < (K + L)\log 2 - K\log 3 < \sum_{i=1}^{2} \frac{1}{a_i2^{k_i} - 1}.$$  

**Proof.** Since

$$1 < \frac{a_i - 3^{-k_i}}{a_i - 2^{-k_i}} < \frac{a_i}{a_i - 2^{-k_i}} = \frac{a_i2^{k_i}}{a_i2^{k_i} - 1},$$

it follows from relation (2) that

$$1 < 2^{K+L}/3^K < \prod_{i=1}^{2} \frac{a_i2^{k_i}}{a_i2^{k_i} - 1}.$$  

Taking logs and using $\log(1 + x) < x$ if $x < 1$ leads to

$$0 < (K + L)\log 2 - K\log 3 < \sum_{i=1}^{2} \frac{1}{a_i2^{k_i} - 1},$$

which proves this lemma.

Note that this is a generalization of the result $0 < (k + \ell)\log 2 - k\log 3 < 1/(2^k - 1)$ in Steiner’s proof. From there on Steiner derives a lower bound $k_{\text{min}}$ with the property that if $k > k_{\text{min}}$, then $(k + \ell)/k$ must be a convergent of the continued fraction expression of $\log_2 3$. A generalization is not straightforward,
since if \( K = \sum_{i=1}^{2} k_i \) is large, a single \( k_i \) can still take a small value. However for

\[
\sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1}
\]

an effective upper bound can be derived by exploiting the average values of \( k_i \) and \( \ell_i \).

**Lemma 4.** If a nontrivial 2-cycle exists, then

\[
\sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1} < 1.19 \cdot 2^{(L-K)/2}.
\]

**Proof.** Let \( k = K/2 \). Let \( \bar{a} > 0 \) be defined by

\[
\bar{a}^2 = \prod_{i=1}^{2} \frac{a_i 2^{k_i} - 1}{2^{k_i}}.
\]

Let \( \rho_i \) be defined by

\[
\rho_i \bar{a} 2^k = a_i 2^{k_i} - 1.
\]

Hence

\[
\frac{\rho_1}{\rho_2} = \frac{a_1 2^{k_1} - 1}{a_2 2^{k_2} - 1} < \left( \frac{2}{3} \right)^{k_1} \frac{a_1 3^{k_1} - 1}{a_2 2^{k_2} - 1} = \left( \frac{2}{3} \right)^{k_1} 2^{\ell_1}.
\]

Since \( \rho_1 \rho_2 = 1 \), we have

\[
\rho_1^2 = \frac{\rho_1}{\rho_2} < 2^{k_1 + \ell_1 - k_1 \log_2 3} < 2^{\ell_1 - \frac{3}{2} k_1}.
\]

Let \( \ell = L/2 \). Then we have for \( \rho_1 \) (since \( \frac{1}{2} k_1 + \frac{1}{2} \ell_2 \geq \frac{3}{2} \))

\[
\rho_1 < 2^{\frac{\ell_1 - \frac{3}{2} k_1}{4}} \leq 2^{\ell - \frac{3}{4}}.
\]

In a similar way we can prove this inequality holds for \( \rho_2 \) and consequently we have

\[
\sum_{i=1}^{2} \frac{1}{\rho_i} = \sum_{i=1}^{2} \rho_i < 2^{\ell + \frac{1}{4}}.
\]

For a nontrivial 2-cycle with \( a_1 a_2 \geq 3 \) we have

\[
\bar{a}^2 = \prod_{i=1}^{2} \frac{a_i 2^{k_i} - 1}{2^{k_i}} > \prod_{i=1}^{2} \frac{a_i (2^{k_i} - 1)}{2^{k_i}} > \frac{a_1 \frac{1}{2} 2^{k_1} a_2 \frac{3}{2} 2^{k_2}}{2^{k_1} 2^{k_2}} \geq 1.
\]

It follows that

\[
\sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1} = \sum_{i=1}^{2} \frac{1}{\rho_i \bar{a}^k} < \frac{1}{\bar{a}} 2^{-k + \frac{1}{4}} < 1.19 \cdot 2^{(L-K)/2},
\]

which proves this lemma. \( \square \)

**Lemma 5.** If a nontrivial 2-cycle exists, then \((K + L)/K\) must be a convergent in the continued fraction expansion of \( \log_2 3 \).
Proof. From Lemma 5 we have \(0 < (K + L) \log 2 - K \log 3\); hence
\[
K + L > K \frac{\log 3}{\log 2} > 1.58K.
\]
Suppose \((K + L) > 1.6K\). Then we have
\[
(K + L) \log 2 - K \log 3 > (1.6 \log 2 - \log 3)K > 0.009K.
\]
We computationally checked that for all starting values \(x_0 \leq 100\) the trivial cycle \((1, 2)\) appears and that for all values \(k_1\) and \(k_2\) with \(k_1 + k_2 \leq 100\) no integer solutions of the system (11) of Lemma 1 exist other than \(a_i = k_i = \ell_i = 1\). So we will now explicitly assume that \(K > 100\) and \(x_i = a_i 2^{k_i} - 1 > 100\). From Lemma 3 we have
\[
(K + L) \log 2 - K \log 3 < \sum_{i=1}^{2} \frac{1}{a_i 2^{k_i} - 1},
\]
and thus \((K + L) \log 2 - K \log 3 < 0.02\), which contradicts the lower bound \(0.009K\), and hence \(K + L < 1.6K\). Consequently
\[
1.19 \cdot 2^{(L-K)/2} < 1.19 \cdot 2^{-0.2K} < \frac{\frac{1}{2} \log 2}{K} \text{ if } K > 100.
\]
Substitution of this result in Lemma 5 and Lemma 1 leads to
\[
0 < (K + L) \log 2 - K \log 3 < \frac{\frac{1}{2} \log 2}{K}
\]
or equivalently
\[
0 < \frac{K + L}{K} - \frac{\log 3}{\log 2} < \frac{1}{2K^2},
\]
which proves this lemma. \(\square\)

Lemma 6. If a nontrivial 2-cycle exists, then \(K < 86000\).

Proof. Let \(\Lambda = (K + L) \log 2 - K \log 3\). Then \(\Lambda > 0\) from Lemma 5. According to a theorem on linear forms in two logarithms of Laurent, Mignotte and Nesterenko 5, if \(\Lambda > 0\), then
\[
\log \Lambda \geq -24.34D^4 \log A_1 \log A_2 \max \left\{ \log \left( \frac{K + L}{\log A_2} + \frac{K}{\log A_1} \right) + 0.14, \frac{21}{D} \cdot \frac{1}{2} \right\}^2.
\]
Here \(D = 1\) is the degree of the extension field of \(\mathbb{Q}\), \(A_1 = 3\) and \(A_2 = e\).

We now distinguish two cases for \(T = \log \left( \frac{K + L}{\log 3} + K \right)\).

(a) If \(T \leq 20.86\), since \(K + L > 1.58K\) from Lemma 5 we have \(K < 4.8 \cdot 10^8\). Also if \(T \leq 20.86\), then \(- \log \Lambda \leq 24.34(\log 3)/21^2 < 11800\). From Lemma 5 we have \(- \log \Lambda \geq 0.2K \log 2 - \log 1.19\). Thus \(0.2K \log 2 - \log 1.19 < 11800\); hence \(K < 86000\).

(b) If \(T > 20.86\), since \(K + L < 1.6K\) from Lemma 5 we have \(K > 4.6 \cdot 10^8\). Also if \(T > 20.86\), then \(- \log \Lambda \leq 24.34(\log 3)(T + 0.14)^2 < 26.75(\log K + 1.04)^2\). From Lemma 5 we have \(- \log \Lambda \geq 0.2K \log 2 - \log 1.19\). Thus, \(0.2K \log 2 - \log 1.19 < 26.75(\log K + 1.04)^2\); hence \(K < 24000\), which contradicts the lower bound \(K > 4.6 \cdot 10^8\).

So \(T \leq 20.86\) and \(K < 86000\), which proves this lemma. \(\square\)
Lemma 7. If a nontrivial 2-cycle exists, then the partial quotient \( a_{n+1} \) in the continued fraction expansion of \( \log_2 3 \), corresponding with the solution \((K + L)/K\), is greater than 3.500.

Proof. According to a theorem of Legendre [3, p. 153] we have for the partial quotients \( a_n \) of a possible solution \((K + L)/K\) of Lemma 7 the inequality

\[
\left| \log_2 3 - \frac{K + L}{K} \right| > \frac{1}{(a_{n+1} + 2)K^2}.
\]

From Lemmas 3 and 4 we have \( 0 < (K + L)\log 2 - K \log 3 < 2(\log 2)/2 + \frac{1}{2} \) or equivalently

\[
\left| \log_2 3 - \frac{K + L}{K} \right| < \frac{2(\log 2)/2 + \frac{1}{2}}{(\log 2)K}.
\]

From Lemma 5 we have \( L - K < -0.4K \). Thus we have for \( K \) the inequality

\[
\frac{1}{(a_{n+1} + 2)K^2} < \frac{2^{-0.2K + \frac{1}{4}}}{(\log 2)K}
\]

or equivalently

\[
a_{n+1} > \frac{(\log 2)2^{0.2K - \frac{1}{4}}}{K} - 2 > 3.500
\]

if \( K > 100 \), which proves this lemma. □

Main Theorem 1. There are no nontrivial 2-cycles for the 3x + 1 problem.

Proof. Suppose such a 2-cycle exists. Then according to Lemma 5 the ratio \((K + L)/K\) must be a convergent in the continued fraction expansion of \( \log_2 3 \). According to Lemmas 4 and 6 we only need Steiner’s calculations for the range \( 100 < K < 86,000 \). The only values of \( K \) and \( K + L \) in this range for which \( \Lambda > 0 \) are (306, 485) and (15601, 24727). The corresponding partial quotients in the continued fraction expansion of \( \log_2 3 \) (taken from Steiner) satisfy \( a_{n+1} < 25 \). This upper bound contradicts the lower bound 3.500 of Lemma 7, which proves the theorem. □

3. Remarks

Remark 1. It is known from exterior calculations [2, pp. 215–218], [10, p. 23] that the cycle length \( K + L \) of a possible cycle satisfies \( K + L > 357,638,239 \). Together with the upper bound \( K < 86,000 \) of Lemma 6 this proves the nonexistence of 2-cycles. If for \( m > 2 \) a generalization of Lemma 4 can be found, the upper bound \( K < 86,000 \) of Lemma 6 will increase, so this ad hoc line of proof should vanish for some \( m > 2 \).

Remark 2. The nonexistence of nontrivial 2-cycles can alternatively be proved by applying a result of de Weger [9, p. 108]. He uses a result of Waldschmidt [8] to derive upper bounds for linear forms of the type \( a \log 2 - b \log 3 \). In particular he (implicitly) proves that the equation \( 1 < 2^{k+1}/3^k < 1 + 3^{-0.1k} \) has for \( k \geq 32 \) no solutions. This can shorten the proof for the nonexistence of 2-cycles (and also Steiner’s proof). The result of de Weger can be reformulated as \( 0 < (k + \ell)\log 2 - k \log 3 < 2^{0.2K + \frac{1}{4}}(\log 2 - 0.158K) \), so nontrivial 2-cycles cannot exist. De Weger’s method can be applied for any coefficient \( 0 < \alpha < 1 \) in
1 + 3^{−αk}. If for \( m > 2 \) a generalization of Lemma 4 can be found, the coefficient 0.2 in the exponent \(-0.2K + \frac{1}{4}\) will decrease, so this line of proof can in principle be generalized for \( m > 2 \).

**Remark 3.** There is no straightforward generalization to prove the nonexistence of \( m \)-cycles (\( m > 2 \)) for the \( 3x + 1 \) problem. We will sketch a trial proof for \( m = 3 \) to demonstrate this. It is easily verified that Lemmas 1, 2 and 3 can be generalized for \( m > 2 \).

An alternative approach to generalize the results for \( m \)-cycles (and 1-cycles) for the \( px + (p−2)r \) problem with \( p \geq 3 \). A similar reasoning as used in Lemmas 1, 2 and 3 leads to the generalized inequality

\[
0 < (K + L) \log 2 - K \log 3 < \sum_{i=1}^{3} \frac{1}{a_i 2^{k_i} - 1}.
\]

We now have to find an upper bound for the right-hand part of this inequality as is done in Lemma 4 for 2-cycles. Let \( k = K/3 \) and let \( \bar{a} \) and \( ρ_i \), respectively, be defined by

\[
\bar{a}^3 = \prod_{i=1}^{3} \frac{a_i 2^{k_i} - 1}{2^{k_i}},
\]

\[
ρ_i \bar{a} 2^k = a_i 2^{k_i} - 1.
\]

Then \( \bar{a}^3 > 0.375 \). Substitution into the generalized inequality leads to

\[
0 < (K + L) \log 2 - K \log 3 < \sum_{i=1}^{3} \frac{1}{a_i 2^{k_i} - 1} < \sum_{i=1}^{3} \frac{1}{ρ_i \bar{a} 2^k}.
\]

In a similar way as is done in Lemma 4 we can derive upper bounds for \( ρ_i^{-3} \):

\[
ρ_1^{-3} < 2^{-\frac{1}{2}k_2 - k_3 + \ell_2 + 2\ell_3},
\]

\[
ρ_2^{-3} < 2^{-\frac{1}{2}k_3 - k_1 + \ell_3 + 2\ell_1},
\]

\[
ρ_3^{-3} < 2^{-\frac{1}{2}k_1 - k_2 + \ell_1 + 2\ell_2}.
\]

If we assume that \( k_i \) and \( \ell_i \) satisfy

\[
-\frac{1}{2}k_2 - k_3 + \ell_2 + 2\ell_3 \leq \frac{1}{2}(l_1 + l_2 + l_3),
\]

\[
-\frac{1}{2}k_3 - k_1 + \ell_3 + 2\ell_1 \leq \frac{1}{2}(l_1 + l_2 + l_3),
\]

\[
-\frac{1}{2}k_1 - k_2 + \ell_1 + 2\ell_2 \leq \frac{1}{2}(l_1 + l_2 + l_3).
\]

then we have \( ρ_i^{-1} < 2^{\frac{1}{2}L} \) and consequently

\[
0 < (K + L) \log 2 - K \log 3 < 1.39 \cdot 2^{\frac{1}{2}L - K}.
\]

This is a result similar to the result of Lemma 4 for 2-cycles and the proof could continue. The exception class of \( k_i \) and \( \ell_i \) values which do not satisfy all these relations is, however, large. Let \( k_1 = 7M + N_1 \) and \( \ell_3 = 7M + N_2 \) (\( N_2 > N_1 \)) and all other \( k_i = \ell_i = M \). Then only the first inequality is not satisfied. These arbitrarily chosen values are not necessarily a solution of the original system of diophantine equations, but all possible solutions must be checked separately. An alternative approach to generalize the results for \( m \)-cycles (\( m > 2 \)) is discussed in a forthcoming paper of Simons and de Weger [6].

**Remark 4.** This line of reasoning can also be used to (dis)prove the nonexistence of 2-cycles (and 1-cycles) for the \( px + (p−2)r \) problem with \( p \geq 3 \). A similar reasoning as used in Lemmas 1, 2 and 3 leads to the generalized inequality

\[
0 < (K + L) \log 2 - K \log 3 < \sum_{i=1}^{2} \frac{r}{a_i 2^{k_i} - r}.
\]
It can be proved that only “small” \( m \)-cycles can exist; however, the class of small cycles for the \( px + (p - 2)r \) problem does contain several \( m \)-cycles. The \( 3x + 5 \) problem has for instance the 1-cycles \((1, 1, 4)\) and \((19, 31, 49, 76, 38)\), the 2-cycle \((23, 37, 58, 29, 46)\) and the 6-cycle \((187, \ldots, 427, \ldots, 1091, \ldots, 1847, \ldots, 781, \ldots, 374)\) with period 27. From such calculations a lower bound can be derived for the continued fraction approximation, and Laurent’s theorem gives an upper bound for the period length of any possible \( m \)-cycle, \( m \) fixed.

**Remark 5.** The remark of Lagarias about the weakness of the result remains valid.

**Acknowledgments**

The author wishes to thank Dr. B.M.M. de Weger (University of Eindhoven), Prof. Dr. R. Tijdeman (University of Leiden), and the referee for valuable comments on an earlier version.

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