OPTIMAL ERROR ESTIMATE
OF THE PENALTY FINITE ELEMENT
METHOD
FOR THE TIME-DEPENDENT
NAVIER-STOKES EQUATIONS

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Abstract. A fully discrete penalty finite element method is presented for the
two-dimensional time-dependent Navier-Stokes equations. The time discretiza-
tion of the penalty Navier-Stokes equations is based on the backward Euler
scheme; the spatial discretization of the time discretized penalty Navier-Stokes
equations is based on a finite element space pair \((X_h, M_h)\) which satisfies some
approximate assumption. An optimal error estimate of the numerical velocity
and pressure is provided for the fully discrete penalty finite element method
when the parameters \(\epsilon, \Delta t, h\) are sufficiently small.

1. Introduction

In this article, we consider the time-dependent Navier-Stokes equations
\[
\begin{align*}
\frac{u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= f, \quad \text{div} u = 0, \quad (x, t) \in \Omega \times (0, T], \\
u u &= 0, \quad (x, t) \in \partial \Omega \times (0, T], u(x, 0) = u_0(x), \quad x \in \Omega,
\end{align*}
\]
where \(\Omega\) is an open bounded set in \(\mathbb{R}^2\) with a smooth boundary \(\partial \Omega\) being of class
\(C^2\), or \(\Omega\) is a plane convex polygon, \(u = u(x, t) = (u_1(x, t), u_2(x, t))\) represents the
velocity vector of a viscous incompressible fluid, \(p = p(x, t)\) the pressure, \(f = f(x, t)\)
the prescribed body force, \(u_0(x)\) the initial velocity, \(\nu > 0\) the viscosity, and \(T > 0\)
a finite time.

We note that the velocity \(u\) and the pressure \(p\) in (1.1)-(1.2) are coupled to-
gether by the incompressibility constraint “\(\text{div} u = 0\)”, which makes the system
difficult to solve numerically. A popular strategy to overcome this difficulty is to
relax the incompressibility constraint in an appropriate way, resulting in a class of
pseudo-compressibility methods, among which are the penalty method, the artifi-
cial compressibility method, the pressure stabilization method and the projection
method (see for instance \([3, 4, 5, 7, 8, 9, 12, 14, 16, 17, 18, 19, 20, 21]\)).

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where in (1.3)-(1.4) can be eliminated to obtain a penalty system of

\begin{equation}
\frac{\partial u}{\partial t} - \nu \Delta u + B(u, u_e) + \nabla p_e = f, \quad \text{div} u_e + \frac{\epsilon}{\nu} p_e = 0, \quad (x, t) \in \Omega \times (0, T],
\end{equation}

(1.4) \quad u_e = 0, \quad (x, t) \in \partial \Omega \times (0, T], \quad u_e(x, 0) = u_0(x), \quad x \in \Omega,

where \( B(u, v) = (u \cdot \nabla) v + \frac{\epsilon}{\nu} (\text{div} u) v \) is the modified bilinear term, introduced by Temam \[19\] to ensure the dissipativity of equations (1.3)-(1.4). We note also that \( p_e \) in (1.3)-(1.4) can be eliminated to obtain a penalty system of \( u_e \) only, which is much easier to solve than the original equations (1.1)-(1.2). Hence the penalty method has been widely used in many areas of computational fluid dynamics (see for instance \[1, 13\]). It is well known \[19\] that \( \lim_{\epsilon \to 0} (u_e(t), p_e(t)) = (u(t), p(t)) \), the solution of (1.1)-(1.2). It has also been known \[2\] that the attractors generated by the penalty equations (1.3)-(1.4) converge to the attractor of the Navier-Stokes equations (1.1)-(1.2). The error bound of \( (u_e, p_e) \) to \( (u, p) \), to the author’s knowledge, has been provided by Shen \[17\] and Huang and Li \[12\] and the error bound is

\begin{equation}
\sup_{0 \leq t \leq T} \| u(t) - u_e(t) \|_{L^2} + \left( \int_0^T \left( \| u - u_e \|_{H^1}^2 + \| p - p_e \|_{L^2}^2 \right) dt \right)^{1/2} \leq \kappa \epsilon^{1/2},
\end{equation}

where \( \kappa > 0 \) is a general positive constant depending on the data \((\nu, u_0, f, \Omega, T)\), which may stand for different values at its different occurrences.

However, the best error estimate available, to the author’s knowledge, is

\begin{equation}
\sup_{0 \leq t \leq T} (\tau^{1/2}(t) \| u(t) - u_{eh}(t) \|_{L^2} + \tau(t) \| u(t) - u_{eh}(t) \|_{H^1})
\end{equation}

(1.5)

\begin{equation}
+ \left( \int_0^T \tau^2(t) \| p(t) - p_{eh} \|_{L^2}^2 dt \right)^{1/2} \leq \kappa \epsilon,
\end{equation}

where \( \tau(t) = \min\{t, 1\} \); the reader can refer to Shen \[16\] for the detail. Furthermore, when the backward Euler scheme is applied to the penalty Navier-Stokes equations (1.3)-(1.4), Shen \[16\] has provided the following optimal error estimate:

\begin{equation}
\sup_{1 \leq n \leq N} (\tau^{1/2}(t_n) \| u(t_n) - u_n^{eh} \|_{L^2} + \tau(t_n) \| u(t_n) - u_n^{eh} \|_{H^1})
\end{equation}

(1.6)

\begin{equation}
+ \left( \Delta t \sum_{n=1}^N \tau^2(t_n) \| p(t_n) - p_n^{eh} \|_{L^2}^2 \right)^{1/2} \leq \kappa (\epsilon + \Delta t),
\end{equation}

where \( 0 < \Delta t < 1 \) is the time-step size, \( t_n = n\Delta t, \quad t_N = T, \quad (u_n^{eh}, p_n^{eh}) \) is an approximation of \( (u, p) \) at time \( t_n \).

In this paper, we aim to extend the work of Shen \[16\] to the case of a fully discrete penalty finite element method for the time-dependent Navier-Stokes equations and provide the optimal error estimates for the penalty finite element solution \( (u_n^{eh}, p_n^{eh}) \).

Under the assumption (A1) about the data \((u_0, f)\) and the assumption (A2) about the finite element space pair \((X_h, M_h)\), we provide the following optimal error estimate:

\begin{equation}
\sup_{1 \leq n \leq N} \tau(t_n) \| u(t_n) - u_n^{eh} \|_{H^1} + \left( \Delta t \sum_{n=1}^N \tau^2(t_n) \| p(t_n) - p_n^{eh} \|_{L^2}^2 \right)^{1/2}
\end{equation}

(1.7)

\begin{equation}
\leq \kappa (\epsilon + h + \Delta t),
\end{equation}

for sufficient small \( \epsilon, \Delta t \) and \( h \).
The remainder of the paper is organized as follows. In the next section, we introduce some notation and preliminary results for the time-dependent penalty Navier-Stokes equations (1.3)-(1.4). In §3, we provide some regularity results for the time discretized penalty Navier-Stokes equations with the Euler backward scheme. The fully discrete finite element method of the penalty Navier-Stokes equations (1.3)-(1.4) is presented in §4 and some boundedness results of the numerical solution \((u^n_h, p^n_h), 1 \leq n \leq N\), are provided in this section. The optimal error estimate is obtained for the fully discrete penalty finite element method in §5.

2. Preliminaries

In this section, we aim to describe some of the notation and results which will be frequently used in this paper. For the mathematical setting of the Navier-Stokes equations (1.1)-(1.2) and the penalty Navier-Stokes equations (1.3)-(1.4), we introduce the Hilbert spaces

\[ X = H^1_0(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = \{q \in L^2(\Omega); \int_\Omega q dx = 0\}. \]

The spaces \(L^2(\Omega)^m, m = 1, 2, 4\), are endowed with the \(L^2\)-scalar product and \(L^2\)-norm denoted by \((\cdot, \cdot)\) and \(\| \cdot \|_0\), respectively. The space \(X\) is equipped with the usual scalar product \((\nabla u, \nabla v)\) and norm \(\|\nabla u\|_0\).

We define \(Au = -\Delta u\) and \(A_e u = -\Delta u - \frac{1}{\epsilon} \nabla \text{div} u\), which are the operators associated with the Navier-Stokes equations and the penalty Navier-Stokes equations. They are the positive self-adjoint operators from \(D(A) = H^2(\Omega)^2 \cap X\) onto \(Y\) and the powers \(A^\alpha\) and \(A^\alpha_e\) of \(A\) and \(A_e\) (\(\alpha \in R\)) are well defined. In particular,

\[(A^{1/2}u, A^{1/2}v) = (\nabla u, \nabla v), \quad (A^1_e u, A^{1/2}v) = (A^{1/2}u, A^{1/2}v) + \frac{1}{\epsilon}(\text{div} u, \text{div} v)\]

hold for all \(u, v \in X\).

It is well known that the following Gagliardo-Nirenberg inequalities hold:

\[
\begin{align*}
(2.1) \quad \|v\|_{L^4} & \leq c \|v\|_{H^1} \|A^{1/2}v\|_{L^2}^{1/2}, \quad \|v\|_0 \leq c \|A^{1/2}v\|_0, \quad \forall v \in X, \\
(2.2) \quad \|\nabla v\|_{L^4} & \leq c \|A^{1/2}v\|_0 \|Av\|_0^{1/2}, \quad \|A^{1/2}v\|_0 \leq c \|Av\|_0, \quad \forall v \in D(A), \\
(2.3) \quad \|v\|_{L^\infty} & \leq c \|v\|_0 \|Av\|_0^{1/2}, \quad \forall v \in D(A),
\end{align*}
\]

where \(c\) is a general positive constant depending only on \(\Omega\), which may stand for different values at its different occurrences.

Furthermore, we recall the following lemma given in [2] [10].

**Lemma 2.1.** There exists a constant \(c_0 > 0\) depending only on \(\Omega\) and such that if \(\epsilon_0 \leq 1\),

\[
(2.4) \quad \|Av\|_0 \leq c_0 \|A_ev\|_0, \quad \|A^{1/2}v\|_0 \leq c_0 \|A^{1/2}v\|_0.
\]

As for the time-dependent Navier-Stokes equations (1.1)-(1.2) and the time-dependent penalty Navier-Stokes equations (1.3)-(1.4), we define the continuous bilinear forms

\[
(u, v) = \nu(A^{1/2}u, A^{1/2}v), \quad a_e(u, v) = \nu(A^1_e u, A^{1/2}v), \quad \forall u, v \in X,
\]

\[
d(v, q) = (\text{div} v, q), \quad \forall v \in X, \quad q \in M,
\]
respectively. We also introduce a continuous trilinear form on \( X \times X \times X \)
\[
b(u,v,w) = \langle B(u,v),w \rangle_{X'} = ((u \cdot \nabla)v,w) + \frac{1}{2}((\text{div}u)v,w)
\]
\[
= \frac{1}{2}((u \cdot \nabla)v,w) - \frac{1}{2}((u \cdot \nabla)w,v) \forall u,v,w \in X.
\]
It is easy to verify that \( b \) satisfies the following important property:
(2.5)
\[
b(u,v,w) = -b(u,w,v), \forall u,v,w \in X.
\]
We usually make the following assumption on the prescribed data \((u_0,f)\): 

(A1) The initial velocity \( u_0(x) \in D(A) \) with \( \text{div}u_0 = 0 \) and the forcing function \( f(x,t) \in H^{1,\infty}(0,T;Y) \) satisfy
\[
\|Au_0\|_0 + \sup_{t \in [0,T]} \{ \|f(t)\|_0 + \|f_t(t)\|_0 \} \leq C,
\]
for some positive constant \( C \).

With the above notation, the Navier-Stokes formulation related to (1.1)–(1.2) and the penalty Navier-Stokes formulation related to (1.3)–(1.4) are defined, respectively, as follows: find \((u,p) \in L^\infty(0,T;X) \cap L^2(0,T;X) \times L^2(0,T;M)\) such that
(2.6) \[
( u_t + a(u,v) - d(\nu,v) + d(u,u), v ) = (f,v), \forall (v,q) \in (X,M),
\]
and find \((u_\epsilon,p_\epsilon) \in L^\infty(0,T;Y) \cap L^2(0,T;X) \times L^2(0,T;M)\) such that for all \((v,q) \in X \times M\)
(2.7) \[
( u_{\epsilon t} + a(u,\epsilon) - d(\nu,\epsilon) + d(u,\epsilon), \epsilon ) + \frac{\epsilon}{\nu} (p_\epsilon,q) + b(u,\epsilon,\epsilon,\epsilon) = (f,v),
\]
with the initial conditions \( u(0) = u_0 \) and \( u_\epsilon(0) = u_0 \), respectively.

Now, let us consider the time discretization of the penalized Navier-Stokes formulation (2.7) by the backward Euler scheme
(2.8) \[
( d_t u^n_\epsilon + a(u^n_\epsilon,v) - d(\nu,v) + d(u^n_\epsilon,u^n_\epsilon), v ) + \frac{\epsilon}{\nu} (p^n_\epsilon,v) + b(u^n_\epsilon,u^n_\epsilon,v) = (f(t_n),v),
\]
for all \((v,q) \in X \times M\) and \( 1 \leq n \leq N \), where \( 0 < \Delta t < 1 \) is the time-step size, \( t_n = n\Delta t \), \( t_N = T \), \((u^n_0,p^n_0) = (u_0,0)\) and \( d_t u^n_\epsilon = \frac{\Delta t}{\nu} (u^n_\epsilon - u^{n-1}_\epsilon) \) for \( 1 \leq n \leq N \), and \( d_t u_0^n \) is defined to satisfy
\[
( d_t u_0^n + a(u_0,\epsilon) + ((u_0 \cdot \nabla)u_0), v ) = (f(0),v), \forall v \in X \text{ with } \text{div}v = 0.
\]
Hence, by using (2.3), it holds that
(2.9) \[
\|d_t u_0^n\|_0 \leq \nu \|Au_0\|_0 + \|((u_0 \cdot \nabla)u_0)\|_0 + \|f(0)\|_0
\]

Theorem 2.2. Suppose that (A1) and \( c\epsilon_0 \leq 1 \) are valid. The following error estimate holds:
(2.10) \[
\tau^2(t_m) \|A^{1/2}(u(t_m) - u^n_m)\|^2 + \Delta t \sum_{n=1}^m \tau^2(t_n) \|p(t_n) - p^n\|^2 \leq \kappa (\epsilon^2 + \Delta t^2),
\]
for all \( 1 \leq m \leq N \).

We refer to Shen [16] for the proof of this result.

In this paper, we will frequently use a discrete version of the Gronwall lemmas used in [11] [18].
Lemma 2.3. Let $C$ and $a_n, b_n, d_n$, for integers $1 \leq m \leq N$, be nonnegative numbers such that
\begin{equation}
(2.11) \quad a_m + \Delta t \sum_{n=1}^{m} b_n \leq \Delta t \sum_{n=0}^{m-1} a_n d_n + C, \quad \forall 1 \leq m \leq N.
\end{equation}
Assume that $d_n \Delta t \leq \frac{1}{2}$, $\forall 1 \leq n \leq N$. Then
\begin{equation}
(2.12) \quad a_m + \Delta t \sum_{n=1}^{m} b_n \leq C \exp(2\Delta t \sum_{n=1}^{m-1} d_n), \quad \forall 1 \leq m \leq N.
\end{equation}

3. Regularity

In order to consider the error bound of the finite element solution related to the penalty Navier-Stokes formulation (2.8), we need the following regularity of functions $\{u^n_m\}_{n=1}^{N}$ and $\{p^n_m\}_{n=1}^{N}$.

**Theorem 3.1.** Under the assumptions of Theorem 2.2, there is a constant $\kappa_0 > 0$ such that if $\Delta t \kappa_0 \leq 1$, then
\begin{equation}
(3.1) \quad \|A^{1/2} u^n_m\|_0^2 + \Delta t \sum_{n=1}^{m} (\|d_t u^n_m\|_0^2 + \|A_e u^n_m\|_0^2 + \|p^n_m\|_0^2) \leq \kappa,
\end{equation}
\begin{equation}
(3.2) \quad \|d_t u^n_m\|_0^2 + \|A_e u^n_m\|_0^2 + \|p^n_m\|_0^2 + \Delta t \sum_{n=1}^{m} \|A^{1/2} d_t u^n_m\|_0^2 \leq \kappa,
\end{equation}
for all $1 \leq m \leq N$, where $\| \cdot \|_1$ denotes the norm of the Sobolev space $H^1(\Omega)$.

**Proof.** Taking $(v, q) = 2(u^n_m, p^n_m) \Delta t$ in (2.8), using (2.5) and the relation
\begin{equation}
(3.3) \quad 2(u - v, u) = \|u\|_0^2 - \|v\|_0^2 + \|u - v\|_0^2, \quad \forall u, v \in Y,
\end{equation}
we have
\begin{equation}
\|u^n_m\|_0^2 - \|u^{n-1}_m\|_0^2 + 2\nu \|A^{1/2} u^n_m\|_0^2 \Delta t \leq 2\nu \|f(t_n)\|_0 \|u^n_m\|_0 \Delta t.
\end{equation}
Summing this inequality from 1 to $m$ and using (3.1) and the Young inequality, we obtain
\begin{equation}
(3.4) \quad \|u^n_m\|_0^2 + \nu \Delta t \sum_{n=1}^{m} \|A^{1/2} u^n_m\|_0^2 \leq \|u_0\|_0^2 + \nu^{-1} \epsilon \Delta t \sum_{n=1}^{N} \|f(t_n)\|_0^2 \leq \kappa.
\end{equation}

Next, we can derive from (2.8) that
\begin{equation}
(3.5) \quad d_t u^n_m + \nu A_e u^n_m + B(u^n_m, u^n_m) = f(t_n).
\end{equation}
Taking the scalar product of (3.5) with $(\nu^{-1} d_t u^n_m + A_e u^n_m) \Delta t$ in $Y$ and using the relation
\begin{equation}
(3.6) \quad 2(A^{1/2}(u - v), A^{1/2}u) = \|A^{1/2} u\|_0^2 - \|A^{1/2} v\|_0^2 + \|A^{1/2} (u - v)\|_0^2, \quad \forall u, v \in X,
\end{equation}
we get
\begin{equation}
\|A^{1/2} u^n_m\|_0^2 - \|A^{1/2} u^{n-1}_m\|_0^2 + \nu^{-1} \|d_t u^n_m\|_0^2 \Delta t + \nu \|A_e u^n_m\|_0^2 \Delta t
\end{equation}
\begin{equation}
(3.7) \quad + b(u^n_m, u^n_m, \nu^{-1} d_t u^n_m + A_e u^n_m) \Delta t = (f(t_n), \nu^{-1} d_t u^n_m + A_e u^n_m) \Delta t.
\end{equation}
By using (2.3)-(2.4), we have
\[ |b(u^n, u^n_0, u^{n-1}_e, v, u^n_e)| \leq c \|A^{1/2}u^n_0\|_0 \|u^n_0\|_{L^\infty} \|u^{n-1}_e\|_0 + A_e u^n_e \|_0 \]
\[ \leq \frac{1}{4} \|d_t u^n_e\|_0^2 + \frac{\nu}{4} \|A_e u^n_e\|_0^2 + \nu^{-1} c \|u^n_e\|_0^2 \|A^{1/2}u^n_0\|_0^2 \|A^{1/2}u^n_e\|_0^2, \]
\[ \|f(t_n), u^{n-1}_e\|_0 \]\[
\leq \frac{1}{4} \|d_t u^n_e\|_0^2 + \frac{\nu}{4} \|A_e u^n_e\|_0^2 + \nu^{-1} c \|f(t_n)\|_0^2.
\]
Combining these estimates with (3.7) yields
\[ \|A^{1/2}u^n_e\|_0^2 - \|A^{1/2}u^{n-1}_e\|_0^2 + \frac{1}{2} \nu^{-1} \|d_t u^n_e\|_0^2 + \nu \|A_e u^n_e\|_0^2) \Delta t \]
\[ \leq d_n \|A^{1/2}u^n_0\|_0^2 \Delta t + \nu^{-1} c \|f(t_n)\|_0^2 \Delta t,
\]
where \( d_n = \nu^{-1} c(1 + \nu^{-2}) \|u^n_0\|_0^2 \|A^{1/2}u^n_0\|_0^2. \) Summing this inequality from 1 to \( m \) and noting \( A^{1/2}u^0 = A^{1/2}u_0 \), we obtain
\[ \|A^{1/2}u^m\|_0^2 - \|A^{1/2}u^0\|_0^2 + \Delta t \sum_{n=1}^m (\nu^{-1} \|d_t u^n_e\|_0^2 + \nu \|A_e u^n_e\|_0^2) \]
\[ \leq \|A^{1/2}u^0\|_0^2 + \Delta t \sum_{n=1}^m d_n \|A^{1/2}u^n_e\|_0^2 + \nu^{-1} c \Delta t \sum_{n=1}^N \|f(t_n)\|_0^2 \]
\[ \leq \kappa + \Delta t \sum_{n=1}^m d_n \|A^{1/2}u^n_e\|_0^2.
\]
If we choose \( \Delta t \) such that \( d_n \Delta t \leq \frac{1}{2} \), by applying Lemma 2.3 to (3.8), it then holds that
\[ \|A^{1/2}u^m\|_0^2 + \frac{1}{2} \Delta t \sum_{n=1}^m (\nu^{-1} \|d_t u^n_e\|_0^2 + \nu \|A_e u^n_e\|_0^2) \leq k \exp(2 \Delta t \sum_{n=1}^m d_n), \]
for all \( 1 \leq m \leq N \). From (2.3), (3.4) and (3.9), there exists a constant \( \kappa_0 > 0 \) such that
\[ 2 \Delta t \sum_{n=1}^m d_n \leq \kappa, \quad d_n = \nu^{-1} c(1 + \nu^{-2}) \|u^n_0\|_0^2 \|A^{1/2}u^n_0\|_0^2 \leq \frac{1}{2} k_0,
\]
for all \( 1 \leq m \leq N \).

Moreover, we derive from (2.3) that
\[ (d_t u^n_e, v) + a_e d_t u^n_e + b d_t u^n_e, u^n_e, v) + b(u^{n-1}_e, d_t u^n_e, v) \]
\[ = \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} f_e(t) dt, v \right), \quad \forall v \in X.
\]
By taking \( v = 2d_t u^n_e \Delta t \) in (3.11) and using (2.5) and (3.3), we get
\[ \|d_t u^n_e\|_0^2 - \|d_t u^{n-1}_e\|_0^2 + 2\nu \|A^{1/2}d_t u^n_e\|_0^2 \Delta t + 2b d_t u^n_e, u^n_e, d_t u^n_e \Delta t \]
\[ \leq 2 \left( \int_{t_{n-1}}^{t_n} f_e(t) dt, d_t u^n_e \right). \]
Due to (2.11) and (2.14), we have
\[
2|b(d_tu^n, u^n_c, d_tu^n)| \leq c|d_tu^n||A^{1/2}d_tu^n||A^{1/2}u^n_n||0 + c|d_tu^n||A^{1/2}d_tu^n||A^{3/2}u^n_n||0\frac{1}{2} \leq \frac{\nu}{2}A^2d_tu^n_0^2 + d_n|d_tu^n|^2 .
\]
Combining these inequalities with (3.12) yields
\[
\|d_tu^n_0^2 \| d_eu^n_0 + \nu|A^2d_tu^n|\frac{2}{0} + \nu^{-1}|f(t_n)||0\)
(3.13)
\[
\leq d_n|d_tu^n|_0^2 + \nu^{-1}|f(t_n)|_0^2 dt .
\]
Summing (3.13) from 1 to m and using (2.3), (3.9) - (3.10), we get
\[
(3.14) \|d_tu^n_0^2 + \nu\Delta t \sum_{n=1}^{m} |A^{1/2}d_tu^n|_0^2 \leq \kappa, \forall 1 \leq m \leq N .
\]
Finally, we derive from (2.8), (3.5) and the inf-sup condition [7] that
\[
(3.15) \nu|A^n u_c^n|_0 \leq \|d_tu^n_0 + \|B(u^n_c, u^n_c)|_0 + \|f(t_n)|_0,
\]
(3.16) \[
\|p^n_1 \|_1 \leq c|d_tu^n_0|_0 + \nu c|A^n u_c^n|_0 + \nu c|B(u^n_c, u^n_c)|_0 + \|f(t_n)|_0.
\]
Using (2.3), (2.4), we have
\[
\|B(u^n_c, u^n_c)|_0 \leq c|A^{1/2}u^n_c|_0 u^n_c|_L^\infty \leq c|A^{1/2}u^n_c|_0 u^n_c|_0^{1/2} \leq \nu \frac{1}{2}|A^n u^n_c|_0 + \nu^{-1}|u^n_c|_0 A^{1/2}u^n_0^2 .
\]
Combining these inequalities with (3.15) and (3.16) and using (2.4), we get
\[
(3.17) \nu|A^n u_c^n|^2 \leq \nu^{-1}|d_tu^n_0|^2 + \nu^{-3}|c|u^n_c|^2_0 A^{1/2}u^n_0^2 + \nu^{-1} c|f(t_n)|^2_0 ,
\]
(3.18) \[
\|p^n_1 \|^2 \leq c\|d_tu^n_0^2 + \nu^2|A^n u_c^n|^2_0 + \nu^{2-2}|u^n_c|^2_0 A^{1/2}u^n_0^4 + \|f(t_n)|^2_0 ,
\]
\[
\Delta t \sum_{n=1}^{m} |p^n_1|^2 \leq c\Delta t \sum_{n=1}^{m} (\|d_tu^n_0^2 + \nu^2|A^n n_c^n|^2_0 )
\]
\[
+ c\Delta t \sum_{n=1}^{m} (\nu^{-2}|u^n_c|^2_0 A^{1/2}u^n_0^4 + \|f(t_n)|^2_0 ) .
\]
Combining (3.17) - (3.19) with (3.9) - (3.10) and (3.14), we completed the proof of Theorem 3.1. \]
\[
(3.20) \tau(t_m)|A^{1/2}d_tu^n_m|^2_0 + \Delta t \sum_{n=1}^{m} \tau(t_n)(|d_tu^n_0|^2_0 + |A_c d_tu^n_c|^2_0 + |d_tu^n|^2_0) \leq \kappa ,
\]
for all 1 \leq m \leq N.
Proof. Taking $v = 2\nu^{-1}d_t u^n \Delta t$ in (3.11) yields
\begin{align}
2(A^{1/2}_{t} d_t u^n, A^{1/2}_{t} (d_t u^n - d_t u^{n-1})) + 2\nu^{-1} ||d_t u^n||_0^2 \Delta t + 2\nu^{-1} b(d_t u^n, u^n, d_t u^n) \Delta t
(3.21)
+ 2\nu^{-1} b(u^n, d_t u^n, d_t u^n) \Delta t = 2\nu^{-1} \left( \int_{t_{n-1}}^{t_n} f_t(t) dt, d_t u^n \right).
\end{align}
Due to (2.1)-(2.4), we have
\begin{align}
2\nu^{-1} |b(d_t u^n, u^n, d_t u^n)| + 2\nu^{-1} |b(d_t u^n, u^n, d_t u^n)|
\leq c\nu^{-1} ||A^{1/2}_{t} d_t u^n||_0 ||A u^n||_0 ||d_t u^n||_0
\leq \frac{1}{4\nu} ||d_t u^n||_0^2 + \nu^{-1} c ||A u^n||_0^2 ||A^{1/2}_{t} d_t u^n||_0^2,
\end{align}

\begin{align}
2\nu^{-1} \left( \int_{t_{n-1}}^{t_n} f_t(t) dt, d_t u^n \right)
\leq \frac{1}{4\nu} ||d_t u^n||_0^2 \Delta t + \nu^{-1} c \int_{t_{n-1}}^{t_n} \|f_t(t)\|_0^2 dt.
\end{align}
Combining these inequalities with (3.21) yields
\begin{align}
2\tau(t_n)(A^{1/2}_{t} d_t u^n, A^{1/2}_{t} (d_t u^n - d_t u^{n-1})) + \nu^{-1} \tau(t_n) ||d_t u^n||_0^2 \Delta t
(3.22)
\leq \nu^{-1} c ||A u^n||_0^2 ||A^{1/2}_{t} d_t u^n||_0^2 + \nu^{-1} c \int_{t_{n-1}}^{t_n} \|f_t(t)\|_0^2 dt.
\end{align}
Summing (3.22) from 1 to $m$ and noting
\begin{align}
2\tau(t_n)(A^{1/2}_{t} d_t u^n, A^{1/2}_{t} (d_t u^n - d_t u^{n-1}))
\geq \tau(t_n) ||A^{1/2}_{t} d_t u^n||_0^2 - \tau(t_{n-1}) ||A^{1/2}_{t} d_t u^{n-1}||_0^2
- ||A^{1/2}_{t} d_t u^{n-1}||_0^2 \Delta t, \quad 2 \leq n \leq N,
2\tau(t_1)(A^{1/2}_{t} d_t u^n, A^{1/2}_{t} (d_t u^n - d_t u^n))
\geq \tau(t_1) ||A^{1/2}_{t} d_t u^n||_0^2 - ||A u^n - Au_0||_0 ||d_t u^n||_0,
\end{align}
we obtain
\begin{align}
\tau(t_m) ||A^{1/2}_{t} d_t u^m||_0^2 + \nu^{-1} \Delta t \sum_{n=1}^{m} \tau(t_n) ||d_t u^n||_0^2
\leq ||A u^1 - Au_0||_0 ||d_t u^0||_0 + \Delta t \sum_{n=1}^{m} ||A^{1/2}_{t} d_t u^n||_0^2
(3.23)
+ \nu^{-1} c \Delta t \sum_{n=1}^{m} ||A u^n||_0^2 ||A^{1/2}_{t} d_t u^n||_0^2 + \nu^{-1} c \int_{0}^{T} \|f_t(t)\|_0^2 dt.
\end{align}
Combining (3.23) with (3.2) yields
\begin{align}
\tau(t_m) ||A^{1/2}_{t} d_t u^m||_0^2 + \nu^{-1} \Delta t \sum_{n=1}^{m} \tau(t_n) ||d_t u^n||_0^2 \leq \kappa, \quad \forall 1 \leq m \leq N.
(3.24)
\end{align}
Finally, we derive from (2.8) and the inf-sup condition [7] that
\begin{align}
\nu ||A d_t u^n||_0 \leq ||d_t u^n||_0 + ||B(d_t u^n, u^n)||_0 + ||B(u^n, d_t u^n)||_0 + ||f_t(t)||_0,
\end{align}
\begin{align}
||d_t u^n||_1 \leq c ||d_t u^n||_0 + \nu c ||A d_t u^n||_0 + c ||B(d_t u^n, u^n)||_0
+ c ||B(u^n, d_t u^n)||_0 + c ||f(t)||_0.
(3.26)
\end{align}
Using (2.1)-(2.4), we have
\[ \| B(d_t u^n, u^n) \|_0 + \| B(d_t u^n, u^n) \|_0 \leq c \| A_r u^n \|_0 \| A^{1/2}_t d_t u^n \|_0. \]
Combining this inequality with (3.25) and (3.26), we get
\[ \nu^2 \Delta t \sum_{n=1}^{m} \tau(t_n) \| A_r d_t u^n \|_0^2 \leq c \Delta t \sum_{n=1}^{m} \tau(t_n) \| d_t u^n \|_0^2 \]
\[ + \ c \Delta t \sum_{n=1}^{m} \| A_r u^n \|_0^2 \| A_r^{1/2} d_t u^n \|_0^2 + c \Delta t \sum_{n=1}^{m} \| f(t_n) \|_0^2, \]
(3.27)
\[ \Delta t \sum_{n=1}^{m} \tau(t_n) \| d_t p^n \|_1^2 \leq c \Delta t \sum_{n=1}^{m} \tau(t_n) \| d_t u^n \|_0^2 + \nu^2 \| A_r d_t u^n \|_0^2 \]
\[ + \ c \Delta t \sum_{n=1}^{m} \| A_r u^n \|_0^2 \| A_r^{1/2} d_t u^n \|_0^2 + \| f(t_n) \|_0^2, \]
(3.28)
Combining (3.27)-(3.28) with (3.24) and using (5.2), we have completed the proof of Theorem 3.2. \qed

4. Finite element penalty method of the Navier-Stokes equations

Let \( h > 0 \) be a real positive parameter. The finite element space pair \((X_h, M_h)\) of \((X, M)\) is characterized by \( J_h = J_h(\Omega) \), a partitioning of \( \Omega \) into triangles \( K \), assumed to be uniformly regular as \( h \to 0 \). For further details, the reader can refer to Ciarlet [6] and Girault and Raviart [7].

Let \( \rho_h : M \to M_h \) denote the \( L^2 \)-orthogonal projections defined by
\[ (\rho_h q, q_h) = (q, q_h), \ \forall q \in M, \ \rho_h \in M_h. \]
For the finite element space pair \((X_h, M_h)\), we will make the following assumption.

(A2) There exists a mapping \( r_h : D(A) \cap X \to X_h \) such that
\[ (\text{div}(u - r_h u), q_h) = 0, \ \forall q_h \in M_h, \]
\[ \| A^{1/2}(u - r_h u) \|_0 \leq c h \| Au \|_0, \ \| p - \rho_h p \|_0 \leq c h \| p \|_1, \ \forall p \in H^1(\Omega) \cap M, \]
and the inverse inequality
\[ \| A^{1/2} v_h \|_0 \leq c h^{-1} \| v_h \|_0, \ \forall v_h \in X_h, \]
holds as well as the discrete inf-sup condition
\[ \| q_h \|_0 \leq c \sup_{v_h \in X_h} \frac{(\text{div} v_h, q_h)}{\| A^{1/2} v_h \|_0}, \ \forall q_h \in M_h. \]

Example 4.1 (Girault-Raviart [7]). If we set
\[ X_h = \{ v_h \in C^0(\Omega)^2 \cap X; \ v_h|_K \in P_2(K)^2, \ \forall K \in J_h(\Omega) \}, \]
\[ M_h = \{ q_h \in M; \ q_h|_K \in P_0(K), \ \forall K \in J_h(\Omega) \}, \]
then \((X_h, M_h)\) satisfies (A2).
Now, we consider the finite element discretization of (2.8). We define \( \{ u^n_ch \}_{n=1}^N \subset X_h \) and \( \{ p^n_ch \}_{n=1}^N \subset M_h \) as the finite element approximations of \( \{ u^n \}_{n=1}^N \subset X \) and \( \{ p^n_e \}_{n=1}^N \subset M \), which satisfy the recursive linear equation

\[
(d_t u^n_ch, v_h) + a(u^n_ch, v_h) - d(v_h, p^n_ch) + d(u^n_ch, q_h) + \frac{c}{\nu} (p^n_ch, q_h) + b(u^n_ch, u^n_ch, v_h) \quad (4.5)
\]

where \( u^n_ch = r_h u_0, \; p^n_ch = 0 \).

**Theorem 4.1.** Assume that (A1), (A2) and \( cc_0 \leq 1, \; \Delta t \kappa_0 \leq 1 \) are valid. Then it holds that

\[
(4.6) \quad \| u^n_ch \|_0^2 + \Delta t \sum_{n=1}^m \| A^{1/2} u^n_ch \|_0^2 \leq \kappa,
\]

\[
(4.7) \quad \| A^{1/2} u^n_ch \|_0^2 + \Delta t \sum_{n=1}^m \| d_t u^n_ch \|_0^2 \leq \kappa + \kappa h^{-2} \Delta t \sum_{n=1}^m \| A^{1/2} (u^n_ch - u^n_ch) \|_0^2 \| u^n_ch \|_0^2,
\]

for all \( 1 \leq m \leq N \).

**Proof.** The proof of (4.6) is exactly similar to that of (3.4) in the proof of Theorem 3.1. It can be omitted. Moreover, we can obtain from (4.5) that

\[
(4.8) \quad (d_t u^n_ch, v_h) + a(u^n_ch, v_h) - d(v_h, p^n_ch)
\]

\[
+ d(d_t u^n_ch, q_h) + \frac{c}{\nu} (d_t p^n_ch, q_h) + b(u^n_ch, u^n_ch, v_h)
\]

\[
= (f(t_n), v_h), \; \forall (v_h, q_h) \in (X_h, M_h).
\]

Taking \( (v_h, q_h) = 2(d_t u^n_ch, p^n_ch) \Delta t \) in (4.8) and using (5.6), we obtain

\[
(4.9) \quad 2 \| d_t u^n_ch \|_0^2 + \nu \| A^{1/2} u^n_ch \|_0^2 - \nu \| A^{1/2} u^n_ch - u^n_ch \|_0^2 + \frac{c}{\nu} \| p^n_ch \|_0^2 \| p^n_ch - p^n_ch \|_0^2
\]

By using (2.1)–(2.3), (2.5) and (4.3), we have

\[
2 | b(u^n_ch, u^n_ch, d_t u^n_ch) | \leq 2 | b(u^n_ch - u^n_ch, u^n_ch, d_t u^n_ch) | + 2 | b(u^n_ch, u^n_ch, d_t u^n_ch) |,
\]

\[
2 | b(u^n_ch, u^n_ch, d_t u^n_ch) | \leq c \| Au^n_ch \|_0 \| A^{1/2} u^n_ch \|_0 \| d_t u^n_ch \|_0
\]

\[
\leq \frac{1}{4} \| d_t u^n_ch \|_0^2 + c \| Au^n_ch \|_0^2 \| A^{1/2} u^n_ch \|_0^2
\]

\[
2 | b(u^n_ch - u^n_ch, u^n_ch, d_t u^n_ch) | \leq c \| u^n_ch - u^n_ch \|_0 \| u^n_ch \|_0 \| d_t u^n_ch \|_0
\]

\[
+ c \| A^{1/2} (u^n_ch - u^n_ch) \|_0 \| u^n_ch \|_0 \| A^{1/2} d_t u^n_ch \|_0
\]

\[
\leq \frac{1}{4} \| d_t u^n_ch \|_0^2 + c \| f(t_n) \|_0^2
\]

\[
2 \| (f(t_n), d_t u^n_ch) \| \leq \frac{1}{4} \| d_t u^n_ch \|_0^2 + c \| f(t_n) \|_0^2.
\]
Combining these estimates with (4.9) yields

\[
\nu A^{1/2} u_h^n \quad - \quad \nu A^{1/2} u_h^{n-1} \quad + \quad d_t u_h^n \quad + \quad \frac{c}{\nu} (||p_h^n||_0^2 - ||p_h^{n-1}||_0^2) \\
\leq c |A| u_h^n \quad + \quad \nu A^{1/2} u_h^n \quad + \quad \frac{c}{\nu} (||p_h^n||_0^2 - ||p_h^{n-1}||_0^2) \\
+ \quad ||u_h^n - u_h^{n-1}||_0^2 + \frac{c}{\nu} ||u_h^n||_0^2 \Delta t + c \|f(t_n)||_0^2 \Delta t.
\]

(4.10)

Summing this inequality from 1 to \( m \) and using Theorem 3.1 and (2.4), we obtain

In order to derive the error estimates of the finite element penalty method, we also need the Galerkin projection \( R_h : (X, M) \rightarrow X_h \), \( Q_h : (X, M) \rightarrow M_h \) defined by

\[
a(u - R_h(u, p), v_h) - d(v_h, p - Q_h(u, p)) + d(u - R_h(u, p), q_h) \\
+ \frac{c}{\nu} (p - Q_h(u, p), q_h) = 0, \quad \forall (v_h, q_h) \in (X_h, M_h),
\]

(4.11)

for all \( (u, p) \in (X, M) \) with \( \text{div} u + \frac{c}{\nu} p = 0 \).

By using a similar argument to that used by Layton and Tobiska in [15], the following approximate properties can be proved.

**Lemma 4.2.** Under the assumptions of Theorem 3.1, the Galerkin projection \((R_h, Q_h)\) satisfies

\[
||u - R_h(u, p)||_0 + h||A^{1/2}(u - R_h(u, p)||_0 + h||p - Q_h(u, p)||_0 \\
\leq c_\nu h^2(||A^{1/2} u||_0 + ||p||_0), \forall (u, p) \in (X, M),
\]

(4.12)

with \( \text{div} u + \frac{c}{\nu} p = 0 \), and

\[
||u - R_h(u, p)||_0 + h||A^{1/2}(u - R_h(u, p)||_0 + h||p - Q_h(u, p)||_0 \\
\leq c_\nu h^2(||A^{1/2} u||_0 + ||p||_0), \forall (u, p) \in (X, M) \cap M, 
\]

(4.13)

with \( \text{div} u + \frac{c}{\nu} p = 0 \), where \( c_\nu \) is a general positive constant depending only on \( \Omega \) and \( \nu \), which may stand for different values at its different occurrences.

**Proof.** The stability of the Galerkin projection follows simply by (4.3) and (4.11), namely

\[
\nu||A^{1/2}R_h(u, p)||_0^2 \leq \frac{\nu}{2} ||A^{1/2}R_h(u, p)||_0^2 + c_\nu (||A^{1/2} u||_0^2 + ||p||_0^2),
\]

(4.14)

\[
||Q_h(u, p)||_0 \leq c_\nu (||A^{1/2} R_h(u, p)||_0 + ||A^{1/2} u||_0 + ||p||_0).
\]

(4.15)

Now (4.14) and (4.15) and the triangle inequality give

\[
||A^{1/2}(u - R_h(u, p))||_0 + ||p - Q_h(u, p)||_0 \leq c_\nu (||A^{1/2} u||_0 + ||p||_0),
\]

(4.16)

for all \( (u, p) \in (X, M) \) with \( \text{div} u + \frac{c}{\nu} p = 0 \).

Next, we introduce the dual problem: find \((\Phi, \Psi) \in (X, M)\) such that for all \((v, q) \in (X, M)\)

\[
a(v, \Phi) + d(v, \Psi) - d(\Phi, q) + \frac{c}{\nu} (q, \Psi) = (v, u - R_h(u, p)).
\]

(4.17)

In view of Lemma 2.1, we can prove that problem (4.17) admits a unique solution \((\Phi, \Psi)\) satisfying

\[
||A\Phi||_0 + ||\Psi||_1 \leq c_\nu ||u - R_h(u, p)||_0.
\]

(4.18)
Proof. Subtracting (4.5) from (2.8) with (4.21) holds for all \( n \), we find
\[
|u - R_h(u, p)|^2 = a(u - R_h(u, p), \Phi - r_h \Phi) + d(u - R_h(u, p), \Psi - q_h \Psi) - d(\Phi - r_h \Phi, p - Q_h(u, p)) + \frac{\epsilon}{\nu} (p - Q_h(u, p), \Psi - q_h \Psi)
\]
(4.19) \( \leq c_v h \|A^{1/2}(u - R_h(u, p))\|_0 + \|p - Q_h(u, p)\|_0 (\|A\Phi\|_0 + \|\Psi\|_1) \]
\( \leq c_v h \|A^{1/2}(u - R_h(u, p))\|_0 + \|p - Q_h(u, p)\|_0 \|u - R_h(u, p)\|_0. \)

Combining (4.19) with (4.10) yields (4.12). Let \((u, p) \in (D(A), H^1(\Omega) \cap M)\) with \( \text{div} u + \frac{\epsilon}{\nu} p = 0 \). Then, we derive from (4.11) and (4.2) that
\[
\|A^{1/2}(r_h u - R_h(u, p))\|_0 + \|\rho_h p - Q_h(u, p)\|_0 \leq c_v \|A^{1/2}(u - r_h u)\|_0 \|\rho_h p - p\|_0 \leq c_v h (\|Au\|_0 + \|p\|_1).
\]
Thus the triangles inequality and (4.2) give
\[
(4.20) \quad \|A^{1/2}(u - R_h(u, p))\|_0 + \|p - Q_h(u, p)\|_0 \leq c_v h (\|Au\|_0 + \|p\|_1).
\]
It now follows from (4.20) and (4.19) that
\[
(4.21) \quad \|u - R_h(u, p)\|_0 \leq c_v h^2 (\|Au\|_0 + \|p\|_1).
\]
Thus, (4.20) and (4.21) imply (4.13). \( \square \)

5. Optimal error analysis

In this section, our aim is to estimate some bounds for the error \((u^n_h - u^n, p^n_h - p^n)\)
and then to obtain the optimal bound of the error \((u(t_n) - u^n_h, p(t_n) - p^n_h)\).

Lemma 5.1. Under the assumptions of Theorem 4.1, the error estimate
\[
\|u^n_h - u^n\|^2_0 + \Delta t \sum_{n=1}^m \|A^{1/2}(u^n_h - u^n)\|^2_0 \leq kh^2
\]
holds for all \( 1 \leq m \leq N. \)

Proof. Subtracting (4.1) from (2.8) with \((v, q) = (v_h, q_h)\) and using (4.1), we obtain
\[
(\text{I} - r_h) d_t u^n_h + d \epsilon^n + v_h) + a(I - r_h) u^n_h + e^n, v_h)
\]
\( - d(v_h, (I - \rho_h)p^n + \eta^n) + d\epsilon^n, q_h)
\]
\( + b((I - r_h) u^n + e^n, u^n, v_h) + b(u^n_h, (I - r_h) u^n + e^n, v_h)
\]
\( + \frac{\epsilon}{\nu}((I - \rho_h)p^n + \eta^n, q_h) = 0, \quad \forall (v_h, q_h) \in (X_h, M_h), \)

where \((e^n, \eta^n) = (r_h u^n_e - u^n_e, \rho_h p^n_e - p^n_e). \) Taking \((v_h, q_h) = 2(e^n, \eta^n) \Delta t \) in (5.2) and using (2.5) and (3.3), we get
\[
\|e^n\|^2_0 + \|e^{n-1}\|_0^2 + 2\nu \|A^{1/2} e^n\|^2_0 \Delta t + \frac{\epsilon}{\nu} \|\eta^n\|_0^2 \Delta t + 2b(e^n, u^n, e^n) \Delta t
\]
\( + 2b((I - r_h) u^n + e^n, e^n) \Delta t + 2b(u^n_h, (I - r_h) u^n + e^n) \Delta t
\)
\( \leq \frac{\epsilon}{\nu} \|((I - \rho_h)p^n + \eta^n)\|^2_0 \Delta t + 2d(e^n, (I - \rho_h)p^n) \Delta t
\]
\( - 2a((I - r_h) u^n + e^n) \Delta t - 2((I - r_h) d_t u^n + e^n) \Delta t. \)
Using (2.1) and (1.2), we have
\[ 2|b(e^n, u^n, e^n)| \leq c\|e^n\|_{L^4}^2 \|A^{1/2}u^n\|_0 + c\|A^{1/2}e^n\|_0 \|u^n\|_{L^4} \|e^n\|_{L^4} \]
\[ \leq \frac{\nu}{4} \|A^{1/2}e^n\|_0^2 + d_n \|e^n\|_0^2, \]
\[ 2|b((I - r_h)u^n, u^n, e^n)| + 2|b(u^n_{ch}, (I - r_h)u^n, e^n)| \]
\[ \leq c\|A^{1/2}(I - r_h)u^n\|_0 \|e^n\|_0 \|A^{1/2}u^n\|_0 + \|A^{1/2}u^n_{ch}\|_0 \|A^{1/2}e^n\|_0 \]
\[ \leq \frac{\nu}{4} \|A^{1/2}e^n\|_0^2 + c_r h^2 \|A u^n\|_0^2 \|A u^n\|_0, \]
\[ \frac{c}{\nu}(I - \rho_h)p^n_e\|_0^2 + 2|d(e^n, (I - \rho_h)p^n_e)| \leq \frac{\nu}{4} \|A^{1/2}e^n\|_0^2 + c_r h^2 \|p^n_e\|_1^2, \]
\[ 2|a((I - r_h)u^n, e^n)| \leq \frac{\nu}{8} \|A^{1/2}e^n\|_0^2 + c_r h^2 \|A u^n\|_0^2, \]
\[ 2|((I - r_h)d_t u^n, e^n)| \leq \frac{\nu}{8} \|A^{1/2}e^n\|_0^2 + c_r h^2 \|A^{1/2}d_t u^n\|_0^2. \]

Combining these estimates with (5.3) yields
\[ \|e^n\|_0^2 - \|e^{n-1}\|_0^2 + \nu \|A^{1/2}e^n\|_0^2 \Delta t \leq d_n \|e^n\|_0^2 \Delta t \]
\[ + c_r h^2 \left( \|A^{1/2}u^n\|_0^2 + \|A^{1/2}u^n_{ch}\|_0^2 \right) \|A u^n\|_0^2 \Delta t \]
\[ + c_r h^2 \|A u^n\|_0^2 + \|A^{1/2}d_t u^n\|_0^2 + \|p^n_e\|_1^2 \Delta t, \quad \forall 1 \leq n \leq N. \]

Summing (5.3) from 1 to m and using Theorem 5.1 Theorem 4.1 and (2.4), we obtain
\[ \|e^m\|_0^2 + \nu \Delta t \sum_{n=1}^m \|A^{1/2}e^n\|_0^2 \leq \|e^n\|_0^2 + \kappa h^2. \]

Due to (3.10), we can apply Lemma (2.3) to (5.5) and obtain
\[ \|e^m\|_0^2 + \nu \Delta t \sum_{n=1}^m \|A^{1/2}e^n\|_0^2 \leq \kappa h^2. \]

Moreover, by using (2.4), (4.2) and Theorem 5.1 we have
\[ \|(I - r_h)u^m_t\|_0^2 + \nu \Delta t \sum_{n=1}^m \|A^{1/2}(I - r_h)u^n_t\|_0^2 \leq c h^2 \|A^{1/2}u^n\|_0^2 + \nu \Delta t \sum_{n=1}^m \|A u^n\|_0^2 \]
\[ \leq \kappa h^2. \]

Combining (5.6) with (5.7) has completed the proof of Lemma 5.1 \(\square\)

**Lemma 5.2.** Under the assumptions of Theorem 4.1, we have
\[ \|A^{1/2}(u^m - u^n)\|_0^2 + \Delta t \sum_{n=1}^m \|p^n_e - p^n_{ch}\|_0^2 \leq \kappa h^2, \quad \forall 1 \leq m \leq N. \]

**Proof.** Subtracting (1.5) from (2.8) with \((v, q) = (v_h, q_h)\) and setting \(e^n = R^n_h - u^n_{ch}, \; \eta^n = Q_h - p^n_{ch}, \; R^n_h = R_h(u^n, p^n), \; Q_h = Q_h(u^n, p^n), \) we deduce
\[ (d^n e^n, v_h) + a(e^n, v_h) - d(v_h, \eta^n) + d(d^n e^n, q_h) + \frac{c}{\nu} (d e^n, q_h) \]
\[ + b(u^n_e - R^n_h, e^n, v_h) + b(u^n_e, u^n_e - R^n_h + e^n, v_h) \]
\[ - b(u^n_e - u^n_{ch}, u^n_e - u^n_{ch}, v_h) = -(d^n u^n_e - R_h(d_t u^n_e, d_t p^n_e), v_h), \]
\[ (5.8) \]

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for all \((v_h, q_h) \in (X_h, M_h)\) with \(1 \leq n \leq N\). Taking \((v_h, q_h) = 2(d_t e_n, \eta^n) \Delta t\) in (5.3), we get

\[
2\|d_t e_n\|_0^2 \Delta t + \nu(\|A^{1/2} e_n\|_0^2 - \|A^{1/2} e_{n-1}\|_0^2) + \frac{\kappa}{\nu} (\|\eta^n\|_0^2 - \|\eta^{n-1}\|_0^2) + 2b(u^n - R^n_h + e^n, d_t e_n) \Delta t + 2v(u^n - R^n_h + e^n, d_t e_n) \Delta t \\
- 2b(u^n - u^n_{ch}, u^n - u^n_{ch}, v_h) \Delta t = -2(d_t u^n - R_h(d_t u^n, d_t p^n), d_t e_n) \Delta t.
\]

Due to \((u^n, p^n) = (u_0, 0)\), we derive from (12.2) and (1.9) that

\[
\begin{align*}
&\|A^{1/2}(u^n_0 - R_h(u^n_0, p^n_0))\|_0 + \|p^n_0 - Q_h(u^n_0, p^n_0)\|_0 \leq c_h \|A u_0\|_0. \\
&\|A^{1/2} e_0\|_0 + \|\eta^0\|_0 \leq \|A^{1/2}(u_0 - R_h(u_0, p_0))\|_0 + \|A^{1/2}(u^n_0 - R_h(u^n_0, p^n_0))\|_0 \leq \|Q_h(u^n_0, p^n_0)\|_0 \leq c_h \|A u_0\|_0.
\end{align*}
\]

Using again (2.1)-(2.3) and (4.3), it follows that

\[
\begin{align*}
2b(e^n, u^n - d_t e^n) + 2b(u^n - e^n, d_t e^n) &\leq c\|A u^n\|_0 \|A^{1/2} e^n\|_0 \|d_t e^n\|_0 \\
&\leq \frac{1}{4} \|d_t e^n\|_0^2 + c\|A u^n\|_0^2 \|A^{1/2} e^n\|_0^2, \\
2b(u^n - R^n_h, u^n - d_t e^n) + b(u^n, u^n - R^n_h, d_t e^n) &\leq c\|A u^n\|_0 \|A^{1/2}(u^n - R^n_h)\|_0 \|d_t e^n\|_0 \\
&\leq \frac{1}{4} \|d_t e^n\|_0^2 + c\|A u^n\|_0^2 \|A^{1/2}(u^n - R^n_h)\|_0^2, \\
2b(u^n - u^n_{ch}, u^n - u^n_{ch}, d_t e^n) &\leq c(h^{-1}) \|A^{1/2}(u^n - u^n_{ch})\|_0 \|u^n - u^n_{ch}\|_0 \\
&\leq c(h^{-2}) \|A^{1/2}(u^n - u^n_{ch})\|_0^2 \|u^n - u^n_{ch}\|_0^2 \\
&\leq \frac{1}{4} \|d_t e^n\|_0^2 + c(h^{-2}) \|A^{1/2}(u^n - u^n_{ch})\|_0^2 \|u^n - u^n_{ch}\|_0^2 \\
+ c(h^{-2}) \|A^{1/2}(u^n - u^n_{ch})\|_0^2 \|u^n - u^n_{ch}\|_0^2 \\
2|\langle d_t (u - R_h(u, p)), d_t e^n \rangle| &\leq \frac{1}{4} \|d_t e^n\|_0^2 + \|d_t u^n - R_h(d_t u^n, d_t p^n)\|_0^2.
\end{align*}
\]

Combining these inequalities with (5.10) and using Lemma 4.2 results in (5.13)

\[
\begin{align*}
\nu \|A^{1/2} e_n\|_0^2 - \nu \|A^{1/2} e_{n-1}\|_0^2 + \|d_t e^n\|_0^2 \Delta t + \frac{\kappa}{\nu} (\|\eta^n\|_0^2 - \|\eta^{n-1}\|_0^2) &\leq c\|A u^n\|_0^2 (\|A^{1/2} e^n\|_0^2 + h^2 \|A u^n\|_0^2 + h^2 \|p^n\|_0^2) \Delta t \\
&\leq c\|A u^n\|_0^2 (\|A^{1/2} e^n\|_0^2 + h^2 \|A u^n\|_0^2 + h^2 \|p^n\|_0^2) \Delta t \\
&+ c(h^{-2}) \|A^{1/2}(u^n - u^n_{ch})\|_0^2 \|u^n - u^n_{ch}\|_0^2 + \|A^{1/2}(u^n - u^n_{ch})\|_0^2) \Delta t.
\end{align*}
\]

From Theorems 3.1 and 4.1 Lemma 5.1 and the fact that \(h^2 \leq c_1 \Delta t \leq c_1 \tau(t_n)\), we derive from (5.13) that

\[
\begin{align*}
\nu \|A^{1/2} e_n\|_0^2 - \nu \|A^{1/2} e_{n-1}\|_0^2 + \|d_t e^n\|_0^2 \Delta t + \frac{\kappa}{\nu} (\|\eta^n\|_0^2 - \|\eta^{n-1}\|_0^2) &\leq \kappa (\|A^{1/2} e^n\|_0^2 + h^2 \|A u^n\|_0^2 + h^2 \|p^n\|_0^2) \Delta t \\
&\leq \kappa (\|A^{1/2} e^n\|_0^2 + h^2 \|A u^n\|_0^2 + h^2 \|p^n\|_0^2) \Delta t \\
&+ c\|A^{1/2}(u^n - u^n_{ch})\|_0^2 \|u^n - u^n_{ch}\|_0^2 + \|A^{1/2}(u^n - u^n_{ch})\|_0^2) \Delta t.
\end{align*}
\]
Summing (5.14) from 1 to \( m \) and using Theorems 3.1 and 3.2, Lemma 5.1 and (5.12) leads to

\[
\nu \| A^{1/2} e^m \|_0^2 + \Delta t \sum_{n=1}^m \| d_e e^n \|_0^2 \leq \kappa h^2, \forall 1 \leq m \leq N.
\]  

Moreover, by using Lemma 4.2 Theorems 3.1 and 3.2 and the fact that \( h^2 \leq c_1 \tau (t_n) \), we have

\[
\nu \| A^{1/2} (u^n_c - R_h(u^n_c, p^n)) \|_0^2 + \Delta t \sum_{n=1}^m \| d_e u^n_c - R_h(d_e u^n_c, d_t p^n) \|_0^2 \\
\leq c_2 h^2 (\| A u^n_c \|_0^2 + \| p^n \|_1^2) + c_2 h^2 \Delta t \sum_{n=1}^m \tau (t_n) (\| A d_t u^n_c \|_0^2 + \| d_t p^n \|_1^2)
\]  

\[
\leq \kappa h^2, \forall 1 \leq m \leq N.
\]  

Combining (5.15) with (5.16) yields

\[
\| A^{1/2} (u^n_c - u^n_{c,h}) \|_0^2 + \Delta t \sum_{n=1}^m \| d_t u^n_c - d_t u^n_{c,h} \|_0^2 \leq \kappa h^2, \forall 1 \leq m \leq N.
\]  

Finally, by using (5.2), (4.1) and (2.1), we obtain

\[
\| \rho_k p^n_c - p^n_{c,h} \|_0 \leq c \| d_t u^n_c - d_t u^n_{c,h} \|_0 + c \nu \| A^{1/2} (u^n_c - u^n_{c,h}) \|_0 \\
+ c \| A^{1/2} (u^n_c - u^n_{c,h}) \|_0 (\| A^{1/2} u^n_c \|_0 + \| A^{1/2} u^n_{c,h} \|_0) \\
+ c \| (I - \rho_h) p^n_c \|_0.
\]  

Using Theorem 3.1 Theorem 4.1 and Lemma 4.2 we derive from (5.18) that

\[
\| p^n_c - p^n_{c,h} \|_0^2 \Delta t \leq c \| d_t u^n_c - d_t u^n_{c,h} \|_0^2 \Delta t + \kappa \| A^{1/2} (u^n_c - u^n_{c,h}) \|_0^2 \Delta t \\
+ c \| (I - \rho_h) p^n_c \|_0 \Delta t.
\]  

Summing (5.19) from 1 to \( m \) and using (5.17), (4.2) and Theorem 3.1 we find

\[
\Delta t \sum_{n=1}^m \| p^n_c - p^n_{c,h} \|_0^2 \leq \kappa h^2, \forall 1 \leq m \leq N.
\]  

Combining (5.20) with (5.17) yields (5.8).

By combining Lemma 5.2 with Theorem 2.2 we obtain the optimal error estimate result.

**Theorem 5.3.** Under the assumptions of Theorem 4.1, the optimal error estimate

\[
\tau^2 (t_m) \| A^{1/2} (u(t_m) - u^n_{c,h}) \|_0^2 + \Delta t \sum_{n=1}^m \tau^2 (t_n) \| p(t_n) - p^n_{c,h} \|_0^2 \\
\leq \kappa (c^2 + \Delta t^2 + h^2), \forall 1 \leq m \leq N
\]  

holds.

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References


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