MATHEMATICS OF COMPUTATION
Volume 74, Number 252, Pages 2081–2094
S 0025-5718(05)01752-7
Article electronically published on February 14, 2005

AN ALGORITHM OF INFINITE SUMS REPRESENTATIONS AND TASOEV CONTINUED FRACTIONS

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Abstract. For any given real number, its corresponding continued fraction is unique. However, given an arbitrary continued fraction, there has been no general way to identify its corresponding real number. In this paper we shall show a general algorithm from continued fractions to real numbers via infinite sums representations. Using this algorithm, we obtain some new Tasoev continued fractions.

1. Introduction

For any real number α its continued fraction $[a_0; a_1, a_2, \dots]$ can be uniquely expressed by the algorithm

$$\begin{aligned} \alpha &= a_0 + (1/\alpha_1), \qquad a_0 = \lfloor \alpha \rfloor, \\ \alpha_n &= a_n + (1/\alpha_{n+1}), \quad a_n = \lfloor \alpha_n \rfloor \quad (n \geq 1). \end{aligned}$$

But, on the contrary, for any given sequence of partial quotients $a_0, a_1, a_2, \dots$ there is no general algorithm to find the real number yielding the continued fraction $[a_0; a_1, a_2, \dots]$. If the sequence of partial quotients is finite, it is possible to find its corresponding rational number. If it is infinite but periodic, it is still possible to find its corresponding quadratic irrational number. But in the other cases it is too hard to find an explicit or recognizable form of its corresponding real number from the continued fraction.

Some relations between the power series and the general continued fractions are known. Euler's identity and Viskovatov's method are such cases ([5, pp. 256–259]). But, they do not always provide a way to find its corresponding infinite sums' representation from any given simple continued fraction. An easy relation $[a_0; a_1, a_2, \dots, a_n] = a_0 + \sum_{k=0}^{n-1} (-1)^k/(q_k q_{k+1})$ is also well known.

In this paper we shall describe an algorithm to obtain the infinite sums representation from a given continued fraction. This algorithm enables us to obtain the new continued fraction expansions as well as well-known continued fraction expansions. As application, we shall show some new Tasoev continued fractions, which were not reduced through the known relations.

Received by the editor November 5, 2003 and, in revised form, June 15, 2004.
2000 *Mathematics Subject Classification.* Primary 11A55, 11J70, 11Y16.
Key words and phrases. Continued fractions, infinite sums, Tasoev continued fractions.
This work was supported in part by a grant from the Sumitomo Foundation (No. 030110).

2. General continued fractions

We denote general continued fractions by

$$a_0^* + \frac{b_1^*}{a_1^*} + \frac{b_2^*}{a_2^*} + \frac{b_3^*}{a_3^*} + \cdots = a_0^* + \cfrac{b_1^*}{a_1^* + \cfrac{b_2^*}{a_2^* + \cfrac{b_3^*}{a_3^* + \dots}}}.$$

In this paper we treat the continued fractions

$$\frac{1}{a_1^*} + \frac{\epsilon_2^*}{a_2^*} + \frac{\epsilon_3^*}{a_3^*} + \cdots = \cfrac{1}{a_1^* + \cfrac{\epsilon_2^*}{a_2^* + \cfrac{\epsilon_3^*}{a_3^* + \dots}}}$$

because this type of continued fractions can be transformed to the simple continued fractions.

Let $\epsilon_k = 1$ or -1 $(k = 2, 3, \dots)$, and let $S(k) = \epsilon_{k+1}/(a_k a_{k+1})$ $(k = 1, 2, \dots)$.

Theorem A. *If both $\sum_{n=0}^{\infty} c_n$ and $\sum_{n=0}^{\infty} c'_n$ are convergent, then*

$$\frac{1}{a_1} + \frac{\epsilon_2}{a_2} + \frac{\epsilon_3}{a_3} + \cdots = \frac{\sum_{n=0}^{\infty} c'_n}{\sum_{n=0}^{\infty} c_n},$$

where for $n = 0, 1, 2, \dots$

$$c'_n = \frac{1}{a_1}(c_n - c_{n-1}s_{1,1} + c_{n-2}s_{2,1} - \cdots + (-1)^{n-1}c_1 s_{n-1,1} + (-1)^n c_0 s_{n,1})$$

and for $k \geq 1$

$$s_{n,k} = s_{1,k}(s_{n-1,1} + s_{n-1,2} + \cdots + s_{n-1,k+1}) \quad (n \geq 2)$$

with $s_{1,k} = S(k)$ and $s_{0,k} = 1$.

The general continued fraction on the left-hand side does not seem simple at first look. But in each specific case it can be transformed into a simple continued fraction.

If the denominator sum on the right-hand side is chosen at will, then the numerator sum is determined in terms of the given continued fraction expansion. Hence, if one chooses the denominator carefully, then one can obtain a recognizable numerator.

This algorithm works well in any case if the infinite sum converges. Here is one of the conditions for convergence.

Lemma 1. *If $a_k \geq 2$ for all $k \geq k_0$, then the series in Theorem A converges.*

Proof. It is sufficient to prove the case where $\epsilon_k = 1$ for all $k \geq 2$. If $a_{k_0-1} = 1$ is the last partial quotient where 1 appears, then it is sufficient to consider the convergence of $[0; a_{k_0}, a_{k_0+1}, \dots]$ instead. Hence, we assume that $a_k \geq 2$ for all $k \geq 1$ without loss of generality.

We shall show for all integers n and k with $n \geq 1$ and $k \geq 1$

$$s_{n,k} \leq \frac{(k+1)(k+2n-2)!}{4^n (n-1)!(k+n)!}.$$

It is clear that for all $k \geq 1$

$$s_{1,k} = \frac{1}{a_k a_{k+1}} \leq \frac{1}{4}.$$

By induction we can have

$$\begin{aligned} s_{n+1,k} = s_{1,k}\sum_{i=1}^{k+1} s_{n,i} &\le \frac{1}{4}\sum_{i=1}^{k+1}\frac{(i+1)(i+2n-2)!}{4^n(n-1)!(i+n)!} \\ &= \frac{(k+1)(k+2n)!}{4^{n+1}n!(k+n+1)!}, \end{aligned}$$

for, if we assume that

$$\sum_{i=1}^{k}\frac{(i+1)(i+2n-2)!}{(i+n)!} = \frac{k(k+2n-1)!}{n(k+n)!}\,,$$

then

$$\begin{aligned} \sum_{i=1}^{k+1}\frac{(i+1)(i+2n-2)!}{(i+n)!} &= \frac{k(k+2n-1)!}{n(k+n)!} + \frac{(k+2)(k+2n-1)!}{(k+n+1)!} \\ &= \frac{(k+1)(k+2n)!}{n(k+n+1)!}\,. \end{aligned}$$

Hence, for $n \ge 1$, we have $0 < s_{n,1} \le D_n$, where

$$D_n = \frac{2(2n-1)!}{4^n(n-1)!(n+1)!}\,.$$

Since

$$n\left(1 - \frac{D_{n+1}}{D_n}\right) = \frac{3n}{2(n+2)} \to \frac{3}{2} > 1 \quad (n \to \infty),$$

$\sum_{n=1}^{\infty} D_n$ converges, and so does $\sum_{n=1}^{\infty} s_{n,1}$.

3. Proof of Theorem A

Define $S(i_1, i_2, \ldots, i_{n-1}, i_n)$ recursively by

$$S(i_1, i_2, \ldots, i_{n-1}, i_n) = S(i_n)\sum_{i_{n-1}=1}^{i_n+1} S(i_1, i_2, \ldots, i_{n-1})$$

with

$$S(i) = \frac{\epsilon_{i+1}}{a_i a_{i+1}}\,.$$

If we put $s_{n,k} = S(\underbrace{i_1, i_2, \ldots, i_{n-1}}_{n-1}, k)$, then

$$\begin{aligned} s_{n,k} &= \frac{\epsilon_{k+1}}{a_k a_{k+1}}\sum_{i_{n-1}=1}^{k+1} S(i_1, i_2, \ldots, i_{n-1}) \\ &= S(k)\sum_{i=1}^{k+1} s_{n-1,i} \quad (n \ge 2) \end{aligned}$$

with

$$s_{1,k} = S(k) = \frac{\epsilon_{k+1}}{a_k a_{k+1}}\,.$$

Now, consider the formal power series for $k = 0, 1, 2, \ldots$

$$f_k(z) = c_{k,0} + c_{k,1}z + c_{k,2}z^2 + \cdots\,,$$

satisfying

$$f_k(z) = a_{k+1} f_{k+1}(z) + \epsilon_{k+2} z f_{k+2}(z)\,,$$

where $\epsilon_{k+2} = 1$ or -1. Then by

$$\frac{f_k(z)}{f_{k+1}(z)} = a_{k+1} + \frac{\epsilon_{k+2} z}{\frac{f_{k+1}(z)}{f_{k+2}(z)}}\,,$$

we can have

$$\frac{f_1(1)}{f_0(1)} = \frac{\sum_{n=0}^{\infty} c_{1,n}}{\sum_{n=0}^{\infty} c_{0,n}} = \frac{1}{a_1} \underset{+}{} \frac{\epsilon_2}{a_2} \underset{+}{} \frac{\epsilon_3}{a_3} \underset{+}{} \cdots\,.$$

For simplicity, we put $c_n = c_{0,n}$ and $c'_n = c_{1,n}$ $(n \geq 0)$.

By comparing the constant and the coefficient of z^n $(n \geq 1)$, $c_{k,n}$ must satisfy the recurrence relations

$$\begin{aligned} c_{k,0} &= a_{k+1} c_{k+1,0}, \\ c_{k,n} &= a_{k+1} c_{k+1,n} + \epsilon_{k+2} c_{k+2,n-1} \quad (n \geq 1)\,. \end{aligned}$$

By the first relation we have $c_{0,0} = \left(\prod_{\nu=1}^{k} a_\nu\right) c_{k,0}$. Hence,

$$c_{k,0} = \left(\prod_{\nu=1}^{k} a_\nu^{-1}\right) c_{0,0}\,. \tag{1}$$

Setting $k = 1$, we have $c'_0 = c_0/a_1$.

By the second relation we have

$$c_{0,n} = \left(\prod_{\nu=1}^{k} a_\nu\right) c_{k,n} + \sum_{i=1}^{k} \left(\prod_{\nu=1}^{i-1} a_\nu\right) \epsilon_{i+1} c_{i+1,n-1}\,. \tag{2}$$

For $n = 1$ by (1) and (2) we have

$$\begin{aligned} c_{0,1} &= \left(\prod_{\nu=1}^{k} a_\nu\right) c_{k,1} + \sum_{i=1}^{k} \left(\prod_{\nu=1}^{i-1} a_\nu\right) \epsilon_{i+1} \left(\prod_{\mu=1}^{i+1} a_\mu^{-1}\right) c_0 \\ &= \left(\prod_{\nu=1}^{k} a_\nu\right) c_{k,1} + c_0 \sum_{i=1}^{k} S(i)\,. \end{aligned}$$

Thus,

$$c_{k,1} = \left(\prod_{\nu=1}^{k} a_\nu^{-1}\right) \left(c_1 - c_0 \sum_{i=1}^{k} S(i)\right)\,. \tag{3}$$

Setting $k = 1$, we have

$$c'_1 = \frac{1}{a_1}\left(c_1 - \frac{c_0 \epsilon_2}{a_1 a_2}\right) = \frac{1}{a_1}(c_1 - c_0 s_{1,1})\,.$$

For $n = 2$ by (2) and (3) we have

$$\begin{aligned} c_{0,2} &= \left(\prod_{\nu=1}^{k} a_\nu\right) c_{k,2} \\ &\quad + \sum_{i=1}^{k} \left(\prod_{\nu=1}^{i-1} a_\nu\right) \epsilon_{i+1} \left(\prod_{\mu=1}^{i+1} a_\mu^{-1}\right) \left(c_1 - c_0 \sum_{j=1}^{i+1} S(j)\right) \\ &= \left(\prod_{\nu=1}^{k} a_\nu\right) c_{k,2} + \sum_{i=1}^{k} S(i) \left(c_1 - c_0 \sum_{j=1}^{i+1} S(j)\right). \end{aligned}$$

Thus,

$$c_{k,2} = \left(\prod_{\nu=1}^{k} a_\nu^{-1}\right) \left(c_2 - c_1 \sum_{i=1}^{k} S(i) + c_0 \sum_{i=1}^{k} S(j,i)\right).$$

Setting $k = 1$, we have

$$c_2' = \frac{1}{a_1}(c_2 - c_1 s_{1,1} + c_0 s_{2,1}).$$

Assume that

$$\begin{aligned} (4) \quad c_{k,n} = \left(\prod_{\nu=1}^{k} a_\nu^{-1}\right) \Bigg(c_n - c_{n-1} \sum_{i_n=1}^{k} S(i_n) + c_{n-2} \sum_{i_n=1}^{k} S(i_{n-1}, i_n) \\ - \cdots + (-1)^n c_0 \sum_{i_n=1}^{k} S(i_1, \ldots, i_{n-1}, i_n)\Bigg). \end{aligned}$$

Then together with (2) we have

$$\begin{aligned} c_{0,n+1} &= \left(\prod_{\nu=1}^{k} a_\nu\right) c_{k,n+1} \\ &\quad + \sum_{i_{n+1}=1}^{k} S(i_{n+1}) \Bigg(c_n - c_{n-1} \sum_{i_n=1}^{i_{n+1}+1} S(i_n) \\ &\quad + c_{n-2} \sum_{i_n=1}^{i_{n+1}+1} S(i_{n-1}, i_n) - \cdots + (-1)^n c_0 \sum_{i_n=1}^{i_{n+1}+1} S(i_1, \ldots, i_{n-1}, i_n)\Bigg) \\ &= \left(\prod_{\nu=1}^{k} a_\nu\right) c_{k,n+1} + \Bigg(c_n \sum_{i_{n+1}=1}^{k} S(i_{n+1}) - c_{n-1} \sum_{i_{n+1}=1}^{k} S(i_n, i_{n+1}) \\ &\quad + c_{n-2} \sum_{i_{n+1}=1}^{k} S(i_{n-1}, i_n, i_{n+1}) - \cdots + (-1)^n c_0 \sum_{i_{n+1}=1}^{k} S(i_1, \ldots, i_n, i_{n+1})\Bigg). \end{aligned}$$

Therefore, by induction (4) holds for every general $n = 1, 2, \ldots$.

Setting $k=1$ in (4), we obtain

$$\begin{aligned} c'_n &= \frac{1}{a_1}\left(c_n - c_{n-1}S(1) + c_{n-2}S(i_{n-1},1) - \cdots + (-1)^n c_0 S(i_1,\dots,i_{n-1},1)\right) \\ &= \frac{1}{a_1}\left(c_n - c_{n-1}s_{1,1} + c_{n-2}s_{2,1} - \cdots + (-1)^n c_0 s_{n,1}\right). \end{aligned}$$

4. An application to Hurwitz continued fractions

First, we shall lead a well-known continued fraction expansion by using Theorem A. Hurwitz continued fractions, quasi-periodic simple continued fractions, have the form

$$\begin{aligned} &[a_0; a_1, \dots, a_n, \overline{Q_1(k), \dots, Q_p(k)}]_{k=1}^{\infty} \\ &\qquad = [a_0; a_1, \dots, a_n, Q_1(1), \dots, Q_p(1), Q_1(2), \dots, Q_p(2), Q_1(3), \dots], \end{aligned}$$

where a_0 is an integer, $a_1, \dots, a_n$ are positive integers, $Q_1, \dots, Q_p$ are polynomials with rational coefficients which take positive integral values for $k = 1, 2, \dots$ and at least one of the polynomials is not constant. As an application, we shall show

$$\tan\frac{1}{a} = [0; a-1, \overline{1, (2k+1)a-2}]_{k=1}^{\infty} \quad (a > 1),$$

which is one of the well-known examples.

Let $\epsilon_k = -1$ $(k \geq 2)$. Then, the general continued fraction can be transformed into a simple continued fraction expansion by the following rule (see also [4, Lemma 2] or [6]).

Lemma 2.

$$\begin{aligned} &\frac{1}{a'_1} - \frac{1}{a'_2} - \frac{1}{a'_3} - \frac{1}{a'_4} - \frac{1}{a'_5} - \frac{1}{a'_6} - \frac{1}{a'_7} - \cdots \\ &= [0; a'_1 - 1, 1, a'_2 - 2, 1, a'_3 - 2, 1, a'_4 - 2, 1, a'_5 - 2, 1, a'_6 - 2, 1, a'_7 - 2, 1, \dots]. \end{aligned}$$

Proof. First, we apply the equivalence transformation in [1, (2.3.23), p. 35] or [4, Lemma 1]. Namely,

$$[a_0; a_1, a_2, \dots] = a_0^* + \frac{b_1^*}{a_1^*} + \frac{b_2^*}{a_2^*} + \frac{b_3^*}{a_3^*} + \cdots$$

iff $a_0 = a_0^*$, $a_1 = a_1^*/b_1^*$ and for $k = 1, 2, \dots$

$$a_{2k} = \frac{b_{2k-1}^* b_{2k-3}^* \cdots b_1^*}{b_{2k}^* b_{2k-2}^* \cdots b_2^*} a_{2k}^* \quad \text{and} \quad a_{2k+1} = \frac{b_{2k}^* b_{2k-2}^* \cdots b_2^*}{b_{2k+1}^* b_{2k-1}^* \cdots b_1^*} a_{2k+1}^*.$$

Then we apply

$$[\dots, a, -b, \gamma] = [\dots, a-1, 1, b-1, -\gamma]$$

in [7, Section 6]. Hence, we have

$$\begin{aligned} &\frac{1}{a'_1} - \frac{1}{a'_2} - \frac{1}{a'_3} - \frac{1}{a'_4} - \frac{1}{a'_5} - \frac{1}{a'_6} - \frac{1}{a'_7} - \cdots \\ &= [0; a'_1, -a'_2, a'_3, -a'_4, a'_5, -a'_6, a'_7, \dots] \\ &= [0; a'_1 - 1, 1, a'_2 - 1, -a'_3, a'_4, -a'_5, a'_6, -a'_7, \dots] \\ &= [0; a'_1 - 1, 1, a'_2 - 2, 1, a'_3 - 1, -a'_4, a'_5, -a'_6, a'_7, \dots] \\ &= \cdots \\ &= [0; a'_1 - 1, 1, a'_2 - 2, 1, a'_3 - 2, 1, a'_4 - 2, 1, a'_5 - 2, 1, a'_6 - 2, 1, a'_7 - 2, 1, \dots]. \quad \square \end{aligned}$$

Example 1. Set $a_k = (2k-1)a$ in Theorem A, where a is an integer with $a > 1$. When

$$c_k = \frac{(-1)^k}{(2k)!a^{2k}} \quad (k = 0, 1, 2, \ldots, n),$$

we can get

$$c'_n = \frac{(-1)^n}{(2n+1)!a^{2n+1}}. \tag{5}$$

Thus, by Lemma 2

$$\begin{aligned}&[0; a-1, \overline{1, (2k+1)a-2}]_{k=1}^{\infty}\\ &= \frac{\sum_{n=0}^{\infty} c'_n}{\sum_{n=0}^{\infty} c_n} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!a^{2n+1}}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!a^{2n}}} = \frac{\sin\frac{1}{a}}{\cos\frac{1}{a}} = \tan\frac{1}{a}.\end{aligned}$$

We shall prove (5). By induction, it is shown that

$$\begin{aligned}s_{n,k} = \frac{(-1)^n}{a^{2n}}\Bigg(&\frac{t_1^{(n)}}{(2k-1)(2k+1)} - \frac{t_2^{(n)}}{(2k-1)(2k+1)(2k+3)}\\ &+ \cdots + (-1)^{n-1}\frac{t_n^{(n)}}{(2k-1)(2k+1)\cdots(2k+2n-1)}\Bigg),\end{aligned}$$

where

$$t_1^{(n)} = \sum_{i=1}^{n-1}(-1)^{i-1}\frac{t_i^{(n-1)}}{2i}\frac{1}{1\cdot 3\cdot 5\cdots(2i-1)} = \sum_{i=1}^{n-1}(-1)^{i-1}\frac{2^{i-1}(i-1)!}{(2i)!}t_i^{(n-1)}$$

and

$$t_{i+1}^{(n)} = \frac{t_i^{(n-1)}}{2i} \quad (i = 1, 2, \ldots, n-1)$$

with $t_1^{(1)} = 1$. Since

$$t_i^{(n)} = \frac{t_1^{(n-i+1)}}{2^{i-1}(i-1)!} \quad (i = 1, 2, \ldots, n),$$

we have

$$t_1^{(n)} = \sum_{i=1}^{n-1}(-1)^{i-1}\frac{t_1^{(n-i)}}{(2i)!}.$$

Notice that

$$\begin{aligned}s_{n,1} &= \frac{(-1)^n}{a^{2n}}\sum_{i=1}^{n}(-1)^{i-1}\frac{t_i^{(n)}}{1\cdot 3\cdots(2i+1)} = \frac{(-1)^n}{a^{2n}}\sum_{i=1}^{n}(-1)^{i-1}\frac{2^i i!}{(2i+1)!}t_i^{(n)}\\ &= \frac{2(-1)^n}{a^{2n}}\sum_{i=1}^{n}(-1)^{i-1}\frac{i}{(2i+1)!}t_1^{(n-i+1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{a^{2n}}{2(-1)^n}(s_{n,1}-c_1s_{n-1,1}+c_2s_{n-2,1}-\cdots+(-1)^{n-2}c_{n-2}s_{2,1}+(-1)^{n-1}c_{n-1}s_{1,1})\\
&=\sum_{k=0}^{n-1}\frac{(-1)^k}{(2k)!}\sum_{j=1}^{n-k}\frac{(-1)^{j-1}j}{(2j+1)!}t_1^{(n-j+1)}\\
&=\sum_{j=1}^{n-1}\frac{(-1)^{j-1}j}{(2j+1)!}\left(t_1^{(n-j+1)}-\sum_{i=1}^{n-j}(-1)^{i-1}\frac{t_1^{(n-i-j+1)}}{(2i)!}\right)+(-1)^{n-1}\frac{n}{(2n+1)!}t_1^{(1)}\\
&=(-1)^{n-1}\frac{n}{(2n+1)!}\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
c_n'&=\frac{1}{a}(c_n-c_{n-1}s_{1,1}+c_{n-2}s_{2,1}-\cdots+(-1)^{n-1}c_1s_{n-1,1}+(-1)^nc_0s_{n,1})\\
&=\frac{1}{a}\left(\frac{(-1)^n}{(2n)!a^{2n}}+\frac{2}{a^{2n}}(-1)^{n-1}\frac{n}{(2n+1)!}\right)\\
&=\frac{(-1)^n}{(2n+1)!a^{2n+1}}\,. \quad \square
\end{aligned}$$

Similarly, one can obtain many known examples. For example, if we set $\epsilon_2=\epsilon_4=\cdots=-1$, $\epsilon_3=\epsilon_5=\cdots=1$ and $a_{2k-1}=s(2k-1)$ $(s>1)$, $a_{2k}=2$ $(k\ge 1)$ and choose

$$c_k=(2s)^{-2k}\big((2k)!\big)^{-1}-(2s)^{-2k-1}\big((2k+1)!\big)^{-1}\quad (k=0,1,\dots,n)\,,$$

then we have

$$c_n'=2(2s)^{-2n-1}\big((2n+1)!\big)^{-1}\,.$$

It follows that

$$\begin{aligned}
&[0;\overline{s(2k-1)-1,1,1}]_{k=1}^{\infty}\\
&=\frac{2\sum_{n=0}^{\infty}(2s)^{-2n-1}\big((2n+1)!\big)^{-1}}{\sum_{n=0}^{\infty}\left((2s)^{-2n}\big((2n)!\big)^{-1}-(2s)^{-2n-1}\big((2n+1)!\big)^{-1}\right)}\\
&=\frac{e^{1/(2s)}-e^{-1/(2s)}}{e^{-1/(2s)}}=e^{1/s}-1
\end{aligned}$$

as is well known.

5. Tasoev continued fractions

Tasoev continued fractions ([8], [9]) are also systematic but have seldom been known before. They are also quasi-periodic like Hurwitz continued fractions, but $Q_j(k)$ includes exponentials in k instead of polynomials. The author obtained the closed form of $[0;\overline{\underbrace{a^k,\dots,a^k}_{m}}]_{k=1}^{\infty}$ in [2] and found some more general forms by

applying a similar method in [3]. Namely,

$$[0;\overline{ua^k}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-2n-1}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty} u^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}},$$

$$[0;ua-1,\overline{1,ua^{k+1}-2}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(-1)^n u^{-2n-1}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-2n}a^{-n^2}\prod_{i=1}^{n}(a^{2i}-1)^{-1}},$$

$$[0;\overline{ua^k,va^k}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^{n}(a^{i}-1)^{-1}}{\sum_{n=0}^{\infty} u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^{n}(a^{i}-1)^{-1}},$$

and

$$[0;ua-1,1,va-2,\overline{1,ua^{k+1}-2,1,va^{k+1}-2}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(-1)^n u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^{n}(a^{i}-1)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^{n}(a^{i}-1)^{-1}}.$$

As application of Tasoev continued fractions, we obtain the following new expansions.

Let u and v be rational, and let r and s be positive integers so that ur and vs take positive integral values with $ur > 1$, $vs > 1$ and $(r,s) \neq (1,1)$.

Theorem 1.

$$[0;\overline{ur^k-1,1,vs^k-1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(ur)^{-n-1}(vs)^{-n}B_n\prod_{i=1}^{n}(r^is^i-1)^{-1}}{\sum_{n=0}^{\infty} u^{-n}v^{-n}A_n\prod_{i=1}^{n}(r^is^i-1)^{-1}},$$

where $A_n = r^{-n}s^{-n+1}B_{n-1} - A_{n-1}$ *and* $B_n = sB_{n-1} - A_{n-1}$ $(n \geq 1)$ *with* $A_0 = B_0 = 1$.

The right-hand side still includes the recurrence relations, but the recognizable forms can be led for some specific cases. See examples below.

Theorem 2.

$$[0;\overline{ur^{2k-1}-1,1,vs^{2k-1}-1,ur^{2k},vs^{2k}}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(ur)^{-n-1}(vs)^{-n}(\beta_n+\beta'_n)}{\sum_{n=0}^{\infty} u^{-n}v^{-n}(\alpha_n-\alpha'_n)},$$

where for $n \geq 1$

$$\alpha_n = \frac{1}{(rs)^n-1}\left(\frac{\beta_{n-1}}{r^ns^{n-1}} - \alpha'_{n-1}\right), \quad \alpha'_n = \frac{1}{(rs)^n+1}\left(\alpha_{n-1} + \frac{\beta'_{n-1}}{r^ns^{n-1}}\right),$$

$$\beta_n = \frac{1}{(rs)^n-1}\left(s\beta_{n-1} - \alpha'_{n-1}\right), \qquad \beta'_n = \frac{1}{(rs)^n+1}\left(\alpha_{n-1} - s\beta'_{n-1}\right)$$

with $\alpha_0 = \beta_0 = 1$ *and* $\alpha'_0 = \beta'_0 = 0$.

Theorem 3.

$$[0;ur-1,\overline{1,us^{2k-1}-2,1,vr^{2k}-1,us^{2k}-1,1,ur^{2k+1}}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty}(-1)^n(ur)^{-n-1}(vs)^{-n}(\beta_n-\beta'_n)}{\sum_{n=0}^{\infty}(-1)^n u^{-n}v^{-n}(\alpha_n+\alpha'_n)},$$

where α_n, α'_n, β_n *and* β'_n $(n \geq 0)$ *are the same as in Theorem 2.*

Theorem 4.

$$[0;\overline{ur^{2k-1}-1,1,vs^{2k-1}-2,1,ur^{2k}-1,vs^{2k}}]_{k=1}^{\infty}$$
$$=\frac{\sum_{n=0}^{\infty}(-1)^n\big((ur)^{-2n-1}(vs)^{-2n}\lambda_{2n}-(ur)^{-2n-2}(vs)^{-2n-1}\lambda_{2n+1}\big)}{\sum_{n=0}^{\infty}(-1)^n(u^{-2n}v^{-2n}\kappa_{2n}-u^{-2n-1}v^{-2n-1}\kappa_{2n+1})},$$

where

$$\kappa_{2n+1}=\frac{1}{(rs)^{2n+1}+1}\left(\frac{\lambda_{2n}}{r^{2n+1}s^{2n}}+\kappa_{2n}\right)\quad(n\ge 0),$$
$$\kappa_{2n}=\frac{1}{(rs)^{2n}-1}\left(\kappa_{2n-1}+\frac{\lambda_{2n-1}}{r^{2n}s^{2n-1}}\right)\quad(n\ge 1),$$
$$\lambda_{2n+1}=\frac{1}{(rs)^{2n+1}+1}\left(s\lambda_{2n}-\kappa_{2n}\right)\quad(n\ge 0),$$
$$\lambda_{2n}=\frac{1}{(rs)^{2n}-1}\left(\kappa_{2n-1}+s\lambda_{2n-1}\right)\quad(n\ge 1)$$

with $\kappa_0=\lambda_0=1$.

6. Proof of Theorems 1 to 4

Proof of Theorem 1. Let $\epsilon_{2k}=-1$ and $\epsilon_{2k+1}=1$ $(k\ge 1)$. In a similar way to Lemma 2, we get

$$\begin{aligned}&\frac{1}{a_1'}-\frac{1}{a_2'}+\frac{1}{a_3'}-\frac{1}{a_4'}+\frac{1}{a_5'}-\frac{1}{a_6'}+\frac{1}{a_7'}-\cdots\\&=[0;a_1',-a_2',-a_3',a_4',a_5',-a_6',-a_7',\dots]\\&=[0;a_1'-1,1,a_2'-1,a_3'-1,1,a_4'-1,a_5'-1,1,a_6'-1,a_7'-1,\dots]\\&=[0;\overline{a_{2k-1}'-1,1,a_{2k}'-1}]_{k=1}^{\infty}.\end{aligned}$$

If we choose

$$c_k=u^{-k}v^{-k}A_k\prod_{i=1}^{k}(r^is^i-1)^{-1}\quad(k=0,1,\dots,n),$$

then we obtain

$$c_n'=(ur)^{-n-1}(vs)^{-n}B_n\prod_{i=1}^{n}(r^is^i-1)^{-1}. \tag{6}$$

Putting $a_{2k-1}=ur^k$ and $a_{2k}=vs^k$ $(k\ge 1)$ entails that

$$[0;\overline{ur^k-1,1,vs^k-1}]_{k=1}^{\infty}=\frac{\sum_{n=0}^{\infty}c_{1,n}}{\sum_{n=0}^{\infty}c_{0,n}}$$
$$=\frac{\sum_{n=0}^{\infty}(ur)^{-n-1}(vs)^{-n}B_n\prod_{i=1}^{n}(r^is^i-1)^{-1}}{\sum_{n=0}^{\infty}u^{-n}v^{-n}A_n\prod_{i=1}^{n}(r^is^i-1)^{-1}}.$$

We shall prove (6). By induction we have

$$s_{n,2k-1} = \frac{(-1)^n}{u^n v^n r^{n-1}} \sum_{i=1}^{n} (-1)^{i-1} \frac{\sigma_{n,i}}{(rs)^{ik}} \quad (k \geq 1)\,,$$

$$s_{n,2k} = \frac{(-1)^{n-1}}{u^n v^n r^n} \sum_{i=1}^{n} (-1)^{i-1} \frac{\tau_{n,i}}{(rs)^{ik}} \quad (k \geq 0)\,,$$

where for $n \geq 2$

$$\sigma_{n,i} = \begin{cases} \sum_{l=1}^{n-1} (-1)^{l-1} \sigma_{n,l+1} & \text{if } i = 1, \\ \frac{r\sigma_{n-1,i-1} + (-1)^{i-1}\tau_{n-1,i-1}}{(rs)^{i-1}-1} & \text{if } 2 \leq i \leq n \end{cases}$$

and

$$\tau_{n,i} = \begin{cases} \sigma_{n,i} & \text{if } i = 1, \\ (-1)^i \left(\frac{\sigma_{n-1,i-1}}{r^{i-2}s^{i-1}} - \sigma_{n,i} \right) & \text{if } 2 \leq i \leq n \end{cases}$$

with $\sigma_{1,1} = \tau_{1,1} = 1$. By Theorem A and

$$\sigma_{j,i} = \frac{\sigma_{j-i+1,1}(-r)^{i-1} A_{i-1}}{\prod_{l=1}^{i-1}(r^l s^l - 1)} = \sigma_{j-i+1,1}\sigma_{i,i} \quad (1 \leq i \leq j)\,,$$

we have

$$\begin{aligned} urc'_n - c_n &= \sum_{j=1}^{n} (-1)^j c_{n-j} s_{j,1} \\ &= \sum_{j=1}^{n} (-1)^j \frac{A_{n-j}}{(uv)^{n-j} \prod_{i=1}^{n-j}(r^i s^i - 1)} \frac{(-1)^j}{u^j v^j r^{j-1}} \sum_{i=1}^{j} (-1)^{i-1} \frac{\sigma_{j,i}}{(rs)^i} \\ &= \frac{1}{(uv)^n r^{n-1}} \sum_{j=1}^{n} (-1)^{n-j} \sigma_{n-j+1,n-j+1} \sum_{i=1}^{j} (-1)^{i-1} \frac{\sigma_{j,i}}{(rs)^i} \\ &= \frac{(-1)^n}{(uv)^n r^{n-1}} \sum_{i=1}^{n} \left(\frac{-1}{rs}\right)^i \sigma_{i,i} \sum_{j=i}^{n} (-1)^{j+1} \sigma_{n-j+1,n-j+1}\sigma_{j-i+1,1} \\ &= \frac{1}{(uv)^n r^{n-1}} \frac{1}{r^n s^n} \frac{(-r)^{n-1} A_{n-1}}{\prod_{i=1}^{n-1}(r^i s^i - 1)} (-1)^{n+1} \\ &= \frac{B_n - r^n s^n A_n}{(uvrs)^n \prod_{i=1}^{n}(r^i s^i - 1)}\,. \end{aligned}$$

We used the fact that $1 \leq i \leq n$:

$$\begin{aligned} \sigma_{n-i+1,1}\sigma_{1,1} = \sigma_{n-i+1,1} &= \sum_{l=1}^{n-i} (-1)^{l-1} \sigma_{n-i+1,l+1} \\ &= \sum_{l=1}^{n-i} (-1)^{l-1} \sigma_{n-i-l+1,1}\sigma_{l+1,l+1}\,. \end{aligned}$$

Therefore,

$$\begin{aligned} c'_n &= \frac{1}{ur}\left(\frac{A_n}{u^n v^n \prod_{i=1}^n (r^i s^i - 1)} + \frac{B_n - r^n s^n A_n}{(uvrs)^n \prod_{i=1}^n (r^i s^i - 1)}\right) \\ &= \frac{B_n}{(ur)^{n+1}(vs)^n \prod_{i=1}^n (r^i s^i - 1)}\,. \end{aligned}$$

Proof of Theorem 2. Let $\epsilon_{2k} = (-1)^k$ and $\epsilon_{2k+1} = 1$ $(k \ge 1)$. Notice that

$$\begin{aligned} &\frac{1}{a'_1} - \frac{1}{a'_2} + \frac{1}{a'_3} + \frac{1}{a'_4} + \frac{1}{a'_5} - \frac{1}{a'_6} + \frac{1}{a'_7} + \frac{1}{a'_8} + \frac{1}{a'_9} - \frac{1}{a'_{10}} + \frac{1}{a'_{11}} + \frac{1}{a'_{12}} + \cdots \\ &= [0; a'_1, -a'_2, -a'_3, -a'_4, -a'_5, a'_6, a'_7, a'_8, a'_9, -a'_{10}, -a'_{11}, -a'_{12}, -a'_{13}, a'_{14}, \dots] \\ &= [0; a'_1 - 1, 1, a'_2 - 1, a'_3, a'_4, a'_5 - 1, 1, a'_6 - 1, a'_7, a'_8, a'_9 - 1, 1, a'_{10} - 1, \dots]\,. \end{aligned}$$

Putting $a_{2k-1} = ur^k$ and $a_{2k} = vs^k$ $(k \ge 1)$ entails that

$$\begin{aligned} [0; \overline{ur^{2k-1} - 1, 1, vs^{2k-1} - 1, ur^{2k}, vs^{2k}}]_{k=1}^{\infty} &= \frac{\sum_{n=0}^{\infty} c_{1,n}}{\sum_{n=0}^{\infty} c_{0,n}} \\ &= \frac{\sum_{n=0}^{\infty} (ur)^{-n-1}(vs)^{-n}(\beta_n + \beta'_n)}{\sum_{n=0}^{\infty} u^{-n} v^{-n}(\alpha_n - \alpha'_n)}\,. \end{aligned}$$

Proof of Theorem 3. Let $\epsilon_{2k} = (-1)^k$ and $\epsilon_{2k+1} = -1$ $(k \ge 1)$. Notice that

$$\begin{aligned} &\frac{1}{a'_1} - \frac{1}{a'_2} - \frac{1}{a'_3} + \frac{1}{a'_4} - \frac{1}{a'_5} - \frac{1}{a'_6} - \frac{1}{a'_7} + \frac{1}{a'_8} - \frac{1}{a'_9} - \frac{1}{a'_{10}} - \frac{1}{a'_{11}} + \frac{1}{a'_{12}} - \cdots \\ &= [0; a'_1, -a'_2, a'_3, a'_4, -a'_5, a'_6, -a'_7, -a'_8, a'_9, -a'_{10}, a'_{11}, a'_{12}, -a'_{13}, a'_{14}, \dots] \\ &= [0; a'_1 - 1, 1, a'_2 - 2, 1, a'_3 - 1, a'_4 - 1, 1, a'_5 - 2, 1, a'_6 - 2, 1, a'_7 - 1, \\ &\qquad a'_8 - 1, 1, a'_9 - 2, 1, a'_{10} - 2, 1, a'_{11} - 1, a'_{12} - 1, 1, \dots] \\ &= [0; a'_1 - 1, \overline{1, a'_{4k-2} - 2, 1, a'_{4k-1} - 1, a'_{4k} - 1, 1, a'_{4k+1} - 2}]_{k=1}^{\infty}\,. \end{aligned}$$

Putting $a_{2k-1} = ur^k$ and $a_{2k} = vs^k$ $(k \ge 1)$ entails that

$$\begin{aligned} [0; ur - 1, \overline{1, us^{2k-1} - 2, 1, vr^{2k} - 1, us^{2k} - 1, 1, ur^{2k+1}}]_{k=1}^{\infty} &= \frac{\sum_{n=0}^{\infty} c_{1,n}}{\sum_{n=0}^{\infty} c_{0,n}} \\ &= \frac{\sum_{n=0}^{\infty} (-1)^n (ur)^{-n-1}(vs)^{-n}(\beta_n - \beta'_n)}{\sum_{n=0}^{\infty} (-1)^n u^{-n} v^{-n}(\alpha_n + \alpha'_n)}\,. \end{aligned}$$

Proof of Theorem 4. Let $\epsilon_{2k} = \epsilon_{2k+1} = (-1)^k$ $(k \ge 1)$. Notice that

$$\begin{aligned} &\frac{1}{a'_1} - \frac{1}{a'_2} - \frac{1}{a'_3} + \frac{1}{a'_4} + \frac{1}{a'_5} - \frac{1}{a'_6} - \frac{1}{a'_7} + \frac{1}{a'_8} + \frac{1}{a'_9} - \frac{1}{a'_{10}} - \frac{1}{a'_{11}} + \frac{1}{a'_{12}} - \cdots \\ &= [0; a'_1, -a'_2, a'_3, a'_4, a'_5, -a'_6, a'_7, a'_8, a'_9, -a'_{10}, a'_{11}, \dots] \\ &= [0; a'_1 - 1, 1, a'_2 - 2, 1, a'_3 - 1, a'_4, a'_5 - 1, 1, a'_6 - 2, 1, a'_7 - 1, \\ &\qquad a'_8, a'_9 - 1, 1, a'_{10} - 2, 1, \dots] \\ &= [0; \overline{a'_{4k-3} - 1, 1, a'_{4k-2} - 2, 1, a'_{4k-1} - 1, a'_{4k}}]_{k=1}^{\infty}\,. \end{aligned}$$

Putting $a_{2k-1} = ur^k$ and $a_{2k} = vs^k$ $(k \geq 1)$ entails that

$$[0; \overline{ur^{2k-1} - 1, 1, vs^{2k-1} - 2, 1, ur^{2k} - 1, vs^{2k}}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} c_{1,n}}{\sum_{n=0}^{\infty} c_{0,n}}$$
$$= \frac{\sum_{n=0}^{\infty}(-1)^n\big((ur)^{-2n-1}(vs)^{-2n}\lambda_{2n} - (ur)^{-2n-2}(vs)^{-2n-1}\lambda_{2n+1}\big)}{\sum_{n=0}^{\infty}(-1)^n(u^{-2n}v^{-2n}\kappa_{2n} - u^{-2n-1}v^{-2n-1}\kappa_{2n+1})}.$$

7. Examples

Example 2. Put $r = s = a^2$ $(a > 1)$ and u is replaced by ua^{-1} in Theorem 1. Then for $n \geq 0$,

$$A_n = (-1)^n a^{-n(n+1)} \prod_{i=1}^{n} \big(a^{2i} + (-1)^i\big)$$

and

$$B_n = a^{-n(n-1)} \prod_{i=1}^{n} \big(a^{2i} + (-1)^i\big).$$

Hence, we obtain

$$[0; \overline{ua^{2k-1} - 1, 1, va^{2k} - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1}v^{-n}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i} - (-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}v^{-n}a^{-n^2}\prod_{i=1}^{n}(a^{2i} - (-1)^i)^{-1}}.$$

Example 3. Put $r = s = a$ $(a > 1)$ in Theorem 1. Then for $n \geq 0$,

$$A_n = (-1)^n a^{-n(n+1)/2} \prod_{i=1}^{n} \big(a^{i} + (-1)^i\big)$$

and

$$B_n = a^{-n(n-1)/2} \prod_{i=1}^{n} \big(a^{i} + (-1)^i\big).$$

Hence, we obtain

$$[0; \overline{ua^{k} - 1, 1, va^{k} - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} u^{-n-1}v^{-n}a^{-(n+1)(n+2)/2}\prod_{i=1}^{n}(a^{i} - (-1)^i)^{-1}}{\sum_{n=0}^{\infty}(-1)^n u^{-n}v^{-n}a^{-n(n+1)/2}\prod_{i=1}^{n}(a^{i} - (-1)^i)^{-1}}.$$

Example 4. Put $r = a$ $(a > 1)$ and $s = 1$ in Theorem 1. Then for $n \geq 1$,

$$A_{2n-1} = -a^{-n^2} \prod_{i=1}^{n} (a^{2i-1} - 1), \qquad B_{2n-1} = 0$$

and

$$A_{2n} = B_{2n} = a^{-n^2} \prod_{i=1}^{n} (a^{2i-1} - 1).$$

Hence, we obtain

$$[0; \overline{ua^{k} - 1, 1, v - 1}]_{k=1}^{\infty}$$
$$= \frac{\sum_{n=0}^{\infty} u^{-2n-1}v^{-2n}a^{-(n+1)^2}\prod_{i=1}^{n}(a^{2i} - 1)^{-1}}{\sum_{n=0}^{\infty}\big((uv)^{-2n}a^{-n^2} - (uv)^{-2n-1}a^{-(n+1)^2}\big)\prod_{i=1}^{n}(a^{2i} - 1)^{-1}}.$$

8. Comments

In our theorems we deal with the cases where two patterns of geometric series, ur, ur^2, $\dots$, ur^n, $\dots$ and vs, vs^2, $\dots$, vs^n, $\dots$, appear. Of course, it is possible to obtain the case where only one pattern of geometric series appears. However, as seen in [2], it becomes much more difficult to treat the case where more than two patterns of geometric series appear.

Acknowledgment

The author thanks Professor Edward Burger for his reviewing of the original manuscript.

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