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POLYHARMONIC SPLINES ON GRIDS $\mathbb{Z} \times a\mathbb{Z}^n$ AND THEIR LIMITS

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Abstract. Radial Basis Functions (RBF) have found a wide area of applications. We consider the case of polyharmonic RBF (called sometimes polyharmonic splines) where the data are on special grids of the form $\mathbb{Z} \times a\mathbb{Z}^n$ having practical importance. The main purpose of the paper is to consider the behavior of the polyharmonic interpolation splines I_a on such grids for the limiting process $a \to 0$, $a > 0$. For a large class of data functions defined on $\mathbb{R} \times \mathbb{R}^n$ it turns out that there exists a limit function I. This limit function is shown to be a *polyspline* of order p on strips. By the theory of polysplines we know that the function I is smooth up to order $2(p-1)$ everywhere (in particular, they are smooth on the hyperplanes $\{j\} \times \mathbb{R}^n$, which includes existence of the normal derivatives up to order $2(p-1)$) while the RBF interpolants I_a are smooth only up to the order $2p-n-1$. The last fact has important consequences for the data smoothing practice.

1. Introduction

Let us recall that a *polyharmonic cardinal spline* of order p is a tempered distribution u on the euclidean space $\mathbb{R}^n$ that is $2p-n-1$ continuously differentiable and such that

$$\Delta^p u(x) = 0 \text{ for all } x \in \mathbb{R}^n \setminus \mathbb{Z}^n.$$

Here $\mathbb{Z}^n$ is the lattice of points in $\mathbb{R}^n$ all of whose coordinates are integers, Δ^p is the p-th iterate of the Laplace operator defined as usual by $\Delta u = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2}$, and p is a natural number ≥ 1. Such distributions were considered in the fundamental work [14] of Madych and Nelson (see also [15]) where their basic properties were provided. One of the key results is the existence and uniqueness of solutions of the cardinal interpolation problem for fixed $p \in \mathbb{Z}$ with $2p \geq n+1$: given a sequence of numbers d_m, $m \in \mathbb{Z}^n$, of polynomial growth there exists a unique polyharmonic spline u of order p such that $u(m) = d_m$ for all $m \in \mathbb{Z}^n$.

For notational reasons it is more convenient for us to work in the euclidean space $\mathbb{R}^{n+1}$ instead of $\mathbb{R}^n$. It is well known, and we will provide the basic techniques further on, that the interpolation result of Madych and Nelson can be generalized to the situation where the lattice $\mathbb{Z}^{n+1}$ is replaced by a lattice Γ_a of the form

$$\Gamma_a := \mathbb{Z} \times a\mathbb{Z}^n, \tag{1}$$

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where a is a positive real number and $a\mathbb{Z}^n$ is the set $\{am : m \in \mathbb{Z}^n\}$. For a discrete subset Γ of $\mathbb{R}^{n+1}$ we define the set $SH^p\left(\mathbb{R}^{n+1}, \Gamma\right)$ of all tempered distributions u on $\mathbb{R}^{n+1}$ that are $2p-n-2$ continuously differentiable and such that

$$\Delta^p u(x) = 0 \qquad \text{for all } x \in \mathbb{R}^{n+1} \setminus \Gamma.$$

The main question we want to address is the following: what happens with the interpolation problem if $a > 0$ tends to zero? More precisely, assume that the data functions $d_j : \mathbb{R}^n \to \mathbb{R}$ with $j \in \mathbb{Z}$ are given. By the above there exists $I_a \in SH^p\left(\mathbb{R}^{n+1}, \mathbb{Z} \times a\mathbb{Z}^n\right)$ such that

$$I_a(j, am) = d_j(am) \qquad \text{for all } m \in \mathbb{Z}^n, j \in \mathbb{Z}. \tag{2}$$

We ask for convergence of the type

$$I_a(x) \to I(x) \qquad \text{for } a \to 0,$$

where $I(x)$ is an appropriate function of $x \in \mathbb{R}^{n+1}$. Assuming that $d_j = 0$ for $|j| > N$ and that the data functions d_j and their Fourier transforms $\widehat{d_j}$ for $j = -N, \ldots, N$ are decaying fast enough (consult Theorem 11) we will establish the existence of a limit function $I(x)$ where the convergence is pointwise. In the case of $N = 0$, i.e., if the data are nonzero on a single hyperplane, the limit function is described by the formula (further we use the notation $x = (t, y) \in \mathbb{R} \times \mathbb{R}^n$)

$$I(t, y) = \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \frac{e^{i\langle y, \xi\rangle} e^{its} \widehat{d_0}(\xi)}{\left(s^2 + |\xi|^2\right)^p \sum_{k \in \mathbb{Z}} \frac{1}{\left((s+2\pi k)^2 + |\xi|^2\right)^p}} ds d\xi.$$

By our results in [1], [2] (see also [7, Section 9.1]), the above integral is a *cardinal polyspline of order p on strips* in the sense of the following definition.

Definition 1. The continuous function $u(t, y)$ defined on $\mathbb{R}^{n+1}$ is called a *cardinal polyspline of order p on strips* if $u(t, y)$ is $2p-2$ times continuously differentiable on the whole $\mathbb{R}^{n+1}$ (in particular, this means that the derivatives in the normal direction, coinciding with t, are everywhere continuous up to order $2p-2$), and for all $(t, x) \in (j, j+1) \times \mathbb{R}^n$ the function $u(t, y)$ satisfies the equation $\Delta^p u(t, x) = 0$.

First of all, let us note an immediate consequence of the above definition: according to the local regularity theorem for elliptic equations, $u \in C^\infty$ inside every strip $(j, j+1) \times \mathbb{R}^n$; cf., e.g., [5].

The conditions that we invoke and the data sets in Theorem 11 are slightly stronger than the conditions necessary to guarantee the interpolation results for polysplines. For the convenience of the reader we shall provide the interpolation result for polysplines from the references [1], [2], although the whole result itself is not needed in the paper, except for the conclusion that the limit function is C^{2p-2}.

For formulating the result we need the spaces $B_s(\mathbb{R}^n)$ of all tempered distributions f whose Fourier transforms $\widehat{f}$ are measurable functions and satisfy

$$\|f\|_s := \int_{\mathbb{R}^n} \left|\widehat{f}(\xi)\right| (1 + |\xi|^s)\, d\xi < \infty$$

(see Definition 10.1.6 in Hörmander [5]).

Theorem 2. *Let the data functions f_j be given such that $f_j \in B_{2p-2}(\mathbb{R}^n) \cap C^{2p-2}(\mathbb{R}^n)$, and assume that the growth condition*

$$\|f_j\|_{2p-2} \le C\left(1 + |j|^\gamma\right) \qquad \textit{for all } j \in \mathbb{Z} \tag{3}$$

holds for some $\gamma \geq 0$. Then there exists a polyspline S of order p on strips, satisfying the interpolation conditions

$$S(j,y) = f_j(y) \qquad \textit{for all } y \in \mathbb{R}^n,$$

as well as the growth estimate

$$|S(t,y)| \leq D\left(1+|t|^{\gamma}\right) \qquad \textit{for all } y \in \mathbb{R}^n.$$

We mention also Theorem 9.3 and Theorem 9.4 in [7] where the case of compactly supported and periodic data functions f_j in Sobolev and Hölder spaces has been studied.

The main question treated in the present paper concerning the relation between the polyharmonic splines and polysplines is motivated by the existence of many practical data sets which are collected by satellites, airplanes, scanners, etc. (see [7, Chapter 6], [12]), where the sample points lie on whole curves (in particular, on parallel lines). Usually such data are dense enough on the data tracks, and if one applies the interpolation polyharmonic splines to them, the above convergence question for $a \longrightarrow 0$ makes sense. The practical experience with polyharmonic and other Radial Basis Functions shows that there appear artifacts (called sometimes pock marks) in the immediate vicinity of the data points which are apparently due to the lower smoothness of the polyharmonic splines. On the other hand, the interpolation polysplines do not exhibit such effects; see the comparison in [7, Chapter 6].

Let us introduce some notation and terminology: the Fourier transform of a function $f : \mathbb{R}^n \to \mathbb{C}$ is defined by

$$\widehat{f}(\omega) := \int_{\mathbb{R}^n} e^{-i\langle x,\omega\rangle} f(x)\, dx.$$

We shall often apply the *Poisson summation formula* for a function $f : \mathbb{R}^n \to \mathbb{C}$, which reads as

$$\frac{1}{(2\pi b)^n} \sum_{m\in\mathbb{Z}^n} \widehat{f}\left(\frac{m}{b}\right) e^{\frac{i}{b}\langle m,\xi\rangle} = \sum_{m\in\mathbb{Z}^n} f(\xi + 2\pi b m).$$

Let us recall that the Poisson summation formula holds (see [16], p. 252, Corollary 2.6) if

$$\begin{cases} |f(x)| \leq A(1+|x|)^{-n-\delta} \text{ and } \left|\widehat{f}(\omega)\right| \leq A(1+|\omega|)^{n-\delta}, \\ f \text{ is continuous.} \end{cases} \tag{4}$$

In particular, for $a > 0$ condition (4) implies

$$\sum_{m\in\mathbb{Z}^n} |f(am)| \leq A \sum_{m\in\mathbb{Z}^n} (1+|am|)^{-n-\delta} < \infty. \tag{5}$$

Throughout the paper it will be assumed that $2p \geq n+2$ since this is the condition providing existence of interpolation polyharmonic splines [14]. On the other hand, let us remark that the existence of the interpolation polysplines needs no such restriction; see [1], [7].

2. Interpolation with polyharmonic splines

We say that $L_{p,a}$ is a *fundamental polyharmonic spline of order p for the grid* Γ_a whenever $L_{p,a}$ is in $SH_p\left(\mathbb{R}^{n+1},\Gamma_a\right)$ and

$$L_{p,a}(0)=1 \text{ and } L_{p,a}(\gamma)=0 \text{ for all } \gamma\in\Gamma_a\setminus\{0\}.$$

Let us recall how fundamental polyharmonic splines can be constructed by Fourier analysis, [14]: the fundamental solution of Δ^p is given by

$$E_p(x)=\begin{cases} c_{n+1,p}\,|x|^{2p-(n+1)} & \text{for odd } n+1,\\ c_{n+1,p}\,|x|^{2p-(n+1)}\log|x| & \text{for even } n+1,\end{cases}$$

where the norming constant $c_{n+1,p}$ is chosen such that $\Delta^p E_p(x)=\delta(x)$. The generalized Fourier transform of E_p is given up to a factor by

$$\widehat{\varphi}(\omega):=|\omega|^{-2p}. \tag{6}$$

For the lattice $\Gamma_a=\mathbb{Z}\times a\mathbb{Z}^n$ (assuming $a>0$) define the dual lattice Γ_a^* of Γ_a by

$$\Gamma_a^*=2\pi\mathbb{Z}\times\frac{2\pi}{a}\mathbb{Z}^n.$$

Fundamental for interpolation problems is the function

$$S_{\varphi,a}(\omega):=\sum_{\gamma^*\in\Gamma_a^*}\widehat{\varphi}(\omega+\gamma^*). \tag{7}$$

A basic theorem in the theory of Radial Basis Functions tells us that

$$L_{p,a}(x)=\frac{a^n}{(2\pi)^{n+1}}\int_{\mathbb{R}^{n+1}}e^{i\langle x,\omega\rangle}\frac{\widehat{\varphi}(\omega)}{S_{\varphi,a}(\omega)}d\omega \tag{8}$$

provides a fundamental polyharmonic spline of order p for the grid Γ_a; see, e.g., [14], [3], [4], [6]. Moreover, $L_{p,a}$ is of exponential decay: there exist constants $C>0$ and $\eta>0$ such that

$$|L_{p,a}(x)|\le Ce^{-\eta|x|} \text{ for all } x\in\mathbb{R}^{n+1}.$$

The last rests upon the fact that for each $a>0$ there exists $\varepsilon>0$ such that the functions

$$\omega\mapsto\frac{1}{S_{\varphi,a}(\omega)} \text{ and } \omega\mapsto\frac{\widehat{\varphi}(\omega)}{S_{\varphi,a}(\omega)}$$

defined for $\omega\in\mathbb{R}^{n+1}\setminus\Gamma_a^*$ can be extended analytically to the strip

$$S(\varepsilon):=\left\{z=(z_1,\dots,z_{n+1})\in\mathbb{C}^{n+1}:|\operatorname{Im}z_k|<\varepsilon \text{ for } k=1,\dots,n+1\right\}.$$

In particular, there exists a constant $M_a>0$ such that for all $\omega\in\mathbb{R}^{n+1}$,

$$0\le\frac{1}{S_{\varphi,a}(\omega)}\le M_a \tag{9}$$

holds.

Assume now that $(d_\gamma)_{\gamma\in\mathbb{Z}\times a\mathbb{Z}^n}$ is a sequence of polynomial growth: define

$$I_a(x):=\sum_{\gamma\in\Gamma_a}d_\gamma L_{p,a}(x-\gamma). \tag{10}$$

By standard techniques provided in the above references it follows that I_a belongs to $SH^p\left(\mathbb{R}^{n+1},\mathbb{Z}\times a\mathbb{Z}^n\right)$ and satisfies $I_a(\gamma)=d_\gamma$ for all $\gamma\in\Gamma_a=\mathbb{Z}\times a\mathbb{Z}^n$, and I_a is of polynomial growth as well.

We specialize our discussion to the case of data functions $d_j : \mathbb{R}^n \to \mathbb{C}$ equal to zero for $j \in \mathbb{Z}, j \neq 0$ and we put $f := d_0$. Then we define $L_{p,a,f}$ as the polyharmonic spline of order p for the grid Γ_a satisfying

$$L_{p,a,f}\left((0, am)\right) = f\left(am\right) \text{ and } L_{a,p,f}\left((j, am)\right) = 0 \tag{11}$$

for $j \in \mathbb{Z} \setminus \{0\}$ and for all $m \in \mathbb{Z}^n$. By (10), $L_{p,a,f}$ is given by

$$L_{p,a,f}\left(t, y\right) := \sum_{m \in \mathbb{Z}^n} f\left(am\right) L_{p,a}\left((t, y - am)\right). \tag{12}$$

Proposition 3. *Suppose that $f : \mathbb{R}^n \to \mathbb{C}$ satisfies (4) and let $a > 0$. Then the polyharmonic spline $L_{p,a,f}$ of order p for the grid Γ_a given in (12) satisfies the equality*

$$L_{p,a,f}\left(t, y\right) = \frac{1}{2\pi} \int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{its} e^{i\langle y, \xi\rangle} \left(\sum_{m \in \mathbb{Z}^n} \widehat{f}\left(\xi + \frac{2\pi}{a} m\right)\right) \frac{\widehat{\varphi}\left(s, \xi\right)}{S_{\varphi,a}\left(s, \xi\right)} ds d\xi. \tag{13}$$

Proof. By (8) and (12) it follows that

$$L_{p,a,f}\left(t, y\right) = \sum_{m \in \mathbb{Z}^n} f\left(am\right) \frac{a^n}{\left(2\pi\right)^{n+1}} \int_{\mathbb{R}^{n+1}} e^{i\langle (t, y - am), \omega\rangle} \frac{\widehat{\varphi}\left(\omega\right)}{S_{\varphi,a}\left(\omega\right)} d\omega.$$

We put $\omega = (s, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and by the theorem of Fubini we obtain

$$L_{p,a,f}\left(t, y\right) = \frac{a^n}{\left(2\pi\right)^{n+1}} \sum_{m \in \mathbb{Z}^n} f\left(am\right) \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{its} e^{i\langle y, \xi\rangle} e^{-i\langle am, \xi\rangle} \frac{\widehat{\varphi}\left(s, \xi\right)}{S_{\varphi,a}\left(s, \xi\right)} d\xi ds.$$

By estimate (5) we can interchange summation and integration; hence

$$L_{p,a,f}\left(t, y\right) = \frac{a^n}{\left(2\pi\right)^{n+1}} \int_{\mathbb{R}^n} e^{i\langle y, \xi\rangle} \left(\sum_{m \in \mathbb{Z}^n} f\left(am\right) e^{-i\langle am, \xi\rangle}\right) \int_{\mathbb{R}} e^{its} \frac{\widehat{\varphi}\left(s, \xi\right)}{S_{\varphi,a}\left(s, \xi\right)} ds d\xi. \tag{14}$$

The Poisson summation formula shows that

$$\frac{a^n}{\left(2\pi\right)^n} \sum_{m \in \mathbb{Z}^n} f\left(am\right) e^{-i\langle am, \xi\rangle} = \sum_{m \in \mathbb{Z}^n} \widehat{f}\left(\xi + \frac{2\pi}{a} m\right). \tag{15}$$

Inserting (15) in (14) implies (13). The proof is complete. □

In the following we want to give a compact formula for $L_{p,a,f}$.

We provide the following definition:

Definition 4. Assume that $2p \geq n + 1$. For all $(s, \xi) \in \mathbb{R}^{n+1} \setminus \left(\{0\} \times \frac{2\pi}{a}\mathbb{Z}^n\right)$ we define the function

$$B_{a,p}\left(s, \xi, y\right) := \sum_{m \in \mathbb{Z}^n} \frac{1}{\left(s^2 + \left|\xi - 2\pi a^{-1} m\right|^2\right)^p} e^{-\frac{2\pi}{a} i\langle y, m\rangle}. \tag{16}$$

Clearly, for each $s \neq 0$ the function $\xi \mapsto B_{a,p}\left(s, \xi, y\right)$ is well-defined since $2p \geq n + 1$. Note that $B_{a,p}\left(s, \xi, y\right)$ has poles at $\{0\} \times \frac{2\pi}{a}\mathbb{Z}^n$. Since

$$\left|B_{a,p}\left(s, \xi, y\right)\right| \leq B_{a,p}\left(s, \xi, 0\right) \leq S_{\varphi,a}\left(s, \xi\right)$$

we have for all $\xi \in \mathbb{R}^n$ and $s \neq 0$,

$$0 \leq \left| \frac{B_{a,p}\left(s, \xi, y\right)}{S_{\varphi,a}\left(s, \xi\right)} \right| \leq \frac{B_{a,p}\left(s, \xi, 0\right)}{S_{\varphi,a}\left(s, \xi\right)} \leq 1. \tag{17}$$

The following theorem contains a basic result, which has the meaning that the map $f \mapsto L_{p,a,f}$ is in fact a *pseudo-differential operator.* For its proof we need some subtle estimates, proved later in Theorem 6, which show the integrability of the symbol (of the operator)

$$\frac{B_{a,p}(s,\xi,y)}{S_{\varphi,a}(s,\xi)}$$

with respect to the variable $s \in \mathbb{R}$.

Theorem 5. *Suppose that $f : \mathbb{R}^n \to \mathbb{C}$ satisfies* (4)*, let $a > 0$, and let $B_{a,p}$ be as in* (16)*. Then the polyharmonic spline $L_{p,a,f}$ of order p for the grid Γ_a in* (12) *satisfies the equality*

$$L_{p,a,f}(t,y) = \frac{1}{2\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{its}e^{i\langle y,\xi\rangle}\widehat{f}(\xi)\frac{B_{a,p}(s,\xi,y)}{S_{\varphi,a}(s,\xi)}dsd\xi. \tag{18}$$

Proof. We interchange summation and integration in (13) and obtain

$$L_{p,a,f}(t,y) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{i\langle y,\xi\rangle}\widehat{f}\left(\xi+\frac{2\pi}{a}m\right)\ e^{its}\frac{\widehat{\varphi}(s,\xi)}{S_{\varphi,a}(s,\xi)}dsd\xi.$$

Substituting $\zeta = \xi + \frac{2\pi}{a}m$ yields

$$L_{p,a,f}(t,y) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{i\left\langle y,\zeta-\frac{2\pi}{a}m\right\rangle}e^{its}\widehat{f}(\zeta)\frac{\widehat{\varphi}\left(s,\zeta-\frac{2\pi}{a}m\right)}{S_{\varphi,a}\left(s,\zeta-\frac{2\pi}{a}m\right)}dsd\zeta. \tag{19}$$

By periodicity, $S_{\varphi,a}\left(s,\zeta-\frac{2\pi}{a}m\right) = S_{\varphi,a}(s,\zeta)$. Again we want to interchange summation and integration, which will now be a more subtle problem: for $m = (m_1,\ldots,m_n)$ we define

$$B_N(s,\xi,y) := \sum_{m\in\mathbb{Z}^n,|m_i|\le N}\frac{1}{\left(s^2+|\xi-2\pi a^{-1}m|^2\right)^p}e^{-\frac{2\pi}{a}i\langle y,m\rangle},$$

which converges pointwise to $B_{a,p}(s,\xi,y)$ for fixed $a > 0$ and $y \in \mathbb{R}^n$. Then

$$L_{p,a,f}(t,y) = \lim_{N\to\infty}\frac{1}{2\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{i\langle y,\zeta\rangle}e^{its}\widehat{f}(\xi)\frac{B_N(s,\xi,y)}{S_{\varphi,a}(s,\xi)}dsd\xi.$$

We have to find an integrable majorant for the integrand. Clearly,

$$h_N(s,\xi,y) := \left|e^{i\langle y,\zeta\rangle}e^{its}\widehat{f}(\xi)\frac{B_N(s,\xi,y)}{S_{\varphi,a}(s,\xi)}\right| \le \frac{B_{a,p}(s,\xi,0)}{S_{\varphi,a}(s,\xi)}\left|\widehat{f}(\xi)\right|.$$

In Theorem 6 in the next section we show that there exists a constant $C > 0$ such that for all $s \neq 0$, all $\xi, y \in \mathbb{R}^n$, and all $0 < a \le 1$,

$$|B_{a,p}(s,\xi,0)| \le \frac{C}{s^{2p}}(1+|s|)^n.$$

By (9) and (17) we obtain

$$h_N(s,\xi,y) \le CM_a\left|\widehat{f}(\xi)\right|(1+|s|)^n s^{-2p} \qquad \text{for } |s|\ge 1, \quad \xi\in\mathbb{R}^n; \tag{20}$$

for $|s| \le 1$ we apply inequality (9). Since the right-hand side of (20) is integrable the proof is complete. □

3. Estimates of the function $B_a(s,\xi,0)$

In this section we assume that p is a positive integer such that $2p \geq n+1$. From (16) follows immediately

$$|B_{a,p}(s,\xi,y)| \leq B_{a,p}(s,\xi,0) = \sum_{m\in\mathbb{Z}^n} \frac{1}{\left(s^2+|\xi-2\pi a^{-1}m|^2\right)^p}.$$

The following estimate is crucial for the proof of our main result.[1]

Theorem 6. *Let $p \in \mathbb{N}$ satisfy $2p \geq n+1$. Then there exists a constant $C > 0$ such that for all $|s| > 0$, all $\xi \in \mathbb{R}^n$, and for all $0 < a \leq 1$,*

$$|B_{a,p}(s,\xi,y)| \leq \frac{C}{s^{2p}}(1+|s|)^n$$

holds.

Proof. Let $\xi = (\xi_1,\ldots,\xi_n)$. Since $\xi \longmapsto B_{a,p}(s,\xi,0)$ is $2\pi a^{-1}\mathbb{Z}^n$-periodic we can assume that $|\xi_i| \leq \pi a^{-1}$ for $i = 1,\ldots,n$. Then $|\xi| \leq \sqrt{n}\cdot\pi a^{-1}$ and

$$\left|\xi - 2\pi a^{-1}m\right| \geq 2\pi a^{-1}|m| - \sqrt{n}\pi a^{-1} = \pi a^{-1}\left(2|m| - \sqrt{n}\right).$$

Hence for $m \in \mathbb{Z}^n$ with $|m| \geq \sqrt{n}$ we have $\left|\xi - 2\pi a^{-1}m\right|^2 \geq \pi^2 a^{-2}|m|^2$. Therefore $|B_{a,p}(s,\xi,0)| \leq I_1 + I_2$ where $I_1 := \sum_{m\in\mathbb{Z}^n,|m|<\sqrt{n}} \frac{1}{\left(s^2+|\xi-2\pi a^{-1}m|^2\right)^p}$ and $I_2 := \sum_{m\in\mathbb{Z}^n,|m|\geq\sqrt{n}} \frac{1}{\left(s^2+\pi^2a^{-2}|m|^2\right)^p}$. Since $s^2 + \left|\xi - 2\pi a^{-1}m\right|^2 \geq s^2$ we can estimate

$$\frac{1}{\left(s^2+|\xi-2\pi a^{-1}m|^2\right)^p} \leq \frac{1}{s^{2p}},$$

and this gives a simple estimate for the finite sum I_1. The sum I_2 can be estimated by

$$I_2 \leq \frac{1}{s^{2p}} \sum_{m\in\mathbb{Z}^n,m\neq 0} \frac{1}{\left(1+\pi^2(as)^{-2}|m|^2\right)^p}.$$

A comparison via an integral will give us the second estimate: by a symmetry argument it suffices to consider $m \in \mathbb{Z}^n$ with nonnegative components. Since the function $x \longmapsto \left(1+\pi^2(as)^{-2}x^2\right)^{-p}$ is decreasing one obtains an estimate

$$\sum_{m\in\mathbb{Z}^n,m_i>0 \text{ for all } i=1,..,n} \frac{1}{\left(1+\pi^2(as)^{-2}|m|^2\right)^p} \leq \int_{\mathbb{R}^n} \frac{1}{\left(1+\pi^2(as)^{-2}|x|^2\right)^p}dx.$$

By a simple substitution argument we can estimate the integral by $C(1+s)^n$. The summation over all $m \in \mathbb{Z}^n$ such that $m_i = 0$ for fixed i can be reduced to a lower-dimensional case. □

[1]We thank the anonymous referee for providing us with the present simple and elegant proof.

4. The main result

In this section we prove our main result. Let f be a function representing the data on the hyperplane $\mathbb{R}^n$ (here we identify $\mathbb{R}^n$ with $\{0\}\times\mathbb{R}^n$). Recall that $L_{p,a,f}$ is a polyharmonic spline of order p for the grid Γ_a given by (18):

$$L_{p,a,f}(t,y) := \frac{1}{2\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{i\langle y,\zeta\rangle} e^{its}\widehat{f}(\xi)\frac{B_{a,p}(s,\xi,y)}{S_{\varphi,a}(s,\xi)}dsd\xi.$$

Moreover for $\omega=(s,\xi)$ with $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^n$, by (7) it follows that

$$S_{\varphi,a}(\omega) = \sum_{\gamma^*\in\Gamma_a^*}\widehat{\varphi}(\omega+\gamma^*) = \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n}\frac{1}{\left((s+2\pi k)^2+|\xi-2\pi a^{-1}m|^2\right)^p}.$$

It is clear that for $\omega\notin\Gamma_a^*$ the convergence of the sum is locally uniform since $2p\geq n+2$.

For $(s,\xi)\in\mathbb{R}^{n+1}\setminus(\mathbb{Z}\times\{0\})$ let us define a function S_p by putting

$$S_p(s,\xi) := \sum_{k\in\mathbb{Z}}\frac{1}{\left((s+2\pi k)^2+|\xi|^2\right)^p}.$$

Our main result, Theorem 9, says that the polyharmonic splines $L_{p,a,f}(t,y)$ converge pointwise to the function

$$L_f(t,y) = \frac{1}{2\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{i\langle y,\xi\rangle}e^{its}\frac{\widehat{f}(\xi)}{\left(s^2+|\xi|^2\right)^p S_p(s,\xi)}dsd\xi \tag{21}$$

provided that

$$\|f\|_{2p} := \int_{\mathbb{R}^n}\left|\widehat{f}(\xi)\right|\left(1+|\xi|^{2p}\right)d\xi<\infty.$$

In [1], [2] we have shown that L_f is indeed a polyspline of order p (see the Introduction); in particular, L_f is $2p-2$ times continuously differentiable. Furthermore, the following interpolation property holds: for all $y\in\mathbb{R}^n$,

$$L_f(0,y)=f(y) \text{ and } L_f(j,y)=0 \qquad j\in\mathbb{Z},\ j\neq 0. \tag{22}$$

We mention that the polyspline L_f satisfies the following decay estimate: for every multi-index α such that $|\alpha|\leq 2(p-1)$, the partial derivative D^α satisfies

$$|D^\alpha L_f(t,y)|\leq Ce^{-\eta|t|}\|f\|_{|\alpha|} \qquad \text{for all } t\in\mathbb{R},\ y\in\mathbb{R}^n,$$

for some constant $C>0$ and $\eta>0$; see [2].

Lemma 7. *For $a\longrightarrow 0$ the function $B_{a,p}(s,\xi,y)$ converges uniformly on compact sets (of the variables (s,ξ,y)) to $\widehat{\varphi}(s,\xi,y)$.*

The proof is straightforward.

Theorem 8. *Suppose that $2p>n+1$. Then $S_{\varphi,a}(s,\xi)$ converges uniformly on compacta K in $\mathbb{R}^{n+1}$ such that $K\cap(2\pi\mathbb{Z}\times\mathbb{R}^n)=\varnothing$ to the function*

$$S_p(s,\xi) = \sum_{k\in\mathbb{Z}}\frac{1}{\left((s+2\pi k)^2+|\xi|^2\right)^p}.$$

Proof. Let us put $S_{\varphi,a}(s,\xi) = \sum_{m\in\mathbb{Z}^n} S_{p,m}(s,\xi,a)$, where

$$S_{p,m}(s,\xi,a) := \sum_{k\in\mathbb{Z}} \frac{1}{\left((s+2\pi k)^2 + |\xi + 2\pi m a^{-1}|^2\right)^p}.$$

Note that $S_{p,m}(s,\xi,a) = B_{1,p}\left(\left|\xi + 2\pi m a^{-1}\right|, s, 0\right)$, where $n = 1$. By Theorem 6 (applied to the case $n = 1$) there exists $C > 0$ such that for all $s \in \mathbb{R}$ and for all $\left|\xi + 2\pi m a^{-1}\right| \neq 0$ (where $\xi \in \mathbb{R}^n$, $m \in \mathbb{Z}^n$, $a > 0$)

$$|S_{p,m}(s,\xi,a)| \leq C \frac{1 + \left|\xi + 2\pi m a^{-1}\right|}{|\xi + 2\pi m a^{-1}|^{2p}}. \tag{23}$$

Let $K \subset \mathbb{R}^{n+1}$ be a compact set such that $K \cap (2\pi\mathbb{Z} \times \mathbb{R}^n) = \varnothing$; choose a small $a_0 > 0$ such that $|a\xi| < \frac{1}{2}$ for all $0 < a \leq a_0 < \frac{\pi}{2}$ and $(s,\xi) \in K$. Then for $|m| \geq 1$ we have $|a\xi| \leq \frac{1}{2}|2\pi m|$, which implies

$$\left|2\pi m a^{-1} + \xi\right| \geq a^{-1}(2\pi|m| - |a\xi|) \geq a^{-1}\pi|m| > 2.$$

Hence, for all $(s,\xi) \in K$, all $0 < a < a_0$, and all $m \in \mathbb{Z}^n, m \neq 0$,

$$|S_{p,m}(s,\xi,a)| \leq D \frac{a^{2p-1}}{|m|^{2p}}(a + |a\xi + 2\pi m|) \leq D_2 \frac{a^{2p-1}}{|m|^{2p-1}}$$

holds, where D_2 is a suitable constant. It follows that

$$\left| S_{\varphi,a}(s,\xi) - \sum_{k\in\mathbb{Z}} \frac{1}{\left((s+2\pi k)^2 + |\xi|^2\right)^p} \right| \leq a^{2p-1} D_2 \sum_{m\in\mathbb{Z}^n, m\neq 0} \frac{1}{|m|^{2p-1}}.$$

Since $2p - 1 > n$, the last series converges. The proof is finished. □

Theorem 9. *Suppose that $f : \mathbb{R}^n \to \mathbb{C}$ satisfies condition* (4) *and that*

$$\widehat{f}(\xi)\left(1 + |\xi|^2\right)^p$$

is integrable. Then the polyharmonic splines $L_{p,a,f}$ of order p for the grid Γ_a defined in (12) *converge pointwise for $a \to 0$ to the function defined in* (21)*. Moreover, the function $L_{p,f}$ in* (21) *is a polyspline in the sense of Definition* 1.

Proof. By Theorem 8 we know that, uniformly on compact sets for $a \longrightarrow 0$,

$$h_a(s,\xi) := e^{i\langle y,\zeta\rangle} e^{its} \widehat{f}(\zeta) \frac{B_{a,p}(s,\xi,y)}{S_{\varphi,a}(s,\xi)} \to \frac{e^{i\langle y,\zeta\rangle} e^{its} \widehat{f}(\zeta)}{\left(s^2 + |\xi|^2\right)^p S_p(s,\xi)}$$

holds. The proof will be finished by an application of Lebesgue's convergence theorem for $a \to 0$, where the majorant for $h_a(s,\xi)$ will be provided below. By Theorem 6 there exists a constant $C > 0$ such that for all $|s| \neq 0$, all $\xi \in \mathbb{R}^n$, and all $a > 0$,

$$|B_{a,p}(s,\xi,y)| \leq C s^{-2p}(1 + |s|)^n.$$

Furthermore, by Proposition 10, $S_{\varphi,a}(s,\xi) \geq S_p(s,\xi) \geq \left(\pi^2 + |\xi|^2\right)^{-p}$ for all $s \in \mathbb{R}\setminus\{0\}$ and for all $\xi \in \mathbb{R}^n$. Using (17) for $|s| \leq 1$ and the above estimates for $|s| \geq 1$ one obtains the result to be proved. □

Proposition 10. *For all $\xi \in \mathbb{R}^n$ and $s \in \mathbb{R}$ with $s \neq 0$ we have*

$$\left| \frac{1}{S_p(s,\xi)} \right| \leq \left(\pi^2 + |\xi|^2 \right)^p .$$

The proof is trivial and uses the 2π-periodicity in s.

We now turn to the problem when finitely many data functions $d_j : \mathbb{R}^n \to \mathbb{C}$ are given, i.e., for some N we have $d_j = 0$ for $|j| > N$. Then

$$I_{p,a}(t,y) := \sum_{j=-N}^{N} L_{p,a,d_j}(t-j,y), \tag{24}$$

is a polyharmonic spline of order p on the grid Γ_a such that

$$I_{p,a}(j,am) = d_j(am)$$

for all $j = -N, \ldots, N$ and for all $m \in \mathbb{Z}^n$. For $|j| > N$ we have clearly $I_{p,a}(j,am) = 0$ for all $m \in \mathbb{Z}^n$. Similarly, we can define

$$J_p(t,y) := \sum_{j=-N}^{N} L_{p,d_j}(t-j,y)$$

which is a polyspline interpolating the data functions $d_j : \mathbb{R}^n \to \mathbb{C}$ for $j = -N, \ldots, N$, i.e.,

$$J_p(j,y) = d_j(y)$$

for all $y \in \mathbb{R}^n$ and for $j = -N, \ldots, N$. Since, by Theorem 9, each summand $L_{p,a,d_j}(t-j,y)$ converges to $L_{p,d_j}(t-j,y)$ the proof of the next theorem is obvious.

Theorem 11. *Suppose $d_j : \mathbb{R}^n \to \mathbb{C}$ satisfies condition* (4) *and that*

$$\widehat{d_j}(\xi) \left(1 + |\xi|^2 \right)^p$$

are integrable for $j = -N, \ldots, N$, and $d_j = 0$ for $|j| > N$. Then the polyharmonic splines $I_{p,a,,f}$ of order p on the grid Γ_a defined in (24) *converge pointwise for $a \to 0$ to the polyspline $J_p(t,y)$.*

The results of the present paper may be further improved in at least two directions. In the first direction, one may consider data $d_j(x)$ that are nonzero for infinitely many $j \in \mathbb{Z}$. This brings so far some new technical problems, which will be resolved in a forthcoming paper.

In the second direction, one recalls that the existence of Madych's interpolation polyharmonic splines has been proved in [15] for data that have a polynomial growth. It is very natural to ask if we are able to prove the results of the present paper for data $d_j(y)$ that have a polynomial growth in y. This problem is not trivial even in the case of finitely many nonzero d_j's (i.e., nonzero for $|j| \leq N$ for some fixed N), and it is intimately related to studying solutions of elliptic PDEs on noncompact domains. This is another problem that requires a further development.

A lot more advanced program for further research is to consider the convergence problem for data that do not lie on regular grids. The lack of explicit representation for the polyharmonic splines for really scattered data needs to develop other more subtle techniques which would correspond to the "a priori" estimates for elliptic boundary value problems.

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