

**ERROR ESTIMATE FOR THE APPROXIMATION OF  
NONLINEAR CONSERVATION LAWS ON BOUNDED DOMAINS  
BY THE FINITE VOLUME METHOD**

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ABSTRACT. In this paper we derive a priori and a posteriori error estimates for cell centered finite volume approximations of nonlinear conservation laws on polygonal bounded domains. Numerical experiments show the applicability of the a posteriori result for the derivation of local adaptive solution strategies.

1. INTRODUCTION

Let  $\Omega$  be an open convex polygonal bounded domain in  $\mathbb{R}^d$ ,  $d = 2, 3$ , endowed with the Euclidean norm  $|\cdot|$ , and let  $T \in \mathbb{R}^+$ . We consider the following initial boundary value problem for nonlinear scalar conservation laws:

$$\begin{aligned} (1) \quad & c_t + \nabla \cdot \mathbf{F}(x, t, c) = 0 \text{ in } \Omega \times (0, T), \\ (2) \quad & c(\cdot, 0) = c_0 \text{ in } \Omega, \\ (3) \quad & c(x, t) = \bar{c}(t, x) \text{ in } \partial\Omega \times (0, T). \end{aligned}$$

The flux in equation (1) is given by the function  $\mathbf{F} \in \mathcal{C}^1(\Omega \times (0, T) \times \mathbb{R}; \mathbb{R}^d)$ ; the functions  $c_0 \in L^\infty(\Omega)$  and  $\bar{c} \in L^\infty(\partial\Omega \times (0, T))$  are the initial and boundary data of the problem (1)–(3), respectively.

The finite volume methods are known to be well suited for the discretization of conservation laws. A basic account for this claim is the fact that, by construction, they respect the conservation principle which constitutes the root of equation (1). Indeed, the evolution of the discrete unknown  $c_K$  in each control volume  $K$  is given by the equation

$$(4) \quad |K| c_K^{n+1} = |K| c_K^n + \Delta t^n \sum_{\sigma \in \partial K} Q_\sigma^n$$

in which we denote abusively by  $\partial K$  the set of faces of  $K$  and where  $|K|$  is the volume of  $K$ . Equation (4) is the expression of the fact that the discrete evolution of  $c_K$  is governed by the values of the discrete fluxes  $Q_\sigma^n$  across the boundary of  $K$  in the time interval  $[t^n, t^{n+1}]$ . It is the choice of these numerical fluxes  $Q_\sigma^n$  that determine the finite volume method. In what follows, we will specifically consider three point finite volume schemes with monotone fluxes (see (13)–(17)).

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This category of schemes encloses all relevant first order three point finite volume schemes.

Since both (1) and (4) are evolution equations, the main features of the analysis of conservation laws and of their approximations by the finite volume method already appear in the context of the Cauchy problem, i.e.,  $\Omega = \mathbb{R}^d$ , and no boundary conditions have to be taken into account. The order of accuracy of the finite volume method for the Cauchy problem is one of these well known features: the first given a priori error estimate is the (sharp)  $h^{1/2}$  ( $h$  being the size of the mesh) estimate of Kuznetsov [Kuz76] in the 1D case. This estimate remains valid on structured meshes in  $\mathbb{R}^d$  while, for finite volume schemes on unstructured meshes, the lack of an uniform  $BV$  estimate on the numerical solution leads to an error estimate of reduced order  $h^{1/4}$  [KNR95, BCV95, CCL95a, EGH00]. Still, in the context of the Cauchy problem, refined error estimates have been given (and their sharpness analyzed), according to the genuine nonlinearity of the flux, to the structure of the entropy solution to (1)–(2), or to the nature of the waves in the solution. We refer to the discussion and compilation made by T. Tang on that profuse subject [Tan01].

For practical applications, a posteriori error estimates are even more important than just convergence rates. Such estimates allow us to extract error indicator information that can be used in order to derive efficient self-adaptive strategies for the finite volume schemes. A posteriori error estimates for finite volume approximations to the Cauchy problem were first derived by Tadmor [Tad91] in one space dimension, and by Cockburn and Gau [CG95] in the multi-dimensional case. A localized estimate for general flux functions and the derivation of self-adaptive schemes was given in [KO00]. Further results for finite volume approximations to the Cauchy problem were obtained in [GM00, KO03, KKP02], while finite element approximations to the Cauchy problem were studied in [JS95, SH95, HMSW99, HH02]. We emphasize that up to now no a posteriori results are available for approximations of the initial boundary value problem (1)–(3).

Although the study of the finite volume method applied to the Cauchy problem (1), (2) has led to the understanding of most of the mechanisms which govern the accuracy of this numerical method of approximation, the initial boundary value problem (1)–(3) has its own interest (for the simple and major reason that the domains under consideration in practical applications can be bounded domains), and its approximation by finite volume schemes deserves an analysis. With that purpose in mind, notice that a new and characteristic feature of the approximation of the initial boundary value problem (1)–(3) by a finite volume scheme is the possible creation of a numerical boundary layer. This numerical boundary layer is a subproduct of the numerical diffusion effects induced by the scheme. Of course, its presence is also related to the way in which the boundary data are implemented in the scheme. Let us specify this point. We consider here and in the following the implementation of boundary data via “ghost control volumes”. This is a way to compute the numerical fluxes at the boundary of the domain inspired by the design of the fluxes inside the domain. Indeed, if  $\sigma$  is an edge of a control volume  $K$  but also one of the edges of the control volume  $L$ , then the numerical flux  $Q_\sigma^n$  (cf. (4)) is given as a function of the discrete unknowns  $c_K^n$  and  $c_L^n$  by the formula  $Q_\sigma^n = G_\sigma^n(c_K^n, c_L^n)$  where, among other properties, the function  $G_\sigma^n$  is nondecreasing with respect to its first argument and nonincreasing with respect to the second. If  $\sigma$ , edge of a control volume  $K$ , is now located at the boundary of the domain  $\Omega$ ,

then a *ghost* control volume  $L$  such that  $L \subset \mathbb{R}^d \setminus \Omega$  and  $\sigma = \overline{K} \cap \overline{L}$  is introduced and the computation of the numerical flux at the boundary  $Q_\sigma^n$  relies on a numerical flux function  $G_\sigma^n$  (nondecreasing with respect to its first argument and nonincreasing with respect to the second) via the formula  $Q_\sigma^n = G_\sigma^n(c_K^n, c_L^n)$  where the value  $c_L^n$  is a discretization (typically the mean value) of the boundary datum  $\bar{c}$  on  $[t^n, t^{n+1}) \times \sigma$ . This method of computation of the numerical fluxes at the boundary of the domain is classical and ensures the convergence of the finite volume scheme to the entropy solution of the problem (1)–(3) [Sze91, BCV95, CCL95a, Vov02]. Let us also stress that the proposed finite volume discretization is of rather great importance for practical applications (see the discussion on the implementation of numerical boundary conditions in the approximation of two phase flow problems in [EGV03]). Before coming back to our considerations of numerical boundary layers and of their influence on the speed of convergence of the finite volume method, let us observe that when *systems* of conservation laws are considered, the computation of the numerical fluxes at the boundary of the domain by the method of ghost control volumes may be not accurate. Other methods, like reflecting, or absorbing boundary conditions are in use, and, when used, the method of ghost control volumes is associated to the Godunov method for the computation of the flux. In this context, the Godunov method is indeed considered to give the reliable choice of numerical flux functions at the boundary.

The study of the numerical boundary layer has been performed by C. Chainais-Hillairet and E. Grenier [CHG01], in the 1D case and for modified Lax–Friedrichs schemes on cartesian grids in the multi-dimensional case. Such an analysis gives a precise description of the numerical solution and, as a consequence, the speed of convergence of this solution to the entropy solution of the problem (1)–(3). In the noncharacteristic case with smooth exact solutions, this speed of convergence is proved to be of order  $h$  in the  $L^\infty(0, T; L^1(\Omega))$  norm, where  $h$  is the size of the mesh.

Unfortunately, the techniques of numerical boundary layer analysis seem difficult to be set when no selected direction of (discrete) derivation exists, as is the case when finite volume schemes on unstructured meshes are used. For such schemes one can therefore think to adapt the technique developed by Kuznetsov [Kuz76] for the analysis of the Cauchy problem in the framework of the initial boundary value problem to get error estimates, with the drawback that this tool is not accurate at all to take into account the special phenomena at the boundary of the domain. In the specific situation  $\mathbf{F}(x, t, c) = \mathbf{u}(x, t)f(c)$  with  $f$  monotone, this drawback can be overcome, for the reason that the inflow and outflow parts of the boundary are determined a priori by the given velocity field  $\mathbf{u}$ . In [Vig97], Vignal gives an a priori error estimate of order  $h^{1/4}$  for the initial boundary value problem. However, to our knowledge, for general fluxes  $\mathbf{F}$ , and general schemes on possibly unstructured meshes, no results or techniques of error estimates which account for the influence of the boundary condition have been delivered. In order to fill in this gap, we adapt the technique of Kuznetsov [Kuz76] to the proof of uniqueness of the entropy solution given by F. Otto [Ott96, MNRR96], and prove that the error can be estimated by an *a posteriori error bound* which is at least of order  $h^{1/6}$  for meshes with mesh size  $h$  (see Propositions 5.1 and 5.2). The order,  $h^{1/6}$ , of our a priori error estimate also has to be discussed. Our comments are postponed to Remark 5.9.

Since the finite volume methods introduce some numerical diffusion effects in the approximation of the entropy solution to the problem (1)–(3), they are often related to the approximation by the vanishing viscosity method with, say, a viscosity of (small) order  $\varepsilon$ . In [IV03, DIV03] the tools are developed (the notion of a kinetic solution for the initial boundary value problem) and the proof is given of an error estimate of order  $\varepsilon^{1/3}$ .

The article is structured as follows. In Section 2 we give and recall some properties of the entropy solution to the problem (1)–(3). In Section 3 the finite volume schemes under consideration are defined; some of their properties are explained in Section 4 while in Section 5 we prove the error estimates which are the center of our study. Finally, in Section 6 we give numerical experiments to illustrate our analysis. We complete the presentation with the proof of a BV estimate on the entropy solution on convex polygonal bounded domains in Appendix A.

## 2. PROPERTIES AND REGULARITY OF THE EXACT SOLUTION

Problem (1)–(3) for general flux  $\mathbf{F}$ , and in the context of entropy solutions, has first been analyzed by C. Bardos, A.-Y. LeRoux and J.-C. Nédélec [BLN79] in the BV framework. The notion of entropy solution given by the three authors has been extended, in the  $L^\infty$  setting, by F. Otto [Ott96, MNRR96]. We present and use this last definition by using the following semi-Kruzhkov entropy-entropy flux pairs [Ser96, Car99].

*Notation 2.1* (Semi-Kruzhkov entropy-entropy flux pairs). Let  $a \top b$  (resp.  $a \perp b$ ) denote the maximum (resp. the minimum) of  $a$  and  $b$ , set  $s^+ = s \top 0$ ,  $s^- = (-s)^+$  and denote by  $\text{sgn}_\pm(s)$  the derivative of the function  $s^+$  (resp.  $s^-$ ) with the value 0 at  $s = 0$ . We denote by  $\Phi^\pm(s, \kappa)$  the entropy flux associated to the entropy  $(s - \kappa)^\pm$ ; that is to say

$$\Phi^\pm(x, t, s, \kappa) = \text{sgn}_\pm(s - \kappa)(\mathbf{F}(x, t, s) - \mathbf{F}(x, t, \kappa)).$$

We will often drop the dependence of  $\Phi^\pm$  over the variables  $x$  and  $t$  and shorten the notation to  $\Phi^\pm(s, \kappa)$ .

*Notation 2.2.* We denote by  $C_m$  and  $C_M \in \mathbb{R}$  some lower and upper bounds for the data

$$C_m \leq c_0, \bar{c} \leq C_M \quad \text{a.e.},$$

set  $\mathcal{C} = \max(|C_m|, |C_M|)$  and let  $\mathcal{L}$  be a fixed real satisfying

$$(5) \quad \mathcal{L} \geq \max\{|\mathbf{F}_c(x, t, c)|; (x, t, c) \in \Omega \times (0, T) \times [C_m, C_M]\}.$$

**Definition 2.3** (Entropy solution). A function  $c \in L^\infty(\Omega \times (0, T))$  is called an entropy weak solution of (1)–(3) if it satisfies the following entropy inequalities: for all  $\kappa \in [C_m, C_M]$ , for all  $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$  with  $\varphi \geq 0$ ,

$$(6) \quad \int_{\Omega \times (0, T)} (c - \kappa)^\pm \partial_t \varphi + \Phi^\pm(c, \kappa) \cdot \nabla \varphi + \int_{\Omega} (c_0 - \kappa)^\pm \varphi(\cdot, 0) + \mathcal{L} \int_{\partial \Omega \times (0, T)} (\bar{c} - \kappa)^\pm \varphi \geq 0.$$

The space  $L^\infty$  is preserved by equation (1), as well as the space BV, and we have the following theorem.

**Theorem 2.4** (Existence, uniqueness, and regularity). *Let  $c_0 \in L^\infty(\Omega)$ ,  $\bar{c} \in L^\infty(\partial\Omega \times (0, T))$ . Suppose that  $\mathbf{F} \in \mathcal{C}^1(\Omega \times (0, T) \times \mathbb{R})$  and  $\operatorname{div}_x \mathbf{F}(x, t, c) = 0$  for all  $(x, t, c) \in \Omega \times (0, T) \times \mathbb{R}$ . Then there exists a unique entropy weak solution  $c \in L^\infty(\Omega \times (0, T))$  of the problem (1)–(3) which is bounded by the data*

$$\|c(t)\|_{L^\infty(\Omega \times (0, T))} \leq \max\{\|c_0\|_{L^\infty(\Omega)}, \|\bar{c}\|_{L^\infty(\partial\Omega \times (0, T))}\}.$$

*If, furthermore,  $c_0 \in BV(\Omega)$  and  $\bar{c} \in BV(\partial\Omega \times (0, T))$ , then  $c \in BV(\Omega \times (0, T))$  and there exists a constant  $C_{BV} > 0$  which depends on the data and on  $\Omega$  only such that*

$$(7) \quad \|c\|_{BV(\Omega \times (0, T))} \leq C_{BV}.$$

*Proof.* We refer to [BLN79, Ott96, MNRR96, Vov02] for the results of existence and uniqueness of the entropy solution. In [BLN79] a  $BV$  estimate is given on the entropy solution, which requires  $\Omega$  to be  $\mathcal{C}^2$ . The  $BV$  estimate in the case where  $\Omega$  is a polygonal bounded domain is a new result and we give the rather involved and technical proof in Appendix A.  $\square$

*Remark 2.5* (BLN). Under the hypotheses of Theorem 2.4, let  $c$  be the entropy solution of the problem (1)–(3). Suppose that  $c \in BV(\Omega \times (0, T))$  and denote by  $\gamma c$  the trace of the function  $c$  on  $\partial\Omega \times (0, T)$ . Then

- (1)  $c$  satisfies the following entropy inequalities for all  $\kappa \in [C_m, C_M]$

$$(8) \quad \int_{\Omega \times \mathbb{R}^+} (c - \kappa)^\pm \partial_t \varphi + \Phi^\pm(c, \kappa) \nabla \varphi + \int_{\Omega} (c_0 - \kappa)^\pm \varphi(\cdot, 0) - \int_{\partial\Omega \times (0, T)} \Phi^\pm(\gamma c, \kappa) \cdot \mathbf{n} \varphi \geq 0$$

for all  $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$  with  $\varphi \geq 0$ .

- (2) Moreover, the so-called BLN condition [BLN79] is satisfied by  $c$  on the boundary of the domain: for a.e.  $(x, t) \in \partial\Omega \times (0, T)$ , for all  $\kappa$  in the interval with extremities  $\gamma c(x, t)$  and  $\bar{c}(x, t)$

$$(9) \quad \Phi^\pm(\gamma c, \kappa) \cdot \mathbf{n} \geq 0.$$

- (3) The inequality (8) together with (9) implies (6). Indeed, if  $(x, t) \in \partial\Omega \times (0, T)$  and if  $\kappa \leq \bar{c}(x, t)$ , then (9) gives

$$-\Phi^-(\gamma c(x, t), \kappa) \cdot \mathbf{n} \leq 0 = \mathcal{L}(\bar{c}(x, t) - \kappa)^-,$$

while, if  $\kappa > \bar{c}(x, t)$ , then (9) gives  $\Phi^-(\gamma c, \bar{c}) \cdot \mathbf{n} \geq 0$  and

$$-\Phi^-(\gamma c(x, t), \kappa) \cdot \mathbf{n} \leq (\Phi^-(\gamma c, \bar{c}) - \Phi^-(\gamma c, \kappa)) \cdot \mathbf{n} \leq \mathcal{L}(\bar{c}(x, t) - \kappa)^-.$$

- (4) In fact, it is possible to prove (see [MNRR96, Vov02]) that for every function  $w$  which is measurable and bounded a.e. on  $\partial\Omega \times (0, T)$ , one has

$$(10) \quad -\Phi^\pm(\gamma c(x, t), w(x, t)) \cdot \mathbf{n} \leq \mathcal{L}(\bar{c}(x, t) - w(x, t))^\pm$$

for a.e.  $(x, t) \in \partial\Omega \times (0, T)$ .

## 3. NOTATIONS, ASSUMPTION AND THE DEFINITION OF THE SCHEME

In this section we will fix the notations and assumptions and define the finite volume scheme for solving (1)–(3).

*Assumption 3.1.* The data of problem (1)–(3) are supposed to satisfy the following conditions:

$$(11) \quad \begin{aligned} c_0 &\in L^\infty \cap BV(\mathbb{R}^d), & \bar{c} &\in L^\infty \cap BV(\partial\Omega \times (0, T)), \\ \mathbf{F} &\in \mathcal{C}^1(\Omega \times (0, T) \times \mathbb{R}), & \operatorname{div}_x \mathbf{F}(x, t, c) &= 0 \quad \forall (x, t, c) \in \Omega \times (0, T) \times \mathbb{R}. \end{aligned}$$

The initial and boundary data are supposed to belong to the space  $L^\infty \cap BV$ . This makes sense if one has in mind practical applications in which these data are physical or biological quantities. The regularity and divergence-free hypotheses on the flux  $F$  are also in coherence with the possible physical or biological underlying model for equation (1). The divergence free condition in (11) may be removed, and source terms may be considered in equation (1) as well.

Let us now give the description of the meshes and schemes used to solve (1)–(3). Let  $J := \{t_0, \dots, t_N\}$  be a partition of  $[0, T]$  and  $\Delta t^n := t^{n+1} - t^n$  be the step size of  $J$ . For each  $n \in \{0, \dots, N\}$ , let  $\mathcal{T}^n = \{T_j | j \in I_{\text{int}}^n\}$  be a regular triangulation of  $\Omega$ . The joint edge of  $T_j$  and  $T_l$  will be denoted by  $S_{jl}$ .

The set of internal edges  $\mathcal{S}_{\text{int}}^n$  and the oriented set of internal edges  $\mathcal{E}_{\text{int}}^n$  are assimilated to the sets of the corresponding indexes and are defined respectively as

$$\begin{aligned} \mathcal{S}_{\text{int}}^n &:= \{(j, l) \in I_{\text{int}}^n \times I_{\text{int}}^n | S_{jl} \text{ is interior edge of } \mathcal{T}^n\}, \\ \mathcal{E}_{\text{int}}^n &:= \{(j, l) \in \mathcal{S}_{\text{int}}^n | j > l\}. \end{aligned}$$

As mentioned in the Introduction, we use the concept of ghost cells to compute the flux at the boundary. Therefore, we introduce the notations related to the use of this method. Let the index set  $I_{\text{ext}}^n$  be such that  $I_{\text{ext}}^n \cap I_{\text{int}}^n = \emptyset$  and such that for each edge  $S \subset \partial\Omega$  there exists a unique pair of indices  $(j, l) \in I_{\text{int}}^n \times I_{\text{ext}}^n$  with  $\partial T_j \cap S = S$ . In this situation we denote  $S_{jl} := S$ . Accordingly, the set of edges located on the boundary of  $\Omega$  is denoted by

$$\mathcal{S}_{\text{ext}}^n := \{(j, l) \in I_{\text{int}}^n \times I_{\text{ext}}^n | S_{jl} \text{ is an exterior edge of } \mathcal{T}^n\}.$$

We also denote by  $h_{\text{min}}^n := \min_{j \in I^n} \operatorname{diam}(T_j)$  the size of the mesh at time  $t^n$ . The mesh  $\mathcal{T}^n$  satisfies the following structural hypothesis.

*Assumption 3.2.* There exists a real  $\alpha > 0$  such that for all  $h_j := \operatorname{diam}(T_j)$ ,  $j \in I$ ,

$$(12) \quad \alpha h_j^d \leq |T_j|, \quad \alpha |\partial T_j| \leq h_j^{d-1}.$$

In order to design the finite volume approximation, we first define the class of monotone numerical fluxes in use.

**Definition 3.3** (Numerical fluxes). The numerical fluxes  $g_{jl}^n$  are functions in  $\mathcal{C}(\mathbb{R}^2, \mathbb{R})$ , for any  $(j, l) \in \mathcal{S}^n$  and  $t_n \in \mathbb{R}^+$ , satisfying the following conditions

(respectively: monotony, conservativity, regularity, consistency).

$$(13) \quad \begin{aligned} \forall v \in [C_m, C_M], \quad &g_{jl}^n(v, \cdot) \text{ is monotone nonincreasing on } [C_m, C_M], \\ \forall w \in [C_m, C_M], \quad &g_{jl}^n(\cdot, w) \text{ is monotone nondecreasing on } [C_m, C_M], \end{aligned}$$

$$(14) \quad \forall v, w \in [C_m, C_M], \forall (j, l) \in \mathcal{S}_{\text{int}}^n, \quad g_{jl}^n(v, w) = -g_{lj}^n(w, v),$$

$$(15) \quad \forall w, v, w', v' \in [C_m, C_M], \quad |g_{jl}^n(v, w) - g_{jl}^n(v', w')| \leq \mathcal{L}|S_{jl}|(|w - w'| + |v - v'|),$$

and

$$(16) \quad g_{jl}^n(w, w) = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \mathbf{F}(x, t, w) \cdot n_{jl} \, dx \, dt,$$

where  $n_{jl}$  denotes the outer unit normal to  $S_{jl}$  with respect to  $T_j$ .

**Definition 3.4** (Finite volume scheme). Set

$$c_j^0 := \frac{1}{|T_j|} \int_{T_j} c_0, \quad j \in I_{\text{int}}^0, \quad c_l^n := \frac{1}{\Delta t} \frac{1}{|S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{S_{j,l}} \bar{c}(x, t) \, dx \, dt, \quad (j, l) \in \mathcal{S}_{\text{ext}}^n.$$

The discrete evolution of the approximate value  $c_j$  of  $c$  in the cell  $T_j$  is governed by the equation

$$(17) \quad c_j^{n+1} := c_j^n - \frac{\Delta t}{|T_j|} \sum_{l \in N(j)} g_{jl}^n(c_j^n, c_l^n), \quad j \in I_{\text{int}}^n,$$

for all  $n \in \{0, \dots, N\}$ , where  $N(j)$  denotes the index set of the neighboring cells of  $T_j$ , including the ghost indices across the boundaries of the domain  $\Omega$ .

Given the discrete values  $c_j^n$ , we denote by  $c_h$  the approximate solution  $c_h : \Omega \times (0, T) \rightarrow \mathbb{R}$  defined by

$$(18) \quad c_h(x, t) := c_j^n \quad \text{if } x \in T_j, \quad t^n \leq t < t^{n+1}.$$

The stability of the explicit scheme (17) is ensured under the following CFL condition.

*Assumption 3.5* (CFL condition). We assume the following CFL condition for a given  $\xi \in (0, 1)$ :

$$\Delta t^n \leq \frac{(1 - \xi)\alpha^2 h_{\min}^n}{\mathcal{L}}.$$

#### 4. PROPERTIES OF THE DISCRETE SOLUTION

As the entropy solution of problem (1)–(3), the discrete solution is  $L^\infty$  stable. On the contrary, the validity of  $BV$  estimates on  $c_h$  is still an open question (in the case where unstructured meshes are considered): only “weak  $BV$  estimates” are known. These two aspects of the behavior of the discrete solution are detailed in the following two lemmas (see [Vov02] for a proof).

**Lemma 4.1** ( $L^\infty$  stability). *Let  $c_h$  be the discrete solution (17), and let the Assumptions 3.1, 3.2, and 3.5 be fulfilled. Then the function  $c_h$  satisfies the  $L^\infty$  estimates,*

$$\|c_h\|_{L^\infty(\Omega \times [0, T])} \leq \max\{\|c_0\|_{L^\infty(\Omega)}, \|\bar{c}\|_{L^\infty(\partial\Omega \times (0, T))}\}$$

and

$$C_m \leq c_j^n \leq C_M, \quad \text{for all } (T_j, t^n) \in \mathcal{T}^n \times J.$$

**Lemma 4.2** (Weak BV estimate). *Let  $c_h$  be the discrete solution (17) and let the Assumptions 3.1, 3.2, and 3.5 be fulfilled. Then there exists  $C \geq 0$  only depending on  $\Omega, c_0, \bar{c}, \mathcal{L}, T, \alpha$  and  $\xi$  such that*

$$(19) \quad \sum_{t^n \in J(j,l) \in \mathcal{E}_{\text{int}}^n} \sum \Delta t^n \left( \max_{c_i^n \leq a \leq b \leq c_j^n} (g_{jl}^n(b, a) - g_{jl}^n(b, b)) \right. \\ \left. + \max_{c_i^n \leq a \leq b \leq c_j^n} (g_{jl}^n(b, a) - g_{jl}^n(a, a)) \right) \leq \frac{C}{\sqrt{h}},$$

and

$$(20) \quad \sum_{t^n \in J_j \in I^n} \sum |T_j| |c_j^{n+1} - c_j^n| \leq \frac{C}{\sqrt{h}}.$$

**Entropy inequality satisfied by the approximate solution.** In Section 2 we recalled that problem (1) has a unique weak solution conforming to the entropy inequality (6). In this subsection we will show that the approximate solution  $c_h$  fulfills an analog inequality, including a small error term. To compare *discrete* to *continuous* equations, let us introduce the following forms  $E^+$  and  $E_h^+$ .

**Definition 4.3.** For the discrete function  $c_h$  being defined by (3.4) and  $\kappa \in \mathbb{R}$ , we set

$$E_h^+(c_h, \kappa, \varphi) := - \sum_{t^n \in J_j \in I^n} \sum \frac{(c_j^{n+1} - \kappa)^+ - (c_j^n - \kappa)^+}{\Delta t^n} \int_{t^n}^{t^{n+1}} \int_{T_j} \varphi(x, t) \, dx \, dt \\ - \sum_{t^n \in J_j \in I^n} \sum \frac{1}{|T_j|} \int_{t^n}^{t^{n+1}} \int_{T_j} \varphi(x, t) \sum_{l \in N(j)} g_{jl}^n(c_j^n \top \kappa, c_l^n \top \kappa) - g_{jl}^n(\kappa, \kappa) \, dx \, dt, \\ E^+(c_h, \kappa, \varphi) := \int_{\Omega \times \mathbb{R}^+} (c_h(x, t) - \kappa)^+ \partial_t \varphi(x, t) \, dx \, dt \\ + \int_{\Omega \times \mathbb{R}^+} \Phi^+(c_h(x, t), \kappa) \cdot \nabla \varphi(x, t) \, dx \, dt \\ + \int_{\Omega} (c_0(x) - \kappa)^+ \varphi(x, 0) \, dx + \mathcal{L} \int_{\partial \Omega \times (0, T)} (\bar{c}(x, t) - \kappa)^+ \varphi \, d\gamma(x) \, dt,$$

for any  $\varphi \in \mathcal{C}^\infty(\bar{\Omega} \times [0, T])$ .

The discrete (and local) entropy inequality given in Lemma 4.4 is the main account for the approximate continuous entropy inequality detailed in Lemma 4.5.

**Lemma 4.4** (Discrete entropy inequality). *Let  $c_h$  be the discrete solution defined in 3.4, and let Assumptions 3.1, 3.2, and 3.5 be fulfilled. Then we have*

$$(21) \quad E_h^+(c_h, \kappa, \varphi) \geq 0.$$

*Proof.* The discrete entropy inequality (21) follows from the monotonicity properties of the numerical fluxes; see, e.g., [Vov02]. □

**Lemma 4.5** (Continuous entropy estimate). *Let  $c_h$  be the discrete solution defined in Definition 3.4, and let the Assumptions 3.1, 3.2, and 3.5 be fulfilled. Then we*

have

$$\begin{aligned}
 E^+(c_h, \kappa, \varphi) &\geq - \int_{\Omega} |c_h(x, 0) - c_0(x)| \varphi(x, 0) \, dx \\
 &\quad - \sum_{t^n \in J} \sum_{j \in I^n} |c_j^{n+1} - c_j^n| \int_{T_j} \int_{t^n}^{t^{n+1}} |\varphi_t(x, t)| \, dx \, dt \\
 &\quad - \sum_{t^n \in J} \sum_{(j,l) \in \mathcal{E}_{\text{int}}^n} 2 \max_{c_l^n \leq a \leq b \leq c_j^n} (g_{jl}^n(b, a) - g_{jl}^n(b, b)) \langle \mu_{jl}^n, |\nabla \varphi| + |\varphi_t| \rangle \\
 &\quad - \sum_{t^n \in J} \sum_{(j,l) \in \mathcal{E}_{\text{int}}^n} 2 \max_{c_l^n \leq a \leq b \leq c_j^n} (g_{jl}^n(b, a) - g_{jl}^n(a, a)) \langle \mu_{lj}^n, |\nabla \varphi| + |\varphi_t| \rangle \\
 &\quad - \sum_{t^n \in J} \sum_{(j,l) \in \mathcal{E}_{\text{int}}^n} \mathcal{L}(|c_j^n| + |c_l^n|) \langle \nu_{jl}^n, |\nabla \varphi| + |\varphi_t| \rangle \\
 &\quad - \mathcal{L} \int_{\partial\Omega \times (0, T)} (\bar{c}_h - \bar{c})^+ \varphi(x, t) d\gamma(x) \, dt \\
 (22) \quad &\quad - \sum_{t^n \in J} \sum_{(j,l) \in \mathcal{S}_{\text{ext}}^n} \mathcal{L}(c_l + c_j - 2C_m) \langle \bar{\nu}_{lj}^n, |\nabla \varphi| \rangle,
 \end{aligned}$$

where the Radon measures  $\mu_{jl}^n, \nu_{jl}^n, \bar{\nu}_{jl}^n$  are defined as

$$\begin{aligned}
 \langle \mu_{jl}^n, g \rangle &:= \frac{h_j + \Delta t^n}{\Delta t^n |T_j| |S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{T_j} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \int_0^1 \\
 &\quad g(\gamma + \vartheta(x - \gamma), s + \vartheta(t - s)) \, d\vartheta \, d\gamma \, ds \, dx \, dt, \\
 \langle \nu_{jl}^n, g \rangle &:= \frac{(h_j + \Delta t^n)^2}{\Delta t^n |S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \int_0^1 \\
 &\quad g(\xi + \vartheta(\gamma - \xi), \tau + \vartheta(s - \tau)) \, d\vartheta \, d\gamma \, dt \, d\xi \, d\tau, \\
 \langle \bar{\nu}_{jl}^n, g \rangle &:= \frac{h_j}{|T_j|} \int_{t^n}^{t^{n+1}} \int_{T_j} \int_{S_{jl}} \int_0^1 g(\gamma + \vartheta(x - \gamma), t) \, d\vartheta \, d\gamma \, dx \, dt.
 \end{aligned}$$

Here, we have introduced the discrete boundary datum  $\bar{c}_h$  defined by  $\bar{c}_h|_{S_{jl} \times [t^n, t^{n+1})} := c_l^n$  for  $(j, l) \in \mathcal{S}_{\text{ext}}^n$ .

*Proof.* From Lemma 4.4 follows

$$E^+(c_h, \kappa, \varphi) \geq E^+(c_h, \kappa, \varphi) - E_h^+(c_h, \kappa, \varphi).$$

Hence, we aim at deriving a lower estimate for  $E^+(c_h, \kappa, \varphi) - E_h^+(c_h, \kappa, \varphi)$ . In order to do so, we split the addend on the right-hand side into two terms, corresponding to the time derivative and the convection part. This yields

$$E^+(c_h, \kappa, \varphi) - E_h^+(c_h, \kappa, \varphi) = (T_{10} - T_1) + (T_{20} - T_2),$$

where the terms  $T_1, T_{10}, T_2, T_{20}$  are defined by

$$\begin{aligned}
T_1 &= - \sum_{t^n \in J} \sum_{j \in I^n} \frac{(c_j^{n+1} - \kappa)^+ - (c_j^n - \kappa)^+}{\Delta t^n} \int_{t^n}^{t^{n+1}} \int_{T_j} \varphi(x, t) \, dx \, dt, \\
T_{10} &= \int_{\Omega \times \mathbb{R}^+} (c_h(x, t) - \kappa)^+ \partial_t \varphi(x, t), + \int_{\Omega} (c_0(x) - \kappa)^+ \varphi(x, 0) \, dx, \\
T_2 &= - \sum_{t^n \in J} \sum_{j \in I^n} \frac{1}{|T_j|} \int_{t^n}^{t^{n+1}} \int_{T_j} \varphi(x, t) \sum_{l \in N(j)} (g_{jl}^n(c_j^n \top \kappa, c_l^n \top \kappa) - g_{jl}^n(\kappa, \kappa)), \\
T_{20} &= \int_{\Omega \times \mathbb{R}^+} (\mathbf{F}(x, t, c_h(x, t) \top \kappa) - \mathbf{F}(x, t, \kappa)) \nabla \varphi(x, t) \, dx \, dt \\
&\quad - \int_{\partial \Omega \times (0, T)} \widetilde{\gamma \Phi}_h^+(c_h, \bar{c}, \kappa) \varphi(x, t) \, d\gamma(x) \, dt.
\end{aligned}$$

From (11) and (16) it follows that

$$\begin{aligned}
T_2 &= - \sum_{t^n \in J} \sum_{j \in I^n} \frac{1}{|T_j|} \int_{t^n}^{t^{n+1}} \int_{T_j} \varphi(x, t) \\
&\quad \times \sum_{l \in N(j)} \{(g_{jl}^n(c_j^n \top \kappa, c_l^n \top \kappa) - g_{jl}^n(\kappa, \kappa)), -(g_{jl}^n(c_j^n \top \kappa, c_j^n \top \kappa) - g_{jl}^n(\kappa, \kappa))\}.
\end{aligned}$$

The discrete function  $c_h$  is piecewise constant in space and time. Thus, we decompose  $T_{20}$  into sums and integrate by parts locally:

$$\begin{aligned}
T_{20} &= \sum_{t^n \in J} \sum_{j \in I^n} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} (\mathbf{F}(x, t, c_j \top \kappa) - \mathbf{F}(x, t, \kappa)) \cdot n_{jl} \varphi(x, t) \, d\gamma(x) \, dt \\
&\quad - \int_{\partial \Omega \times (0, T)} \widetilde{\gamma \Phi}_h^+(c_h, \bar{c}, \kappa) \varphi(x, t) \, d\gamma(x) \, dt.
\end{aligned}$$

Next we have the summation in  $T_{20}$ , and then  $T_2$  is rearranged in accordance to the following lemma.

**Lemma 4.6.** *Let  $A$  be a function  $\prod_{n \in \mathbb{N}} \{n\} \times \mathcal{S}^n \rightarrow \mathbb{R}$ . Then*

$$\sum_{t^n \in J} \sum_{j \in I^n} \sum_{l \in N(j)} A_{jl}^n = \sum_{t^n \in J} \sum_{(j, l) \in \mathcal{E}_{\text{int}}^n} (A_{jl}^n + A_{lj}^n) + \sum_{t^n \in J} \sum_{(j, l) \in \mathcal{S}_{\text{ext}}^n} A_{jl}^n$$

This reordering of the summations in  $T_2$  and  $T_{20}$  on the edges leads to the decomposition  $T_2 = T_2^{\text{int}} + T_2^{\text{ext}}$  and  $T_{20} = T_{20}^{\text{int}} + T_{20}^{\text{ext}}$ . Then the method used to estimate the term  $|T_{20}^{\text{int}} - T_2^{\text{int}}|$ , as well as the term  $|T_{10} - T_1|$ , is, step by step, the method used in the proof of Theorem 4 in [CH99a] or Theorem 4.1 in [EGGH98].

We refer to these articles for an integral proof of the following results:

$$\begin{aligned}
 |T_{10} - T_1| &\leq \sum_{t^n \in J} \sum_{j \in I^n} |c_j^{n+1} - c_j^n| \int_{T_j} \int_{t^n}^{t^{n+1}} |\varphi_t(x, t)| \, dx \, dt \\
 &\quad + \int_{\Omega} |c_h(x, 0) - c_0(x)| \varphi(x, 0) \, dx, \\
 |T_{20}^{\text{int}} - T_2^{\text{int}}| &\leq \sum_{t^n \in J(j,l) \in \mathcal{E}_{\text{int}}^n} \sum_{l \in \mathcal{E}_{\text{int}}^n} \mathcal{L}(|c_j^n| + |c_l^n|) \langle \nu_{jl}^n, |\nabla \varphi| + |\varphi_t| \rangle \\
 &\quad + \sum_{t^n \in J(j,l) \in \mathcal{E}_{\text{int}}^n} \sum_{l \in \mathcal{E}_{\text{int}}^n} 2 \max_{c_l^n \leq a \leq b \leq c_j^n} (g_{jl}^n(b, a) - g_{jl}^n(b, b)) \langle \mu_{jl}^n, |\nabla \varphi| + |\varphi_t| \rangle \\
 &\quad + \sum_{t^n \in J(j,l) \in \mathcal{E}_{\text{int}}^n} \sum_{l \in \mathcal{E}_{\text{int}}^n} 2 \max_{c_l^n \leq a \leq b \leq c_j^n} (g_{jl}^n(b, a) - g_{jl}^n(a, a)) \langle \mu_{lj}^n, |\nabla \varphi| + |\varphi_t| \rangle,
 \end{aligned}$$

where  $\mu_{jl}^n$  and  $\nu_{jl}^n$  are defined in the statement of the lemma.

A lower estimate on the boundary terms  $T_{20}^{\text{ext}} - T_2^{\text{ext}}$  remains to be proved. We have

$$\begin{aligned}
 T_2^{\text{ext}} &= - \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \frac{1}{|T_j|} \int_{t^n}^{t^{n+1}} \int_{T_j} \varphi(x, t) \{ (g_{jl}^n(c_j^n \top \kappa, c_l^n \top \kappa) - g_{jl}^n(\kappa, \kappa)), \\
 &\quad - (g_{jl}^n(c_j^n \top \kappa, c_j^n \top \kappa) - g_{jl}^n(\kappa, \kappa)) \}, \\
 T_{20}^{\text{ext}} &= \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} (\mathbf{F}(x, t, c_j \top \kappa) - \mathbf{F}(x, t, \kappa)) \cdot n_{jl} \varphi(x, t) \, d\gamma(x) \, dt \\
 &\quad + \mathcal{L} \int_{\partial\Omega \times (0, T)} (\bar{c} - \kappa)^+ \varphi(x, t) \, d\gamma(x) \, dt.
 \end{aligned}$$

First, observe that, by (16), we have

$$\begin{aligned}
 &- \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \frac{1}{|T_j|} \int_{t^n}^{t^{n+1}} \int_{T_j} \varphi(x, t) (g_{jl}^n(c_j^n \top \kappa, c_j^n \top \kappa) - g_{jl}^n(\kappa, \kappa)) \, dx \, dt, \\
 &= - \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} (\mathbf{F}(x, t, c_j \top \kappa) - \mathbf{F}(x, t, \kappa)) \cdot n_{jl} \varphi_j^n \, d\gamma(x) \, dt,
 \end{aligned}$$

where  $\varphi_j^n := \frac{1}{|T_j|} \int_{t^n}^{t^{n+1}} \int_{T_j} \varphi(x, t) \, dx \, dt$ , and hence

$$\begin{aligned}
 T_{20}^{\text{ext}} - T_2^{\text{ext}} &= \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \frac{1}{|T_j|} \int_{t^n}^{t^{n+1}} \int_{T_j} \varphi(x, t) (g_{jl}^n(c_j^n \top \kappa, c_l^n \top \kappa) - g_{jl}^n(\kappa, \kappa)) \\
 &\quad \sum_{l \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} (\mathbf{F}(x, t, c_j \top \kappa) - \mathbf{F}(x, t, \kappa)) \cdot n_{jl} (\varphi(x, t) - \varphi_j^n) \, d\gamma(x) \, dt \\
 &\quad + \mathcal{L} \int_{\partial\Omega \times (0, T)} (\bar{c} - \kappa)^+ \varphi(x, t) \, d\gamma(x) \, dt \\
 &=: U_1^{\text{ext}} + U_2^{\text{ext}} + U_3^{\text{ext}} + U_4^{\text{ext}},
 \end{aligned}$$

where

$$\begin{aligned}
U_1^{\text{ext}} &:= \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \frac{1}{|T_j|} \int_{t^n}^{t^{n+1}} \int_{T_j} \varphi(x,t) (g_{jl}^n(c_j^n \top \kappa, c_l^n \top \kappa) - g_{jl}^n(\kappa, \kappa)) \\
&\quad - \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \frac{1}{|S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \varphi(x,t) (g_{jl}^n(c_j^n \top \kappa, c_l^n \top \kappa) - g_{jl}^n(\kappa, \kappa)), \\
U_2^{\text{ext}} &:= \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \\
&\quad \times (\mathbf{F}(x,t, c_j \top \kappa) - \mathbf{F}(x,t, \kappa)) \cdot n_{jl} (\varphi(x,t) - \varphi_j^n) \, d\gamma(x) \, dt, \\
U_3^{\text{ext}} &:= \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \frac{1}{|S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \varphi(x,t) (g_{jl}^n(c_j^n \top \kappa, c_l^n \top \kappa) - g_{jl}^n(\kappa, \kappa)) \\
&\quad + \mathcal{L} \int_{\partial\Omega \times (0,T)} (\bar{c}_h - \kappa)^+ \varphi(x,t) \, d\gamma(x) \, dt, \\
U_4^{\text{ext}} &:= \mathcal{L} \int_{\partial\Omega \times (0,T)} (\bar{c} - \kappa)^+ \varphi(x,t) \, d\gamma(x) \, dt \\
&\quad - \mathcal{L} \int_{\partial\Omega \times (0,T)} (\bar{c}_h - \kappa)^+ \varphi(x,t) \, d\gamma(x) \, dt.
\end{aligned}$$

Recall that the discrete boundary datum  $\bar{c}_h$  is defined by  $\bar{c}_h|_{S_{jl} \times [t^n, t^{n+1})} := c_l^n$  for  $(j,l) \in \mathcal{S}_{\text{ext}}^n$  which yields

$$\int_{\partial\Omega \times (0,T)} (\bar{c}_h - \kappa)^+ \varphi(x,t) \, d\gamma(x) \, dt = \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \frac{1}{|S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \varphi(x,t) (c_l^n \top \kappa - \kappa)$$

and

$$U_3^{\text{ext}} = \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{l \in \mathcal{S}_{\text{ext}}^n} \frac{1}{|S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \varphi(x,t) (g_{jl}^n(c_j^n \top \kappa, c_l^n \top \kappa) - g_{jl}^n(\kappa, \kappa) + \mathcal{L}(c_l^n \top \kappa - \kappa)).$$

Since  $g_{jl}^n$  is nondecreasing with respect to its first variable, and by (15) we have

$$g_{jl}^n(c_j^n \top \kappa, c_l^n \top \kappa) - g_{jl}^n(\kappa, \kappa) \geq g_{jl}^n(\kappa, c_l^n \top \kappa) \geq -\mathcal{L}(c_l^n \top \kappa - \kappa),$$

we have  $U_3^{\text{ext}} \geq 0$ . Besides, using the 1-Lipschitz continuity of the function  $c \mapsto (c - \kappa)^+$  and the  $\mathcal{L}$ -Lipschitz continuity of the numerical fluxes we derive the estimates

$$\begin{aligned}
U_1^{\text{ext}} &\geq - \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{m \in \mathcal{S}_{\text{ext}}^n} \mathcal{L}(c_l - C_m) \langle \bar{v}_{lj}^n, |\nabla \varphi| \rangle, \\
U_2^{\text{ext}} &\geq - \sum_{t^n \in J(j,l) \in \mathcal{S}_{\text{ext}}^n} \sum_{m \in \mathcal{S}_{\text{ext}}^n} \mathcal{L}(c_j - C_m) \langle \bar{v}_{lj}^n, |\nabla \varphi| \rangle, \\
U_4^{\text{ext}} &\geq -\mathcal{L} \int_{\partial\Omega \times (0,T)} (\bar{c}_h - \bar{c})^+ \varphi(x,t) \, d\gamma(x) \, dt.
\end{aligned}$$

This concludes the proof of Lemma 4.5.  $\square$

5. ERROR ESTIMATES

**Proposition 5.1** (A posteriori estimate). *Let  $c_h$  be the discrete solution defined in Definition 3.4, and let  $c$  be the entropy solution of (1)–(3). Furthermore, let the Assumptions 3.1, 3.2, and 3.5 be fulfilled. Then we have*

$$\|c_h - c\|_{L^1(\Omega \times (0, T))} \leq \eta(c_h),$$

with

$$\eta(c_h) := 2T(N_f + 1) \min_{r, r_d \in \mathbb{R}^+} \left[ \eta_0 + \bar{\eta} + (\eta_t + \eta_c)K_1r + \eta_c K'_1(r + r_d) + K_0 \left( \frac{2}{r} + \frac{r}{r_d} \right) \right].$$

Here  $\eta_0, \eta_t, \eta_c,$  and  $\bar{\eta}$  are defined by

$$\begin{aligned} \eta_0 &= \int_{\Omega} |c_h(x, 0) - c_0(x)| \, dx, \\ \eta_t &= \sum_{t^n \in J} \sum_{j \in I^n} |T_j| \Delta t^n |c_j^{n+1} - c_j^n|, \\ \eta_c &= \sum_{t^n \in J} \sum_{(j, l) \in \mathcal{E}_{\text{int}}^n} \left[ 2(h_{jl} + \Delta t^n) \Delta t^n \left( \max_{c_l^n \leq a \leq b \leq c_j^n} (g_{jl}^n(b, a) - g_{jl}^n(b, b)) \right. \right. \\ &\quad \left. \left. + \max_{c_l^n \leq a \leq b \leq c_j^n} (g_{jl}^n(b, a) - g_{jl}^n(a, a)) \right) \right. \\ &\quad \left. + \mathcal{L}(|c_j^n| + |c_l^n|)(h_{jl} + \Delta t^n)^2 \Delta t^n h_{jl} \right] \\ &\quad + \sum_{t^n \in J} \sum_{(j, l) \in \mathcal{S}_{\text{ext}}^n} \mathcal{L}(c_j + c_l - 2C_m) h_j \Delta t^n h_{jl}, \\ \bar{\eta} &= \int_{\partial\Omega \times (0, T)} |\bar{c}_h - \bar{c}| d\gamma(x) \, dt, \end{aligned}$$

and the constants  $K_1, K'_1$  are defined by (30). We have set

$$(23) \quad K_0 := 2d\mathcal{L}\mathcal{L}(1 + \|w'_0\|_{L^1(\mathbb{R})} + |c_0|_{BV} + |\bar{c}|_{BV} + K'_1 + C_{BV} + C_{\Omega})(1 + T|\Omega|).$$

**Proposition 5.2** (A priori estimate). *Let  $c_h$  be the discrete solution defined in Definition 3.4, and let  $c$  be the entropy solution of (1)–(3). Furthermore, let the Assumptions 3.1, 3.2, and 3.5 be fulfilled. For uniform meshes of mesh size  $h$  we then have the a priori error estimate*

$$(24) \quad \|c_h - c\|_{L^1(\Omega \times (0, T))} \leq \eta(c_h) \leq Kh^{1/6}.$$

Here  $K$  denotes a generic constant independent of the mesh size.

The proof of the error estimates relies on the technique of the doubling of variables [Kru70, Kuz76]. The basic elements of this technique (approximation of the unit, choice of the test function and the so-called “dual forms”) are the object of the following three definitions.

**Definition 5.3** (Approximation of the unit). Let  $w_0 \in C_0^\infty(\mathbb{R}, \mathbb{R})$  be such that

$$\text{supp}(w_0) \subset [0, 1], \quad w_0 \geq 0, \quad \int_{\mathbb{R}} w_0(x) \, dx = 1.$$

For  $r$  and  $r_d \geq 1$ , we define the approximation of the unit  $w$  by  $\forall(x, t) = (\bar{x}, x_d, t) \in \mathbb{R}^d \times (0, T)$ ,

$$w(\bar{x}, x_d, t) = rw_0(rx_1) \cdots w_0(rx_{d-1}) \times r_d w_0(r_d x_d) \times rw_0(rt).$$

**Definition 5.4** (Choice of the test function). Let  $\psi : \mathbb{R}^d \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$  and  $\lambda : \mathbb{R}^d \rightarrow [0, 1]$  denote some smooth nonnegative functions that will be specified later. We suppose  $\text{supp}(\psi) \subset \mathbb{R}^d \times [0, T]$ . We set

$$(25) \quad \varphi(x, t) = \varphi(x, t, y, s) := \psi(y, s)\lambda(y)w(x - y, t - s).$$

*Remark 5.5.* Notice that if  $\alpha \in \{x_1, \dots, x_{d-1}, t\}$ , say  $\alpha = t$  for example, then  $\partial_t w(x, t) = rw_0(rx_1) \cdots w_0(rx_{d-1}) \times r_d w_0(rx_d) \times r^2 w'_0(rt)$  and, since  $\int_{\mathbb{R}} r^2 |w'_0(rt)| dt = r \int_{\mathbb{R}} |w'_0(\sigma)| d\sigma$  and  $\int r w_0(r\sigma) d\sigma = 1$ , we have

$$(26) \quad \forall \alpha \in \{x_1, \dots, x_{d-1}, t\}, \quad \int_{\mathbb{R}^d \times \mathbb{R}^+} \partial_\alpha |w(x, t)| dx dt \leq r \int_{\mathbb{R}} |w'_0(\sigma)| d\sigma \quad \text{and} \\ \int_{\mathbb{R}^d \times \mathbb{R}^+} \partial_{x_d} |w(x, t)| dx dt \leq r_d \int_{\mathbb{R}} |w'_0(\sigma)| d\sigma.$$

We will make mechanical use of these estimates in the following proofs. We will also frequently and without specification use the inequality

$$(27) \quad \int_{\Omega} \int_{\mathbb{R}^+} \psi(y, s)\lambda(y) dy ds \leq T|\Omega| \|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)}.$$

**Definition 5.6** (The form  $\tilde{E}^+$  and the dual forms  $E^{+*}$  and  $\tilde{E}^{+*}$ ). Let  $c$  be the entropy solution of (1)–(3), and  $c_h$  the discrete solution defined by (18). We define  $\tilde{E}^+(c_h, c, \psi\lambda)$  by

$$(28) \quad \tilde{E}^+(c_h, c, \psi\lambda) := \int_{\Omega \times \mathbb{R}^+} E^+(c_h, c(y, s), \varphi(\cdot, \cdot, y, s)) dy ds,$$

where  $\varphi(x, t, y, s)$  is defined by (25) and, corresponding to the forms  $E^+$  and  $\tilde{E}^+$ , we set

$$E^{+*}(\kappa, c, \psi\lambda) := \int_{\Omega \times \mathbb{R}^+} (c(y, s) - \kappa)^- \partial_s \tilde{\varphi}(y, s) dy ds \\ + \int_{\Omega \times \mathbb{R}^+} \Phi^-(c(y, s), \kappa) \cdot \nabla_y \tilde{\varphi}(y, s) dy ds \\ + \int_{\Omega} (c_0(y) - \kappa)^- \tilde{\varphi}(y, 0) dy \\ - \int_{\partial\Omega \times (0, T)} \Phi^-(\gamma c(y, s), \kappa) \cdot \mathbf{n}(y, s) \tilde{\varphi}(y, s) d\gamma(y) ds, \\ \tilde{E}^{+*}(c_h, c, \psi\lambda) := \int_{\Omega \times \mathbb{R}^+} E^{+*}(c_h(x, t), c, \tilde{\varphi}) dx dt,$$

with  $\tilde{\varphi}(y, s) = \varphi(\cdot, \cdot, y, s)$ . Here we denote by  $\gamma c$  the trace of  $c$  on the boundary  $\partial\Omega \times (0, T)$ , a function which is well defined (measurable and bounded a.e.) since  $c \in BV \cap L^\infty(\Omega \times (0, T))$  (see Theorem 2.4).

The proof of the Propositions 5.1 and 5.2 falls into three parts. In the first part, we derive estimates on the quantity  $\tilde{E}^+(c_h, c, \psi\lambda) + \tilde{E}^{+*}(c_h, c, \psi\lambda)$ . In the second part, we analyze and give estimates on terms related to the behavior of  $c$  and  $c_h$  on the boundaries  $\{t = 0\}$  and  $\partial\Omega \times (0, T)$  to deduce, in the third part, the estimates (5.1) and (24).

5.1. **First step.**

**Lemma 5.7.** *Let  $c_h$  be the discrete solution defined by (18), and let  $c$  be the entropy solution of (1)–(3). Then, under the Assumptions 3.1, 3.2, and 3.5, we have*

$$(29) \quad \begin{aligned} & \tilde{E}^+(c_h, c, \psi\lambda) + \tilde{E}^{+*}(c_h, c, \psi\lambda) \\ & \geq -\|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)} \left( \eta_0 + \eta_t [K_1 r] + \eta_c [K_1 r + K'_1 (r + r_d)] + \bar{\eta} \right), \end{aligned}$$

where  $\eta_0, \eta_t, \eta_c, \bar{\eta}$  are defined in Proposition 5.1.

*Sketch of the proof.* By the definition of entropy solution and by (8), we have  $E^{+*}(\kappa, c, \psi) \geq 0$  for all  $\kappa \in [C_m, C_M]$  and, hence,  $\tilde{E}^{+*}(c_h, c, \psi) \geq 0$ . The lower estimate of  $\tilde{E}^+(c_h, c, \psi)$  relies on

- (1) the approximate entropy inequality described in Lemma 4.5, and
- (2) estimates on the  $L^\infty$  norm of the function  $(x, t) \mapsto \int_{\Omega \times \mathbb{R}^+} \varphi(x, t, y, s) \, dy \, ds$  defined by (25).

These last estimates make use of estimates on the  $L^1$  norm of the approximation of the unit  $w$ , which are given by

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}^+} w(x - y, t - s) \, dx \, dt & \leq 1, \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \int_{\mathbb{R}^d \times \mathbb{R}^+} |\nabla w(x - y, t - s)| \, dx \, dt & \leq (r + r_d)K'_1, \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \\ \int_{\mathbb{R}^d \times \mathbb{R}^+} |\partial_t w(x - y, t - s)| \, dx \, dt & \leq rK_1, \quad \forall (x, t) \in \mathbb{R}^d \times \mathbb{R}^+, \end{aligned}$$

where  $K_1, K'_1$  are defined by

$$(30) \quad K_1 := \int_{\mathbb{R}} |\partial_t w_0(t)| \, dt, \quad K'_1 := \int_{\mathbb{R}^d} |\nabla w_0(x_1) \cdots w_0(x_d)| \, dx. \quad \square$$

5.2. **Second step.** The relation of symmetry  $(s - \sigma)^+ = (\sigma - s)^-$ , applied with  $s = c_h$  and  $\sigma = c$ , together with the identities  $(\partial_t + \partial_s)w(x - y, t - s) = 0$  and  $(\nabla_x + \nabla_y)w(x - y, t - s) = 0$  leads to

$$\begin{aligned} & \tilde{E}^+(c_h, c, \varphi) + \tilde{E}^{+*}(c_h, c, \bar{\varphi}) \\ (31) \quad & = \int_{\Omega \times \mathbb{R}^+} \int_{\Omega \times \mathbb{R}^+} \left[ (c_h(x, t) - c(y, s))^+ \psi_s \lambda(y) \right. \\ & \quad \left. + \Phi^+(c_h(x, t), c(y, s)) \cdot \nabla_y(\psi\lambda) \right] w(x - y, t - s) \, dx \, dt \, dy \, ds \\ (32) \quad & + \int_{\Omega \times \mathbb{R}^+} \int_{\Omega} (c_0(x) - c(y, s))^+ \varphi(x, 0, y, s) \, dx \, dt \, dy \\ (33) \quad & - \int_{\Omega \times \mathbb{R}^+} \int_{\partial\Omega \times (0, T)} \Phi^-(\gamma c(y, s), c_h(x, t)) \varphi(x, t, y, s) \, dx \, dt \, d\gamma(y) \, ds. \end{aligned}$$

Our aim is to estimate the term (31). In view of Lemma 5.7, this requires estimates on the terms (32) and (33), respectively, related to the behavior of the discrete and the entropy solution at initial time and at the boundary of the domain.

The term (32) is small with respect to  $1/r + 1/r_d$  because  $c(y, 0+) = c_0(y)$  (the initial trace of the entropy solution coincides everywhere with the initial datum) [CH99a, EGGH98]. This is no longer true on the boundary of the domain  $\partial\Omega \times (0, T)$ . The trace of the entropy solution may be distinct from the boundary datum on a part of the boundary. Therefore, in order to estimate the term (33)

we have to use a specific technique. For that purpose, we introduce some functions of localization in order to give an accurate parametrization of the boundary. The supports of such functions, in particular, are chosen in order to isolate flat parts of the boundary of  $\Omega$  (parts of the boundary of  $\Omega$  which are included in a hyperplane of  $\mathbb{R}^d$ ; recall that  $\Omega$  is a convex polygonal open subset of  $\mathbb{R}^d$ ).

**Definition 5.8** (Localization). Let  $\lambda$  be a function with values in  $[0, 1]$  such that  $\text{supp}(\lambda) \cap \partial\Omega$  is included in a hyperplane of  $\mathbb{R}^d$ , and such that the orthogonal projection of  $\text{supp}(\lambda) \cap \Omega$  on this hyperplane is included in  $\text{supp}(\lambda) \cap \partial\Omega$ . Upon rotating and relabeling the axes (*via* the action of an orthogonal matrix  $A \in \mathcal{O}_d(\mathbb{R})$ ), we can suppose that

$$(34) \quad \text{supp}(\lambda) \cap \Omega \subset \mathbb{R}_+^d \quad \text{and} \quad \text{supp}(\lambda) \cap \partial\Omega \subset \mathbb{R}^{d-1} \times \{0\} = \mathbb{R}^{d-1},$$

where  $\mathbb{R}_+^d = \{x = (\bar{x}, x_d) \in \mathbb{R}^d, x_d > 0\}$ .

Up to now, we set  $\lambda$  to represent this very function of localization in the definition of the mapping

$$\varphi : (x, t, y, s) \mapsto \psi(y, s)\lambda(y)w(x - y, t - s).$$

Besides, we will suppose that  $r \leq r_d$ , and that  $r, r_d$  are large enough to ensure that (34) still holds when  $K_\lambda = \text{supp}(\lambda)$  is replaced by its neighborhood

$$V_{r,r_d}(K_\lambda) = \left\{ x \in \mathbb{R}^d, \text{dist}(x, K_\lambda) < \sqrt{1/r^2 + 1/r_d^2} \right\}.$$

To estimate the term (32), we set

$$(y, s) \mapsto \int_{-\infty}^{-s} \varphi(x, 0, y, -\sigma) d\sigma,$$

and  $\kappa = u_0(x)$  to be, respectively, the test-function and the parameter in the entropy inequality (6) (with negative semi-entropy) satisfied by  $c$ , and we integrate the result with respect to  $x \in \Omega$ . This yields

$$\int_{\Omega \times \mathbb{R}^+} \int_{\Omega} (c_0(x) - c(y, s))^+ \varphi(x, 0, y, s) \, dx \, dt \, dy \leq R_1 + R_2 + R_3,$$

where

$$R_1 = \int_{\Omega} \int_{\Omega \times \mathbb{R}^+} \Phi^+(c_0(x), c(y, s)) \cdot \nabla_y \left( \int_{-\infty}^{-s} \varphi(x, 0, y, -\sigma) d\sigma \right) \, dy \, ds \, dx,$$

$$R_2 = \int_{\Omega} \int_{\Omega} (c_0(x) - c_0(y))^+ \psi(y, s)\lambda(y)\tilde{w}(x - y) \, dy \, dx,$$

$$R_3 = \mathcal{L} \int_{\Omega} \int_{\partial\Omega \times \mathbb{R}^+} (c_0(x) - \bar{c}(y, s))^+ \int_{-\infty}^{-s} \varphi(x, 0, y, -\sigma) d\sigma d\gamma(y) \, ds \, dx.$$

Here  $\tilde{w}$  denotes the approximation of the unit  $x \mapsto rw_0(rx_1) \cdots rw_0(x_{d-1})r_d w_0(x_d)$ .

Since

$$\nabla_y \left( \int_{-\infty}^{-s} \varphi(x, 0, y, -\sigma) d\sigma \right) = -\nabla_x \left( \int_{-\infty}^{-s} \varphi(x, 0, y, -\sigma) d\sigma \right),$$

we have

$$R_1 = - \int_{\Omega} \int_{\Omega \times \mathbb{R}^+} \Phi^+(c_0(x), c(y, s)) \cdot \nabla_x \left( \int_{-\infty}^{-s} \varphi(x, 0, y, -\sigma) d\sigma \right) \, dy \, ds \, dx.$$

Besides, for  $x \in \partial\Omega \cap V_{r,r_d}(K_\lambda)$ ,  $y \in \Omega \cap K_\lambda$ , we have, by (34),  $(x - y)_d \leq 0$  and, since  $\text{supp}(w_0) \subset [0, 1]$ ,  $\varphi(x, 0, y, -\sigma) = 0$ . In particular, by integrating by parts with respect to  $x$ , we have that

$$\int_{\Omega} \int_{\Omega \times \mathbb{R}^+} \Phi^+(c_0(y), c(y, s)) \cdot \nabla_x \left( \int_{-\infty}^{-s} \varphi(x, 0, y, -\sigma) d\sigma \right) dy ds dx = 0$$

and

$$R_1 = \int_{\Omega} \int_{\Omega \times \mathbb{R}^+} (\Phi^+(c_0(y), c(y, s)) - \Phi^+(c_0(x), c(y, s))) \cdot \nabla_x \left( \int_{-\infty}^{-s} \varphi(x, 0, y, -\sigma) d\sigma \right) dy ds dx.$$

Since

$$|(\Phi^+(c_0(y), c(y, s)) - \Phi^+(c_0(x), c(y, s)))| \leq \mathcal{L}|c_0(y) - c_0(x)| \leq \mathcal{L}|c_0|_{BV}|x - y|,$$

we have

$$R_1 \leq \mathcal{L}|c_0|_{BV} \|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)} \int_{\Omega} \int_{\Omega} |x - y| |\nabla_x \tilde{w}(x - y)| dy dx \times \left( \int_{\mathbb{R}^+} \int_{-\infty}^{-s} r w_0(-r\sigma) d\sigma ds \right).$$

The inequality

$$(35) \quad \int_{\mathbb{R}^+} \int_{-\infty}^{-s} r w_0(-r\sigma) d\sigma ds = \frac{1}{r} \int_0^1 \sigma w_0(\sigma) d\sigma \leq \frac{1}{r} \int_0^1 w_0(\sigma) d\sigma = \frac{1}{r}$$

gives

$$R_1 \leq \mathcal{L}|c_0|_{BV} \|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)} \int_{\Omega} \int_{\Omega} |x - y| |\nabla_x \tilde{w}(x - y)| dy dx \frac{1}{r}.$$

Furthermore, by (26) we have

$$(36) \quad \int_{\Omega} \int_{\Omega} |x - y| |\nabla_x \tilde{w}(x - y)| dy dx = \int_{\Omega} \int_{x-\Omega} |z| |\nabla \tilde{w}(z)| dz dx \leq |\Omega| \int_{\mathbb{R}^d} |z| |\nabla \tilde{w}(z)| dz \leq |\Omega| K'_1 \left( d - 1 + \frac{r}{r_d} \right), \quad \text{since } r \leq r_d.$$

Eventually, we have

$$(37) \quad R_1 \leq \mathcal{L}|c_0|_{BV} |\Omega| K'_1 (d - 1) \|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)} \left( \frac{1}{r} + \frac{1}{r_d} \right) \leq K_0 \|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)} \frac{1}{r},$$

where the constant  $K_0$  is defined by (23).

Similar computations (see [CH99a, EGGH98]) lead to the estimates  $R_2, R_3 \leq K_0 \frac{1}{r} \|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)}$ , and we have

$$(38) \quad \int_{\Omega \times \mathbb{R}^+} \int_{\Omega} (c_0(x) - c(y, s))^+ \varphi(x, 0, y, s) dx dt dy \leq 3K_0 \|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^+)} \frac{1}{r}.$$

We now prove the following estimate on (33):

$$(39) \quad \begin{aligned} \bar{I}_\lambda &:= - \int_{\Omega \times \mathbb{R}^+} \int_{\partial\Omega \times (0, T)} \Phi^-(\gamma c(y, s), c_h(x, t)) \cdot \mathbf{n}(y) \varphi(x, t, y, s) \, dx \, dt \, d\gamma(y) \, ds \\ &\leq K_0 \|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} (1 + 2r) \frac{1}{r_d} + \|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} (\eta_0 + \eta_t + \eta_c + \bar{\eta}). \end{aligned}$$

The identity  $(c - \kappa)^- = (c \top w - \kappa)^- + (c - \kappa \top w)^-$  holds for every  $c, \kappa, w \in \mathbb{R}$ . As a counterpart the corresponding entropy fluxes satisfy the relation

$$\Phi^-(c, \kappa) = \Phi^-(c \top w, \kappa) + \Phi^-(c, \kappa \perp w)$$

and, for a.e.  $(x, t, y, s) \in \Omega \times (0, T) \times \partial\Omega \times (0, T)$ , we have

$$\begin{aligned} -\Phi^-(\gamma c(y, s), c_h(x, t)) \cdot \mathbf{n}(y) &= -\Phi^-(\gamma c(y, s) \top \bar{c}(y, s), c_h(x, t)) \cdot \mathbf{n}(y) \\ &\quad -\Phi^-(\gamma c(y, s), c_h(x, t) \perp \bar{c}(y, s)) \cdot \mathbf{n}(y). \end{aligned}$$

By (10), we have

$$-\Phi^-(\gamma c(y, s), c_h(x, t) \perp \bar{c}(y, s)) \cdot \mathbf{n}(y) \leq \mathcal{L}(\bar{c}(y, s) - c_h(x, t) \perp \bar{c}(y, s))^- = 0,$$

and therefore  $\bar{I}_\lambda \leq \bar{J}_\lambda$  with

$$\begin{aligned} \bar{J}_\lambda &:= - \int_{\Omega \times \mathbb{R}^+} \int_{\partial\Omega \times (0, T)} \Phi^-(\gamma c(y, s) \top \bar{c}(y, s), c_h(x, t)) \\ &\quad \cdot \mathbf{n}(y) \varphi_{A_\lambda}(x, t, y, s) \, dx \, dt \, d\gamma(y) \, ds. \end{aligned}$$

From the relation of symmetry  $\Phi^-(a, b) = \Phi^+(b, a)$  it follows that

$$(40) \quad \begin{aligned} \bar{J}_\lambda &= - \int_{\mathbb{R}_+^d \times \mathbb{R}_+} \int_0^T \int_{\mathbb{R}^{d-1}} \Phi^+(c_h(x, t), \gamma c(\bar{y}, s) \top \bar{c}(\bar{y}, s)) \\ &\quad \cdot \mathbf{n}(y) \lambda(y) \varphi(x, t, \bar{y}, s) \, dx \, dt \, d\bar{y} \, ds. \end{aligned}$$

We now make use of the approximate entropy inequality (22) satisfied by  $c_h$  to get an estimate on  $\bar{J}_\lambda$ . Choose  $\kappa := \gamma c \top \bar{c}(\bar{y}, s)$ , and the test function  $(x, t) \mapsto \varphi^*(x, t)$  in (22), where

$$\begin{aligned} \varphi^*(x, t) &:= \lambda(y) \psi(\bar{y}, 0, s) r w_0(r(x_1 - y_1)) \cdots w_0(r(x_{d-1} - y_{d-1})) \\ &\quad \left( \int_{x_d}^{+\infty} r_d w_0(r_d \sigma) d\sigma \right) r w_0(r(t - s)). \end{aligned}$$

Integrate the result with respect to  $(\bar{y}, s)$ . This yields

$$\int_{\mathbb{R}_+^d \times \mathbb{R}_+} E^+(c_h, \gamma c \top \bar{c}(\bar{y}, s), \varphi^*) d\bar{y} \, ds \geq F,$$

where the right hand side  $F$  can be estimated from below by

$$F \geq -\|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} (\eta_0 + \eta_t [K_1 r] + \eta_c [K_1 r + K_1'(r + r_d)] + \bar{\eta}),$$

exactly as in the proof of Lemma 5.7, leading to

$$(41) \quad \begin{aligned} &\int_{\mathbb{R}_+^d \times \mathbb{R}_+} E^+(c_h, \gamma c \top \bar{c}(\bar{y}, s), \varphi^*) d\bar{y} \, ds \\ &\geq -\|\psi\lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} \left( \eta_0 + \eta_t [K_1 r] + \eta_c [K_1 r + K_1'(r + r_d)] + \bar{\eta} \right). \end{aligned}$$

On the other hand, we have

$$(42) \quad \int_{\mathbb{R}_+^d \times \mathbb{R}_+} E^+(c_h, \gamma c(\bar{y}, s) \top \bar{c}(\bar{y}, s), \varphi^*) d\bar{y} ds = A_1 + A_2 + A_3 + A_4,$$

where

$$\begin{aligned} A_1 &= \int_{\mathbb{R}^{d-1} \times \mathbb{R}_+} \int_{\mathbb{R}_+^d \times \mathbb{R}_+} (c_h - \gamma c \top \bar{c}(\bar{y}, s))^+ \partial_t \varphi^* dx dt d\bar{y} ds, \\ A_2 &= \int_{\mathbb{R}^{d-1} \times \mathbb{R}_+} \int_{\mathbb{R}_+^d \times \mathbb{R}_+} \Phi^+((c_h, \gamma c \top \bar{c}(\bar{y}, s)) \cdot \nabla \varphi^* dx dt d\bar{y} ds, \\ A_3 &= \int_{\mathbb{R}^{d-1} \times \mathbb{R}_+} \int_{\mathbb{R}_+^d} (c_0(x) - \gamma c \top \bar{c}(\bar{y}, s))^+ \varphi^*(0, x, s, \bar{y}) dx d\bar{y} ds, \\ A_4 &= \mathcal{L} \int_{\mathbb{R}^{d-1} \times \mathbb{R}_+} \int_{\mathbb{R}^{d-1} \times \mathbb{R}_+} (\bar{c}(\bar{x}, 0) - \gamma c \top \bar{c}(\bar{y}, s))^+ \varphi^*(t, \bar{x}, 0, s, \bar{y}) d\bar{x} dt d\bar{y} ds. \end{aligned}$$

Since  $\text{supp}(w_0) \subset [0, 1]$ , we have  $A_3 = 0$ . Besides, the terms  $A_1, A_4$  are small with respect to  $1/r_d$ . Indeed, we have

$$(43) \quad \int_0^\infty \int_{x_d}^\infty r_d w_0(r_d \sigma) d\sigma dx_d = \frac{1}{r_d} \int_0^1 \sigma w_0(\sigma) d\sigma \leq \frac{1}{r_d},$$

and, by (26), (27),

$$(44) \quad |A_1| \leq 2\mathcal{C} \|w'_0\|_{L^1(\mathbb{R})} T |\Omega| \|\psi \lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} \frac{r}{r_d} \leq K_0 \|\psi \lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} \frac{r}{r_d}.$$

Besides, we have  $(\bar{c}(\bar{x}, t) - \gamma c \top \bar{c}(\bar{y}, s))^+ \leq (\bar{c}(\bar{x}, t) - \bar{c}(\bar{y}, s))^+$ , and, therefore,

$$(45) \quad |A_4| \leq |\bar{c}|_{BV} T |\Omega| \|\psi \lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} \frac{1}{r} \leq K_0 \|\psi \lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} \frac{1}{r}.$$

We now intend to compare  $-A_2$  to  $\bar{\mathcal{J}}_\lambda$ . Since  $\partial_{x_d} \varphi^*(x, t, \bar{y}, s) = \lambda(y) \varphi(x, t, \bar{y}, s)$  and since  $\mathbf{n}(y) = (0, \dots, 0, -1)^T \in \mathbb{R}^d$ , we have, by (40),

$$A_2 + \bar{\mathcal{J}}_\lambda = \int_{\mathbb{R}^{d-1} \times \mathbb{R}_+} \int_{\mathbb{R}_+^d \times \mathbb{R}_+} \Phi^+((c_h, \gamma c \top \bar{c}(\bar{y}, s)) \cdot \bar{\nabla} \varphi^* dx dt d\bar{y} ds,$$

where  $\bar{\nabla} \varphi^* = (\partial_{x_1} \varphi^*, \dots, \partial_{x_{d-1}} \varphi^*, 0)^T \in \mathbb{R}^d$ . By (43) we have

$$(46) \quad \begin{aligned} |A_2 + \bar{\mathcal{J}}_\lambda| &\leq 2\mathcal{L}\mathcal{C}(d-1) \|w'_0\|_{L^1(\mathbb{R})} T |\Omega| \|\psi \lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} \frac{r}{r_d} \\ &\leq K_0 \|\psi \lambda\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}_+)} \frac{r}{r_d}. \end{aligned}$$

From (41), (42), (44), (45), and (46) follows (39).

**5.3. Third step.** Set  $\psi(s, y) = \psi(s) = \frac{T-s}{T} \chi_{(0,T)}(s)$ . Then  $0 \leq \psi \leq 1$  and, collecting (29), (31)–(33), (38), and (39), we have

$$\begin{aligned} &\int_{\Omega \times (0,T)} \int_{\Omega \times (0,T)} \left[ \frac{-1}{T} (c_h(x, t) - c(y, s))^+ \lambda(y) \right. \\ &\quad \left. + \Phi^+(c_h(x, t), c(y, s)) \cdot \nabla_y \lambda \right] w(x - y, t - s) dx dt dy ds \\ &\geq -\|\lambda\|_{L^\infty(\mathbb{R}^d)} \left( 2(\eta_0 + \eta_t [K_1 r] + \eta_c [K_1 r + K'_1(r + r_d)] + \bar{\eta}) + K_0 \left( \frac{4}{r} + 2\frac{r}{r_d} \right) \right), \end{aligned}$$

where  $\lambda$  is defined in Definition 5.8. Since  $0 \leq \lambda \leq 1$ , we have in particular

$$(47) \quad \int_{\Omega \times (0,T)} \int_{\Omega \times (0,T)} \left[ \begin{array}{l} \frac{-1}{T} (c_h(x,t) - c(y,s))^+ \lambda(y) \\ + \Phi^+(c_h(x,t), c(y,s)) \cdot \nabla_y \lambda \end{array} \right] w(x-y, t-s) \, dx \, dt \, dy \, ds \\ \geq - \left( 2(\eta_0 + \eta_t [K_1 r] + \eta_c [K_1 r + K'_1(r+r_d)] + \bar{\eta}) + K_0 \left( \frac{4}{r} + 2 \frac{r}{r_d} \right) \right).$$

We gather those local estimates (47) to get a global estimate. Denote by  $(A_i)_{i=1, \dots, N_f}$  the faces of  $\Omega$ , and by  $n_i$  the outward unit normal to  $\Omega$  along  $A_i$ . Let  $B_i^{\tilde{r}}$  be the subset of all  $x \in \Omega$  such that  $\text{dist}(x, A_i) < \frac{1}{\tilde{r}}$  and  $\text{dist}(x, A_i) < \text{dist}(x, A_j)$  if  $i \neq j$ ; define  $G_i^{\tilde{r}}$  to be the largest cylinder generated by  $n_i$  included in  $B_i^{\tilde{r}}$ , and set  $\Delta_i^{\tilde{r}} = B_i^{\tilde{r}} \setminus G_i^{\tilde{r}}$ ,  $\Omega_{\tilde{r}} = \Omega \setminus (\bigcup_{i=1, N_f} \Delta_i^{\tilde{r}})$  and  $b^{\tilde{r}} = \mathbf{1}_{\Omega_{\tilde{r}/2}} \star \rho_{\tilde{r}/4}$ .

An estimate as (47) remains true if  $\lambda = \lambda_0$ , where  $\lambda_0$  is a localization function with support included in  $\Omega$ . Then no boundary conditions have to be taken into account, and the term (33) can be considered to be zero. Now, we can write  $b^{\tilde{r}} = \sum_{i=0, N_f} \lambda_i$  for such a function  $\lambda_0$  and for functions  $(\lambda_i)_{1, N_f}$  satisfying the hypotheses of Definition 5.8. Therefore, we have that

$$(48) \quad \iint_{(\Omega \times (0,T))^2} \frac{-1}{T} (c_h(x,t) - c(y,s))^+ w(x-y, t-s) \, dx \, dt \, dy \, ds \geq \delta_1 + \delta_2 \\ - (N_f + 1) \left( 2(\eta_0 + \eta_t [K_1 r] + \eta_c [K_1 r + K'_1(r+r_d)] + \bar{\eta}) + K_0 \left( \frac{4}{r} + 2 \frac{r}{r_d} \right) \right),$$

where

$$\delta_1 = \iint_{(\Omega \times (0,T))^2} \frac{-1}{T} (c_h(x,t) - c(y,s))^+ \left( 1 - \sum_{i=0, N_f} \lambda_i \right) w(x-y, t-s) \, dx \, dt \, dy \, ds, \\ \delta_2 = \iint_{(\Omega \times (0,T))^2} \Phi^+(c_h(x,t), c(y,s)) \cdot \nabla \left( 1 - \sum_{i=0, N_f} \lambda_i \right) w(x-y, t-s) \, dx \, dt \, dy \, ds.$$

We have  $|\Omega \setminus \Omega_{\tilde{r}}| \leq \frac{C_\Omega}{\tilde{r}^2}$  where  $C_\Omega$  is a constant which depends only on  $\Omega$  and  $\|\nabla \lambda_i\|_{L^\infty} \leq \tilde{r}$  so that, for  $\tilde{r} < 1$ ,

$$|\delta_1| \leq 2CT|\Omega| \frac{C_\Omega}{\tilde{r}^2} \leq \frac{K_0}{\tilde{r}}, \quad |\delta_2| \leq 2\mathcal{L}CT|\Omega| \frac{C_\Omega}{\tilde{r}} \leq \frac{K_0}{\tilde{r}}.$$

Besides, by the *BV* estimate (7), we have

$$\iint_{(\Omega \times (0,T))^2} (c_h(x,t) - c(y,s))^+ w(x-y, t-s) \, dx \, dt \, dy \, ds - \|(c_h - c)^+\|_{L^1(\Omega \times (0,T))} \\ = \iint_{(\Omega \times (0,T))^2} ((c_h(x,t) - c(y,s))^+ - (c_h(x,t) - c(x,t))^+) w(x-y, t-s) \, dx \, dt \, dy \, ds \\ \geq - \iint_{(\Omega \times (0,T))^2} |c(y,s) - c(x,t)| w(x-y, t-s) \, dx \, dt \, dy \, ds \\ \geq -C_{BV}T|\Omega| \frac{1}{r} \geq -K_0 \frac{1}{r},$$

and, consequently,

$$\begin{aligned} & \| (c_h - c)^+ \|_{L^1(\Omega \times (0, T))} \\ & \leq 2TK_0 \frac{1}{\tilde{r}} \\ & \quad + 2T(N_f + 1) \left[ \eta_0 + \bar{\eta} + (\eta_t + \eta_c)K_1 r + \eta_c K'_1 (r + r_d) + K_0 \left( \frac{2}{r} + \frac{r}{r_d} \right) \right]. \end{aligned}$$

We let  $\tilde{r} \rightarrow +\infty$  in this last result and then minimize the right hand side with respect to  $r$  and  $r_d$  to get the a posteriori error estimate

$$\begin{aligned} & \| (c_h - c)^+ \|_{L^1(\Omega \times (0, T))} \\ & \leq 2T(N_f + 1) \\ & \quad \times \min_{r, r_d \in \mathbb{R}^+} \left[ \eta_0 + \bar{\eta} + (\eta_t + \eta_c)K_1 r + \eta_c K'_1 (r + r_d) + K_0 \left( \frac{2}{r} + 1 \frac{r}{r_d} \right) \right]. \end{aligned}$$

The a priori error estimate now follows with the estimates on  $\eta_0, \bar{\eta}, \eta_t, \eta_c$  (see [CH99a, EGGH98]), i.e.,

$$\eta_0 + \bar{\eta} + \eta_t + \eta_c \leq Kh^{1/2},$$

and choosing, for example,

$$r := \left( \frac{(\eta_t + \eta_c)K_1 + \eta_c K'_1}{K_0} \right)^{-1/3}, \quad r_d := \left( \frac{K_0 r}{\eta_c K'_1} \right)^{1/2}.$$

□

*Remark 5.9* (Nonoptimal order of convergence). The error estimates in Propositions 5.1, and 5.2 are nonoptimal compared to the convergence rate  $h^{1/4}$  that can be proved for finite volume approximations of the Cauchy problem (cf. [CH99a]). The nonoptimality of our result comes from the estimate on the boundary term (39). Let us mention that in the special situation where  $\mathbf{F}(x, t, c) = \mathbf{u}(x, t)f(c)$ , and  $f$  is monotone, this estimate can be improved, and the order  $h^{1/4}$  is recovered (see also [Vig97]). However, the improvement in this special situation makes excessive use of the a priori knowledge of inflow and outflow boundaries and gives no hint to improve our general result.

## 6. ADAPTIVE ALGORITHM AND NUMERICAL EXPERIMENTS

In this subsection we will derive an adaptive algorithm from our theoretical a posteriori result in Proposition 5.1, and we will give some numerical experiments in order to demonstrate the applicability of the resulting adaptive solution scheme. In addition, we will demonstrate that the creation of artificial boundary layers depends on the choice of the numerical flux function.

**6.1. Adaptive algorithm.** The adaptive solution algorithm is derived by localizing the global error estimator  $\eta$  of Proposition 5.1 into local error indicators  $\eta_k^n$ . Here  $n$  denotes the time step and  $k$  the triangle number of the underlying mesh. An equal distribution strategy of the local indicators leads us to the space adaptive algorithm. The adaptive time step is implicitly given through the CFL condition. As the derivation of the adaptive algorithm is a direct generalization of the algorithm on unbounded domains we refer to [KO00], and [Ohl01a, Ohl01b] for further details.



FIGURE 1. Color shading of the exact solution of the linear problem at  $t = \pi/4$ ,  $t = \pi/2$ , and  $t = 2$ .

**6.2. First example: Linear transport problem.** As a first example we choose a linear problem where the inflow and outflow regions are known a priori. The example is chosen, for instance, as it comes with a known exact solution. Thus, we can compare the  $L^1$ -error between the exact and the approximate solution with the error estimator  $\eta$  defined in Proposition 5.1.

We look at the following initial boundary value problem in  $\Omega := (0, 1) \times (0, 1)$ :

$$\begin{aligned} c_t + \nabla \cdot (\mathbf{u}c) &= 0 \text{ in } \Omega \times (0, T), \\ c(\cdot, 0) &= 0 \text{ in } \Omega, \\ c(x, t) &= \bar{c}(t, x) \text{ in } \partial\Omega \times (0, T). \end{aligned}$$

Then  $c(x, t)$  is constant along the streamlines of the prescribed velocity field  $\mathbf{u}(x_1, x_2) := (x_2, -x_1)^\top$ , and therefore only depends on the initial data and the boundary values at the inflow boundary. In our example we set

$$\bar{c}(t, x) := \begin{cases} 1, & \text{if } x \in \{0\} \times (0.4, 0.8), \\ 0, & \text{else.} \end{cases}$$

The exact solution (see Figure 1) of this problem is

$$c(t, x) := \begin{cases} 1, & \text{if } \arcsin\left(\frac{x_1}{|x|}\right) - t \leq 0, \text{ and } |x| \in (0.4, 0.8), \\ 0, & \text{else.} \end{cases}$$



FIGURE 2. Comparison of the boundary layer behavior at the outflow boundary for the full upwind flux (left), and the Lax-Friedrichs flux (right).

TABLE 1. Comparison of the  $L^1$ -error and the convergence rate for upwind and Lax–Friedrichs flux on uniform meshes.

Number of triangles	$L^1$ -error Upwind flux	Order of $L^1$ -error	$L^1$ -error Lax–Friedrichs flux	Order of $L^1$ -error
4096	9.244172E-02		2.496511E-01	
16384	6.307752E-02	0.551	1.845089E-01	0.436
65536	4.460084E-02	0.500	1.342645E-01	0.459
262144	3.174344E-02	0.491	9.676411E-02	0.473
1048576	2.252004E-02	0.495	6.918218E-02	0.484

TABLE 2. Comparison of the error estimator  $\eta$  and its convergence rate for upwind and Lax–Friedrichs flux on uniform meshes.

Number of triangles	$\eta$ Upwind flux	Order of $\eta$	$\eta$ Lax–Friedrichs flux	Order of $\eta$
4096	0.4140817		0.7457754	
16384	0.3256964	0.346	0.6108820	0.288
65536	0.2594602	0.328	0.4922482	0.312
262144	0.2070642	0.325	0.3922287	0.328
1048576	0.1645522	0.332	0.3102630	0.338

In our first numerical experiment we compare the generation of an artificial boundary layer at the outflow boundary for two different numerical flux functions. As Figure 2 clearly shows, an artificial boundary layer is created by using the Lax–Friedrich flux, while no artificial layer is produced with full upwinding (e.g., Engquist–Osher, or Godunov flux). For a detailed study of this boundary layer behavior in one space dimension, we refer to [CHG01]. In addition we remark that in the case of systems of conservation laws it might be necessary to choose a Godunov flux at the boundary in order to get a proper discretization of the boundary conditions. In the scalar case the creation of an artificial boundary layer does not influence the convergence rate of the scheme. For instance, in our example both methods converge with an experimental order of convergence of  $h^{1/2}$  where  $h$  denotes the uniform mesh size. Nevertheless, the absolute error for the upwind method is much smaller than in the Lax–Friedrichs case (see Table 1).

The influence of the choice of the numerical flux function on the adaptive solution algorithm is shown in Figure 3. While in both cases the interior layers are resolved by the adaptive algorithm, the grid is additionally refined in the artificial boundary layer in the case of the Lax–Friedrichs flux. This leads to some extra numerical cost which might lead to an inefficient adaptive numerical scheme.

In Table 2 we give the values of the error estimator  $\eta$  of Proposition 5.1. The numerical order of convergence of the estimator is  $h^{1/3}$  for both choices of the numerical flux (see Table 2). Our theoretical investigation would lead to the same order of convergence if we would be able to prove that the numerical solution is uniformly bounded in BV. Up to now such a bound is only available in one space dimension or for finite volume schemes on structured rectangular grids. In the general case we are only able to show that the BV norm of the approximate solution

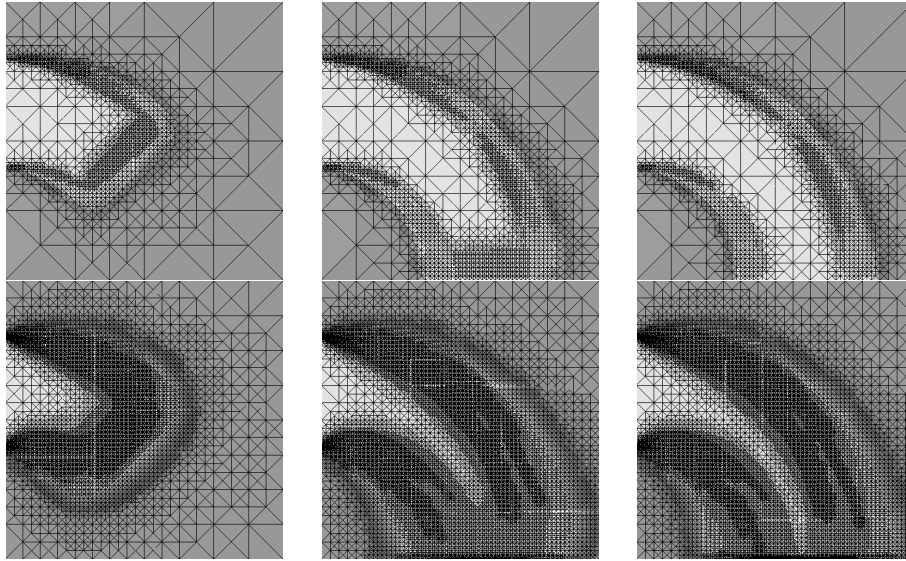


FIGURE 3. Adaptive solution of the linear problem at  $t = \pi/4$ ,  $t = \pi/2$ , and  $t = 2$ . A color shading of the solution together with the adaptive grid is plotted for the full upwind flux (top), and the Lax–Friedrichs flux (bottom).

blows up like  $h^{-1/2}$  which leads to the convergence rate  $h^{1/6}$  instead of  $h^{1/3}$ . In this sense, the numerical experiments coincide with our theoretical a priori error bound of the error estimator  $\eta$ . As the  $L^1$ -error itself converges with a rate  $h^{1/2}$ , it is obvious that our error analysis does not give the optimal rate. Nevertheless,

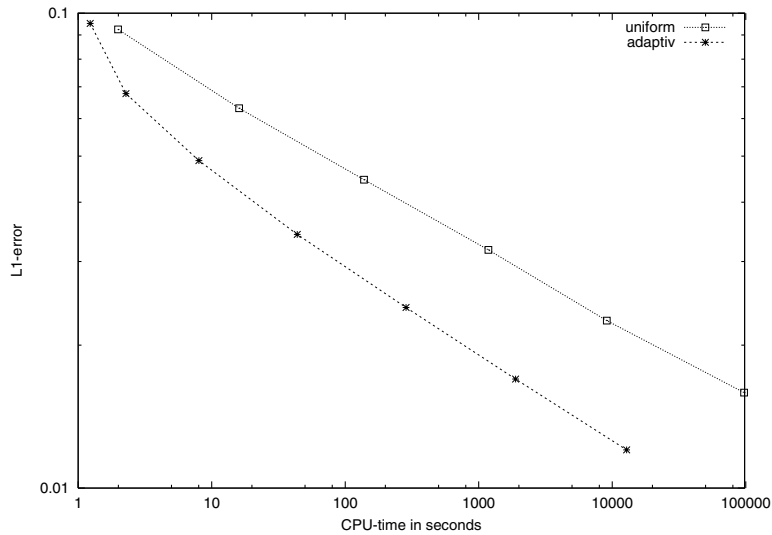


FIGURE 4.  $L^1$  error versus run time for uniform and adaptive computations for the linear problem.

we stress that there was no proof of any convergence rate in the general case of bounded domains before.

In a last experiment we analyze the performance of the adaptive scheme versus the same scheme on a mesh with uniform mesh size. Therefore, in Figure 4 we plot the  $L^1$ -error versus run time for uniform and adaptive computations using the upwind flux. The comparison shows that the adaptive scheme performs much better than the method on uniform grids. In addition, we stress that the adaptive algorithm requires far less storage than the uniform one. For instance, in the finest computations the maximal number of mesh cells in the adaptive case was about 350.000, while 4.200.000 mesh cells were used in the uniform computation.

**6.3. Second example: Burgers' problem.** Next we consider an essential one dimensional problem that we formulate and solve in a two dimensional framework. We look at the following initial boundary value problem in  $\Omega := (0, 2) \times (0, 1)$ :

$$\begin{aligned} c_t + \nabla \cdot (1/2 c^2, 0)^\top &= 0 \text{ in } \Omega \times (0, T), \\ c(\cdot, 0) &= 0 \text{ in } \Omega \\ c(x, t) &= \bar{c}(t, x) \text{ in } \partial\Omega \times (0, T). \end{aligned}$$

We choose  $T = 2.0$ , and  $\bar{c}(t, x)$  is given as

$$\bar{c}(t, x) = \begin{cases} 1 & \text{if } t \in (0, 0.5) \cup (1, 1.5) \\ -1 & \text{else.} \end{cases}$$

For this problem, the exact solution is given as follows:

If  $t \in (0, 0.5)$ ,

$$c(t, x) := \begin{cases} 1, & \text{if } 0 \leq x_1 \leq 0.5t, \\ 0, & \text{else.} \end{cases}$$

If  $t \in (0.5, 1.0)$ ,

$$c(t, x) := \begin{cases} \frac{x_1}{t-0.5}, & \text{if } 0 \leq x_1 < t - 0.5, \\ 1, & \text{if } t - 0.5 \leq x_1 < 0.5t, \\ 0, & \text{if } 0.5t \leq x_1 < 2.0 - 0.5(t - 0.5), \\ -1, & \text{else.} \end{cases}$$

If  $t \in (1.0, 1.5)$ ,

$$c(t, x) := \begin{cases} 1, & \text{if } 0 \leq x_1 < (t - 0.5) - \sqrt{0.5(t - 0.5)}, \\ \frac{x_1}{t-0.5}, & \text{if } (t - 0.5) - \sqrt{0.5(t - 0.5)} \leq x_1 < \sqrt{t - 0.5}, \\ 0, & \text{if } \sqrt{t - 0.5} \leq x_1 < 2 - 0.5(t - 0.5), \\ -1, & \text{if } 2 - 0.5(t - 0.5) \leq x_1 < 3 - t, \\ -\frac{2-x_1}{t-1}, & \text{else.} \end{cases}$$

If  $t \in (1.5, 2.0)$ ,

$$c(t, x) := \begin{cases} \frac{x_1}{t-1.5}, & \text{if } 0 \leq x_1 < t - 1.5, \\ 1, & \text{if } t - 1.5 \leq x_1 < t - 0.5 - \sqrt{0.5(t - 0.5)}, \\ \frac{x_1}{t-0.5}, & \text{if } t - 0.5 - \sqrt{0.5(t - 0.5)} \leq x_1 < \sqrt{0.5(t - 0.5)}, \\ 0, & \text{if } \sqrt{0.5(t - 0.5)} \leq x_1 < 2 - \sqrt{0.5(t - 1)}, \\ -\frac{2-x_1}{t-1}, & \text{if } 2 - \sqrt{0.5(t - 1)} \leq x_1 < 3 - t + \sqrt{0.5(t - 1)}, \\ -1, & \text{else.} \end{cases}$$

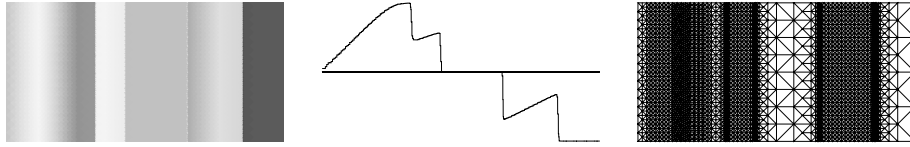


FIGURE 5. Adaptive solution of the Burgers' problem at  $t = 2.0$ . Color shading of the solution (left), profile on a horizontal cut through the domain (middle) and adaptive grid (right).

We chose this test problem in order to demonstrate the behavior of the error estimator and the resulting numerical scheme in the case where the inflow and outflow boundaries change in time, dependent on the boundary values and the solution. Figure 5 shows the adaptive numerical solution and computational grid for  $t = 2.0$  for a moderately prescribed error tolerance. We see that the grid is refined in the shock regions and also moderately in the regions of the rarefaction waves, while a very coarse resolution is needed in the regions of constant states.

Figure 6 demonstrates the adaptive refinement around  $t = 1$ . For  $t$  slightly smaller than one, no heavy refinement is needed near the boundaries which is automatically reflected by the error estimator. At  $t = 1$  the boundary values change, and immediately the boundary zones are heavily refined. This example shows that the adaptive algorithm is capable to detect the sources of errors coming from the approximation of the boundary values automatically.

Finally, we compare the adaptive and uniform algorithms in an *error versus run time* plot (see Figure 7). Although the algorithm on uniform grids converges with a rate of  $h$  in the case of the Burgers' problem, and the error estimator  $\eta$  still converges like  $h^{1/3}$ , the adaptive algorithm performs better than the uniform one, and requires far less storage.

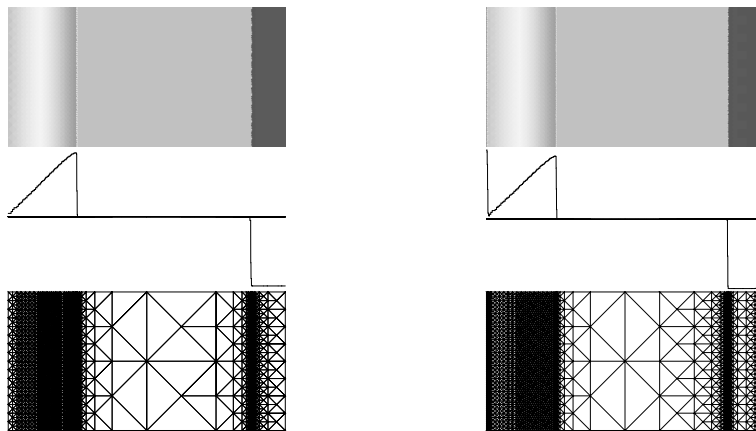


FIGURE 6. Adaptive solution of the Burgers' problem at  $t = 0.997$  (left), and  $t = 1.016$  (right). Color shading of the solution (top), profile on a horizontal cut through the domain (middle) and adaptive grid (bottom).

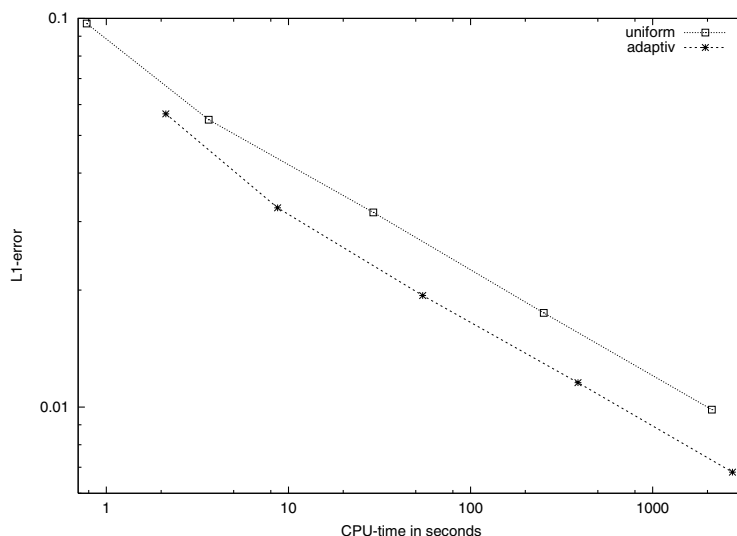


FIGURE 7.  $L^1$  error versus run time for the Burgers' problem.

APPENDIX A.  $BV$  ESTIMATE OF THE EXACT SOLUTION ON CONVEX DOMAINS

Under certain regularity hypotheses on the data and on the boundary of  $\Omega$ , the  $BV$  estimate (7) has been proved by Bardos, LeRoux and Nédélec [BLN79] (see also [MNR96]). They obtain the estimate from a uniform  $BV$  estimate on viscous approximations  $u^\varepsilon$  of problem (1)–(3),  $\varepsilon$  being the viscosity parameter. In our considerations  $\Omega$  is only a polygonal open bounded set, which makes it difficult to adapt the technique developed in [BLN79] (the solution  $u^\varepsilon$  of the viscous approximation of problem (1)–(3) is not regular enough in that case). In order to prove the  $BV$  estimate (7) in our situation, we will follow the approach of Chainais-Hillairet [CH99a] for unbounded domains. Thus, we construct a sequence of finite volume approximations to (1)–(3) on structured meshes and drive a uniform bound on the  $BV$  semi-norm. Technical difficulties arise from the fact that we consider a cartesian grid on a polygonal open set which is possibly not rectangular. For simplicity of our presentation we will suppose that the flux function  $\mathbf{F}$  is constant with respect to  $(x, t)$ , i.e.,  $\mathbf{F}(x, t, u) = \mathbf{F}(u)$ .

**Notation for a structured mesh approximation of  $\Omega$ .** Let  $(\mathbf{e}_i)_{1,d}$  be an orthonormal basis of  $\mathbb{R}^d$ , and  $O \in \mathbb{R}^d$  a fixed origin. Then  $(O, \mathbf{e}_1, \dots, \mathbf{e}_d)$  is an affine basis of  $\mathbb{R}^d$  viewed as an affine space. Given  $h > 0$ , we define the structured mesh  $\mathcal{T}_h(\mathbb{R}^d)$  as

$$\mathcal{T}_h(\mathbb{R}^d) := \{T_j \mid T_j := (j_1h, (j_1 + 1)h) \times \dots \times (j_dh, (j_d + 1)h), j \in \mathbb{Z}^d\}$$

and set

$$\mathcal{T}_h := \mathcal{T}_h(\Omega) := \{T \in \mathcal{T}_h(\mathbb{R}^d) \mid \bar{T} \subset \Omega\}, \quad \bar{\Omega}_h = \bigcup_{T \in \mathcal{T}_h} \bar{T}.$$

Define  $\Omega_h$  as the interior of  $\bar{\Omega}_h$ . Furthermore, the time step  $\Delta t$  is defined as  $\Delta t := \frac{T}{N}$ , where  $N = N(h)$  is given, and we denote  $J := \{t_0, t_1, \dots, t_N\}$  where  $t_n := n\Delta t$ . Using multi-index notation, we define the set of neighboring cells to

a cell  $T_j$  by  $\mathcal{N}(j) := \{T_l \in \mathcal{T}_h(\mathbb{R}^d) | l = j \pm \mathbf{e}_i, i = 1, \dots, d\}$  and denote the face between two neighboring cells by  $S_{j,l} := \overline{T_j} \cap \overline{T_l}$ . Furthermore, set

$$\mathcal{E} := \{S_{j,l} \neq \emptyset | T_j, T_l \in \mathcal{T}_h\}, \quad \partial\mathcal{E} := \{S_{j,l} \neq \emptyset | T_j \in \mathcal{T}_h, T_l \in \mathcal{T}_h(\mathbb{R}^d) \setminus \mathcal{T}_h\}.$$

In the next step we introduce some notation that allows us to link the exterior faces  $S \in \partial\mathcal{E}$  to its orthogonal projection  $P(S)$  on the *closest* part of  $\partial\Omega$ . Therefore, let us denote  $A_1, \dots, A_n$  the distinct faces of  $\Omega$  which generate the hyperplanes  $H_1, \dots, H_n$ . For  $S \in \partial\mathcal{E}$  define  $\mathcal{I}_S = \{i \in \{1, \dots, n\} | \text{dist}(S, A_i) \leq 2D_d h\}$ , where  $D_d := \sqrt{2^{d-1}}$  is the diameter of the unit cube in  $\mathbb{R}^d$ . Let  $P_i$  denote the orthogonal projection on  $H_i$ , and define  $G_i$  as the cylinder generated by  $A_i$ :  $G_i = P_i^{-1}(A_i)$ . We define the set of faces associated with the boundary part  $A_i$  as

$$\partial\mathcal{E}_i := \{S \in \partial\mathcal{E} | \mathcal{I}_S = \{i\} \text{ and } S \subset G_i\},$$

and the remaining boundary faces as

$$\partial\mathcal{E}_\bullet := \partial\mathcal{E} \setminus \bigcup_{i=1}^n \partial\mathcal{E}_i.$$

**Finite volume approximation on  $\mathcal{T}_h \times J$ .** For the finite volume approximation on  $\mathcal{T}_h \times J$ , we first define the discrete initial and boundary values as

$$(49) \quad c_j^0 := \frac{1}{|T_j|} \int_{T_j} c_0 \quad \forall T_j \in \mathcal{T}_h,$$

$$(50) \quad \bar{c}_{jl}^n := \frac{1}{\Delta t |P_i(S_{jl})|} \int_{t^n}^{t^{n+1}} \int_{P_i(S_{jl})} \bar{c}(x, t) d\gamma(x) dt \quad \forall S_{jl} \in \partial\mathcal{E}_i,$$

$$(51) \quad \bar{c}_{jl}^n := 0 \quad \forall S_{jl} \in \partial\mathcal{E}_\bullet.$$

Then for all  $t^{n+1} \in J$  and all  $T_j \in \mathcal{T}_h$ , the values  $c_j^{n+1}$  are defined as

$$(52) \quad c_j^{n+1} := c_j^n - \frac{\Delta t}{h} \sum_{l \in \mathcal{N}(j)} \left( F_{jl}(c_j^n) - F_{jl}(c_{jl}^n) \right), \quad c_{jl}^n := \begin{cases} c_l^n, & \text{if } T_l \in \mathcal{T}_h, \\ \bar{c}_{jl}^n, & \text{else,} \end{cases}$$

where we use a flux splitting approach with monotone fluxes

$$F_{jl}(c) := \frac{1}{2}(-\mathbf{F}(c) \cdot \mathbf{n}_{jl} + \mathcal{L}c),$$

where  $\mathcal{L}$  denotes the Lipschitz constant of  $\mathbf{F}$ . The numerical approximation  $c_h$  of the entropy solution  $c$  is given by

$$(53) \quad c_h(x, t) = \begin{cases} c_j^n & \text{if } (x, t) \in T_j \times [t^n, t^{n+1}), T_j \in \mathcal{T}_h, \\ 0 & \text{otherwise.} \end{cases}$$

Although the value given in (51) is arbitrary, the error in the implementation of the finite volume method induced by such a process does not affect the convergence of the scheme because the number of interfaces  $S$  “close” to a corner of  $\partial\Omega$  is at most of order  $h/|S|$  where  $|S| = h^{d-1}$  is the Lebesgue measure of  $S$ . We specify this point in the following lemma.

**Lemma A.1.** *For sufficiently small  $h$ , the set  $\partial\mathcal{E}_\bullet$  is finite and there exists a constant  $C_\#$  which depends on the dimension  $d$  and on  $\Omega$  only such that the cardinal of  $\partial\mathcal{E}_\bullet$  is bounded by  $\frac{C_\#}{h^{d-2}}$ .*

*Proof.* For  $X \subset \mathbb{R}^d$  and  $a > 0$ , we define the  $a$  neighborhood of  $X$  as  $\mathcal{V}_a(X) := \{y \in \mathbb{R}^d \mid \text{dist}(y, X) < a\}$ . Let  $H$  and  $G$  be two affine hyperplanes of  $\mathbb{R}^d$ , and denote by  $\overline{HG}$  the angle between them. We will show in a first step that for  $H, G$  nonparallel, i.e.,  $\overline{HG} \neq 0$ , it holds that

$$(54) \quad \mathcal{V}_a(H) \cap \mathcal{V}_a(G) \subset \mathcal{V}_{\left(\frac{3\pi}{2\overline{HG}}+1\right)a}(H \cap G).$$

Consider a point  $U \in \mathcal{V}_a(H) \cap \mathcal{V}_a(G)$ . Denote by  $U_{H \cap G}, U_H$  and  $U_G$  the orthogonal projections of  $U$  on  $H \cap G, H$  and  $G$ , respectively, and by  $U_{HG}$  the orthogonal projection of  $U_H$  on  $G$ . We have

$$\begin{aligned} \text{dist}(U, H \cap G) &= |U_{H \cap G} - U| \\ &\leq |U_{H \cap G} - U_H| + |U_H - U| = \frac{|U_H - U_{HG}|}{\sin(\overline{HG})} + |U_H - U| \\ &\leq \frac{|U_H - U| + |U - U_G| + |U_G - U_{HG}|}{\sin(\overline{HG})} + |U_H - U|, \end{aligned}$$

and we conclude (54) by using the estimates

$$|U_G - U_{HG}| \leq |U - U_H| < a, \quad |U - U_G| < a,$$

and

$$\frac{1}{\sin(\overline{HG})} \leq \frac{\pi}{2\overline{HG}}.$$

For

$$h < h_0 := \frac{1}{8D_d} \min_{1 \leq i, j \leq n} \{\text{dist}(A_i, A_j), A_i \cap A_j = \emptyset\},$$

the intersection  $\mathcal{V}_{4D_d h}(A_i) \cap \mathcal{V}_{4D_d h}(A_j)$  is not empty if and only if  $A_i$  and  $A_j$  are adjacent. Furthermore, if  $A_i \neq A_j$ , then  $\overline{A_i A_j} > 0$ , and (54) yields

$$\mathcal{V}_{4D_d h}(A_i) \cap \mathcal{V}_{4D_d h}(A_j) \subset \mathcal{V}_{C_1 h}(A_i \cap A_j),$$

to where  $C_1 := \max_{1 \leq k \neq l \leq n} \left\{ \left( \frac{3\pi}{2A_k A_l} + 1 \right) 4D_d h, A_k \cap A_l \neq \emptyset \right\}$ .

Let  $S \in \partial \mathcal{E}_\bullet$ . We will show that  $S \subset \mathcal{V}_{C_1 h}(A_i \cap A_j)$  for some adjacent  $A_i, A_j$ . This assertion is clearly satisfied if  $\{i, j\} \subset \mathcal{I}_S$ . If there is no such subset  $\{i, j\}$ , there is an  $i \in \mathcal{I}_S$  with  $S \not\subset G_i$ . In that case  $\text{dist}(S, \partial A_i) = \text{dist}(S, A_i) \leq 2D_d h$ , and hence there exists  $x \in S, y \in \partial A_i$  such that  $|x - y| \leq 2D_d h$ . Let  $j \neq i$  be such that  $y \in A_i \cap A_j$ . We have  $x \in \mathcal{V}_{2D_d h}(A_i) \cap \mathcal{V}_{2D_d h}(A_j)$ , and, since  $\text{diam}(S) = D_d h$ ,  $S \subset \mathcal{V}_{4D_d h}(A_i) \cap \mathcal{V}_{4D_d h}(A_j) \subset \mathcal{V}_{C_1 h}(A_i \cap A_j)$ .

As a consequence, every  $S \in \partial \mathcal{E}_\bullet$  is subset of a disk  $\Delta^2$  of radius  $C_1 h$  if  $d = 2$ , of a cylinder  $\Delta^3$  of radius  $C_1 h$  if  $d = 3$ . The cardinal of  $\Delta^d \cap \mathcal{T}$  is of order  $h^{-(d-2)}$  and every cell of  $\mathcal{T}$  has  $2d$  edges. This proves  $\#\partial \mathcal{E}_\bullet \leq \frac{C_\#}{h^{d-2}}$ .  $\square$

In the next lemma we are going to prove some basic properties of the projection  $P_i(S)$  for  $S \in \partial \mathcal{E}_i$ .

**Lemma A.2.** (1) *There exists  $C_0 > 0$  which depends on  $d$  and  $\Omega$  only such that*

$$(55) \quad \forall i \in \{1, \dots, n\} \forall S \in \partial \mathcal{E}_i, \quad |S| \leq C_0 |P_i(S)|.$$

(2) *If  $S, S' \in \partial \mathcal{E}_i$ , then  $P_i(S) \cap P_i(S') = \emptyset$  if, and only if,  $S \cap S' = \emptyset$ .*

*Proof.* (1) Let  $H$  be an affine hyperplane of  $\mathbb{R}^d$ , and let  $P$  denote the orthogonal projection on  $H$ . It is then elementary to show that

$$(56) \quad |P(S)| = |S| \cos(\overline{HS})$$

for all  $S \in \mathcal{E}$ , where  $\overline{HS}$  denotes the angle between  $H$  and the hyperplane  $\langle S \rangle$  generated by  $S$ .

Let  $A_i$  be one of the polygonal boundary faces of  $\Omega$ . First suppose that  $A_i$  is parallel to one of the canonical hyperplanes generated by the basis  $(\mathbf{e}_i)_{1,d}$ . Then any elements of  $\mathcal{E}$  is either orthogonal or parallel to  $H_i$ . Let  $S_{jl} \in \partial\mathcal{E}_i$  and suppose that  $S_{jl}$  is orthogonal to  $H_i$  (in which case  $|P_i(S)| = 0$  by (56)). As  $S_{jl} \in \partial\mathcal{E}_i$ , we have  $\overline{T_l} \cap \partial\Omega \neq \emptyset$ : there exists a  $A_k$  such that  $\overline{T_l} \cap A_k \neq \emptyset$  and, since  $\max_{x \in S_{jl}, y \in \overline{T_l}} |x - y| = D_d$ , we have  $k \in \mathcal{I}_S$ . Furthermore,  $k \neq i$  (for  $T_l$  is a subset of the cylinder generated by  $S$  and the normal  $\mathbf{n}_S$  to the hyperplane  $\langle S \rangle$ , a cylinder which is parallel to  $A_i$ ), and this contradicts the fact that  $S_{jl} \in \partial\mathcal{E}_i$ . Consequently  $S_{jl}$  is parallel to  $H_i$  and  $|P(S_{jl})| = |S_{jl}|$ .

Suppose now that  $H_i$  is not parallel to any of the hyperplanes generated by  $(\mathbf{e}_i)_{1,d}$ , and define

$$\alpha_i := \max \left\{ \overline{H_i((\mathbf{e}_k)_{k \in J})}; J \subset \{1, \dots, d\}, \#J = d - 1 \right\} < \frac{\pi}{2}.$$

Then, for every  $S \in \mathcal{E}$ , we have, by (56),

$$|P_i(S)| \geq |S| \cos(\alpha_i).$$

Consequently, we have shown that (55) holds with  $C_0^{-1} := \min_{1 \leq i \leq n} \cos(\alpha_i) > 0$ .

(2) Let us prove the second part of Lemma A.2. Let  $S_{jl}, S_{nm} \in \partial\mathcal{E}_i$ . Obviously,  $P_i(S_{jl}) \cap P_i(S_{nm}) = \emptyset \Rightarrow S_{jl} \cap S_{nm} = \emptyset$ . We suppose that  $P_i(S_{jl}) \cap P_i(S_{nm}) \neq \emptyset$  and intend to prove that  $S_{jl} \cap S_{nm} \neq \emptyset$ . Let  $x_i \in P_i(S_{jl}) \cap P_i(S_{nm})$ , and let  $D_i$  be the line  $P_i^{-1}(\{x_i\})$ . Let  $z \in D_i \cap S_{jl}$ ,  $z' \in D_i \cap S_{nm}$ . If  $x_i = z = z'$ , we are done. Thus, suppose  $x_i \neq z$  and set  $\mathbf{u} := \frac{z - x_i}{|z - x_i|}$ . Then  $t \mapsto x_i + t\mathbf{u}$  is a parametrization of  $D_i$ . We have  $z = x_i + t_z\mathbf{u}$  with  $t_z = |x_i - z| > 0$ , and  $z' = x_i + t'_z\mathbf{u}$  with  $t'_z \in \mathbb{R}$ . In fact,  $t'_z > 0$ ; otherwise, we would have  $x_i \in [z, z'] \subset \Omega$  (by convexity of  $\Omega$ ) and this contradicts  $x_i \in A_i \subset \partial\Omega$ . Suppose that  $S_{jl} \cap S_{nm} = \emptyset$ . Then  $t'_z \neq t_z$  and, for example,  $t'_z > t_z$ . We will show that  $T_m \in \mathcal{T}_h$ , which contradicts  $S_{nm} \in \partial\mathcal{E}_i$ . Denote by  $H_i^+$  the open affine half-space generated by  $H_i$  and  $\mathbf{u}$ . Let  $J_0 = \{k \in \{1, \dots, d\}, \mathbf{u} \cdot \mathbf{e}_k \neq 0\}$  and  $I_0 = \{1, \dots, d\} \setminus J_0$ . Define  $(\varepsilon_k)_{1,d}$  by  $\varepsilon_k = \text{sgn}(\mathbf{u} \cdot \mathbf{e}_k)$  if  $k \in J_0$ ,  $\varepsilon_k \in \{-1, 1\}$  if  $k \in I_0$ , and set  $Q_{\mathbf{u}} = \prod_{k=1}^d I(0, \varepsilon_k)$ , where  $I(0, \varepsilon_k)$  denotes the interval  $(0, 1)$  if  $\varepsilon_k = 1$ ,  $(-1, 0)$  if  $\varepsilon_k = -1$ . For each  $T_p = p + (0, h)^d \in \mathcal{T}_h(\mathbb{R}^d)$ , there exists a unique  $p_{\mathbf{u}} \in h\mathbb{Z}^d$  such that  $T_p = p_{\mathbf{u}} + hQ_{\mathbf{u}}$  (actually,  $p_{\mathbf{u}}$  reads  $p_{\mathbf{u}} = O + \sum_{k=1}^d (p_k + (\varepsilon_k)^-)\mathbf{e}_k$ ).

These notations being set, we prove that for  $T_p$  with  $\overline{T_p} \cap \{z + t\mathbf{u}, t > 0\} \neq \emptyset$  we have  $\overline{T_p} \subset H_i^+$ . We work in the affine base  $(j_{\mathbf{u}}, \varepsilon_1\mathbf{e}_1, \dots, \varepsilon_d\mathbf{e}_d)$  (recall that  $j_{\mathbf{u}}$  is such that  $T_j = j_{\mathbf{u}} + Q_{\mathbf{u}}$ ), and denote by  $(\tilde{x}_k)_{1,d}$  the coordinate of a point  $x \in \mathbb{R}^d$ . Let  $y \in \overline{T_p} \cap \{z + t\mathbf{u}, t > 0\}$ , set  $Z_y = \{k \in \{1, \dots, d\}, \tilde{y}_k \in h\mathbb{Z}\}$ , and  $T_y = \{x \in \mathbb{R}^d, 0 < \tilde{x}_k - h\beta_k < h\}$ , where  $\beta_k = \lceil \tilde{y}_k/h \rceil$  is the unique integer such that  $\beta_k \leq \tilde{y}_k/h < \beta_k + 1$ . Since  $y \in \overline{T_p}$ , we have  $T_p = T_y$  or  $T_p = T_y - \mathbf{e}_k$  for a  $k \in Z_y$ . Besides,  $y = z + t\mathbf{u}$  for a positive  $t$  and  $\tilde{y}_k = (y - j_{\mathbf{u}}) \cdot \varepsilon_k\mathbf{e}_k$ ; therefore,  $\tilde{y}_k = (z - j_{\mathbf{u}}) \cdot \varepsilon_k\mathbf{e}_k + t\varepsilon_k\mathbf{u} \cdot \mathbf{e}_k$ . In the new coordinates, we have  $\overline{T_j} = \{x \in \mathbb{R}^d, 0 \leq \tilde{x}_k \leq h\}$  and since  $z \in \overline{T_j}$ , the term  $(z - j_{\mathbf{u}}) \cdot \varepsilon_k\mathbf{e}_k$  is nonnegative. By definition of  $\varepsilon_k$ , the term  $t\varepsilon_k\mathbf{u} \cdot \mathbf{e}_k$  is also nonnegative, and, if  $k \in J_0$ , it is positive. In

particular, if  $k \in Z_y \cap J_0$ , then  $\tilde{y}_k \geq h$  and  $\beta_k \geq 1$ . This implies that  $\overline{T_p} \subset C_{\mathbf{u}} := \{x \in \mathbb{R}^d, \forall k \in J_0, \tilde{x}_k \geq 0\}$ . If  $x \in \mathbb{R}^d$ , we have  $(x - j_{\mathbf{u}}) \cdot \mathbf{u}i = \sum_{k=1}^d \tilde{x}_k \varepsilon_k \mathbf{e}_k$ , therefore,  $C_{\mathbf{u}} \subset \{x \in \mathbb{R}^d, (x - j_{\mathbf{u}}) \cdot \mathbf{u} \geq 0\}$ , and this last set is itself a subset of  $H_i^+$  for  $j_{\mathbf{u}} \in H_i^+$  (we have  $T_j \subset H_i^+$ ). This proves  $\overline{T_p} \subset H_i^+$ .

Since  $z' \in \overline{T_m} \cap \{z + t\mathbf{u}, t > 0\}$ , this last intersection is not empty and  $\overline{T_m} \subset H_i^+$ . This entails  $\overline{T_m} \subset \Omega$ , which contradicts the hypothesis that  $T_m \in \mathcal{T}_h(\mathbb{R}^d) \setminus \mathcal{T}_h(\Omega)$ . Therefore, we have  $S_{jl} \cap S_{nm} \neq \emptyset$ .  $\square$

**Proof of a uniform bound on the BV norm of  $c_h$ .** Now we have two tasks. First we show that  $c_h$  is indeed a numerical approximation of  $c$  (Lemma A.3), and second we establish a uniform bound on the BV norm of  $c_h$  (Lemma A.4).

**Lemma A.3.** *Let  $c_h$  be defined by (53). Assume that Assumption 3.1 holds. Then, under the CFL condition*

$$(57) \quad \Delta t \leq \frac{(1 - \xi)}{2(d - 1)\mathcal{L}}h, \quad \xi \in (0, 1),$$

*the sequence  $(c_h)$  converges to the entropy solution  $c$  of the problem (1)–(3) in  $L^1(\Omega \times (0, T))$ .*

*Sketch of the proof.* Recall that, by the maximum principle, the sequence  $(c_h)$  is uniformly bounded with respect to  $h$  in  $L^\infty(\Omega \times (0, T))$ . We first make reference to [Vov02] where the following result is proved. Given a nonnegative function  $\varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^+)$ , there exists  $\varepsilon_h := \varepsilon_h(\varphi)$  such that  $\lim_{h \rightarrow 0} \varepsilon_h = 0$  and

$$(58) \quad \begin{aligned} \int_0^T \int_{\Omega_h} (c_h - \kappa)^\pm \partial_t \varphi &+ \Phi^\pm(c_h, \kappa) \cdot \nabla \varphi + \int_{\Omega_h} (c_h^0 - \kappa)^\pm \varphi(\cdot, 0) \\ &+ \mathcal{L} \sum_{n=0}^N \sum_{S_{jl} \in \partial \mathcal{E}} \Delta t |S_{jl}| (\bar{c}_{jl}^n - \kappa)^\pm \varphi_{jl}^n \geq -\varepsilon_h, \end{aligned}$$

where  $\varphi_{jl}^n := \frac{1}{\Delta t |S_{jl}|} \int_{t^n}^{t^{n+1}} \int_{S_{jl}} \varphi(x, t) \, dx \, dt$ .

We intend to prove the following bound for a constant  $C > 0$ , and  $\tilde{\varepsilon}_h := \tilde{\varepsilon}_h(\varphi)$  such that  $\lim_{h \rightarrow 0} \tilde{\varepsilon}_h = 0$ :

$$(59) \quad \sum_{n=0}^N \sum_{S_{jl} \in \partial \mathcal{E}} \Delta t |S_{jl}| (\bar{c}_{jl}^n - \kappa)^\pm \varphi_{jl}^n \leq C \int_{\partial \Omega \times (0, T)} (\bar{c} - \kappa)^\pm \varphi d\gamma(x) \, dt + \tilde{\varepsilon}_h.$$

Indeed, suppose for a moment that (59) holds. We then conclude the proof with the following arguments. The Lebesgue measure  $|\Omega \setminus \Omega_h| = \mathcal{O}(h)$ , because  $\Omega \setminus \Omega_h \subset \mathcal{V}_{D,h}(\partial \Omega)$ . Consequently, since  $(c_h - \kappa)^\pm \partial_t \varphi + \Phi^\pm(c_h, \kappa) \cdot \nabla \varphi$  and  $(c_h^0 - \kappa)^\pm \varphi(\cdot, 0)$  are uniformly bounded with respect to  $h$  in  $L^\infty(\Omega \times (0, T))$ , and  $L^\infty(\Omega)$ , respectively, we have with (59)

$$(60) \quad \begin{aligned} \int_0^T \int_{\Omega} (c_h - \kappa)^\pm \partial_t \varphi &+ \Phi^\pm(c_h, \kappa) \cdot \nabla \varphi + \int_{\Omega} (c_h^0 - \kappa)^\pm \varphi(\cdot, 0) \\ &+ C\mathcal{L} \int_{\partial \Omega \times (0, T)} (\bar{c} - \kappa)^\pm \varphi d\gamma(x) \, dt \geq -\tilde{\varepsilon}_h, \end{aligned}$$

where  $\lim_{h \rightarrow 0} \tilde{\varepsilon}_h = 0$ . Since  $(c_h^0)$  is converging to  $c_0$  in  $L^1(\Omega)$ , the uniform bound on  $(c_h)$  and (60) are sufficient to get a subsequence of  $(c_h)$  that converges in the non-linear weak- $\star$  sense to an entropy process solution  $\tilde{c}$  of problem (1)–(3). Let us note

that the constant factor in front of the boundary term  $\int_{\partial\Omega \times (0,T)} (\bar{c} - \kappa)^\pm \varphi d\gamma(x)$  dt in the definition of the entropy solution is required to be any bound on the Lipschitz semi-norm of  $\mathbf{F}$ , and not *the* Lipschitz constant of  $\mathbf{F}$ . By a uniqueness result, this entropy process solution turns out to be the entropy solution  $c$  of problem (1)–(3), and this shows that the whole sequence  $(c_h)$  converges to  $c$  in  $L^1(\Omega \times (0, T))$ . We refer to [Vov02] for details on the terminology and the results.

Let us turn to the proof of (59). Denote by  $U_h$  the left hand side of (59). We have  $U_h = \sum_{i=1}^n U_i + U_\bullet$  with

$$U_i = \sum_{n=0}^N \sum_{S_{jl} \in \partial\mathcal{E}_i} \Delta t |S_{jl}| (\bar{c}_{jl}^n - \kappa)^\pm \varphi_{jl}^n, \quad U_\bullet = \sum_{n=0}^N \sum_{S_{jl} \in \partial\mathcal{E}_\bullet} \Delta t |S_{jl}| (\bar{c}_{jl}^n - \kappa)^\pm \varphi_{jl}^n.$$

Since  $(\bar{c}_{jl}^n - \kappa)^\pm \varphi_{jl}^n \leq (|\bar{c}_{jl}^n|_{L^\infty} + |\kappa|) |\varphi|_{L^\infty}$ , we have, by Lemma A.1,  $0 \leq U_\bullet \leq Ch$  for a constant  $C$  independent of  $h$ , and  $\lim_{h \rightarrow 0} \bar{U}_\bullet = 0$ . For  $i \in \{1, \dots, n\}$ , and  $S_{jl} \in \partial\mathcal{E}_i$ , where  $\bar{c}_{jl}^n$  is defined as the mean value of  $\bar{c}$  over  $(A_i \cap P_i(S_{jl})) \times (t^n, t^{n+1})$ . Since  $s \mapsto (s - \kappa)^\pm$  is convex, we have by Jensen’s inequality

$$(\bar{c}_{jl}^n - \kappa)^\pm \leq \frac{1}{\Delta t |A_i \cap P_i(S_{jl})|} \int_{t^n}^{t^{n+1}} \int_{A_i \cap P_i(S_{jl})} (\bar{c}(x, t) - \kappa)^\pm d\gamma(x) dt.$$

Taking (55) into account, we get

$$\bar{U}_i \leq C_1 \sum_{n=0}^N \sum_{S_{jl} \in \partial\mathcal{E}_i} \int_{t^n}^{t^{n+1}} \int_{A_i \cap P_i(S_{jl})} (\bar{c}(x, t) - \kappa)^\pm d\gamma(x) dt \varphi_{jl}^n.$$

By Lemma A.2(2) and the regularity of the function  $\varphi$  we have

$$U_i \leq C \int_0^T \int_{A_i} (\bar{c}(x, t) - \kappa)^\pm \varphi d\gamma(x) dt + \tilde{\varepsilon}_h^i,$$

where  $\tilde{\varepsilon}_h^i \rightarrow 0$  when  $h \rightarrow 0$ . Hence, (59) follows with  $\tilde{\varepsilon}_h = \sum_{i=1}^n \tilde{\varepsilon}_h^i + \bar{U}_\bullet$ . □

**Lemma A.4.** *Under the assumptions of Lemma A.3, there exists a constant  $C_{BV} > 0$ , which depends on the data and on  $\Omega$  only such that*

$$(61) \quad \|c_h\|_{BV(\Omega_h \times (0,T))} \leq C_{BV}.$$

*Proof.* For  $BV$  data, the bounded character of the variation with respect to *time* of the solution of an explicit finite volume scheme with monotone fluxes is ensured independently of the structure of the mesh (see [CCL95b] for example). On the contrary, the bounded character of the variation in *space* of the same numerical solution remains an open question in the case where an unstructured mesh is used. This is the reason why we used a cartesian grid to defined  $c_h$ . We decompose  $\|c_h\|_{BV(\Omega_h \times (0,T))} = BV_t + BV_x$  with

$$(62) \quad BV_t := \sum_{n=0}^N \sum_{T_j \in \mathcal{T}_h} |T_j| |c_j^{n+1} - c_j^n|, \quad BV_x := \sum_{n=0}^N \sum_{S_{jl} \in \mathcal{E}} \Delta t h^{d-1} |c_j^n - c_l^n|.$$

We refer to [CCL95b] for the proof of the estimate

$$(63) \quad BV_t \leq C,$$

where the constant depends on the data only, and we focus on the estimate of  $BV_x$ .

For  $i \in \{1, \dots, d\}$ , denote by  $\mathcal{E}_i$  the set of interior edges orthogonal to the direction  $\langle \mathbf{e}_i \rangle$ , i.e., if  $S_{jl} \in \mathcal{E}_i$ , then either  $l = j + \mathbf{e}_i$  or  $l = j - \mathbf{e}_i$ , and set

$$BV_x = \sum_{n=0}^N \Delta t \sum_{i=1}^d BV_i^n, \quad BV_i^n := \sum_{S_{jl} \in \mathcal{E}_i} h^{d-1} |c_j^n - c_l^n|.$$

Note that the set of neighboring cells to a cell  $T_j$  is given as  $N(j) = \{j * \mathbf{e}_i \mid * \in \{+, -\}, i \in \{1, \dots, d\}\}$ . Furthermore, we have  $F_{j,j \pm \mathbf{e}_i}(c) = F_{k,k \pm \mathbf{e}_i}(c)$ , and  $F_{j,j + \mathbf{e}_i}(c) = -F_{k+\mathbf{e}_i,k}(c)$  for all  $j, k \in \mathbb{Z}^d$ .

If there is no boundary, i.e.,  $\mathcal{T}_h(\mathbb{R}^d) = \mathcal{T}_h(\Omega)$ , then it has been proved in [CH99a] that  $BV_i^{n+1} \leq BV_i^n$ . This result uses the fact that the discrete solution is translation invariant under shifts  $h\mathbf{e}_i, i = 1, \dots, d$ , which yields

$$\begin{aligned} |c_j^{n+1} - c_{j \pm \mathbf{e}_i}^{n+1}| &\leq \left(1 - \sum_{\substack{* \in \{+, -\}, \\ l \in \{1, \dots, d\}}} Q_{j, j * \mathbf{e}_l}\right) |c_j^n - c_{j \pm \mathbf{e}_i}^n| \\ (64) \quad &+ \sum_{\substack{* \in \{+, -\}, \\ l \in \{1, \dots, d\}}} Q_{j * \mathbf{e}_l, j \pm \mathbf{e}_i * \mathbf{e}_l} |c_{j * \mathbf{e}_l}^n - c_{j \pm \mathbf{e}_i * \mathbf{e}_l}^n|, \\ Q_{jl} &:= \frac{\Delta t}{h} \frac{F_{jl}(c_j) - F_{jl}(c_l)}{(c_j - c_l)}, \text{ if } c_j \neq c_l, \text{ and } Q_{jl} := 0 \text{ else.} \end{aligned}$$

Note that the CFL condition (57) and the monotony of  $\mathbf{F}_{jl}$  ensure that  $0 \leq Q_{jl}, \sum_{l \in N(j)} Q_{jl} \leq 1$ . Let us now study the evolution of  $BV_i^n$  for our bounded domain  $\Omega$ . Therefore, we define

$$(65) \quad \overline{BV}_i^n := BV_i^n + \sum_{S_{j,j \pm \mathbf{e}_i} \in \partial \mathcal{E}} h^{d-1} |c_j^n - \bar{c}_{j,j \pm \mathbf{e}_i}^n|.$$

We first note that the inequality (64) remains valid for any  $S_{j,j \pm \mathbf{e}_i} \in \mathcal{E}_i$ . Let us adapt (64) for a given edge  $S_{j,j \pm \mathbf{e}_i} \in \partial \mathcal{E}$ . The values  $\bar{c}_{j,j \pm \mathbf{e}_i}^n$  are then given by (50) and (51). In addition, for  $\mathbf{e}_l \neq \mathbf{e}_i$ , define the values  $c_{j \pm \mathbf{e}_i, j \pm \mathbf{e}_i \pm \mathbf{e}_l}^n$  through

$$c_{j \pm \mathbf{e}_i, j \pm \mathbf{e}_i \pm \mathbf{e}_l}^n := \begin{cases} \bar{c}_{j \pm \mathbf{e}_i, j \pm \mathbf{e}_i \pm \mathbf{e}_l}^n, & \text{if } S_{j \pm \mathbf{e}_i, j \pm \mathbf{e}_i \pm \mathbf{e}_l} \in \mathcal{E} \ (\Rightarrow S_{j \pm \mathbf{e}_i, j \pm \mathbf{e}_i \pm \mathbf{e}_l} \in \partial \mathcal{E}), \\ \bar{c}_{j \pm \mathbf{e}_l, j \pm \mathbf{e}_i \pm \mathbf{e}_l}^n, & \text{if } S_{j \pm \mathbf{e}_l, j \pm \mathbf{e}_i \pm \mathbf{e}_l} \in \partial \mathcal{E}, \\ \bar{c}_{j, j \pm \mathbf{e}_l}^n, & \text{else } (\Rightarrow S_{j, j \pm \mathbf{e}_l} \in \partial \mathcal{E}). \end{cases}$$

From the definition of the scheme, we get

$$(66) \quad c_j^{n+1} = c_j^n - \frac{\Delta t}{h} \sum_{\substack{* \in \{+, -\}, \\ l \in \{1, \dots, d\}}} F_{j, j * \mathbf{e}_l}(c_j^n) - F_{j, j * \mathbf{e}_l}(c_{j, j * \mathbf{e}_l}^n).$$

Furthermore, set

$$\begin{aligned} (67) \quad \delta_{j, j \pm \mathbf{e}_i}^n &:= c_{j, j \pm \mathbf{e}_i}^{n+1} - c_{j, j \pm \mathbf{e}_i}^n \\ &+ \frac{\Delta t}{h} \sum_{\substack{* \in \{+, -\}, \\ l \in \{1, \dots, d\} \setminus \{i\}}} F_{j \pm \mathbf{e}_i, j \pm \mathbf{e}_i * \mathbf{e}_l}(\bar{c}_{j, j \pm \mathbf{e}_i}^n) - F_{j \pm \mathbf{e}_i, j \pm \mathbf{e}_i * \mathbf{e}_l}(c_{j \pm \mathbf{e}_i, j \pm \mathbf{e}_i \pm \mathbf{e}_l}^n). \end{aligned}$$

We then have

$$\begin{aligned}
 c_j^{n+1} - c_{j,j\pm e_i}^{n+1} &= c_j^n - c_{j,j\pm e_i}^n - \frac{\Delta t}{h} \sum_{\substack{* \in \{+, -\}, \\ l \in \{1, \dots, d\}}} F_{j,j^*e_l}(c_j^n) - F_{j,j^*e_l}(c_{j,j^*e_l}^n) \\
 &+ \frac{\Delta t}{h} \sum_{\substack{* \in \{+, -\}, \\ l \in \{1, \dots, d\} \setminus \{i\}}} \left( F_{j\pm e_i, j\pm e_i^*e_l}(c_{j,j\pm e_i}^n) - F_{j\pm e_i, j\pm e_i^*e_l}(c_{j\pm e_i, j\pm e_i^*e_l}^n) \right) \\
 &- \delta_{j,j\pm e_i}^n.
 \end{aligned}$$

Reordering these terms and using the shift invariance leads to the equivalent form (64)

$$\begin{aligned}
 |c_j^{n+1} - c_{j,j\pm e_i}^{n+1}| &\leq \left( 1 - \sum_{\substack{* \in \{+, -\}, \\ l \in \{1, \dots, d\} \setminus \{i\}}} Q_{j,j^*e_l} - Q_{j,j\pm e_i} \right) |c_j^n - c_{j,j\pm e_i}^n| \\
 (68) \quad &+ Q_{j,j\mp e_i} |c_j^n - c_{j,j\mp e_i}^n| + |\delta_{j,j\pm e_i}^n| \\
 &+ \sum_{\substack{* \in \{+, -\}, \\ l \in \{1, \dots, d\} \setminus \{i\}}} Q_{(j,j^*e_l), (j\pm e_i, j\pm e_i^*e_l)} |c_{j,j^*e_l}^n - c_{j\pm e_i, j\pm e_i^*e_l}^n|.
 \end{aligned}$$

Now, let us examine the process of compensation of terms when we sum the right hand side of (68) over  $S_{j,j\pm e_i} \in \partial\mathcal{E}$  and add it to the sum over  $S_{jl} \in \mathcal{E}_i$  of the right hand side of (64). A careful examination leads to the estimate

$$(69) \quad \overline{BV}_i^{n+1} \leq \overline{BV}_i^n + h^{d-1} \sum_{S_{j,j\pm e_i} \in \partial\mathcal{E}} |\delta_{j,j\pm e_i}^n|.$$

Notice that whenever  $F_{j\pm e_i, j\pm e_i^*e_l}(c_{j,j\pm e_i}^n) - F_{j\pm e_i, j\pm e_i^*e_l}(c_{j\pm e_i, j\pm e_i^*e_l}^n) \neq 0$  in (67), it amounts to a difference  $F_{j,j^*e_l}(c_{j,j\pm e_i}^n) - F_{j,j^*e_l}(c_{gh}^n)$  for some  $S_{gh} \in \partial\mathcal{E}$  with  $S_{j,j\pm e_i} \cap S_{gh} \neq \emptyset$ . Since the function  $F_{j,j^*e_l}$  is  $\mathcal{L}$ -Lipschitz continuous, we have  $\sum_{S_{j,j\pm e_i} \in \partial\mathcal{E}} |\delta_{j,j\pm e_i}^n| \leq \Delta_t^n + 2(d-1)\mathcal{L} \frac{\Delta t}{h} \Delta_x^n$  with

$$(70) \quad \Delta_t^n := \sum_{S_{jl} \in \partial\mathcal{E}} |\bar{c}_{jl}^{n+1} - \bar{c}_{jl}^n|, \quad \Delta_x^n := \sum_{S_{jl} \in \partial\mathcal{E}} \sum_{\substack{S_{gh} \in \partial\mathcal{E}, \\ S_{jl} \cap S_{gh} \neq \emptyset}} |\bar{c}_{jl}^n - \bar{c}_{gh}^n|.$$

By (57), we have  $2(d-1)\mathcal{L} \frac{\Delta t}{h} \leq 1$ . Let us estimate  $\Delta_x^n$ . We have  $\Delta_x^n = \Delta_{x,\bullet}^n + \sum_{i=1}^n \Delta_{x,i}^n$  with

$$\begin{aligned}
 \Delta_{x,\bullet}^n &:= \sum_{S_{jl} \in \partial\mathcal{E}_\bullet} \sum_{\substack{S_{gh} \in \partial\mathcal{E}, \\ S_{jl} \cap S_{gh} \neq \emptyset}} |\bar{c}_{jl}^n - \bar{c}_{gh}^n|, \\
 (71) \quad \Delta_{x,i}^n &:= \sum_{S_{jl} \in \partial\mathcal{E}_i} \sum_{\substack{S_{gh} \in \partial\mathcal{E}, \\ S_{jl} \cap S_{gh} \neq \emptyset}} |\bar{c}_{jl}^n - \bar{c}_{gh}^n|.
 \end{aligned}$$

Since  $\bar{c}_{gh}^n = 0$  for every  $S_{gh} \in \partial\mathcal{E}_\bullet$ , we have

$$\Delta_{x,\bullet}^n = \sum_{S_{jl} \in \partial\mathcal{E}_\bullet} \sum_{\substack{S_{gh} \in \partial\mathcal{E} \setminus \partial\mathcal{E}_\bullet, \\ S_{jl} \cap S_{gh} \neq \emptyset}} |\bar{c}_{gh}^n|.$$

As each face  $S_{jl}$  may have at most  $2(d-1)$  neighboring faces, and by Lemma A.1 the cardinal of  $\partial\mathcal{E}_\bullet$  is bounded by  $\frac{C_\#}{h^{d-2}}$ , we have

$$(72) \quad \Delta_{x,\bullet}^n \leq 2(d-1) \frac{C_\#}{h^{d-2}} \|\bar{c}\|_{L^\infty(\partial\Omega \times (0,T))}.$$

Similarly, we have, for  $i \in \{1, \dots, n\}$

$$(73) \quad \sum_{S_{jl} \in \partial\mathcal{E}_i} \sum_{\substack{S_{gh} \in \partial\mathcal{E}_\bullet \\ S_{jl} \cap S_{gh} \neq \emptyset}} |\bar{c}_{jl}^n - \bar{c}_{gh}^n| \leq 2(d-1) \frac{C_\#}{h^{d-2}} \|\bar{c}\|_{L^\infty(\partial\Omega \times (0,T))},$$

and therefore  $\Delta_{x,i}^n \leq \bar{\Delta}_{x,i}^n + \leq 2(d-1) \frac{C_\#}{h^{d-2}} \|\bar{c}\|_{L^\infty(\partial\Omega \times (0,T))}$  with

$$(74) \quad \bar{\Delta}_{x,i}^n := \sum_{S_{jl} \in \partial\mathcal{E}_i} \sum_{\substack{S_{gh} \in \partial\mathcal{E}_i \\ S_{jl} \cap S_{gh} \neq \emptyset}} |\bar{c}_{jl}^n - \bar{c}_{gh}^n|.$$

Since  $P_i$  is contractive, we have  $\text{diam}(P_i(S_{jl}) \cap P_i(S_{gh})) \leq \text{diam}(S_{jl}, S_{gh}) \leq 2h$ , if  $S_{jl}, S_{gh} \in \partial\mathcal{E}_i, S_{jl} \cap S_{gh} \neq \emptyset$ . Together with the nonoverlapping of the projections (Lemma A.2(2)) and the CFL condition (57), this shows that

$$(75) \quad \sum_{t^n \in J} h \bar{\Delta}_{x,i}^n \leq C \|\bar{c}\|_{BV(A_i \times (0,T))}$$

if  $\bar{c} \in BV(A_i \times (0,T))$ . Here and in the following,  $C$  denotes a constant independent of  $h$ , and  $\Delta t$ . Similarly, we have

$$(76) \quad \sum_{t^n \in J} h \Delta_t^n \leq C \|\bar{c}\|_{BV(\partial\Omega \times (0,T))}.$$

Collecting the estimates, we deduce

$$(77) \quad \overline{BV}_i^n \leq \overline{BV}_i^0 + C \quad \forall t^n \in J.$$

Let us bound  $\overline{BV}_i^0$ . We have  $BV_i^0 \leq C \|c_0\|_{BV(\Omega)} \leq C$  such that (cf. (65))

$$\overline{BV}_i^0 \leq C + \sum_{S_{j,j \pm e_i} \in \partial\mathcal{E}} h^{d-1} |c_j^0 - \bar{c}_{j,j \pm e_i}^0|.$$

Since the cardinal of  $\partial\mathcal{E}$  is of order  $\mathcal{O}(h^{d-1})$  and since  $\bar{c}$  and  $c_0$  are bounded, we have  $\sum_{S_{j,j \pm e_i} \in \partial\mathcal{E}} h^{d-1} |c_j^0 - \bar{c}_{j,j \pm e_i}^0| \leq C$ , and  $\overline{BV}_i^n \leq C \quad \forall t^n \in J$ . All in all, we have  $BV_x \leq C$ . Together with (63), this completes the proof of the lemma.  $\square$

**Proof of the BV estimate (7).** Let  $\varphi \in (C_c^\infty(\Omega \times (0,T)))^d$  with  $|\varphi(x,t)| \leq 1$  for all  $(x,t) \in \Omega \times (0,T)$ . Let  $K_x$  and  $K_t$  be some compact subsets of  $\Omega$  and  $(0,T)$ , respectively, such that  $\text{supp}(\varphi) \subset K_x \times K_t$ . Let  $h$  be smaller than  $\text{dist}(K_x, \Omega^c)/2D_d$  such that  $K_x \subset \Omega_h$ . We have by Lemma A.4

$$\begin{aligned} \int_{\Omega \times \mathbb{R}^+} c \nabla \cdot \varphi \, dx \, dt &= \int_{\Omega_h^0} \int_0^T c \nabla \cdot \varphi \, dx \, dt \\ &= \int_{\Omega_h^0} \int_0^T (c - c_h) \nabla \cdot \varphi \, dx \, dt + \int_{\Omega_h^0} \int_0^T c_h \nabla \cdot \varphi \, dx \, dt \\ &\leq \|\nabla \cdot \varphi\|_{L^\infty(\Omega \times (0,T))} \|c - c_h\|_{L^1(\Omega \times (0,T))} + C_{BV}. \end{aligned}$$

At the limit  $h \rightarrow 0$ , by Lemma A.3, we get  $\int_{\Omega \times \mathbb{R}^+} c \nabla \cdot \varphi \, dx \, dt \leq C$ . This is true for an arbitrary function  $\varphi \in (C_c^\infty(\Omega \times (0, T)))^d$  with  $|\varphi(x, t)| \leq 1$  for all  $(x, t) \in \Omega \times (0, T)$ . Therefore  $\|c\|_{BV(\Omega \times (0, T))} \leq C_{BV}$ .  $\square$

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