MATHEMATICS OF COMPUTATION
Volume 75, Number 253, Pages 73–90
S 0025-5718(05)01792-8
Article electronically published on October 3, 2005

HIERARCHICAL DECOMPOSITION OF DOMAINS WITH FRACTURES

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Abstract. We consider the efficient and robust numerical solution of elliptic problems with jumping coefficients occurring on a network of thin fractures. We present an iterative solution concept based on a hierarchical separation of the fractures and the surrounding rock matrix. Upper estimates for the convergence rates are independent of the width of the fractures and of the jumps of the coefficients. Inexact solution of the local subproblems is also considered. The theoretical results are illustrated by numerical experiments.

1. Introduction

Saturated groundwater flow in fractured porous media can be described by linear elliptic problems. Fractures on one or more micro scales are usually represented by effective parameters of corresponding single or multiporosity models based on homogenization techniques. Fractures on macro scales directly enter the geometry of the mathematical model. For lack of data, such fracture networks are typically generated automatically based on stochastic reasoning [20]. The permeability k_F within the fractures is usually some orders of magnitude larger than the permeability of the surrounding rock matrix, while their width ϵ_F might be some orders of magnitude smaller than the overall computational domain. Similar problems occur in other applications: The heat transfer in the human body is dominated by the blood vessels which, depending on their size, are represented by effective parameters [7, 8] or have to be incorporated directly. Another example concerns diffusion-induced drug permeation through stratum corneum [11, 16]. In this case, the lipid “mortar” between the corneocytes plays the role of the fractures.

In order to avoid numerical troubles resulting from small width ϵ_F and large permeability k_F, fractures (or vessels) are often discretized by lower-dimensional elements [3, 12, 13] (or [18]). However, there are also some disadvantages of this approach. For example, outward normal flow and mass conservation across the interface are not incorporated. This motivated recent work on equidimensional discretizations [9, 10, 14, 15].

On this background, we consider the efficient and robust numerical solution of elliptic problems with jumping coefficients occurring on a network of long, thin fractures. Robustness means that the complexity should not depend on the crucial parameters ϵ_F and k_F. To this end, the fracture network is discretized by

Received by the editor October 25, 2004 and, in revised form, February 8, 2005.
2000 *Mathematics Subject Classification.* Primary 65N55.
This work has been funded in part by the Deutsche Forschungsgemeinschaft under contract Ko 1806/2-1 and Ko 1806/2-3.

anisotropic isoparametric finite elements, while usual shape regular elements are used elsewhere. The main part of the paper concentrates on the iterative solution of the resulting discrete problems. We introduce so-called *hierarchical domain decomposition methods* (HDD) based on a hierarchical splitting of the discrete solution space into three subspaces associated with the interior of the fractures, the interface, and the surrounding rock matrix. The matrix space consists of functions which are essentially constant across the fractures. We show that the stability of this decomposition and therefore the convergence speed of the resulting subspace correction method is independent of ϵ_F, k_F and of the mesh size. The proof is based on the assumption that the fractures are "almost" lower dimensional in the sense that their resolution by isotropic elements is not desirable, e.g., for efficiency reasons. This setting allows us to trace back the stability of the hierarchical decomposition to the optimal stability properties of one-dimensional interpolation. As a consequence, our approach is applicable in any space dimension. Exceptional situations like crossings of fractures or fractures ending in the computational domain can be resolved by local estimates on the surrounding elements. The exact solution of the three local subproblems can be replaced by suitable multigrid methods. We present an illustrative example adapting well-known results on hierarchical bases in Section 6. In this case, the number of refinement steps enters only polynomially and independence of ϵ_F and, in particular, of k_F is preserved. However, this particular result is restricted to two space dimensions. In our numerical experiments, we observe similar convergence speed as for classical multigrid methods applied to the Laplace equation.

In a sense, our approach is complementary to the algorithm presented by Heisig et al. [11], which aims at compensating small ε_F by successive anisotropic refinement and accounts for large k_F by ILU-smoothing. Algebraic multigrid methods (see, e.g., [5, 19] or [23] for an overview) usually work reasonably well but mostly suffer from a certain lack of theory. Apel and Schöberl [1] consider a completely different anisotropic problem with a tensor product structure, where line (or plane) smoothers or semi-coarsening can be applied.

For ease of presentation, we restrict ourselves to a two-dimensional domain with two intersecting fractures. However, the basic approach and the mathematical analysis can be transferred to more complicated situations, not only in two, but also in three, space dimensions. In Section 2 we describe the discretization and provide an error estimate. The next two sections concentrate on the stable separation of the subspaces associated with the fractures and with the interface, respectively. These two sections form the core of the paper. In Section 5 we describe the basic HDD algorithm and prove upper bounds for the convergence rates which are robust with respect to ϵ_F and k_F. Afterwards an example for an inexact version is given. Numerical experiments confirm our theoretical findings.

2. A discrete elliptic problem on a domain with fractures

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, e.g., the unit square, with the two fractures

$$\Omega_F^i = \{x \in \Omega \mid x = b_F^i + s d_F^i + t n_F^i,\ s \in \mathbb{R},\ t \in (0, \varepsilon_F)\}, \quad i = 1, 2,$$

each of which is characterized by its position vector b_F^i, direction d_F^i, normal n_F^i and width $\epsilon_F > 0$. We assume that the fractures cross inside of Ω, i.e., that $\overline{\Omega}_c \subset \Omega$, $\Omega_c = \Omega_F^1 \cap \Omega_F^2$. The network of fractures is denoted by $\Omega_F = \Omega_F^1 \cup \Omega_F^2$, while

$\Omega_M = \Omega \setminus \overline{\Omega}_F$ and $\Gamma = \overline{\Omega}_M \cap \overline{\Omega}_F$ represent the rock matrix and the interface, respectively. This leads to the decomposition

$$\Omega = \Omega_F \cup \Gamma \cup \Omega_M. \tag{2.1}$$

We consider the elliptic variational problem

$$u \in H: \qquad a(u,v) = \ell(v) \qquad \forall v \in H \tag{2.2}$$

with the symmetric bilinear form

$$a(v,w) = (K\nabla u, \nabla v)_{L^2(\Omega)}$$

and jumping permeability K,

$$K(x) = \begin{cases} k_F \geq 1, & x \in \Omega_F, \\ 1, & x \in \Omega_M. \end{cases} \tag{2.3}$$

For simplicity, let $H = H_0^1(\Omega)$ and let $\ell \in H'$ be some right-hand side. The energy norm is denoted by $\| \cdot \| = a(\cdot,\cdot)^{1/2}$.

Let $\mathcal{P}_0 = \mathcal{T}_0 \cup \mathcal{Q}_0$ be a subdivision of $\Omega = \overline{\Omega}_M \cup \overline{\Omega}_F$ consisting of the partitions

$$\overline{\Omega}_M = \bigcup_{T \in \mathcal{T}_0} T, \qquad \overline{\Omega}_F = \bigcup_{Q \in \mathcal{Q}_0} Q$$

into triangles T and trapezoidals Q, respectively. We assume that the vertices of each trapezoidal $Q \in \mathcal{Q}_0$ lie on Γ (see the left picture in Figure 2.1). In particular,

$$Q_c := \overline{\Omega}_c \in \mathcal{Q}_0$$

is a parallelogram. We further assume that $\mathcal{P}_0$ is *conforming* in the sense that the intersection of two different elements is either a common edge, a common vertex or empty. Finally, $\mathcal{P}_0$ is supposed to be *shape regular* in the sense that all $P \in \mathcal{P}_0$ have positive area and all $Q \in \mathcal{Q}_0$ have four different vertices. Equivalently, there are positive constants s_0, $\gamma_0 \in \mathbb{R}$ such that

$$s_0 h_Q \leq h'_Q, \quad \forall Q \in \mathcal{Q}_0 \setminus \{Q_c\}, \qquad \gamma_0 \leq \gamma_P \leq \pi - \gamma_0 \quad \forall P \in \mathcal{P}_0 \tag{2.4}$$

holds with $h_Q \geq h'_Q$ denoting the lengths of the two parallel edges contained in Γ and γ_P being an arbitrary interior angle. Note that all edges of Q_c have the same length h_{Q_c}.

Though all results and algorithms to be presented can be directly extended to locally refined grids, we assume for simplicity that the triangulation $\mathcal{T}_1$ is obtained by uniform refinement of $\mathcal{T}_0$. More precisely, each triangle $T \in \mathcal{T}_0$ is subdivided into four similar subtriangles. Connecting the resulting new midpoints of opposite edges of each $Q \in \mathcal{Q}_0 \setminus \{Q_c\}$, we get the set $\mathcal{Q}_1$ of anisotropically refined trapezoidals. Reiteration of this procedure leads to a sequence of refined partitions $\mathcal{P}_j = \mathcal{T}_j \cup \mathcal{Q}_j$, $j = 0, 1, \ldots$. Observe that $\mathcal{P}_j$ is conforming and the shape regularity (2.4) is preserved uniformly in j. In particular, the angle condition in (2.4) implies

$$h_Q - h'_Q \leq \frac{\varepsilon_F}{(\sin \gamma_0)^2} = \mathcal{O}(\varepsilon_F) \qquad \forall Q \in \mathcal{Q}_j \tag{2.5}$$

so that the elements of $\mathcal{Q}_j$ tend to (highly anisotropic) parallelograms as ε_F tends to zero.

We assume that the *fractures are long, thin objects* in the sense that the resolution of the corresponding anisotropy of the elements $Q \in \mathcal{Q}_0$ by the above refinement

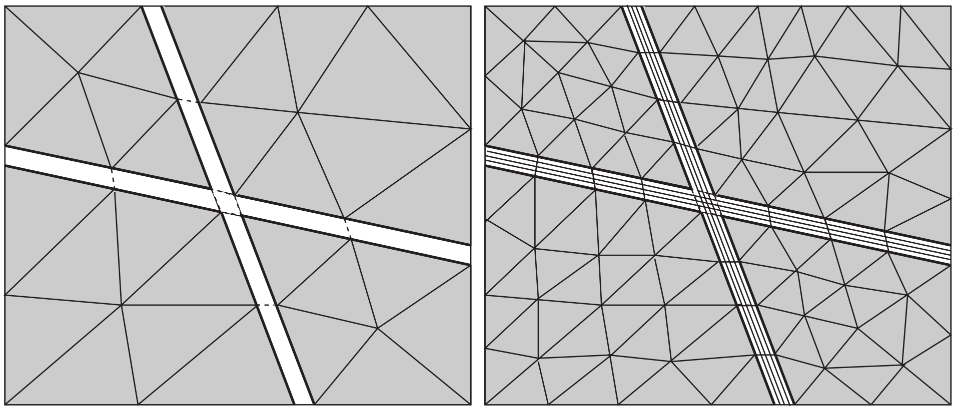

FIGURE 2.1. Initial partition $\mathcal{P}_0$ and refined partition $\mathcal{P}_{12}$

procedure is not desirable, e.g., for efficiency reasons. More precisely, we impose the condition

$$\varepsilon_F \le C_0 h_Q \qquad \forall Q \in \mathcal{Q}_j \tag{2.6}$$

which means that the width of the fractures is bounded by the size of the surrounding mesh.

Anisotropic refinement of $\mathcal{Q}_j$ in the other direction is performed by bisecting and connecting the midpoints of all edges not contained in Γ. Application of k steps of this procedure to $\mathcal{P}_j$ provides the partition $\mathcal{P}_{jk} = \mathcal{T}_j \cup \mathcal{Q}_{jk}$ (see the right picture of Figure 2.1 for $j = 1$, $k = 2$). The refined partitions $\mathcal{P}_{jk}$ are conforming, and the shape regularity (2.4) holds uniformly in $j, k \in \mathbb{N}$. Finally note that $\mathcal{P}_{jk}$ does not depend on the order of the above two types of refinement steps.

For given j, k, let $\mathcal{S}_{jk} \in H_0^1(\Omega)$ be the subspace of functions v such that $v|_T$ is linear and $v|_Q$ is isoparametric bilinear for all $T \in \mathcal{T}_j$ and $Q \in \mathcal{Q}_{jk}$, respectively. Then the corresponding finite element discretization of the continuous problem (2.2) reads as follows:

$$u_{jk} \in \mathcal{S}_{jk}: \qquad a(u_{jk}, v) = \ell(v) \qquad \forall v \in \mathcal{S}_{jk}. \tag{2.7}$$

In preparation for an error estimate, we introduce the weighted Sobolev norms

$$\|v\|_{m,K} = \sum_{|\alpha| \le m} \int_\Omega K(x)(D^\alpha v(x))^2 \, dx, \tag{2.8}$$

for $m = 0, 1, \ldots$, using standard multi-index notation (cf., e.g., [6, Chapter 1]). The obvious norm equivalence

$$\|v\|_{H^m(\Omega)} \le \|v\|_{m,K} \le k_F \|v\|_{H^m(\Omega)}$$

directly extends to the corresponding intermediate norms $\|v\|_{H^s(\Omega)}$ $\|v\|_{s,K}$, $s \in \mathbb{R}$, as obtained by interpolation (cf. Bergh and Löfström [4] or Brenner and Scott [6, Chapter 12]).

Proposition 2.1. *Assume that $u \in H^{1+s}(\Omega)$ with $0 < s \le 1$. Then the finite element solution u_{jk} satisfies the error estimate*

$$\|u - u_{jk}\| \le C h_{jk}^s \|u\|_{1+s,K}, \qquad h_{jk} = \max_{P \in P_{jk}} \operatorname{diam} P, \tag{2.9}$$

with a constant $C = C(s_0, \gamma_0)$ depending only on the shape regularity (2.4) of $\mathcal{P}_0$.

Proof. Utilizing the Lax–Milgram lemma together with standard estimates of the interpolation error on isotropic triangles or trapezoidals and the results of Ženíšek and Vanmaele [22] for the anisotropic case, we obtain the estimate

$$\|u - u_{jk}\| \le c h_{jk} \|u\|_{2,K},$$

provided that $u \in H^2(\Omega)$. Here, the constant c depends only on the shape regularity of $P \in \mathcal{P}_{jk}$ and therefore of $\mathcal{P}_0$. Now the desired estimate (2.9) follows from standard results on interpolated Sobolev spaces [4], [6, Chapter 12]. □

3. Separation of the fractures

We consider the direct splitting

$$\mathcal{S}_{jk} = \mathcal{S}_j^{\overline{M}} \oplus \mathcal{S}_{jk}^F \tag{3.1}$$

of the finite element space $\mathcal{S}_{jk}$ into the fracture space

$$\mathcal{S}_{jk}^F = \left\{ v \in \mathcal{S}_{jk} \mid v|_{\overline{\Omega}_M} = 0 \right\} \tag{3.2}$$

and its complement $\mathcal{S}_j^{\overline{M}}$ consisting of all $v \in \mathcal{S}_{jk}$ such that $v|_Q$ is isoparametric bilinear for all $Q \in \mathcal{Q}_j$. The construction is illustrated in Figure 3.1.

The stability of the splitting (3.1) is equivalent to the stability of the interpolation operator $I_{jk}^{\overline{M}} : \mathcal{S}_{jk} \to \mathcal{S}_j^{\overline{M}}$ defined by

$$I_{jk}^{\overline{M}} v(p) = v(p), \qquad p \in \mathcal{N}_j^{\overline{M}}.$$

Here, $\mathcal{N}_j^{\overline{M}}$ denotes the set of all the vertices of $P \in \mathcal{P}_{jk}$ which lie in $\overline{\Omega}_M$. Stability of $I_{jk}^{\overline{M}}$ will be shown by local estimates. To this end, we first consider transformations to suitable reference elements.

The nodes of $Q = (p_1, \dots, p_4) \in \mathcal{Q}_j$ are ordered anticlockwise and, for $Q \neq Q_c$, in such a way that the lengths h_Q, h'_Q of the edges $[p_1, p_2]$, $[p_3, p_4]$ lying in Γ satisfy $h'_Q \le h_Q$. With each $Q \neq Q_c$ we associate the reference element $\hat{Q} = [0,1] \times [0, \varepsilon]$, where $\varepsilon = \varepsilon_F / h_Q$. Observe that we always have $\varepsilon \le C_0$ by (2.6), but $\varepsilon \to 0$ occurs for $\varepsilon_F \to 0$. For the isotropic crossing Q_c we select the canonical reference element

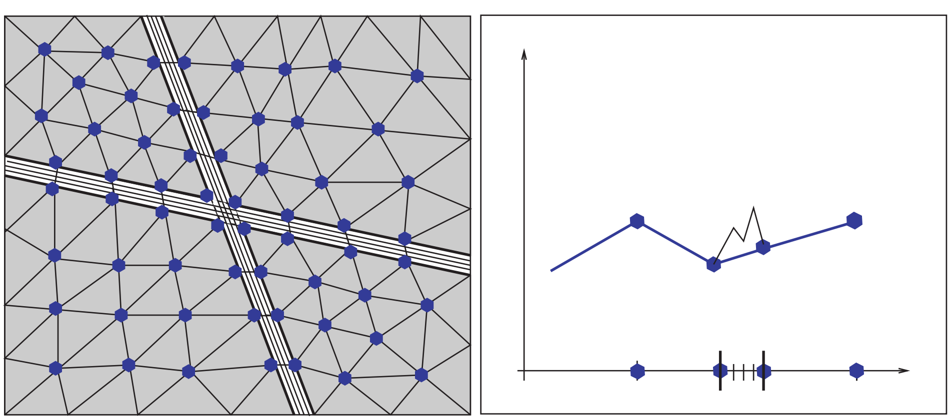

Figure 3.1. The nodes of $\mathcal{S}_j^{\overline{M}}$ and the splitting (3.1) in one dimension

$\hat{Q}_c = [0,1]^2$. Let $\mathcal{F}_Q : \hat{Q} \to Q$ denote the bijective bilinear mapping such that the ordering of the vertices is preserved and $\mathcal{F}_Q(0,0) = p_1$.

Throughout this paper, we write $a \preceq b$ for $a \le Cb$ and $a \asymp b$ for $cb \le a \le Cb$ with some c, C depending only on the constants s_0, γ_0 and C_0 from (2.4) and (2.6), respectively.

Lemma 3.1. *Let $Q \in \mathcal{Q}_j$. Each $v \in H^1(Q)$ and its transformation $\hat{v}$,*

$$\hat{v} = v(\mathcal{F}_Q(\cdot)) : \hat{Q} \to \mathbb{R},$$

satisfy the norm equivalence

$$\|\nabla v\|_{L^2(Q)} \asymp \|\nabla \hat{v}\|_{L^2(\hat{Q})}. \tag{3.3}$$

Proof. As the assertion is well known for $Q = Q_c$, we only have to consider the remaining case $Q \neq Q_c$. The chain rule yields

$$\nabla \hat{v} = \nabla v B_Q$$

denoting $B_Q = \mathcal{F}_Q'$. Elementary calculations utilizing (2.4) and (2.5) provide

$$|B_Q| \preceq (1 + \tfrac{h_Q - h_Q'}{\varepsilon_F}) \max\{h_Q, \frac{\varepsilon_F}{\varepsilon}\} = h_Q,$$

where $|\cdot|$ is the spectral norm. Similarly, we get

$$|B_Q^{-1}| \preceq (1 + \tfrac{h_Q - h_Q'}{\varepsilon_F}) \max\{h_Q^{-1}, \frac{\varepsilon}{\varepsilon_F}\} \preceq h_Q^{-1}$$

and, in addition,

$$|\det B_Q| \le h_Q \frac{\varepsilon_F}{\varepsilon} = h_Q^2, \quad |\det B_Q^{-1}| \preceq h_Q^{-1} \frac{\varepsilon}{\varepsilon_F} = h_Q^{-2},$$

so that the assertion follows from the substitution rule for integrals. □

The constants in the norm equivalence (3.3) depend only on the geometric similarity between the elements Q and $\hat{Q}$ in terms of the regularity of the transformation $\mathcal{F}_Q$, but not on the shape regularity of these elements.

The following technical lemma allows us to extend basic arguments from one-dimensional intervals to anisotropic elements in higher dimensions. In this sense, it is crucial for the rest of this paper.

Lemma 3.2. *Let $w : \hat{Q} = [0,1] \times [0,\varepsilon] \to \mathbb{R}$ be such that $w(\xi,\cdot)$ is absolutely continuous for each fixed $\xi \in [0,1]$, $w(\cdot,\eta)$ is linear for each fixed $\eta \in [0,\varepsilon]$, and $w(\xi,0) = w(\xi,\varepsilon) = 0$ holds for all $\xi \in [0,1]$. Then*

$$\|\nabla w\|^2_{L^2(\hat{Q})} \le (1 + 12\varepsilon^2) \|\tfrac{\partial}{\partial \eta} w\|^2_{L^2(\hat{Q})}. \tag{3.4}$$

Proof. The assertion will be proved in three steps. First, let $f : [0,\varepsilon] \to \mathbb{R}$ be absolutely continuous and let $f(0) = 0$. Then the fundamental theorem of calculus together with Cauchy's inequality yields

$$\int_0^\varepsilon (f(\eta))^2 \, d\eta = \int_0^\varepsilon \left(\int_0^\eta 1 \cdot f'(s) \, ds \right)^2 d\eta \le \varepsilon^2 \int_0^\varepsilon (f'(s))^2 \, ds. \tag{3.5}$$

Next, let $g : [0,1] \to \mathbb{R}$ be linear. Then elementary calculation leads to

$$(g(1) - g(0))^2 \le 12 \int_0^1 (g(\xi))^2 \, d\xi, \tag{3.6}$$

where, in particular, the binomial estimate

$$(a-b)^2 \le 4(a^2+ab+b^2) \qquad \forall a,b \in \mathbb{R}$$

has been used. Inserting $f(\eta) = w(1,\eta) - w(0,\eta)$ and $g(\xi) = \frac{\partial}{\partial\eta} w(\xi,\eta)$ for fixed η, in (3.5) and (3.6), respectively, the assertion follows from

$$\int_0^\varepsilon \int_0^1 \left(\tfrac{\partial}{\partial\xi} w(\xi,\eta)\right)^2 d\xi\, d\eta = \int_0^\varepsilon (w(1,\eta)-w(0,\eta))^2\, d\eta$$
$$\le \varepsilon^2 \int_0^\varepsilon \left(\tfrac{\partial}{\partial\eta}(w(1,\eta)-w(0,\eta))\right)^2 d\eta \le 12\varepsilon^2 \int_0^\varepsilon \int_0^1 \left(\tfrac{\partial}{\partial\eta} w(\xi,\eta)\right)^2 d\xi\, d\eta.$$

□

The proof of Lemma 3.2 solely relies on the assumption that the considered functions are linear in the direction orthogonal to the "small" edge of length ε. Therefore the estimate (3.4) directly extends to abitrary space dimensions. The constant does not deteriorate as the length ε of one (or more) edges tends to zero.

Now we can state a local estimate for anisotropic elements $Q \ne Q_c$.

Lemma 3.3. *The estimate*

$$\|\nabla I_{jk}^{\overline{M}} v\|_{L^2(Q)} \preceq \|\nabla v\|_{L^2(Q)} \tag{3.7}$$

holds for all $v \in \mathcal{S}_{jk}$ *and* $Q \in \mathcal{Q}_j \setminus \{Q_c\}$.

Proof. Transformation of $I_{jk}^{\overline{M}} v$ to the reference element $\hat{Q} = [0,1] \times [0,\varepsilon]$ provides $\hat{I}\hat{v}$, where $\hat{v}$ is the transformation of some $v \in \mathcal{S}_{jk}$ and $\hat{I}$ denotes the bilinear interpolation at the vertices of $\hat{Q}$. It is easily checked that $w = \hat{v} - \hat{I}\hat{v}$ satisfies the assumptions of Lemma 3.2. Hence,

$$\|\nabla(\hat{v} - \hat{I}\hat{v})\|^2_{L^2(\hat{Q})} \le (1+12\varepsilon^2) \|\tfrac{\partial}{\partial\eta}(\hat{v} - \hat{I}\hat{v})\|^2_{L^2(\hat{Q})}.$$

Using the orthogonality

$$\int_0^\varepsilon \tfrac{\partial}{\partial\eta}(\hat{v} - \hat{I}\hat{v}) \tfrac{\partial}{\partial\eta}\hat{I}\hat{v}\, d\eta = 0$$

we get

$$\|\tfrac{\partial}{\partial\eta}(\hat{v} - \hat{I}\hat{v})\|^2_{L^2(\hat{Q})} \le \|\tfrac{\partial}{\partial\eta}\hat{v}\|^2_{L^2(\hat{Q})},$$

so that the assertion follows from Lemma 3.1. □

The proof of Lemma 3.3 extends the stability of nodal interpolation from intervals to anisotropic elements. The stability of nodal interpolation on anisotropic elements in arbitrary space dimensions can be derived in the same way.

The above technique cannot be applied to the isotropic crossing Q_c. The basic idea to treat such exceptional elements is first to estimate the norm of $I_{jk}^{\overline{M}} v$ on Q_c by the norm of $I_{jk}^{\overline{M}} v$ on a neighboring element $Q \in \mathcal{Q}_j \setminus Q_c$ and then to apply the preceding Lemma 3.3. Observe that $v|_{Q_c}$ does not enter the resulting upper bound.

Lemma 3.4. *There is a neighboring element* $Q \in \mathcal{Q}_j$ *of* Q_c *such that the estimate*

$$\|\nabla I_{jk}^{\overline{M}} v\|_{L^2(Q_c)} \preceq \|\nabla v\|_{L^2(Q)} \tag{3.8}$$

holds for all $v \in \mathcal{S}_{jk}$.

Proof. Let $\hat{w}$ be a bilinear function on $\hat{Q} = [0,1] \times [0,\varepsilon]$ interpolating the values $w_1, \ldots, w_4$ in the vertices $(0,0)$, $(1,0)$, $(1,\varepsilon)$, $(0,\varepsilon)$. Elementary calculation yields

$$\begin{aligned}\|\nabla \hat{w}\|^2_{L^2(\hat{Q})} &= \tfrac{1}{3}\varepsilon \left((w_2-w_1)^2 + (w_3-w_4)^2 + (w_2-w_1)(w_3-w_4)\right) \\ &\quad + \tfrac{1}{3}\varepsilon^{-1}\left((w_4-w_1)^2 + (w_3-w_2)^2 + (w_4-w_1)(w_3-w_2)\right).\end{aligned} \tag{3.9}$$

Hence, the binomial estimate

$$\tfrac{4}{3}(a^2+b^2+ab) \geq a^2 \qquad \forall a,b \in \mathbb{R} \tag{3.10}$$

provides

$$4\varepsilon \|\nabla \hat{w}\|^2_{L^2(Q_c)} \geq \max\{(w_4-w_1)^2, (w_3-w_2)^2\}. \tag{3.11}$$

Replacing $\hat{Q}$ by $\hat{Q}_c = [0,1]^2$, formula (3.9) obviously holds with $\varepsilon = 1$. Using $ab \leq \frac{1}{2}(a^2+b^2)$ we easily get

$$\|\nabla \hat{w}\|^2_{L^2(Q_c)} \leq 2 \max_{l=1,\ldots,4} (w_{l+1} - w_l)^2, \tag{3.12}$$

denoting $w_5 = w_1$.

Now let $v \in \mathcal{S}_{jk}$ and $w = I_{jk}^{\overline{M}} v$. Using (3.12) together with Lemma 3.1 we obtain without loss of generality

$$\|\nabla w\|^2_{L^2(Q_c)} \preceq \max_{l=1,\ldots,4} (w_{l+1} - w_l)^2 = (w_4 - w_1)^2.$$

In this case, we select $Q = (p_1, p_2, p_3, p_4) \in \mathcal{Q}_j$ of Q_c such that $p_1, p_4 \in Q \cap Q_c$. Then (3.11) and Lemma 3.1 provide the estimate

$$\|\nabla w\|_{L^2(Q_c)} \preceq \|\nabla w\|_{L^2(Q)},$$

and the assertion follows from the preceding Lemma 3.3. □

Again the idea of the proof of Lemma 3.8 extends directly to arbitrary space dimensions. Estimates of the norm of $I_{jk}^{\overline{M}} v$ on other exceptional elements occurring, e.g., at the end of a fracture, by its norm on neighboring elements can be derived in a similar way. We also refer to the proof of Proposition 4.1 later on.

We are ready for the main result of this section.

Proposition 3.1. *For each $v \in \mathcal{S}_{jk}$ the decomposition $v = v^F + v^{\overline{M}}$ into $v^F \in \mathcal{S}_{jk}^F$ and $v^{\overline{M}} \in \mathcal{S}_j^{\overline{M}}$ satisfies*

$$\|v^F\|^2 + \|v^{\overline{M}}\|^2 \preceq \|v\|^2. \tag{3.13}$$

Proof. As a consequence of Lemma 3.3 and Lemma 3.4 we obtain

$$\|I_{jk}^{\overline{M}} v\|^2 = \sum_{T \in \mathcal{T}_j} \|\nabla v\|^2_{L^2(T)} + \sum_{Q \in \mathcal{Q}_j} k_F \|\nabla I_{jk}^{\overline{M}} v\|^2_{L^2(Q)} \preceq \|v\|^2,$$

so that the assertion follows with $v^{\overline{M}} = I_{jk}^{\overline{M}} v$ and $v^F = v - I_{jk}^{\overline{M}} v$. □

The gradient of a function $v \in \mathcal{S}_j^{\overline{M}}$ with fixed nodal values typically becomes arbitrarily large as $\varepsilon_F \to 0$. This causes problems with the robustness of iterative solvers for local subproblems on $\mathcal{S}_j^{\overline{M}}$ with respect to small ε_F. As a remedy, we now consider a further splitting of $\mathcal{S}_j^{\overline{M}}$ into a space of functions living on the interface Γ and another space of functions which are essentially constant across the fractures.

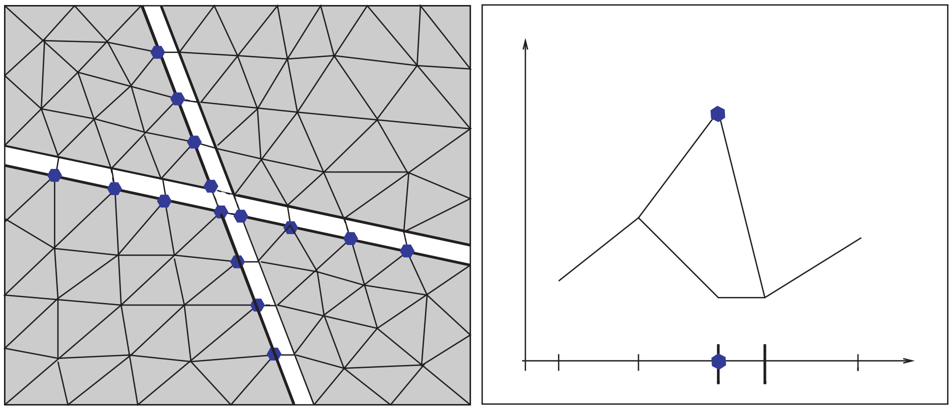

FIGURE 4.1. The nodes of $\mathcal{S}_j^{\Gamma_0}$ and the splitting (4.1) in one dimension

4. Separation of the interface

In the preceding section, we have separated the unknowns inside of the fracture Ω_F from the remaining unknowns living on the interface Γ and inside the rock matrix Ω_M. The next step is to identify the unknowns on both sides of the fractures. In such a way fractures are reduced to lower-dimensional objects which make sense even in the limit case $\varepsilon = 0$. This leads to robustness of the resulting iterative solvers for fractures of arbitrary small width ε.

More precisely, we decompose the interface $\Gamma = \Gamma_0 \cup \Gamma_1$ into its "lower" and "upper" parts consisting of $\Gamma_l = \{x \in \Omega \mid x = b_F^i + sd_F^i + ln_F^i, s \in \mathbb{R}, i = 1, 2\}$ with $l = 0$ and $l = 1$, respectively. Let $\mathcal{N}_j^{\Gamma_0} = \mathcal{N}_j \cap \Gamma_0$ and $\mathcal{E}_j^F = \mathcal{E}_j \cap \overline{\Omega}^F$, with $\mathcal{N}_j$ and $\mathcal{E}_j$ denoting the sets of interior vertices and edges of $P \in \mathcal{P}_j$, respectively. We consider the direct splitting

$$\mathcal{S}_j^{\overline{M}} = \mathcal{S}_j^M \oplus \mathcal{S}_j^{\Gamma_0} \tag{4.1}$$

into the interface space

$$\mathcal{S}_j^{\Gamma_0} = \{v \in \mathcal{S}_j^{\overline{M}} \mid v(p) = 0 \quad \forall p \notin \mathcal{N}_j^{\Gamma_0}\}, \tag{4.2}$$

and its complement $\mathcal{S}_j^M$ consisting of all $v \in \mathcal{S}_j^{\overline{M}}$ such that $v|_E$ is constant for all edges $E \in \mathcal{E}_j^F$. The construction is illustrated in Figure 4.1.

The splitting (4.1) is induced by the interpolation operator $I_j^M : \mathcal{S}_j^{\overline{M}} \to \mathcal{S}_j^M$ defined by

$$I_j^M v(p) = v(p), \qquad p \in \mathcal{N}_j \setminus \mathcal{N}_j^{\Gamma_0}.$$

If $p \in \mathcal{N}_j^{\Gamma_0}$, then $I_j^M v(p) = v(p^*)$, where p^* is the vertex of the edge $E = (p, p^*) \in \mathcal{E}_j^F$ or the vertex of Q_c which is not contained in $\mathcal{N}_j^{\Gamma_0}$. In particular, $I_j^M v$ is constant on Q_c.

We proceed with local stability estimates on $T \in \mathcal{T}_j$ and $Q \in \mathcal{Q}_j$, respectively.

Lemma 4.1. *The estimate*

$$\|\nabla I_j^M v\|_{L^2(T)}^2 \preceq \|\nabla v\|_{L^2(T)}^2 + \sum_{i=1}^{3} \left(I_j^M v(p_i) - v(p_i)\right)^2 \tag{4.3}$$

holds for all $v \in \mathcal{S}_j^{\overline{M}}$ *and all* $T = (p_1, p_2, p_3) \in \mathcal{T}_j$.

Proof. We set $w = I_j^M v$ and $\hat{w}$, $\hat{v}$ denote the usual transformations of w, v to the reference element $T_0 = ((0,0),(1,0),(0,1))$. Elementary calculations provide

$$\|\nabla \hat{w}\|^2_{L^2(T_0)} = \tfrac{1}{2}\left((w_2 - w_1)^2 + (w_3 - w_1)^2\right) \le 2\|\nabla \hat{v}\|^2_{L^2(T_0)} + 4\textstyle\sum_{i=1}^3 (w_i - v_i)^2,$$

and the assertion follows from the well-known estimates $\|\nabla w\|_{L^2(T)} \preceq \|\nabla \hat{w}\|_{L^2(T_0)}$ and $\|\nabla \hat{v}\|_{L^2(T_0)} \preceq \|\nabla v\|_{L^2(T)}$. □

Lemma 4.1 simply expresses the fact that the H^1-seminorm of a linear function over the reference element can be rewritten in terms of scaled differences of her values along the edges. The proof of the following lemma is based on the same dimension-independent arguments as were already used in Lemma 3.4.

Lemma 4.2. *The estimates*

$$\|\nabla I_j^M v\|_{L^2(Q)} \preceq \|\nabla v\|_{L^2(Q)}, \qquad \max_{i=1,\dots,4}\left(I_j^M v(p_i) - v(p_i)\right)^2 \preceq \|\nabla v\|^2_{L^2(Q)} \tag{4.4}$$

hold for all $v \in \mathcal{S}_j^{\overline{M}}$ *and all* $Q = (p_1, p_2, p_3, p_4) \in \mathcal{Q}_j$.

Proof. We set $w = I_j^M v$ and let $\hat{w}$, $\hat{v}$ denote the transformed functions on the reference element $\hat{Q} = [0,1] \times [0,\varepsilon]$. In light of Lemma 3.1 it is sufficient to show that

$$\|\nabla \hat{w}\|_{L^2(\hat{Q})} \preceq \|\nabla \hat{v}\|_{L^2(\hat{Q})}, \qquad \max_{i=1,\dots,4}(w_i - v_i)^2 \preceq \|\nabla \hat{v}\|^2_{L^2(\hat{Q})}, \tag{4.5}$$

respectively. Here we have set $w_i = w(p_i)$, $v_i = v(p_i)$. As $\hat{v}$ is bilinear on $\hat{Q}$, we can use (3.11) to obtain

$$4\varepsilon\|\nabla \hat{v}\|^2_{L^2(\hat{Q})} \ge \max\{(v_4 - v_1)^2, (v_3 - v_2)^2\} = \max_{i=1,\dots,4}(w_i - v_i)^2$$

which is the right estimate in (4.4). Note that

$$\|\nabla \hat{w}\|^2_{L^2(\hat{Q})} = \varepsilon(w_2 - w_1)^2 \le \varepsilon \max_{i,l=1,\dots,4}(v_i - v_l)^2 \le 4\varepsilon \max_{l=1,\dots,4}(v_{l+1} - v_l)^2,$$

where we have set $v_5 = v_1$. Using the binomial estimate (3.10) and the representation (3.9) of $\|\nabla \hat{v}\|^2_{L^2(\hat{Q})}$, we further get

$$\varepsilon \max_{l=1,\dots,4}(v_{l+1} - v_l)^2 \le 4(1+\varepsilon^2)\|\nabla \hat{v}\|^2_{L^2(\hat{Q})}$$

which proves the assertion for $Q \ne Q_c$. In the case $Q = Q_c$ the left estimate in (4.4) is trivial and the right one follows by similar arguments as before. □

Now we are ready to prove stability of the splitting (4.1).

Proposition 4.1. *For each* $v \in \mathcal{S}_j^{\overline{M}}$ *the decomposition* $v = v^M + v^{\Gamma_0}$ *into* $v^M \in \mathcal{S}_j^M$ *and* $v^{\Gamma_0} \in \mathcal{S}_j^{\Gamma_0}$ *satisfies*

$$\|v^M\|^2 + \|v^{\Gamma_0}\|^2 \preceq \|v\|^2. \tag{4.6}$$

Proof. Let $v \in \mathcal{S}_j^{\overline{M}}$. We set $v^M = I_j^M v$ and $v_0^\Gamma = v - I_j^M v$. Utilizing Lemma 4.1 and Lemma 4.2 we get

$$\|\nabla v^M\|^2_{L^2(T)} \preceq \|\nabla v\|^2_{L^2(T)} + \sum_{Q \in \mathcal{Q}_j(T)} \|\nabla v\|^2_{L^2(Q)} \qquad \forall T \in \mathcal{T}_j,$$

denoting $\mathcal{Q}_j(T) = \{Q \in \mathcal{Q}_j \mid Q \cap T \neq \emptyset\}$. As a consequence of the minimal angle condition (2.4) we have

$$\sum_{T \in \mathcal{T}_j} \sum_{Q \in \mathcal{Q}_j(T)} \|\nabla v\|^2_{L^2(Q)} \preceq \sum_{Q \in \mathcal{Q}_j} \|\nabla v\|^2_{L^2(Q)}.$$

Together with $k_F \geq 1$ and Lemma 4.2 these estimates provide

$$\|v^M\|^2 = \sum_{T \in \mathcal{T}_j} \|\nabla v^M\|^2_{L^2(T)} + \sum_{T \in \mathcal{Q}_j} k_F \|\nabla v^M\|^2_{L^2(Q)} \preceq \|v\|^2.$$

The assertion now follows from the triangle inequality. □

Note that Proposition 4.1 implies the stability of the overlapping splitting

$$\mathcal{S}_j^{\overline{M}} = \mathcal{S}_j^M + \mathcal{S}_j^\Gamma, \tag{4.7}$$

where $\mathcal{S}_j^{\Gamma_0}$ is replaced by the larger space

$$\mathcal{S}_j^\Gamma = \left\{ v \in \mathcal{S}_j^{\overline{M}} \mid v(p) = 0 \quad \forall p \in \mathcal{N}_j \cap \Omega_M \right\}. \tag{4.8}$$

Utilizing Proposition 3.1, we also get the stability of the decomposition

$$\mathcal{S}_{jk} = \mathcal{S}_j^M + \mathcal{S}_{jk}^{\overline{F}}, \qquad \mathcal{S}_{jk}^{\overline{F}} = \mathcal{S}_{jk}^F + \mathcal{S}_j^\Gamma.$$

In three space dimensions the interface space can be separated by similar arguments as mentioned above. In this case, both one- or two dimensional objects can occur as the width of the fractures tends to zero.

5. A hierarchical domain decomposition method

Now we are ready to derive and analyse iterative schemes for the discrete problem (2.7). To this end, we use the general framework of multiplicative subspace correction methods (cf. Xu [24] or Yserentant [26]). These methods are based on a decomposition

$$\mathcal{S} = \mathcal{W}_0 + \mathcal{W}_1 + \cdots + \mathcal{W}_J, \quad \mathcal{W}_l \subset \mathcal{S},$$

of the discrete solution space $\mathcal{S} = \mathcal{S}_{jk}$ and symmetric, positive definite bilinear forms $b_l(\cdot,\cdot)$ on $\mathcal{W}_l$. Starting with some given iterate $w_{-1} = u^\nu \in \mathcal{S}$, a sequence of intermediate iterates w_l, $l = 0, \dots, J$, is computed according to

$$\begin{aligned} v_l \in \mathcal{W}_l: \qquad b_l(v_l, v) &= \ell(v) - a(w_{l-1}, v) \qquad \forall v \in \mathcal{W}_l, \\ w_{l+1} &= w_l + v_l, \end{aligned} \tag{5.1}$$

and $u^{\nu+1} = w_J$ is the subsequent iterate. *Hierarchical domain decomposition methods* are obtained from decompositions as presented in the preceding sections.

Theorem 5.1. *Select the splitting*

$$\mathcal{S}_{jk} = \mathcal{S}_j^M + \mathcal{S}_j^\Gamma + \mathcal{S}_{jk}^F \tag{5.2}$$

with $\mathcal{S}_j^M$, $\mathcal{S}_j^\Gamma$ *and* $\mathcal{S}_{jk}^F$ *defined in* (4.1), (4.8) *and* (3.2), *respectively, and the bilinear form* $a(\cdot,\cdot)$ *on all these three spaces.*

Then the iterates u_{jk}^ν *of the resulting hierarchical domain decomposition method converge to the exact solution* u_{jk} *of* (2.7) *and satisfy the error estimate*

$$\|u_{jk} - u_{jk}^{\nu+1}\|^2 \leq \rho \|u_{jk} - u_{jk}^\nu\|^2, \qquad \nu = 0, 1, \dots, \tag{5.3}$$

with $\rho < 1$ depending only on the shape regularity (2.4) *of the initial partition $\mathcal{P}_0$ and on the constant C_0 in* (2.6) *relating the width ε_F of the fractures to the size of the triangles.*

Proof. The proof follows from general convergence results for subspace correction methods (cf. Xu [24] or Yserentant [26]). More precisely, we can apply Theorem 5.1 in [26] with $\omega = 1$, $K_1 \preceq 1$ (cf. Propositions 3.1, 4.1) and $K_2 = 3$. □

We emphasize that Theorem 5.1 implies robust convergence with respect to arbitrary large permeability k_F and arbitrary small width ε_F of the fractures. These results are not restricted to two space dimensions. Utilizing the techniques presented in the preceding sections, robust HDD algorithms with mesh-independent convergence rates can be derived for three space dimensions and for more complicated geometries, such as multiple crossings or fractures ending in the computational domain.

6. An inexact version

The HDD algorithm presented in Theorem 5.1 requires the solution of an anisotropic subproblem on $\mathcal{S}_{jk}^F$ (without jumping coefficients), a lower dimensional subproblem on $\mathcal{S}_{jk}^\Gamma$, and a subproblem with jumping coefficients on $\mathcal{S}_{jk}^M$ (without anisotropies). Thus we have separated the main difficulties of the original discrete problem (2.7). Now existing strategies can be used to derive multilevel methods for the subproblems that allow us to extend the robust convergence (5.3) to corresponding *inexact HDD algorithms*, where the exact solution of the subproblems is replaced by one or more multigrid steps.

As an example, we analyze the multigrid solution of the subproblem on $\mathcal{S}_j^M$ adapting well-known results on hierarchical bases. We emphasize that these considerations are limited to two space dimensions.

Successive refinement in the rock matrix gives rise to the sequence of nested subspaces

$$\mathcal{S}_0^M \subset \cdots \subset \mathcal{S}_{j-1}^M \subset \mathcal{S}_j^M, \tag{6.1}$$

where $\mathcal{S}_l^M$ consists of all functions $v \in H_0^1(\Omega)$ such that $v|_T$ is linear for all $T \in \mathcal{T}_l$, $v|_Q$ is isoparametric bilinear for all $Q \in \mathcal{Q}_l$, and $v|_E$ is constant for all edges $E \in \mathcal{E}_l$. Recall that $\mathcal{E}_l$ and $\mathcal{N}_l$ are denoting the sets of interior edges and vertices of $P \in \mathcal{P}_l$, respectively. The underlying refinement rules imply

$$c2^{-l} \leq \operatorname{diam}(T) \leq C2^{-l} \qquad \forall T \in \mathcal{T}_l$$

with constants c, C depending only on the initial triangulation $\mathcal{T}_0$. From now on constants hidden by the shortcuts $'\asymp'$ and $'\preceq'$ may additionally depend on c, C.

The nodal interpolation $I_l : \mathcal{S}_j \to \mathcal{S}_l$ defined by

$$I_l v(p) = v(p), \qquad p \in \mathcal{N}_l, \tag{6.2}$$

gives rise to the direct splitting

$$\mathcal{S}_j^M = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_j \tag{6.3}$$

into the subspaces

$$\mathcal{V}_0 = \mathcal{S}_0^M, \qquad \mathcal{V}_l = (I_l - I_{l-1})\mathcal{S}_j^M, \quad l = 1, \ldots, j. \tag{6.4}$$

We state a variant of the well-known stability result of Yserentant [25]. Recall the weighted L^2-norm $\|\cdot\|_{0,K}$ defined in (2.8).

Proposition 6.1. *For each* $v \in \mathcal{S}_j^M$ *the decomposition* $v = \sum_{l=0}^{j} v_l$ *into* $v_l \in V_l$ *satisfies*

$$\|v_0\|^2 + \sum_{l=1}^{j} 4^l \|v_l\|_{0,K}^2 \preceq (1+j)^2 \|v\|^2. \tag{6.5}$$

Proof. Let $v \in \mathcal{S}_j^M$. First note that

$$4^l \|(I_l - I_{l-1})v\|_{0,K}^2 \preceq \|I_l v\|^2, \quad l = 1, \dots, j,$$

can be shown by transformation to reference elements $\hat{T}$ or $\hat{Q}$ and exploiting the equivalence of norms on 2- or 5-dimensional quotient spaces. Hence, it is sufficient to prove

$$\|I_l v\|^2 \preceq (1 + j - l)\|v\|^2, \qquad l = 0, \dots, j. \tag{6.6}$$

Obviously, (6.6) is a consequence of the local estimate

$$\|\nabla I_l v\|_{L^2(P)}^2 \preceq (1 + j - l)\|\nabla v\|_{L^2(P)}^2 \qquad \forall P \in \mathcal{P}_l = \mathcal{T}_l \cup \mathcal{Q}_l,$$

which is well known for $T \in \mathcal{T}_l$ (cf. Yserentant [25]). In order to show

$$\|\nabla I_l v\|_{L^2(Q)}^2 \preceq \|\nabla v\|_{L^2(Q)}^2 \quad \forall Q \in \mathcal{Q}_l, \tag{6.7}$$

we set $w = I_l v$ and let $\hat{w}$, $\hat{v}$ denote the transformed functions on the reference element $\hat{Q} = [0,1] \times [0, \varepsilon]$. As $\frac{\partial}{\partial \eta}\hat{w} = \frac{\partial}{\partial \eta}\hat{v} = 0$ the orthogonality

$$\int_0^1 \tfrac{\partial}{\partial \xi}\hat{w} \tfrac{\partial}{\partial \xi}(\hat{v} - \hat{w}) \, d\xi$$

implies

$$\|\nabla \hat{v}\|_{L^2(\hat{Q})}^2 = \|\nabla \hat{w}\|_{L^2(\hat{Q})}^2 + \|\nabla(\hat{v} - \hat{w})\|_{L^2(\hat{Q})}^2 \geq \|\nabla \hat{w}\|_{L^2(\hat{Q})}^2$$

so that (6.7) follows from Lemma 3.1. This proves the assertion. □

Next we extend a strengthened Cauchy–Schwarz inequality from classical finite element spaces to $\mathcal{S}_l^M$. Recall the weighted L^2-norm $\|\cdot\|_{0,K}$ as introduced in (2.8).

Proposition 6.2. *Let* $0 \leq k < l \leq j$. *Then the estimate*

$$a(v, w) \preceq \left(\tfrac{1}{\sqrt{2}}\right)^{l-k} \|v\| 2^l \|w\|_{0,K} \tag{6.8}$$

holds for all $v \in \mathcal{S}_k^M$ *and* $w \in \mathcal{S}_l^M$.

Proof. It is sufficient to show the local estimate

$$(\nabla v, \nabla w)_{L^2(P)} \preceq (\frac{1}{\sqrt{2}})^{l-k} \|\nabla v\|_{L^2(P)} 2^l \|w\|_{L^2(P)} \qquad \forall P \in \mathcal{T}_k \cup \mathcal{Q}_k. \tag{6.9}$$

As (6.9) is well known for $T \in \mathcal{T}_l$ (cf., e.g., [26, p. 313]), we only have to consider the case $P = Q \in \mathcal{Q}_k$. Then the transformation to the associated reference element $\hat{Q}$ leads to

$$(\nabla v, \nabla w)_{L^2(Q)} = \varepsilon \int_0^1 \alpha(\xi) \hat{v}_\xi \hat{w}_\xi \, d\xi$$

because v and w are constant along edges $E \in \mathcal{E}_k$. Here $\alpha(\xi)$ is a quadratic polynomial with the properties

$$\alpha(\xi) \asymp 1, \quad |\alpha'(\xi)| \preceq 1 \qquad \forall \xi \in [0,1].$$

Now (6.9) follows from a one-dimensional version of the arguments used in the proof of Lemma 6.1 in [26]. □

We now present an inexact variant of HDD which requires a multigrid V-cycle and the exact solution of local problems on $\mathcal{S}_j^\Gamma$ and $\mathcal{S}_{jk}^F$ in each iteration step.

Theorem 6.1. *Select the splitting*

$$\mathcal{S}_{jk} = \left(\sum_{l=0}^{j} \mathcal{S}_l^M\right) + \mathcal{S}_j^\Gamma + \mathcal{S}_{jk}^F$$

with $\mathcal{S}_l^M$, $\mathcal{S}_j^\Gamma$ *and* $\mathcal{S}_j^F$ *defined in* (6.1), (4.8), *and* (3.2), *respectively. Let* $b_l(\cdot,\cdot)$, *be generated by symmetric Gauß–Seidel smoothers on* $\mathcal{S}_l^M$, $l = 1,\dots,j$, *and choose the bilinear form* $a(\cdot,\cdot)$ *on the three remaining spaces.*

Then the iterates u_{jk}^ν *of the resulting hierarchical domain decomposition method converge to the exact solution* u_{jk} *of* (2.7) *and satisfy the error estimate*

$$\|u_{jk} - u_{jk}^{\nu+1}\|^2 \le \left(1 - C(1+j)^{-3}\right) \|u_{jk} - u_{jk}^{\nu}\|^2, \qquad \nu = 0,\dots,$$

with C *depending only on the initial partition* $\mathcal{P}_0$ *and the constant* C_0 *from* (2.6).

Proof. We select the direct splitting

$$\mathcal{S}_{jk} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_J, \quad J = j+2, \tag{6.10}$$

into the subspaces $\mathcal{V}_l \subset \mathcal{W}_l$, $l = 0,\dots,j$, defined in (6.4), and $\mathcal{V}_{j+1} = \mathcal{S}_j^{\Gamma_0} \subset \mathcal{W}_{j+1}$ and $\mathcal{V}_{j+2} = \mathcal{S}_{jk}^F = \mathcal{W}_{j+2}$, defined in (4.2) and (3.2), respectively. The smoothing property (2.33) in [26] holds with $\omega = 1$ for the Gauß–Seidel smoothers $b_l(\cdot,\cdot)$, $l = 1,\dots,j$, and is trivial otherwise. The stability (5.2) in [26] of the decomposition (6.10) with $K_1 \preceq (1+j)^2$ follows from Propositions 3.1, 4.1, and 6.1, utilizing the norm equivalence

$$b_l(v,v) \asymp 4^l \|v\|_{0,K}^2 \quad \forall v \in \mathcal{S}_l^M, \qquad l = 1,\dots,j. \tag{6.11}$$

Note that the refinement condition (2.6) is used in the proof of (6.11). Inserting (6.11) into (6.8) and using the smoothing property, we obtain the Cauchy–Schwarz-type inequality

$$a(w_k, w_l) \preceq \left(\frac{1}{\sqrt{2}}\right)^{|l-k|} b_k(w_k,w_k)^{\frac12} b_l(w_l,w_l)^{\frac12} \quad \forall w_k \in \mathcal{W}_k, w_l \in \mathcal{W}_l.$$

This inequality directly leads to

$$\|\sum_{l=0}^{j} w_l\|^2 \preceq \sum_{l=0}^{j} b_l(w_l,w_l) \qquad \forall w_l \in \mathcal{W}_l, \quad l = 0,\dots,j,$$

and the Cauchy-Schwarz inequality provides condition (5.7) in [26] with $K_2 \preceq 1$. Now the assertion follows from Theorem 5.4 in [26]. □

The proof of Theorem 6.1 directly applies to other smoothers satisfying (6.11). Even suitable nonsymmetric smoothers, e.g., the Gauß–Seidel method, are allowed (cf. Neuss [17]).

According to Theorem 6.1 the inexact version of HDD preserves robustness with respect to ε_F and, as a consequence of the local properties of the interpolation operators I_l, also with respect to k_F. On the other hand, the stability of I_l and therefore Theorem 6.1 is restricted to two space dimensions. Replacing I_l by L^2-type projections, similar results can be obtained for three dimensions at the expense of robustness with respect to k_F. More sophisticated techniques for elliptic problems with jumping coefficients in three space dimensions can be found in [21].

The exact solution of the local subproblems associated with the fracture space $\mathcal{S}_{jk}^F$ can also be replaced by one step of a multigrid method. Indeed, Lemma 3.2 provides the stability of an hierarchical splitting of the fracture space $\mathcal{S}_{jk}^F$ into subspaces of $\mathcal{S}_{jl}^F$, $l = 0, \dots, k$, of functions $v \in \mathcal{S}_{jk}^F$ which are linear on all edges $E \in \mathcal{E}_{jl} \cap \overline{\Omega}^F$. Here, $\mathcal{E}_{jl}$ denotes the edges of $P \in \mathcal{P}_{jl}$. The convergence rate of the resulting hierarchical multigrid method with line Gauß–Seidel smoother is robust for $\varepsilon \to 0$ and independent of the mesh size. For similar results we refer to Bramble and Zhang [2] and the references cited therein.

Finally note that the "robust" smoothers proposed by Gebauer et al. [10] for the multigrid solution of problems on $\mathcal{S}_j^{\overline{M}} = \mathcal{S}_j^\Gamma + \mathcal{S}_j^M$ can be interpreted in terms of a suitable multilevel splitting of $\mathcal{S}^\Gamma$.

7. Numerical results

We consider the model problem

$$\nabla \cdot (K \nabla u) = 0 \qquad \text{on} \qquad \Omega = (0,6) \times (0,6) \tag{7.1}$$

with K defined in (2.3), $u(0,y) = 2$, $u(6,y) = 1$ for $y \in [0,6]$ and homogeneous Neumann data elsewhere. Obviously, (7.1) can be written in weak form (2.2) with suitable H and ℓ. The fracture network Ω_F, together with the initial partition $\mathcal{P}_0$, is shown in the left picture of Figure 2.1 for a comparatively large width $\varepsilon_F = 0.2$. Corresponding partitions for smaller ε_F are obtained by shifting the nodes lying on the interface Γ towards the centerlines of the fractures. In the limit case $\varepsilon_F = 0$ the fractures disappear and the problem reduces to the Laplace equation.

In order to illustrate the robustness of the hierarchical domain decomposition method presented in Theorem 6.1, we consider the corresponding discretized problem (2.7) for $j = 6$ and $k = 2$. In the left picture of Figure 7.1 we depict the convergence rates for fixed $\varepsilon_F = 10^{-5}$ and increasing permeability k_F. More precisely, the convergence rates are approximated by

$$\rho = \frac{\|u_{jk}^{\nu_0+1} - u_{jk}^{\nu_0}\|}{\|u_{jk}^{\nu_0} - u_{jk}^{\nu_0-1}\|},$$

where ν_0 is chosen such that $\|u_{jk}^{\nu_0+1} - u_{jk}^{\nu_0}\| \leq 10^{-12}$. As expected from the theoretical findings, the convergence speed is hardly influenced by the size of the jump. The right picture shows (approximate) convergence rates for fixed $k_F = 1$ and decreasing ε_F. The convergence rates are almost the same for $10^{-9} \leq \varepsilon_F \leq 10^{-2}$. They scarcely differ from the convergence rates of classical multigrid for the reduced Laplace problem which are indicated by the horizontal line. Note that we have $C_0 \approx 40$ in condition (2.6) for $\varepsilon_F = 10^{-1}$, which explains the unsatisfying convergence speed for this value.

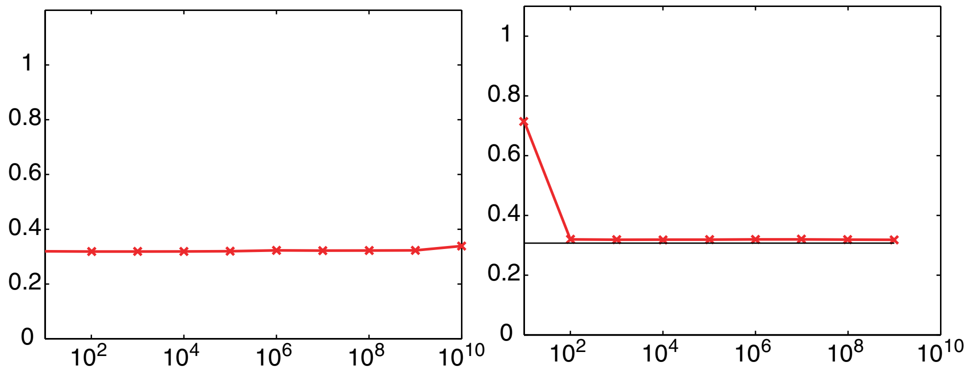

FIGURE 7.1. Robustness with respect to increasing k_F and vanishing ε_F

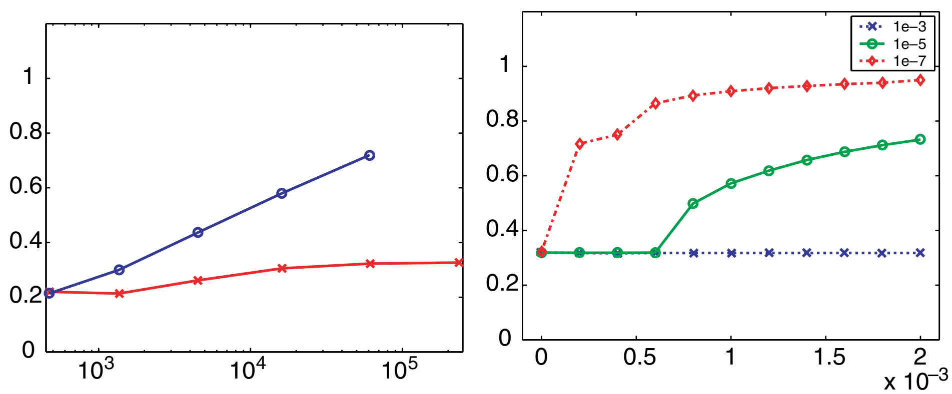

FIGURE 7.2. Influence of large ε_F/h_Q and small interior angles γ

We now compare the convergence rates for fixed $k_F = 10^6$ and increasing number of refinement steps j. The left picture in Figure 7.2 shows that the convergence speed rapidly deteriorates for "large" $\varepsilon_F = 10^{-1}$ (upper curve) and is hardly affected for "small" $\varepsilon_F = 10^{-5}$ (lower curve). Note that $C_0 \approx 2 \cdot 10^{-3}$ in the latter case. The right picture illustrates the influence of decreasing interior angles. The length of an edge $E \in \Gamma$ of an element $Q \in \mathcal{Q}_0$ is shifted by a fixed factor s (independent of ε_F). This leads to an interior angle $\gamma \approx \arctan(\varepsilon_F/s)$ which obviously tends to zero for increasing s and small ε_F. It is interesting to see how convergence rates branch off for increasing s or, equivalently, for γ becoming too small. These two experiments complement our analysis in the sense that moderate constants in the conditions (2.4) and (2.6) also seem to be necessary for fast convergence.

Let us remark in closing that comparisons with the algebraic multigrid method by Ruge and Stüben [19] in Gebauer [9] confirm the superiority of hierarchical domain decomposition not only from a theoretical but also from a practical point of view.

Acknowledgments

Special thanks to R. Helmig (Uni Stuttgart), R. Hinkelmann (TU Berlin), L. Neunhäuserer (Uni Stuttgart), and S. Ochs (Uni Stuttgart) for stimulating discussions and fruitful cooperation on flow and transport in fractured porous media.

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