NEW IRRATIONALITY MEASURES FOR $q$-LOGARITHMS

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Abstract. The three main methods used in diophantine analysis of $q$-series are combined to obtain new upper bounds for irrationality measures of the values of the $q$-logarithm function

$$\ln_q(1 - z) = \sum_{\nu=1}^{\infty} \frac{z^\nu q^\nu}{1 - q^\nu}, \quad |z| \leq 1,$$

when $p = 1/q \in \mathbb{Z} \setminus \{0, \pm 1\}$ and $z \in \mathbb{Q}$.

1. Introduction

The main purpose of this article is to improve the earlier irrationality measures of the values of the $q$-logarithm function

$$\ln_q(1 - z) = \sum_{\nu=1}^{\infty} \frac{z^\nu q^\nu}{1 - q^\nu}, \quad |z| \leq 1.$$ 

In order to improve the earlier results we shall combine the following three major methods used in diophantine analysis of $q$-series:

1. a general hypergeometric construction of rational approximations to the values of $q$-logarithms vs. the $q$-arithmetic approach ([Z1]);
2. a continuous iteration procedure for additional optimization of analytic estimates ([Bo], [MV]);
3. introducing the cyclotomic polynomials for sharpening least common multiples of the constructed linear forms in the case when $z$ is a root of unity ([BV], [As], [MP]).

Also, some standard analytic tools (i.e., from [Ha]) for deducing irrationality measures will be required. We underline that in the corresponding arithmetic study of the values of the ordinary logarithm (cf. [Ru] for log 2 and [Ha] for other logarithms) only feature (1) is mainly applied, but in particular feature (3) has no ordinary analogues. Thus the present $q$-problems invoke new attractions in arithmetic questions.

We present the bounds for irrationality measures by means of certain estimates for irrationality exponents. Recall that the irrationality exponent of a real irrational
number $\gamma$ is defined by the relation

$$\mu = \mu(\gamma) = \inf\{c \in \mathbb{R} : \text{the inequality } |\gamma - a/b| \leq |b|^{-c} \text{ has}
\text{only finitely many solutions in } a, b \in \mathbb{Z}\}.$$ 

Our main results include the case of general rational $z$ satisfying $|z| \leq 1$ as well as the case $z = -1$ of $\ln_q(2)$. Another special case, $z = 1$ in [1], of the $q$-harmonic series, is considered in [22]. Our present methods do not allow us to sharpen the result in [22], where the arithmetic group structure approach (specific for $z = 1$) is used.

**Theorem 1.** Let $z \in \mathbb{Q}$ be such that $0 < |z| \leq 1$. Then the irrationality exponent of $\ln_q(1 - z)$ satisfies the estimate

$$\mu(\ln_q(1 - z)) \leq 3.76338419 \cdots,$$

where $q = p^{-1}$ and $p \in \mathbb{Z} \setminus \{0, \pm 1\}$.

**Theorem 2.** The irrationality exponent of $\ln_q(2)$ satisfies the estimate

$$\mu(\ln_q(2)) \leq 2.93832530 \cdots,$$

where $q = p^{-1}$ and $p \in \mathbb{Z} \setminus \{0, \pm 1\}$.

The estimate in Theorem 1 improves corresponding results of [BV], [MV]; the estimate in Theorem 2 sharpens results in [As], [Z1].

One important part in the proof of Theorem 2 is the precise knowledge of the least common multiple $D_n(x, z)$ of the polynomials $x - z, x^2 - z, \ldots, x^n - z$ at $z = -1$. This is a special case of a general algebraic result on $D_n(x, \omega)$ with a root of unity $\omega$. The proof of this result, the following Theorem 3, seems to be an interesting application of cyclotomic polynomials.

**Theorem 3.** Let $\omega$ denote a primitive $r$-th root of unity for some $r \geq 2$. Then in the polynomial ring $\mathbb{Z}[\omega][x]$ the following estimate is valid:

$$\deg_x D_n(x, \omega) = \frac{3n^2}{\pi^2} \prod_{p|r} p^2 - 1 \sum_{l}^* \frac{1}{l^2} + O(n \log^2 n) \quad \text{as } n \to \infty,$$

where $\sum_l^*$ stands for summation over integers $l$ in the interval $1 \leq l \leq r$ and coprime with $r$.

To the end of Section 3, the integer $p$ stands for $1/q$. We recall some standard $q$-notation:

$$(a; q)_n := \prod_{\nu=1}^{n} (1 - aq^{\nu-1}),$$

$$[n]_q! := \frac{(q; q)_n}{(1 - q)^n}, \quad \left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q! \cdot [n - k]_q!} = \frac{(q; q)_n}{(q; q)_k \cdot (q; q)_{n-k}},$$

where $k = 0, 1, \ldots, n$ and $n = 0, 1, 2, \ldots$. 

2. Hypergeometric construction

Let $n_0$, $n_1$, $n_2$, and $m$ be positive integers satisfying $n_1 \geq n_0$, $n_2 \geq n_0$. The additional condition $n_2 - n_0 \leq m \leq n_2$ will be required to further simplify the explanation (the choices $m < n_2 - n_0$ and $m > n_2$ do not correspond to nice approximations to the $q$-logarithm). First, consider the rational function

$$\bar{R}_q(T) = \frac{\prod_{k=1}^{n_0} (1 - q^k T)}{\prod_{k=1}^{n_0} (1 - q^k)} \cdot \frac{\prod_{k=1}^{n_2} (1 - q^{k+n_1+1} T)}{\prod_{k=0}^{n_2} (1 - q^{k+n_1+1} T)} \cdot T^{m_2-n_0} \cdot \frac{(q T; q)_{n_0}}{(q; q)_{n_0}} \cdot \frac{(q; q)_{n_2}}{(q^{n_1+1}; q)_{n_2+1}} \cdot T^{m_2-n_0},$$

which is of order $O(T^{-1})$ as $T \to \infty$. This may be decomposed into the sum of partial fractions:

$$\bar{R}_q(T) = \sum_{k=0}^{n_2} \frac{A_k(q)}{1 - q^{k+n_1+1} T},$$

where the standard procedure of determining coefficients gives us

$$A_k(q) = (-1)^{n_0} q^{n_0(n_0+1)/2 - n_0(k+n_1+1)} \left[ k + n_1 \atop n_0 \right] \cdot \frac{p_{n_2}}{k} \cdot q^{-(n_2-n_0)(k+n_1+1)} \cdot \left[ k + n_1 \atop n_0 \right] \cdot \frac{p_n}{k} \cdot q^{-(n_2-n_0)(k+n_1+1)}$$

for $k = 0, 1, \ldots, n_2$. Setting $R_q(T) = \bar{R}_q(T) \cdot T^{m_2+1}$, where $m_0 = m - n_2 + n_0$, we introduce the quantity

$$I_q(z) = z^{n_1+1} \sum_{t=0}^{\infty} z^t R_q(T) \bigg|_{T=q^t}.$$

Since $R_q(T)$ has zeros at the points $T = q^{-1}, q^{-2}, \ldots, q^{-n_0}$, after reordering of the summation we may write

$$I_q(z) = \sum_{k=0}^{n_2} A_k(q) q^{-(k+n_1+1)(m_0+1)} z^{-k} \sum_{l=-n_0}^{\infty} \frac{z^l q^{(l+k+n_1+1)(m_0+1)}}{1 - q^{l+k+n_1+1}}$$

for $k = 0, 1, \ldots, n_2$. Setting $R_q(T) = \bar{R}_q(T) \cdot T^{m_2+1}$, where $m_0 = m - n_2 + n_0$, we introduce the quantity

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for $k = 0, 1, \ldots, n_2$. Setting $R_q(T) = \bar{R}_q(T) \cdot T^{m_2+1}$, where $m_0 = m - n_2 + n_0$, we introduce the quantity

$$I_q(z) = z^{n_1+1} \sum_{t=0}^{\infty} z^t R_q(T) \bigg|_{T=q^t}.$$

The last inner sum may be computed as follows:

$$\sum_{l=k+n_1-n_0+1}^{\infty} \frac{z^l q^{l(m_0+1)}}{1 - q^l} = \sum_{l=k+n_1-n_0+1}^{\infty} \frac{z^l q^l}{1 - q^l} - \sum_{l=k+n_1-n_0+1}^{\infty} \frac{z^l q^{l(m_0+1)}}{1 - q^l};$$

writing the first sum on the right-hand side as

$$\sum_{l=1}^{\infty} \frac{z^l q^l}{1 - q^l} - \sum_{l=1}^{k+n_1-n_0} \frac{z^l q^l}{1 - q^l} = \ln_q(1 - z) - \sum_{l=1}^{k+n_1-n_0} \frac{z^l q^l}{1 - q^l};$$

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and the second sum as
\[\sum_{l=k+n_1-n_0+1}^{m_0} z^l \sum_{j=1}^{m_0} q_j^l = \sum_{j=1}^{m_0} (q_j z)^{k+n_1-n_0+1} \frac{1 - q_j^2 z}{1 - q_j^2},\]
we finally obtain
\[I_q(z) = A(p, z) \ln_q(1 - z) + A'(p, z) + A''(p, z),\]
where
\[A(p, z) = \sum_{k=0}^{n_2} A_k(q)p^{(k+n_1+1)(m_0+1)} z^{-k},\]
\[A'(p, z) = \sum_{k=0}^{n_2} A_k(q)p^{(k+n_1+1)(m_0+1)} z^{-k} \sum_{l=1}^{m_0} \frac{z^l}{p^l},\]
\[A''(p, z) = \sum_{k=0}^{n_2} A_k(q)p^{(k+n_1+1)(m_0+1)} z^{n_1-n_0+1} \sum_{j=1}^{m_0} \frac{1}{p^j - z},\]

\[(\text{the last step uses Lemma 3 from [ZI]).}\]

Since \(m \leq n_2\), we have
\[M_1 = \frac{n_0(n_0+1)}{2} + (m+1)(n_1+1) + \min_{0 \leq k \leq n_2} \left\{-n_2k + (m+1)k + \frac{k(k-1)}{2} \right\},\]
\[= \frac{n_0(n_0+1)}{2} + (m+1)(n_1+1) - \frac{(n_2-m)(n_2-m-1)}{2} ;\]
set also
\[M_2 = \frac{n_0(n_0+1)}{2} + (n_0+1)(m+1) + (n_2+1)(n_1-n_0),\]
and by \( D_n(p, z) \) denote the least common multiple of the polynomials \( p - z, p^2 - z, \ldots, p^n - z \). Then the above formulae yield the inclusions

\[
p^{-M_1}z^{n_2} \cdot A(p, z) \in \mathbb{Z}[p, z], \quad p^{-M_1}z^{n_2}D_{n_1+n_2-n_0}(p,1) \cdot A'(p, z) \in \mathbb{Z}[p, z],
\]

\[
p^{-M_2}D_{n_0}(p, z) \cdot A''(p, z) \in \mathbb{Z}[p, z]
\]

(by noticing that \((p^{m-j} p^{-1})_{n_2-n_0+k} = 0\) if \( m - j - n_2 + n_0 - k \leq 0\); hence

\[
p^{-M} \tilde{D}_{n_1+n_2-n_0,m_0}(p, z) \cdot I_q(z) \in \mathbb{Z}[p, z] \ln_q(1 - z) + \mathbb{Z}[p, z],
\]

where \( M = \min\{M_1, M_2\} = M_1 \) and \( \tilde{D}_{n,m}(p, z) \) denotes a common multiple of the polynomials \( D_n(p) = D_n(p,1) \) and \( D_m(p, z) \). It is known \([\text{Ge}]\) that the polynomial \( D_n(p) \) is the product of the first \( n \) cyclotomic polynomials

\[
\Phi_l(p) = \prod_{k=1}^{l} (p - e^{2\pi i k/l}) \in \mathbb{Z}[p], \quad l = 1, 2, 3, \ldots,
\]

so that the usual choice of \( \tilde{D}_{n,m}(p, z) \) is as follows:

\[
\tilde{D}_{n,m}(p, z) = D_n(p) \cdot \prod_{j=1}^{m} (p^j - z).
\]

However, if \( z \) is a root of unity, there is a better choice instead; we discuss this type of question in Sections 3 and 4 below.

Finally, we would like to mention that the quantity \( I_q(z) \) is in fact the value of the Heine series,

\[
I_q(z) = z^{n_1+1} \cdot \left( \frac{q; q}{q} \right)_{n_1} \cdot \frac{q; q_{n_2}}{q; q_{n_1+n_2+1}} \cdot \phi_1 \left( \frac{q^{n_0+1}}{q^{n_1+n_2+2}} \mid q, q^{m+1} z \right)
\]

(see \([\text{GR}]\)), and that the construction in \([\text{MV}]\) corresponds to the following choice of the parameters: \( n_0 = n_2 = n, n_1 = n + 1, \) and \( m = K - 1 \).

3. Analytic and arithmetic valuation

Writing

\[
A(p, z) = (-1)^{n_0} p^{-n_0(n_0+1)/2 + (n_0+m+1)(n_1+1)} \times \sum_{k=0}^{n_2} (-1)^k p^{(n_0+m+1)k - k(k+1)/2} \left[ \frac{k + n_1}{n_0} \right] \left[ \frac{n_2}{k} \right] \left[ \frac{n_2}{q} \right] z^{-k}
\]

and using

\[
\max_{0 \leq k \leq n_2} \left\{ (n_0 + m + 1)k - \frac{k(k + 1)}{2} \right\} = (n_0 + m + 1)n_2 - \frac{n_2(n_2 + 1)}{2}
\]

(since \( n_0 + m + 1 > n_2 \)), we conclude that

\[
|A(p, z)| = |p|^{-n_0(n_0+1)/2 - n_2(n_2+1)/2 + (n_0+m+1)(n_1+n_2+1)+O(n_0+n_1+n_2+m)},
\]

where the constant in \( O \) depends on \( z \) only. Similarly,

\[
|I_q(z)| = |p|^{O(n_0+n_1+n_2+m)}.
\]
The general asymmetry of our construction yields the existence of a common divisor \( \Pi(p) = \Pi_{n_0,n_1,n_2}(p) \in \mathbb{Z}[p] \) of the polynomials
\[
\begin{bmatrix}
  k + n_1 \\
  n_0
\end{bmatrix}_p \begin{bmatrix}
  n_2 \\
  k
\end{bmatrix}_p, \quad k = 0, 1, \ldots, n_2,
\]
and hence of the coefficients \( A(p,z), A'(p,z), A''(p,z) \) after multiplication by \( p^{-M} \cdot \tilde{D}_{n_1+n_2-n_0,m_0}(p,z) \) in \( \mathbb{Z}[p] \). Namely, using representations
\[
\begin{bmatrix}
  k + n_1 \\
  n_0
\end{bmatrix}_p \begin{bmatrix}
  n_2 \\
  k
\end{bmatrix}_p = \frac{[n_1]_p! [n_2]_p!}{[n_0]_p! [n_1+n_2-n_0]_p!} \begin{bmatrix}
  k + n_1 \\
  k
\end{bmatrix}_p \begin{bmatrix}
  n_1 + n_2 - n_0 \\
  n_2 - k
\end{bmatrix}_p,
\]
and the knowledge that \( p \)-binomial coefficients are polynomials from \( \mathbb{Z}[p] \) having only cyclotomic polynomials as irreducible factors, we may take
\[
\Pi(p) = \prod_{l=1}^{n_1+n_2-n_0} \Phi_l(p)^{\varpi(l)},
\]
where
\[
\varpi(l) = \max \left\{ 0, \left\lfloor \frac{n_1}{l} \right\rfloor + \left\lfloor \frac{n_2}{l} \right\rfloor - \left\lfloor \frac{n_0}{l} \right\rfloor - \left\lfloor \frac{n_1+n_2-n_0}{l} \right\rfloor \right\}
\]
and \( \lfloor \cdot \rfloor \) denotes the integer part of a number (see [Z1], the proof of Lemma 5).

These arguments allow us to sharpen the inclusions (3) as follows:
\[
p^{-M} \tilde{D}_{n_1+n_2-n_0,m_0}(p,z) \cdot \Pi_{n_0,n_1,n_2}(p)^{-1} \cdot I_q(z) \in \mathbb{Z}[p,z] [\ln_q(1-z) + \mathbb{Z}[p,z]].
\]

Finally, set
\[
n_0 = \alpha_0 n, \quad n_1 = \alpha_1 n, \quad n_2 = \alpha_2 n, \quad m = \lfloor \alpha n \rfloor,
\]
where the parameter \( n \) tends to \( \infty \). Then
\[
\lim_{n \to \infty} \frac{\log |A(p,z)|}{n^2 \log |p|} = C_1, \quad \lim_{n \to \infty} \frac{\log |I_q(z)|}{n^2 \log |p|} = 0
\]
by (5), (7), and
\[
\lim_{n \to \infty} \frac{\log |p^{-M} \tilde{D}_{n_1+n_2-n_0,m_0}(p,z)^{-1} \cdot \Pi_{n_0,n_1,n_2}(p)|}{n^2 \log |p|} = C_0
\]
with the choice (5), where
\[
C_1 = C_1(\alpha) = -\frac{\alpha_0^2 + \alpha_2^2}{2} + (\alpha_0 + \alpha)(\alpha_1 + \alpha_2),
\]
\[
C_0 = C_0(\alpha) = \frac{\alpha_0^2}{2} + \alpha_1 \alpha - \frac{(\alpha_2 - \alpha)^2}{2}
\]
\[
\quad - \frac{3}{\pi^2} \left( (\alpha_1 + \alpha_2 - \alpha_0)^2 - \int_0^1 \varpi_0(x) d(-\psi'(x)) \right) - \frac{(\alpha - \alpha_2 + \alpha_0)^2}{2}
\]
and
\[
\varpi_0(x) = \max \{ 0, \lfloor \alpha_1 x \rfloor + \lfloor \alpha_2 x \rfloor - \lfloor \alpha_0 x \rfloor - \lfloor (\alpha_1 + \alpha_2 - \alpha_0) x \rfloor \}.
\]
Then \(\mu(\ln q(1-z)) \leq C_1(\alpha)/C_0(\alpha)\) provided that \(\alpha_2 - \alpha_0 \leq \alpha \leq \alpha_2\) and \(C_0(\alpha) > 0\). It is important that the parameters \(\alpha_0, \alpha_1, \alpha_2\) should be positive integers to ensure validity of the above formula for \(C_0(\alpha)\) (namely, its integration part due to [ZL], Lemma 1). Thus after making a suitable choice for these three parameters we can minimize the quantity \(C_1(\alpha)/C_0(\alpha)\) with respect to the remaining parameter \(\alpha\), which may take any (even irrational) value in the interval \(\alpha_2 - \alpha_0 \leq \alpha \leq \alpha_2\). This idea comes from [MV], and, as in that work, there is no difficulty in minimizing \(C_1(\alpha)/C_0(\alpha)\) since \(C_1(\alpha)\) depends linearly and \(C_0(\alpha)\) quadratically on the parameter \(\alpha\).

**Proof of Theorem 1.** Taking \(\alpha_0 = 6, \alpha_1 = \alpha_2 = 7\), so that \(\varpi_0(x) = 1\) for \(x \in [0,1)\) lying in the following set:

\[
[\frac{1}{7}, \frac{6}{7}) \cup [\frac{7}{7}, \frac{12}{7}) \cup [\frac{13}{7}, \frac{18}{7}) \cup [\frac{19}{7}, \frac{24}{7}) \cup [\frac{25}{7}, \frac{30}{7}),
\]

and then \(\alpha = 5.63997199\ldots\), we arrive at the estimate

\[
\mu(\ln q(1-z)) \leq 3.76338419\ldots
\]

of the theorem. \(\square\)

4. **Cyclotomic Background**

We will agree from the beginning to deal with the cyclotomic polynomials \(\Phi_l(x)\) and least common multiples \(D_n(x,z)\) and \(\hat{D}_{n,m}(x,z)\) as polynomials in the variable \(x\), and to keep the substitution \(x = p \in \mathbb{Z} \setminus \{0, \pm1\}\) for final arithmetic results. As follows from definition [4], \(\deg \Phi_l(x) = \varphi(l)\), Euler’s totient function. Therefore, the degree of the polynomial \(D_n(x) = D_n(x, 1) = \prod_{l=1}^n \Phi_l(x)\) may be computed by application of Mertens’ formula

\[
\deg D_n(x) = \sum_{1 \leq l \leq n} \varphi(l) = \frac{3}{\pi^2} n^2 + O(n \log n) \quad \text{as } n \to \infty;
\]

hence

\[
\lim_{n \to \infty} \frac{\log |D_n(p)|}{n^2 \log |p|} = \frac{3}{\pi^2}.
\]

This is the formula used in computing the right-hand side of (8). We will also require the following summation formulae for Euler’s totient function:

\[
\sum_{1 \leq j \leq n} \varphi(2j) = \frac{4}{\pi^2} n^2 + O(n \log n), \quad \sum_{0 \leq j \leq n} \varphi(2j + 1) = \frac{8}{\pi^2} n^2 + O(n \log n)
\]

as \(n \to \infty\) (for \(n\) real and not necessarily integral); see also the general formula [12] below.

**Lemma 1.** In the polynomial ring \(\mathbb{Z}[x]\) the following estimate is valid:

\[
\deg D_n(x, -1) = \frac{4}{\pi^2} n^2 + O(n \log n) \quad \text{as } n \to \infty.
\]

First proof. Since \(x^k - 1 = \prod_{l | k} \Phi_l(x)\), we have

\[
x^k + 1 = \frac{x^{2k} - 1}{x^k - 1} = \frac{\prod_{l | 2k} \Phi_l(x)}{\prod_{l | 2k} \Phi_l(x)} = \prod_{l | 2k} \Phi_l(x) = \prod_{l | k \text{ is odd}} \Phi_2(x), \quad k = 1, \ldots, n.
\]
Therefore, $x^k + 1$ divides $\prod_{l=1}^{n} \Phi_{2l}(x)$ for $k = 1, \ldots, n$ and, clearly, $\Phi_{2l}(x)$ divides $x^l + 1$ for $l = 1, \ldots, n$. Thus $D_n(x, -1) = \prod_{l=1}^{n} \Phi_{2l}(x)$ and application of the first formula in (11) leads to the desired result. \hfill \Box

Second proof. This proof follows the ideas of proving Lemma 2 in [MP]; we indicate it to make clear the ideas of proving Theorem 3 below.

For each $n > 0$ (not necessarily integral!), denote by $L_n(x)$ the least common multiple of the polynomials $x^k + 1$, where $k$ runs over positive odd integers in the interval $1 \leq k \leq n$. Since $x^k + 1 = -((-x)^k - 1) = -\prod_{|k|} \Phi_{l}(x)$ for $k$ odd, we obtain

$$L_n(x) = \prod_{1 \leq l \leq n \atop l \text{ is odd}} \Phi_{l}(-x) = \prod_{j=0}^{\lfloor n/2 \rfloor} \Phi_{2j+1}(-x);$$

hence

(13) \quad \deg L_n(x) = \frac{2}{\pi^2} n^2 + O(n \log n) \quad \text{as } n \to \infty,

by the second formula in (11). Clearly, $L_{n/2}(x^2)$ gives the least common multiple of the polynomials $x^k + 1$, where $k$ runs over positive even integers in the interval $1 \leq k \leq n$ not divisible by 4; then $L_{n/4}(x^4)$ gives the least common multiple of the polynomials $x^k + 1$, where $k \equiv 4 \pmod{8}$ runs in the interval $1 \leq k \leq n$, and so on. If exponents of 2 in the prime decompositions of the numbers $k$ and $j$ are different, then polynomials $x^k + 1$ and $x^j + 1$ have no common complex roots; hence they are coprime over $\mathbb{C}[x]$ and as a consequence over $\mathbb{Z}[x]$ as well. Therefore, we arrive at the formula

$$D_n(x, -1) = L_n(x) L_{n/2}(x^2) L_{n/4}(x^4) L_{n/8}(x^8) \cdots,$$

where the product on the right contains only a finite number $O(\log n)$ of factors, and the (almost desired) estimate for the degree of $D_n(x, -1)$,

$$\deg D_n(x, -1) = \frac{4}{\pi^2} n^2 + O(n \log^2 n) \quad \text{as } n \to \infty,$$

follows from an accurate substitution of formula (13). \hfill \Box

Corollary. If $n/2 \leq m \leq n$, then a common multiple $\hat{D}_{n,m}(x, -1)$ (over $\mathbb{Z}[x]$) of the polynomials $D_n(x)$ and $D_m(x, -1)$ may be taken in such a way that

$$\deg \hat{D}_{n,m}(x, -1) = \frac{1}{\pi^2} (2n^2 + 4m^2) + O(n \log n) \quad \text{as } n \to \infty.$$

Proof. The polynomials $x^k + 1$ for $1 \leq k \leq n/2$ divide both $D_n(x)$ and $D_m(x, -1)$. Therefore we may take

$$\hat{D}_{n,m}(x, -1) = \frac{D_n(x) D_m(x, -1)}{D_{n/2}(x, -1)},$$

and estimates (10), (12) give the desired result. \hfill \Box

Remark. The above choice of $\hat{D}_{n,m}(x, -1)$ sharpens the choice in [Z1], Lemma 8.
Proof of Theorem 2. Using the above corollary of Lemma 1 we may replace the constant $C_0$ in (9) by

$$C'_0 = C'_0(\alpha) = \frac{\alpha_0^2}{2} + \alpha_1 \alpha - \frac{(\alpha_2 - \alpha_0)^2}{2} - \frac{1}{\pi^2} \left( 2(\alpha_1 + \alpha_2 - \alpha_0)^2 + 4(\alpha - \alpha_2 + \alpha_0)^2 \right)$$

where $1 \leq \alpha \leq 2$, and $\pi(\ln(2)) \leq C_1/C'_0 \leq 2.93832530 \cdots$ obtained by using the values $\alpha_0 = 4, \alpha_1 = \alpha_2 = 5, \alpha = 4.09112737 \cdots$. In this case, $\omega_0(x) = 1$ for $x \in [0,1)$ belonging to the following set:

$$\left[ \frac{1}{5}, \frac{1}{4} \right) \cup \left[ \frac{2}{5}, \frac{1}{2} \right) \cup \left[ \frac{3}{5}, \frac{2}{3} \right) \cup \left[ \frac{4}{5}, \frac{3}{4} \right).$$

This proves Theorem 2.

5. Common multiples involving cyclotomic polynomials

The number $p$ will be used to denote a prime. We will require the asymptotic formula

$$\sum_{j=0}^{n} \varphi(rj + b) = \frac{3r}{\pi^2} n^2 \prod_{p|r} p^2 + O(n \log n) \quad \text{as } n \to \infty,$$

where $1 \leq b \leq r$ and $(b, r) = 1$ (see [Ba] and [MP]).

Proof of Theorem 3. For each $n > 0$ (not necessarily integral!) and any integer $b$ satisfying $1 \leq b \leq r$ and $(b, r) = 1$, denote by $L_{n,b}(x)$ the least common multiple of the polynomials $x^k - \omega$, where $k$ runs over integers in the interval $1 \leq k \leq n$ satisfying $k \equiv b \pmod{r}$. The polynomials $x^k - \omega$ and $x^j - \omega$, where $k$ and $j$ are integers coprime with $r$ and $k \neq j \pmod{r}$, have no common roots; hence these polynomials are coprime over $\mathbb{C}[x]$. This, in particular, yields that the $\varphi(r)$ polynomials $L_{n,b}(x)$, $1 \leq b \leq r$, $(b, r) = 1$, are pairwise coprime over $\mathbb{C}[x]$ and over $\mathbb{Z}[\omega][x] \subset \mathbb{C}[x]$ as well; hence

$$L_n(x) = \prod_{\substack{1 \leq k \leq r \\ (b, r) = 1}} L_{n,b}(x)$$

is the least common multiple of the polynomials $x^k - \omega$, where $k$ runs over integers satisfying $1 \leq k \leq n$ coprime with $r$. Having this common multiple and concluding as in the second proof of Lemma 1, we obtain

$$D_n(x, \omega) = \prod_{s_1=0}^{\infty} \cdots \prod_{s_m=0}^{\infty} L_{n,(p_1^{s_1} \cdots p_m^{s_m})}(x^{p_1^{s_1} \cdots p_m^{s_m}}),$$

where $p_1, \ldots, p_m$ are all distinct prime divisors of the number $r$. Note that, in spite of infinite products in (16), only a finite number $[O(\log n)]$ of the factors differ from 1.

In order to compute the polynomials $L_{n,b}(x)$, we start by noting the formula

$$x^{r_j + b} - \omega = \omega \left((\omega^a x)^{r_j + b} - 1\right) = \omega \prod_{d|r_j + b} \Phi_d(\omega^a x),$$
where \( ab \equiv -1 \pmod{r} \). Therefore, assigning the numbers \( b_l \) in the interval \( 1 \leq b_l \leq r \) to each \( l \), \( 1 \leq l \leq r \), \( (l, r) = 1 \), by the rule \( lb_l \equiv b \pmod{r} \) (as in [MP]) we obtain

\[
\prod_{1 \leq j \leq r} \Phi_{rj+b_l}(\omega^a x) \left| L_{n,b}(x) \right| \prod_{1 \leq j \leq r} \Phi_{rj+b_l}(\omega^a x)
\]

(where “\( \lfloor \)" means “divides”, as before); hence

\[
deg_x L_{n,b} = \sum_l \left( \sum_{j=0}^{[n/(rj)]} \phi(rj+b_l) + O(n \log n) \right)
\]

\[
= \sum_l \left( \frac{3r}{\pi^2} \left( \frac{n}{rl} \right)^2 \prod_{p|r} \frac{p^2}{p^2-1} + O(n \log n) \right)
\]

\[
= \frac{3n^2}{\pi^2r} \prod_{p|r} \frac{p^2}{p^2-1} \sum_l \frac{1}{l^2} + O(n \log n)
\]

as \( n \to \infty \), by (14). Using (15) we obtain

\[
deg_x L_n = \frac{3n^2\phi(r)}{\pi^2r} \prod_{p|r} \frac{p^2}{p^2-1} \sum_l \frac{1}{l^2} + O(n \log n)
\]

as \( n \to \infty \).

Finally, computing the degree of the polynomial \( D_n(x, \omega) \) in (16) with the help of the relation

\[
\sum_{s_1=0}^\infty \cdots \sum_{s_n=0}^\infty \frac{1}{p_1^{s_1} \cdots p_n^{s_n}} = \left( 1 - \frac{1}{p_1} \right)^{-1} \cdots \left( 1 - \frac{1}{p_n} \right)^{-1} = \frac{r}{\phi(r)}
\]

gives the desired result (2). This proves Theorem 3. \( \square \)

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