COMPUTING THE EHRHART QUASI-POLYNOMIAL
OF A RATIONAL SIMPLEX

ALEXANDER BARVINOK

Abstract. We present a polynomial time algorithm to compute any fixed number of the highest coefficients of the Ehrhart quasi-polynomial of a rational simplex. Previously such algorithms were known for integer simplices and for rational polytopes of a fixed dimension. The algorithm is based on the formula relating the $k$th coefficient of the Ehrhart quasi-polynomial of a rational polytope to volumes of sections of the polytope by affine lattice subspaces parallel to $k$-dimensional faces of the polytope. We discuss possible extensions and open questions.

1. Introduction and main results

Let $P \subset \mathbb{R}^d$ be a rational polytope, that is, the convex hull of a finite set of points with rational coordinates. Let $t \in \mathbb{N}$ be a positive integer such that the vertices of the dilated polytope

$$tP = \{tx : x \in P\}$$

are integer vectors. As is known (see, for example, Section 4.6 of [27]), there exist functions $e_i(P; n) : \mathbb{N} \rightarrow \mathbb{Q}$, $i = 0, \ldots, d$, such that

$$e_i(P; n + t) = e_i(P; n) \quad \text{for all } n \in \mathbb{N}$$

and

$$\left|nP \cap \mathbb{Z}^d\right| = \sum_{i=0}^{d} e_i(P; n)n^i \quad \text{for all } n \in \mathbb{N}.$$

The function on the right-hand side is called the Ehrhart quasi-polynomial of $P$. It is clear that if $\dim P = d$, then $e_d(P; n) = \text{vol} P$. In this paper, we are interested in the computational complexity of the coefficients $e_i(P; n)$.

If the dimension $d$ is fixed in advance, the values of $e_i(P; n)$ for any given $P$, $n$, and $i$ can be computed in polynomial time by interpolation, as implied by a polynomial time algorithm to count integer points in a polyhedron of a fixed dimension [4], [6].

If the dimension $d$ is allowed to vary, it is an NP-hard problem to check whether $P \cap \mathbb{Z}^d \neq \emptyset$, let alone to count integer points in $P$. This is true even when $P$ is a rational simplex, as exemplified by the knapsack problem; see, for example, Section 16.6 of [25]. If the polytope $P$ is integral, then the coefficients $e_i(P; n) = e_i(P)$
do not depend on \( n \). In that case, for any \( k \) fixed in advance, computation of the Ehrhart coefficient \( e_{d-k}(P) \) reduces in polynomial time to computation of the volumes of the \((d-k)\)-dimensional faces of \( P \). The algorithm is based on efficient formulas relating \( e_{d-k}(P) \), volumes of the \((d-k)\)-dimensional faces, and cones of feasible directions at those faces; see [22], [6], and [23]. In particular, if \( P = \Delta \) is an integer simplex, there is a polynomial time algorithm for computing \( e_{d-k}(\Delta) \) as long as \( k \) is fixed in advance.

In this paper, we extend the last result to rational simplices (a \( d \)-dimensional rational simplex is the convex hull in \( \mathbb{R}^d \) of \((d+1)\) affinely independent points with rational coordinates).

- Let us fix an integer \( k \geq 0 \). The paper presents a polynomial time algorithm, which, given an integer \( d \geq k \), a rational simplex \( \Delta \subset \mathbb{R}^d \), and a positive integer \( n \), computes the value of \( e_{d-k}(\Delta; n) \).

We present the algorithm in Section 7 and discuss its possible extensions in Section 8.

This is in contrast to the case of an integral polytope, for a general rational polytope \( P \) computation of \( e_i(P; n) \) cannot be reduced to computation of the volumes of faces and some functionals of the “angles” (cones of feasible direction) at the faces. A general result of McMullen [19] (see also [21] and [20]) asserts that the contribution of the \( i \)-dimensional face \( F \) of a rational polytope \( P \) to the coefficient \( e_i(P; n) \) is a function of the volume of \( F \), the cone of feasible directions of \( P \) at \( F \), and the translation class of the affine hull \( \text{aff}(F) \) of \( F \) modulo \( \mathbb{Z}^d \).

Our algorithm is based on a new structural result, Theorem 1.1 below, relating the coefficient \( e_{d-k}(P; n) \) to volumes of sections of \( P \) by affine lattice subspaces parallel to faces \( F \) of \( P \) with \( \dim F \geq d-k \). Theorem 1.1 may be of interest in its own right.

1.1. Valuations and polytopes. Let \( V \) be a \( d \)-dimensional real vector space and let \( \Lambda \subset V \) be a lattice, that is, a discrete additive subgroup which spans \( V \). A polytope \( P \subset V \) is called a \( \Lambda \)-polytope or a lattice polytope if the vertices of \( P \) belong to \( \Lambda \). A polytope \( P \subset V \) is called \( \Lambda \)-rational or just rational if \( tP \) is a lattice polytope for some positive integer \( t \).

For a set \( A \subset V \), let \([A] : V \rightarrow \mathbb{R}\) be the indicator of \( A \):

\[
[A](x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}
\]

A complex-valued function \( \nu \) on rational polytopes \( P \subset V \) is called a valuation if it preserves linear relations among indicators of rational polytopes:

\[
\sum_{i \in I} \alpha_i [P_i] = 0 \implies \sum_{i \in I} \alpha_i \nu(P_i) = 0,
\]

where \( P_i \subset V \) is a finite family of rational polytopes and \( \alpha_i \) are rational numbers. We consider only \( \Lambda \)-valuations or lattice valuations \( \nu \) that satisfy

\[
\nu(P + u) = \nu(P) \quad \text{for all } u \in \Lambda;
\]

see [21] and [20].

A general result of McMullen [19] states that if \( \nu \) is a lattice valuation, \( P \subset V \) is a rational polytope, and \( t \in \mathbb{N} \) is a number such that \( tP \) is a lattice polytope, then
there exist functions $\nu_i(P; \cdot) : \mathbb{N} \to \mathbb{C}$, $i = 0, \ldots, d$, such that

$$\nu(nP) = \sum_{i=0}^{d} \nu_i(P; n)n^i \quad \text{for all } n \in \mathbb{N}$$

and

$$\nu_i(P; n + t) = \nu_i(P; n) \quad \text{for all } n \in \mathbb{N}.$$  

Clearly, if we compute $\nu(mP)$ for $m = n, n + t, \ldots, n + td$, we can obtain $\nu_i(P; n)$ by interpolation.

We are interested in the counting valuation $E$, where $V = \mathbb{R}^d$, $\Lambda = \mathbb{Z}^d$, and

$$E(P) = |P \cap \mathbb{Z}^d|$$

is the number of lattice points in $P$.

The idea of the algorithm is to replace valuation $E$ by some other valuation, so that the coefficients $e_d(P; n), \ldots, e_{d-k}(P; n)$ remain intact, but the new valuation can be computed in polynomial time on any given rational simplex $\Delta$, so that the desired coefficient $e_{d-k}(\Delta; n)$ can be obtained by interpolation.

1.2. Valuations $E_L$. Let $L \subset \mathbb{R}^d$ be a lattice subspace, that is, a subspace spanned by the points $L \cap \mathbb{Z}^d$. Suppose that $\dim L = k$ and let $pr : \mathbb{R}^d \to L$ be the orthogonal projection onto $L$. Let $P \subset \mathbb{R}^d$ be a rational polytope, let $Q = pr(P)$, $Q \subset L$, be its projection, and let $\Lambda = pr(\mathbb{Z}^d)$. Since $L$ is a lattice subspace, $\Lambda \subset L$ is a lattice.

Let $L^\perp$ be the orthogonal complement of $L$. Then $L^\perp \subset \mathbb{R}^d$ is a lattice subspace. We introduce the volume form $\nu_{d-k}$ on $L^\perp$ which differs from the volume form inherited from $\mathbb{R}^d$ by a scaling factor chosen so that the determinant of the lattice $\mathbb{Z}^d \cap L^\perp$ is 1. Consequently, the same volume form $\nu_{d-k}$ is carried by all translations $x + L^\perp$, $x \in \mathbb{R}^d$.

We consider the following quantity

$$E_L(P) = \sum_{m \in \Lambda} \nu_{d-k}(P \cap (m + L^\perp)) = \sum_{m \in Q \cap \Lambda} \nu_{d-k}(P \cap (m + L^\perp))$$

(clearly, for $m \notin Q$ the corresponding terms are 0).

In words, we take all lattice translates of $L^\perp$, select those that intersect $P$, and add the volumes of the intersections.

Clearly, $E_L$ is a lattice valuation, so

$$E_L(nP) = \sum_{i=0}^{d} e_i(P; L; n)n^i$$

for some periodic functions $e_i(P; L; \cdot)$. If $tP$ is an integer polytope for some $t \in \mathbb{N}$, then

$$e_i(P; L; n + t) = e_i(P; L; n) \quad \text{for all } n \in \mathbb{N}$$

and $i = 0, \ldots, d$.

Note that if $L = \{0\}$, then $E_L(P) = \text{vol } P$ and if $L = \mathbb{R}^d$, then $E_L(P) = |P \cap \mathbb{Z}^d|$, so the valuations $E_L$ interpolate between the volume and the number of lattice points as dim $L$ grows.

We prove that $e_{d-k}(P; n)$ can be represented as a linear combination of $e_{d-k}(P; L; n)$ for some lattice subspaces $L$ with dim $L \leq k$.
Theorem 1.1. Let us fix an integer $k \geq 0$. Let $P \subset \mathbb{R}^d$ be a full-dimensional rational polytope and let $t$ be a positive integer such that $tP$ is an integer polytope. For a $(d-k)$-dimensional face $F$ of $P$ let $\text{lin}(F) \subset \mathbb{R}^d$ be the $(d-k)$-dimensional subspace parallel to the affine hull $\text{aff}(F)$ of $F$ and let $L^F = (\text{lin} F) \perp$ be its orthogonal complement, so $L^F \subset \mathbb{R}^d$ is a $k$-dimensional lattice subspace.

Let $\mathcal{L}$ be a finite collection of lattice subspaces which contains the subspaces $L^F$ for all $(d-k)$-dimensional faces $F$ of $P$ and is closed under intersections. For $L \in \mathcal{L}$ let $\mu(L)$ be integer numbers such that the identity

$$\left[ \bigcup_{L \in \mathcal{L}} L \right] = \sum_{L \in \mathcal{L}} \mu(L)[L]$$

holds for the indicator functions of the subspaces from $\mathcal{L}$.

Let us define

$$\nu(nP) = \sum_{L \in \mathcal{L}} \mu(L)E_L(nP) \quad \text{for } n \in \mathbb{N}.$$ 

Then there exist functions $\nu_i(P; \cdot) : \mathbb{N} \rightarrow \mathbb{Q}$, $i = 0, \ldots, d$, such that

(1) $$\nu(nP) = \sum_{i=0}^d \nu_i(P; n)n^i \quad \text{for all } n \in \mathbb{N},$$

(2) $$\nu_i(P; n + t) = \nu_i(P; n) \quad \text{for all } n \in \mathbb{N},$$

and

(3) $$e_{d-i}(P; n) = \nu_{d-i}(P; n) \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad i = 0, \ldots, k.$$ 

We prove Theorem 1.1 in Section 4 after some preparations in Sections 2 and 3.

Remark 1.2. Valuation $E$ clearly does not depend on the choice of the scalar product in $\mathbb{R}^d$. One can observe that valuation $\nu$ of Theorem 1.1 admits a dual description which does not depend on the scalar product. Instead of $\mathcal{L}$, we consider the set $\mathcal{L}'$ of subspaces containing the subspaces $\text{lin}(F)$ and closed under taking sums of subspaces, and for $L \in \mathcal{L}'$ we define $E^\vee_L(\cdot)$ as the sum of the volumes of sections of the polytope by the lattice affine subspaces parallel to $L$. Then

$$\nu = \sum_{L \in \mathcal{L}'} \mu^\vee(L)E^\vee_L,$$

where $\mu^\vee$ are some integers computed from the set $\mathcal{L}'$, partially ordered by inclusion.

However, using the explicit scalar product turns out to be more convenient.

The advantage of working with valuations $E_L$ is that they are more amenable to computations.

• Let us fix an integer $k \geq 0$. We present a polynomial time algorithm, which, given an integer $d \geq k$, a $d$-dimensional rational simplex $\Delta \subset \mathbb{R}^d$, and a lattice subspace $L \subset \mathbb{R}^d$ such that $\dim L \leq k$, computes $E_L(\Delta)$.

We present the algorithm in Section 6 after some preparations in Section 5.
1.3. The main ingredient of the algorithm to compute \(e_{d-k}(\Delta; n)\). Theorem 1.1 allows us to reduce the computation of \(e_{d-k}(\Delta; n)\) to that of \(E_L(\Delta)\), where \(L \subset \mathbb{R}^d\) is a lattice subspace and \(\dim L \leq k\). Let us choose a particular lattice subspace \(L\) with \(\dim L = j \leq k\).

If \(P = \Delta\) is a simplex, then the description of the orthogonal projection \(Q = \text{pr}(\Delta)\) of \(\Delta\) onto \(L\) can be computed in polynomial time. Moreover, one can compute in polynomial time a decomposition of \(Q\) into a union of non-intersecting polyhedral pieces \(Q_i\), such that \(\text{vol}_{d-j}(\text{pr}^{-1}(x))\) is a polynomial on each piece \(Q_i\). Thus computing of \(E_L(\Delta)\) reduces to computing of the sum

\[
\sum_{m \in Q_i \cap \Lambda} \phi(m),
\]

where \(\phi\) is a polynomial with \(\deg \phi = d - j\), \(Q_i \subset L\) is a polytope with \(\dim Q_i = j \leq k\), and \(\Lambda \subset L\) is a lattice. The sum is computed by applying the technique of “short rational functions” for lattice points in polytopes of a fixed dimension; cf. [7], [6], and [12].

The algorithm for computing the sum of a polynomial over integer points in a polytope is discussed in Section 5.

2. The Fourier expansions of \(E\) and \(E_L\)

Let \(V\) be a \(d\)-dimensional real vector space with the scalar product \(\langle \cdot, \cdot \rangle\) and the corresponding Euclidean norm \(\| \cdot \|\). Let \(\Lambda \subset V\) be a lattice and let \(\Lambda^* \subset V\) be the dual or the reciprocal lattice

\[
\Lambda^* = \{ x \in V : \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Lambda \}.\]

For \(\tau > 0\), we introduce the theta function

\[
\theta_{\Lambda}(x, \tau) = \tau^{d/2} \sum_{m \in \Lambda} \exp \left\{ -\pi \tau \|x - m\|^2 \right\} = (\det \Lambda)^{-1} \sum_{l \in \Lambda^*} \exp \left\{ -\pi \|l\|^2 / \tau + 2\pi i \langle l, x \rangle \right\}, \quad \text{where } x \in V.
\]

The last inequality is the reciprocity relation for theta series (essentially, the Poisson summation formula); see, for example, Section 69 of [9].

For a polytope \(P\), let \(\text{int} P\) denote the relative interior of \(P\) and let \(\partial P = P \setminus \text{int} P\) be the boundary of \(P\).

**Lemma 2.1.** Let \(P \subset V\) be a full-dimensional polytope such that \(\partial P \cap \Lambda = \emptyset\). Then

\[
|P \cap \Lambda| = \lim_{\tau \to +\infty} \int_P \theta_{\Lambda}(x, \tau) \, dx = (\det \Lambda)^{-1} \lim_{\tau \to +\infty} \sum_{l \in \Lambda^*} \exp \left\{ -\pi \|l\|^2 / \tau \right\} \int_P \exp \{2\pi i \langle l, x \rangle \} \, dx.
\]

**Proof.** As is known (cf., for example, Section B.5 of [17]), as \(\tau \to +\infty\), the function \(\theta_{\Lambda}(x, \tau)\) converges in the sense of distributions to the sum of the delta-functions concentrated at the points \(m \in \Lambda\). Therefore, for every smooth function \(\phi : \mathbb{R}^d \to \mathbb{R}\) with a compact support, we have

\[
\lim_{\tau \to +\infty} \int_{\mathbb{R}^d} \phi(x) \theta_{\Lambda}(x, \tau) \, dx = \sum_{m \in \Lambda} \phi(m). \tag{2.1}
\]
Since $\partial P \cap \Lambda = \emptyset$, we can replace $\phi$ by the indicator function $[P]$ in (2.1).

\[\square\]

Remark 2.2. If $\partial P \cap \Lambda \neq \emptyset$, the limit still exists but then it counts every lattice point $m \in \partial P$ with the weight equal to the “solid angle” of $m$ at $P$, since every term $\exp \left\{ -\pi \tau \|x - m\|^2 \right\}$ is spherically symmetric about $m$. This connection between the solid angle valuation and the theta function was described by the author in the unpublished paper [2] (the paper is very different from paper [5] which has the same title) and independently discovered by Diaz and Robins [13]. Diaz and Robins used a similar approach based on Fourier analysis to express coefficients of the Ehrhart polynomial of an integer polytope in terms of cotangent sums [14]. Banaszczyk [1] obtained asymptotically optimal bounds in transference theorems for lattices by using a similar approach with theta functions, with the polytope $P$ replaced by a Euclidean ball.

The formula of Lemma 2.1 can be considered as the Fourier expansion of the counting valuation.

We need a similar result for valuation $E_L$ defined in Section 1.2.

Lemma 2.3. Let $P \subset \mathbb{R}^d$ be a full-dimensional polytope and let $L \subset \mathbb{R}^d$ be a lattice subspace with $\dim L = k$. Let $pr : \mathbb{R}^d \rightarrow L$ be the orthogonal projection onto $L$, let $Q = pr(P)$, and let $\Lambda = pr(\mathbb{Z}^d)$, so $\Lambda \subset L$ is a lattice in $L$. Suppose that $\partial Q \cap \Lambda = \emptyset$.

Then

\[E_L(P) = \lim_{\tau \rightarrow +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{ -\pi \frac{\|l\|^2}{\tau} \right\} \int_P \exp \{2\pi i \langle l, x \rangle \} \, dx.\]

Proof. We observe that $L \cap \mathbb{Z}^d = \Lambda^*$. For a vector $x \in \mathbb{R}^d$, let $x_L$ be the orthogonal projection of $x$ onto $L$. Applying the reciprocity relation for theta functions in $L$, we write

\[\sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{ -\pi \frac{\|l\|^2}{\tau} + 2\pi i \langle l, x \rangle \right\} = \sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{ -\pi \frac{\|l\|^2}{\tau} + 2\pi i \langle l, x_L \rangle \right\} = (\det \Lambda)^{k/2} \sum_{m \in \Lambda} \exp \left\{ -\pi \tau \|x_L - m\|^2 \right\}.\]

As is known (cf., for example, Section B.5 of [17]), as $\tau \rightarrow +\infty$, the function

\[g_\tau(x) = \tau^{k/2} \sum_{m \in \Lambda} \exp \left\{ -\pi \tau \|x_L - m\|^2 \right\}\]

converges in the sense of distributions to the sum of the delta-functions concentrated on the subspaces $m + L^\perp$ (this is the set of points where $x_L = m$) for $m \in \Lambda$.

Therefore, for every smooth function $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$ with a compact support, we have

\[\lim_{\tau \rightarrow +\infty} \int_{\mathbb{R}^d} \phi(x) g_\tau(x) \, dx = \sum_{m \in \Lambda} \int_{m + L^\perp} \phi(x) \, d_{L^\perp} x,\]

where $d_{L^\perp} x$ is the Lebesgue measure on $m + L^\perp$ induced from $\mathbb{R}^d$. 

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Since $\partial Q \cap \Lambda = \emptyset$, each subspace $m + L^\perp$ for $m \in \Lambda$ either intersects the interior of $P$ or is at least some distance $\epsilon = \epsilon(P, L) > 0$ away from $P$. Hence we may replace $\phi$ by the indicator $[P]$ in (2.2).

Recall from Section 1.2 that measuring volumes in $m + L^\perp$, we scale the volume form in $L^\perp$ induced from $\mathbb{R}^d$ so that the determinant of the lattice $L^\perp \cap \mathbb{Z}^d$ is 1.

One can observe that $\det \Lambda$ provides the required normalization factor, so

$$(\det \Lambda) \int_{m + L^\perp} [P](x) \ d_{L^\perp}(x) = \text{vol}_{d-k} \left( P \cap (m + L^\perp) \right).$$

The proof now follows. \hfill $\square$

**Remark 2.4.** If $\partial Q \cap \Lambda \neq \emptyset$, the limit still exists, but then for $m \in \partial Q \cap \Lambda$ the volume $\text{vol}_{d-k} \left( P \cap (m + L^\perp) \right)$ is counted with the weight defined as follows: we find the minimal (under inclusion) face $F$ of $P$ such that $m + L^\perp$ is contained in $\text{aff}(F)$ and the weight is equal to the solid angle of $P$ at $F$.

### 3. Exponential valuations

Let $V$ be a $d$-dimensional Euclidean space, let $\Lambda \subset V$ be a lattice, and let $\Lambda^*$ be the reciprocal lattice. Let us choose a vector $l \in \Lambda^*$ and let us consider the integral

$$\Phi_l(P) = \int_{P} \exp \{2 \pi i (l, x)\} \ dx,$$

where $dx$ is the Lebesgue measure in $V$. Note that for $l = 0$ we have $\Phi_l(P) = \Phi_0(P) = \text{vol} P$. We have

$$\Phi_l(P + a) = \exp \{2 \pi i (l, a)\} \Phi_l(P) \quad \text{for all} \ a \in V.$$

It follows that $\Phi_l$ is a $\Lambda$-valuation on rational polytopes $P \subset V$.

If $l \neq 0$, then the following lemma (essentially, Stokes’ formula) shows that $\Phi_l$ can be expressed as a linear combination of exponential valuations on the facets of $P$. The proof can be found, for example, in [3].

**Lemma 3.1.** Let $P \subset V$ be a full-dimensional polytope. For a facet $\Gamma$ of $P$, let $d_{\Gamma}x$ be the Lebesgue measure on $\text{aff}(\Gamma)$, and let $p_\Gamma$ be the unit outer normal to $\Gamma$. Then, for every $l \in V \setminus 0$, we have

$$\int_{P} \exp \{2 \pi i (l, x)\} \ dx = \sum_{\Gamma} \frac{(l, p_\Gamma)}{2 \pi i \|l\|^2} \int_{\Gamma} \exp \{2 \pi i (l, x)\} \ d_{\Gamma}x,$$

where the sum is taken over all facets $\Gamma$ of $P$.

Let $F \subset P$ be an $i$-dimensional face of $P$. Recall that by $\text{lin}(F)$ we denote the $i$-dimensional subspace of $\mathbb{R}^d$ that is parallel to the affine hull $\text{aff}(F)$ of $F$. We need the following result.

**Theorem 3.2.** Let $P \subset V$ be a rational full-dimensional polytope and let $t$ be a positive integer such that $tP$ is a lattice polytope. Let $\epsilon \geq 0$ be a rational number and let $a \in V$ be a vector. Let us choose $l \in \Lambda^*$. Then there exist functions $f_i(P, \epsilon, a, l; \cdot) : \mathbb{N} \to \mathbb{C}$, $i = 0, \ldots, d$, such that

(1) $$\Phi_l((n + \epsilon)P + a) = \sum_{i=0}^{d} f_i(P, \epsilon, a, l; n) n^i \quad \text{for all} \ n \in \mathbb{N}$$

and
Suppose that \( f_{d-k}(P,\epsilon,a,l;n) \neq 0 \) for some \( n \). Then there exists a \((d-k)\)-dimensional face \( F \) of \( P \) such that \( l \) is orthogonal to \( \text{lin}(F) \).

Proof. Since

\[
\Phi_l(P+a) = \exp \{ 2\pi i (l,a) \} \Phi_l(P),
\]

without loss of generality we assume that \( a = 0 \). We will denote \( f_i(P,\epsilon,0,l;n) \) just by \( f_i(P,\epsilon,l;n) \).

We proceed by induction on \( d \). For \( d = 0 \) the statement of the theorem obviously holds. Suppose that \( d \geq 1 \). If \( l = 0 \), then \( \Phi_l((n+\epsilon)P) = (n+\epsilon)^d \text{vol} P \) and the statement holds as well.

Suppose that \( l \neq 0 \). For a facet \( \Gamma \) of \( P \), let \( \Lambda_\Gamma = \Lambda \cap \text{lin}(\Gamma) \) and let \( l_\Gamma \) be the orthogonal projection of \( l \) onto \( \text{lin}(\Gamma) \). Thus \( \Lambda_\Gamma \) is a lattice in the \((d-1)\)-dimensional Euclidean space \( \text{lin}(\Gamma) \) and \( l_\Gamma \in \Lambda_\Gamma \), so we can define valuations \( \Phi_{l_\Gamma} \) on \( \text{lin}(\Gamma) \). Since \( tP \) is a lattice polytope, for every facet \( \Gamma \) there is a vector \( u_\Gamma \in V \) such that

\[
\text{lin}(\Gamma) = \text{aff}(t\Gamma) - tu_\Gamma \quad \text{and} \quad tu_\Gamma \in \Lambda.
\]

Let \( \Gamma' = \Gamma - u_\Gamma \), so \( \Gamma' \subset \text{lin}(\Gamma) \) is a \( \Lambda_\Gamma \)-rational \((d-1)\)-dimensional polytope such that \( t\Gamma' \) is a \( \Lambda_\Gamma \)-polytope. We have

\[
(n+\epsilon)\Gamma = (n+\epsilon)\Gamma' + (n+\epsilon)u_\Gamma.
\]

Applying Lemma 3.1 to \((n+\epsilon)P\), we get

\[
\Phi_l((n+\epsilon)P) = \sum_{\Gamma} \psi(\Gamma,l;n) \Phi_{l_\Gamma}((n+\epsilon)\Gamma'),
\]

where

\[
\psi(\Gamma,l;n) = \frac{\langle l, p_\Gamma \rangle}{2\pi i \|l\|^2} \exp \{ 2\pi i (n+\epsilon)\langle l, u_\Gamma \rangle \}
\]

and the sum is taken over all facets \( \Gamma \) of \( P \).

Since \( tu_\Gamma \in \Lambda \) and \( l \in \Lambda^* \), we have

\[
\psi(\Gamma,l;n+t) = \psi(\Gamma,l;n) \quad \text{for all } n \in \mathbb{N}.
\]

Hence, applying the induction hypothesis, we may write

\[
f_i(P,\epsilon,l;n) = \sum_{\Gamma} \psi(\Gamma,l;n) f_i(\Gamma',\epsilon,l_\Gamma;n) \quad \text{for all } n \in \mathbb{N}
\]

and \( i = 0, \ldots, d-1 \) and \( f_d(P,\epsilon,l;n) \equiv 0 \). Hence (1)–(2) follows by the induction hypothesis.

If \( f_{d-k}(P,\epsilon,l;n) \neq 0 \), then there is a facet \( \Gamma \) of \( P \) such that \( f_{d-k}(\Gamma',\epsilon,l_\Gamma;n) \neq 0 \).

By the induction hypothesis, there is a face \( F' \) of \( \Gamma' \) such that \( \dim F' = d-k \), and \( l_\Gamma \) is orthogonal to \( \text{lin}(F') \). Then \( F = F' + u_\Gamma \) is a \((d-k)\)-dimensional face of \( P \), \( \text{lin}(F') = \text{lin}(F) \), and \( l \) is orthogonal to \( \text{lin}(F) \), which completes the proof. \( \square \)
4. Proof of Theorem 1.1

First, we discuss some ideas relevant to the proof.

4.1. Shifting a valuation by a polytope. Let \( V \) be a \( d \)-dimensional real vector space, let \( \Lambda \subset V \) be a lattice, and let \( \nu \) be a \( \Lambda \)-valuation on rational polytopes. Let us fix a rational polytope \( R \subset V \). McMullen [19] observed that the function \( \mu \) defined by

\[
\mu(P) = \nu(P + R)
\]

is a \( \Lambda \)-valuation on rational polytopes \( P \). Here “+” stands for the Minkowski sum:

\[
P + R = \{ x + y : x \in P, y \in R \}.
\]

This result follows since the transformation \( P \mapsto -\rightarrow P + R \) preserves linear dependencies among indicators of polyhedra; cf. [21].

Let \( t \) be a positive integer such that \( tP \) is a lattice polytope. McMullen [19] deduced that there exist functions \( \nu_i(P, R; \cdot) : \mathbb{N} \to \mathbb{C}, i = 0, \ldots, d \), such that

\[
\nu(nP + R) = \sum_{i=0}^{d} \nu_i(P, R; n)n^i \quad \text{for all } n \in \mathbb{N}
\]

and

\[
\nu_i(P, R; n + t) = \nu_i(P, R; n) \quad \text{for all } n \in \mathbb{N}.
\]

4.2. Continuity properties of valuations \( E \) and \( E_L \). Let \( R \subset \mathbb{R}^d \) be a full-dimensional rational polytope containing the origin in its interior. Then for every polytope \( P \subset \mathbb{R}^d \) and every \( \epsilon > 0 \) we have \( P \subset \big( P + \epsilon R \big) \). We observe that

\[
|(P + \epsilon R) \cap \mathbb{Z}^d| = |P \cap \mathbb{Z}^d|,
\]

for all sufficiently small \( \epsilon > 0 \). If \( P \) is a rational polytope, the supporting affine hyperplanes of the facets of \( nP \) for \( n \in \mathbb{N} \) are split among finitely many translation classes modulo \( \mathbb{Z}^d \). Therefore, there exists \( \delta = \delta(P, R) > 0 \) such that

\[
|(nP + \epsilon R) \cap \mathbb{Z}^d| = |nP \cap \mathbb{Z}^d| \quad \text{for all } 0 < \epsilon < \delta \quad \text{and all } n \in \mathbb{N}.
\]

We also note that for every rational subspace \( L \subset \mathbb{R}^d \), we have

\[
\lim_{\epsilon \to 0^+} E_L(P + \epsilon R) = E_L(P).
\]

We will use the perturbation \( P \mapsto P + \epsilon R \) to push valuations \( E \) and \( E_L \) into a sufficiently generic position, so that we can apply Lemmas 2.1 and 2.3 without having to deal with various boundary effects. This is somewhat similar in spirit to the idea of [8].

4.3. Linear identities for quasi-polynomials. Let us fix positive integers \( t \) and \( d \). Suppose that we have a possibly infinite family of quasi-polynomials \( p_l : \mathbb{N} \to \mathbb{C} \) of the type

\[
p_l(n) = \sum_{i=0}^{d} p_i(l; n)n^i \quad \text{for all } n \in \mathbb{N},
\]

where functions \( p_i(l; \cdot) : \mathbb{N} \to \mathbb{C}, i = 0, \ldots, d \), satisfy

\[
p_i(l; n) = p_i(l; n + t) \quad \text{for all } n \in \mathbb{N}.
\]
Suppose further that $p : \mathbb{N} \to \mathbb{C}$ is yet another quasi-polynomial

$$p(n) = \sum_{i=0}^{d} p_i(n)n^i \quad \text{where } p_i(n+t) = p_i(n) \quad \text{for all } n \in \mathbb{N}.$$ 

Finally, suppose that $c_i(\cdot) : \mathbb{R}_+ \to \mathbb{C}$ is a family of functions and that

$$p(n) = \lim_{\tau \to +\infty} \sum_{l} c_i(\tau)p_l(n) \quad \text{for all } n \in \mathbb{N}$$

and that the series converges absolutely for every $n \in \mathbb{N}$ and every $\tau > 0$.

Then we claim that for $i = 0, \ldots, d$ we have

$$p_i(n) = \lim_{\tau \to +\infty} \sum_{l} c_i(\tau)p_l(n; n) \quad \text{for all } n \in \mathbb{N}$$

and that the series converges absolutely for every $n \in \mathbb{N}$ and every $\tau > 0$.

This follows since $p_i(n)$, respectively $p_i(l; n)$, can be expressed as linear combinations of $p(m)$, respectively $p_i(m)$, for $m = n, n+t, \ldots, n+td$ with the coefficients depending on $m, n, t$, and $d$ only.

Now we are ready to prove Theorem 1.1.

4.4. Proof of Theorem 1.1. Let us fix a rational polytope $P \subset \mathbb{R}^d$ as defined in the statement of the theorem. For $L \in \mathcal{L}$ let $P_L \subset L$ be the orthogonal projection of $P$ onto $L$ and let $\Lambda_L \subset L$ be the orthogonal projection of $\mathbb{Z}^d$ onto $L$.

Let $a \in \text{int } P$ be a rational vector and let

$$R = P - a.$$ 

Hence $R$ is a rational polytope containing the origin in its interior. Let $R_L$ denote the orthogonal projection of $R$ onto $L$.

Since $P$ is a rational polytope and $\mathcal{L}$ is a finite set of rational subspaces, there exists $\delta = \delta(P, R) > 0$ such that for all $0 < \epsilon < \delta$ and all $n \in \mathbb{N}$, we have

$$\bigl(nP + \epsilon R\bigr) \cap \mathbb{Z}^d = nP \cap \mathbb{Z}^d \quad \text{and} \quad \partial\bigl(nP + \epsilon R\bigr) \cap \mathbb{Z}^d = \emptyset \quad \text{for all } n \in \mathbb{N}$$

and for all $L \in \mathcal{L}$, we have

$$(nP_L + \epsilon R_L) \cap \Lambda_L = nP_L \cap \Lambda_L \quad \text{and} \quad \partial\bigl(nP_L + \epsilon R_L\bigr) \cap \Lambda_L = \emptyset \quad \text{for all } n \in \mathbb{N};$$

cf. Section 4.2. Let us choose any rational $0 < \epsilon < \delta$.

Because of (4.1), we can write

$$\bigl\| (nP + \epsilon R) \cap \mathbb{Z}^d \bigr\| = \sum_{i=0}^{d} c_i(P; n)n^i \quad \text{for all } n \in \mathbb{N}$$

and by Lemma 2.1 we get

$$\bigl\| (nP + \epsilon R) \cap \mathbb{Z}^d \bigr\| = \lim_{\tau \to +\infty} \sum_{l \in \mathbb{Z}^d} \exp\{-\pi\|l\|^2/\tau\} \Phi_l(nP + \epsilon R),$$

where $\Phi_l$ are the exponential valuations of Section 3.

Since $\Phi_l$ is a $\mathbb{Z}^d$-valuation, by Section 4.1 there exist functions $f_i(P, \epsilon, l; \cdot) : \mathbb{N} \to \mathbb{C}, i = 0, \ldots, d$, such that

$$\Phi_l(nP + \epsilon R) = \sum_{i=0}^{d} f_i(P, \epsilon, l; n)n^i \quad \text{for } n \in \mathbb{N}$$
and

\begin{equation}
\label{eq:4.6}
f_i(P, \epsilon, l; n + t) = f_i(P, \epsilon, l; n) \quad \text{for all } n \in \mathbb{N}.
\end{equation}

Moreover, we can write

\[nP + \epsilon R = nP + (P - a) = (n + \epsilon)P - \epsilon a.\]

Therefore, by Theorem 3.2, for \(i \leq k\) we have \(f_{d-i}(P, \epsilon, l; n) = 0\) unless \(l \in L^F\) for some face \(F\) of \(P\) with \(\dim F = d - k\). Therefore, combining (4.3)–(4.6) and Section 4.3, we obtain for all \(0 \leq i \leq k\) and all \(n \in \mathbb{N}\)

\[e_{d-i}(P; n) = \lim_{\tau \to +\infty} \sum_{l \in \mathbb{Z}^d} \exp \left\{-\pi \|l\|^2/\tau \right\} f_{d-i}(P, \epsilon, l; n)
\]

\[= \lim_{\tau \to +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{-\pi \|l\|^2/\tau \right\} f_{d-i}(P, \epsilon, l; n),\]

since vectors \(l \in \mathbb{Z}^d\) outside of subspaces \(L \in \mathcal{L}\) contribute 0 to the sum. Therefore, for \(0 \leq i \leq k\) and all \(n \in \mathbb{N}\)

\begin{equation}
\label{eq:4.7}
e_{d-i}(P; n) = \lim_{\tau \to +\infty} \sum_{L \in \mathcal{L}} \mu(L) \sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{-\pi \|l\|^2/\tau \right\} f_{d-i}(P, \epsilon, l; n)
\end{equation}

On the other hand, because of (4.2), by Lemma 2.3 we get for all \(L \in \mathcal{L}\) and all \(n \in \mathbb{N}\)

\begin{equation}
\label{eq:4.8}
E_L(nP + \epsilon R) = \lim_{\tau \to +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{-\pi \|l\|^2/\tau \right\} \Phi_l(nP + \epsilon R).
\end{equation}

Since \(E_L\) are \(\mathbb{Z}^d\)-valuations, by Section 4.1 there exist functions \(e_i(P, \epsilon, L; \cdot) : \mathbb{N} \to \mathbb{Q}\), \(i = 0, \ldots, d\), such that

\begin{equation}
\label{eq:4.9}
E_L(nP + \epsilon R) = \sum_{i=0}^{d} e_i(P, \epsilon, L; n)n^i \quad \text{for all } n \in \mathbb{N}
\end{equation}

and

\begin{equation}
\label{eq:4.10}
e_i(P, \epsilon, L; n + t) = e_i(P, \epsilon, L; n) \quad \text{for all } n \in \mathbb{N}.
\end{equation}

Combining (4.5)–(4.6) and (4.8)–(4.10), by Section 4.3 we conclude

\[e_{d-i}(P, \epsilon, L; n) = \lim_{\tau \to +\infty} \sum_{l \in L \cap \mathbb{Z}^d} \exp \left\{-\pi \|l\|^2/\tau \right\} f_{d-i}(P, \epsilon, l; n) \quad \text{for all } n \in \mathbb{N}.
\]

Therefore, by (4.7), for \(0 \leq i \leq k\) we have

\begin{equation}
\label{eq:4.11}
e_{d-i}(P; n) = \sum_{L \in \mathcal{L}} \mu(L)e_{d-i}(P, \epsilon, L; n) \quad \text{for all } n \in \mathbb{N}.
\end{equation}

Since \(E_L\) is a \(\mathbb{Z}^d\)-valuation, there exist functions \(e_i(P, L; \cdot) : \mathbb{N} \to \mathbb{Q}\), \(i = 0, \ldots, d\), such that

\begin{equation}
\label{eq:4.12}
E_L(nP) = \sum_{i=0}^{d} e_i(P, L; n)n^i \quad \text{for all } n \in \mathbb{N}
\end{equation}

and

\[e_i(P, L; n + t) = e_i(P, L; n) \quad \text{for all } n \in \mathbb{N}.
\]

Let us choose an \(m \in \mathbb{N}\). Substituting \(n = m, m + t, \ldots, m + td\) in (4.12), we obtain \(e_i(P, L; m)\) as a linear combination of \(E_L(nP)\) with coefficients depending on \(n, m,\)
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t, and d only. Similarly, substituting \( n = m, m + t, \ldots, m + td \) in (4.9), we obtain \( e_i(P, \epsilon, L; m) \) as the same linear combination of \( E_L(nP + \epsilon R) \). Since volumes are continuous functions, in view of (4.2) (see also Section 4.2), we get

\[
\lim_{\epsilon \to 0^+} E_L(nP + \epsilon R) = E_L(nP) \quad \text{for } n = m, m + t, \ldots, m + td.
\]

Therefore,

\[
\lim_{\epsilon \to 0^+} e_i(P, \epsilon, L; m) = e_i(P, L; m) \quad \text{for all } m \in \mathbb{N}.
\]

Taking the limit as \( \epsilon \to 0^+ \) in (4.11), we obtain for \( 0 \leq i \leq k \)

\[
e_{d-i}(P; n) = \sum_{L \in \mathcal{L}} \mu(L) e_{d-i}(P, L; n) \quad \text{for all } n \in \mathbb{N}.
\]

To complete the proof, we note that

\[
\nu_{d-i}(P, L; n) = \sum_{L \in \mathcal{L}} \mu(L) e_{d-i}(P, L; n).
\]

5. Summing up a polynomial over integer points in a rational polytope

Let us fix a positive integer \( k \) and let us consider the following situation. Let \( Q \subset \mathbb{R}^k \) be a rational polytope, let \( \text{int} \ Q \) be the relative interior of \( Q \), and let \( f : \mathbb{R}^k \to \mathbb{R} \) be a polynomial with rational coefficients. We want to compute the value

\[
\sum_{m \in \text{int} \ Q \cap \mathbb{Z}^k} f(m).
\]

We claim that as soon as the dimension \( k \) of the polytope \( Q \) is fixed, there is a polynomial time algorithm to do that. We assume that the polytope \( Q \) is given by the list of its vertices and the polynomial \( f \) is given by the list of its coefficients.

For an integer point \( m = (\mu_1, \ldots, \mu_k) \), let

\[
x^m = x_1^{\mu_1} \cdots x_k^{\mu_k} \quad \text{for } x = (x_1, \ldots, x_k)
\]

be the Laurent monomial in \( k \) variables \( x = (x_1, \ldots, x_k) \). We use the following result [6].

5.1. The short rational function algorithm. Let us fix \( k \). There is a polynomial time algorithm, which, given a rational polytope \( Q \subset \mathbb{R}^k \), computes the generating function (Laurent polynomial)

\[
S(Q; x) = \sum_{m \in \text{int} \ Q \cap \mathbb{Z}^k} x^m
\]

in the form

\[
S(Q; x) = \sum_{i \in I} \frac{x^{a_i}}{(1 - x^{b_{i1}}) \cdots (1 - x^{b_{ik}})}.
\]

where \( a_i \in \mathbb{Z}^k \), \( b_{ij} \in \mathbb{Z}^k \setminus \{0\} \), and \( \epsilon_i \in \mathbb{Q} \). In particular, the number \( |I| \) of fractions is bounded by a polynomial in the input size of \( Q \).

Our first step is computing the generating function

\[
S(Q, f; x) = \sum_{m \in \text{int} \ Q \cap \mathbb{Z}^k} f(m)x^m.
\]
Our approach is similar to that of [12], although we obtain better complexity bounds (our algorithm is polynomial in deg \( f \) whereas the algorithm of [12] is exponential in deg \( f \)).

5.2. The algorithm for computing \( S(Q, f; x) \). We observe that

\[
S(Q, f; x) = f \left( x_1 \frac{\partial}{\partial x_1}, \ldots, x_k \frac{\partial}{\partial x_k} \right) S(Q; x).
\]

We compute \( S(Q; x) \) as in Section 5.1.

Let \( a = (\alpha_1, \ldots, \alpha_k) \) be an integer vector, let \( b_j = (\beta_{j1}, \ldots, \beta_{jk}) \) be non-zero integer vectors for \( j = 1, \ldots, k \), and let \( \gamma_1, \ldots, \gamma_k \) be positive integers. Then

\[
\left( x_1 \frac{\partial}{\partial x_1} \right)^{\alpha_1} \frac{x^a}{(1 - x^{b_1})^{\gamma_1} \cdots (1 - x^{b_k})^{\gamma_k}} = \alpha_1 \frac{x^a}{(1 - x^{b_1})^{\gamma_1} \cdots (1 - x^{b_k})^{\gamma_k}} + \sum_{j=1}^{k} \frac{\gamma_j \beta_{j1}}{\gamma_j + 1} \prod_{s \neq j} \frac{1}{(1 - x^{b_s})^{\gamma_s}}.
\]

Consecutively applying the above formula and collecting similar fractions, we compute

\[
f \left( x_1 \frac{\partial}{\partial x_1}, \ldots, x_k \frac{\partial}{\partial x_k} \right) \frac{x^a}{(1 - x^{b_1}) \cdots (1 - x^{b_k})}
\]
as an expression of the type

\[
\sum_j \rho_j \frac{x^{a_j}}{(1 - x^{b_{1j}}) \cdots (1 - x^{b_{kj}})^{\gamma_{jk}}},
\]

where \( \rho_j \in \mathbb{Q} \), \( \gamma_{j1}, \ldots, \gamma_{jk} \) are non-negative integers satisfying \( \gamma_{j1} + \cdots + \gamma_{jk} \leq k + \text{deg} \ f \) and \( a_j \) are vectors of the type

\[
a_j = a + \mu_1 b_1 + \cdots + \mu_k b_k,
\]

where \( \mu_i \) are non-negative integers and \( \mu_1 + \cdots + \mu_k \leq \text{deg} \ f \). The number of terms in (5.2) is bounded by \( (\text{deg} \ f)^O(k) \), which shows that for a \( k \) fixed in advance, the algorithm runs in polynomial time.

Consequently, \( S(Q, f; x) \) is computed in polynomial time.

Formally speaking, to compute the sum (5.1), we have to substitute \( x_i = 1 \) into the formula for \( S(Q, f; x) \). This, however, cannot be done in a straightforward way since \( x = (1, \ldots, 1) \) is a pole of every fraction in the expression for \( S(Q, f; x) \). Nevertheless, the substitution can be done via efficient computation of the relevant residue of \( S(Q, f; x) \) as described in [4] and [7].

5.3. The algorithm for computing the sum. The output of Algorithm 5.2 represents \( S(Q, f; x) \) in the general form

\[
S(Q, f; x) = \sum_{i \in I} \epsilon_i \frac{x^{a_i}}{(1 - x^{b_{i1}}) \cdots (1 - x^{b_{ik}})^{\gamma_{ik}}},
\]

where \( \epsilon_i \in \mathbb{Q}, a_i \in \mathbb{Z}^k, b_{ij} \in \mathbb{Z}^k \setminus \{0\}, \) and \( \gamma_{ij} \in \mathbb{N} \) are such that \( \gamma_{i1} + \cdots + \gamma_{ik} \leq k + \text{deg} \ f \) for all \( i \in I \).

Let us choose a vector \( l \in \mathbb{Q}^k, l = (\lambda_1, \ldots, \lambda_k) \), such that \( (l, b_{ij}) \neq 0 \) for all \( i, j \) (such a vector can be computed in polynomial time; cf. [4]). For a complex \( \tau \), let

\[
x(\tau) = (e^{\tau \lambda_1}, \ldots, e^{\tau \lambda_k}).
\]
We want to compute the limit
\[ \lim_{\tau \to 0} G(\tau) \] for \( G(\tau) = S(Q, f; x(\tau)) \).
In other words, we want to compute the constant term of the Laurent expansion of \( G(\tau) \) around \( \tau = 0 \).

Let us consider a typical fraction
\[ \frac{x^a}{(1 - x^{b_1})^{\gamma_1} \cdots (1 - x^{b_k})^{\gamma_k}}. \]
Substituting \( x(\tau) \), we get the expression
\[ (5.3) \quad e^{\alpha \tau} (1 - e^{\tau \beta_1})^{\gamma_1} \cdots (1 - e^{\tau \beta_k})^{\gamma_k}, \]
where \( \alpha = \langle a, l \rangle \) and \( \beta_i = \langle b_i, l \rangle \) for \( i = 1, \ldots, k \). The order of the pole at \( \tau = 0 \) is \( D = \gamma_1 + \cdots + \gamma_k \leq k + \deg f \). To compute the constant term of the Laurent expansion of (5.3) at \( \tau = 0 \), we do the following.

We compute the polynomial
\[ q(\tau) = \sum_{i=0}^{D} \frac{\alpha^i}{i!} \tau^i \]
that is the truncation at \( \tau^D \) of the Taylor series expansion of \( e^{\alpha \tau} \). For \( i = 1, \ldots, k \) we compute the polynomial \( p_i(\tau) \) with \( \deg p_i = D \) such that
\[ \tau \frac{1}{1 - e^{\tau \beta_i}} = p_i(\tau) + \text{terms of higher order in } \tau \]
at \( \tau = 0 \). Consecutively multiplying polynomials mod \( \tau^{D+1} \), we compute a polynomial \( u(\tau) \) with \( \deg u = D \) such that
\[ q(\tau) p_1^{\gamma_1}(\tau) \cdots p_k^{\gamma_k}(\tau) \equiv u(\tau) \mod \tau^{D+1}. \]
The coefficient of \( \tau^D \) in \( u(\tau) \) is the desired constant term of the Laurent expansion.

6. Computing \( E_L(\Delta) \)

Let us fix a positive integer \( k \). Let \( \Delta \subset \mathbb{R}^d \) be a rational simplex given by the list of its vertices and let \( L \subset \mathbb{R}^d \) be a rational subspace given by its basis and such that \( \dim L = k \). In this section, we describe a polynomial time algorithm for computing the value of \( E_L(\Delta) \) as defined in Section 1.2.

Let \( pr : \mathbb{R}^d \rightarrow L \) be the orthogonal projection. We compute the vertices of the polytope \( Q = pr(\Delta) \) and a basis of the lattice \( \Lambda = pr(\mathbb{Z}^d) \). For basic lattice algorithms see [25] and [16].

As is known, as \( x \in \Delta \) varies, the function
\[ \phi(x) = \text{vol}_{d-k}(P_x) \quad \text{where } P_x = (\Delta \cap (x + L^\perp)) \]
is a piecewise polynomial on \( Q \). Our first step consists of computing a decomposition
\[ (6.1) \quad Q = \bigcup_i C_i \]
such that \( C_i \subset Q \) are rational polytopes (chambers) with pairwise disjoint interiors and polynomials \( \phi_i : L \rightarrow \mathbb{R} \) such that \( \phi_i(x) = \phi(x) \) for \( x \in C_i \).

We observe that every vertex of \( P_x \) is the intersection of \( x + L^\perp \) and some \( k \)-dimensional face \( F \) of \( \Delta \).
For every face $G$ of $\Delta$ with $\dim G = k - 1$ and such that $\text{aff}(G)$ is not parallel to $L^\perp$, let us compute

$$A_G = \{ x \in L : x + L^\perp \cap \text{aff}(G) \neq \emptyset \}.$$ 

Then $A_G$ is an affine hyperplane in $L$. The number of different hyperplanes $A_G$ is $dO(k)$ and hence they cut $Q$ into at most $dO(k^2)$ polyhedral chambers $C_i$; cf. Section 6.1 of [18]. As long as $x$ stays within the relative interior of a chamber $C_i$, the strong combinatorial type of $P_x$ does not change (the facets of $P_x$ move parallel to themselves) and hence the restriction $\phi_i$ of $\phi$ onto $C_i$ is a polynomial; cf. Section 5.1 of [24]. Since in the $(d-k)$-dimensional space $x + L^\perp$ the polytope $P_x$ is defined by $d$ linear inequalities, $\phi_i$ can be computed in polynomial time; see [15] and [3].

The decomposition (6.1) gives rise to the formula

$$|Q| = \sum_j |Q_j|,$$

where $Q_j$ are open faces of the chambers $C_i$ (the number of such faces is bounded by a polynomial in $d$); cf. Section 6.1 of [18]. Hence we have

$$E_L(\Delta) = \sum_j \sum m \in Q_j \cap \Lambda \phi(m).$$

Each inner sum is the sum of a polynomial over lattice points in a polytope of dimension at most $k$. By a change of the coordinates, it becomes the sum over integer points in a rational polytope and we compute it as described in Section 5.

7. Computing $e_{d-k}(\Delta; n)$

Let us fix an integer $k \geq 0$. We describe our algorithm, which, given a positive integer $d \geq k$, a rational simplex $\Delta \subset \mathbb{R}^d$ (defined, for example, by the list of its vertices), and a positive integer $n$, computes the number $e_{d-k}(\Delta; n)$.

We use Theorem 1.1.

7.1. Computing the set $L$ of subspaces. We compute subspaces $L$ and numbers $\mu(L)$ described in Theorem 1.1. Namely, for each $(d-k)$-dimensional face $F$ of $\Delta$, we compute a basis of the subspace $L^F = (\text{lin } F)^\perp$. Hence $\dim L^F \leq k$. Clearly, the number of distinct subspaces $L^F$ is $dO(k)$. We let $L$ be the set consisting of the subspaces $L^F$ and all other subspaces obtained as intersections of $L^F$. We compute $L$ in $k$ (or fewer) steps. Initially, we let

$$L := \{ L^F : F \text{ is a } (d-k)\text{-dimensional face of } \Delta \}.$$ 

Then, at every step, we consider the previously constructed subspaces $L \in L$, consider the pairwise intersections $L \cap L^F$ as $F$ ranges over the $(d-k)$-dimensional faces of $\Delta$, and add the obtained subspace $L \cap L^F$ to the set $L$ if it is not already there. If no new subspaces are obtained, we stop. Clearly, in the end of this process, we will obtain all subspaces $L$ that are intersections of different $L^F$. Since $\dim L^F = k$, each subspace $L \in L$ is an intersection of some $k$ subspaces $L^{F_i}$. Hence the process stops after $k$ steps and the total number $|L|$ of subspaces is $dO(k^2)$.

Having computed the subspaces $L \in L$, we compute the numbers $\mu(L)$ as follows.
For each pair of subspaces $L_1, L_2 \in \mathcal{L}$ such that $L_1 \subset L_2$, we compute the number $\mu(L_1, L_2)$ recursively: if $L_1 = L_2$, we let $\mu(L_1, L_2) = 1$. Otherwise, we let
\[
\mu(L_1, L_2) = - \sum_{L \in \mathcal{L} \mid L \subset L_2 \subset L_1 \neq L_2} \mu(L_1, L).
\]
In the end, for each $L \in \mathcal{L}$, we let
\[
\mu(L) = \sum_{L \subset L_1 \in \mathcal{L}} \mu(L, L_1).
\]

Hence $\mu(L_i, L_j)$ are the values of the Möbius function on the set $\mathcal{L}$ partially ordered by inclusion, so
\[
\mu(L)\left[\bigcup_{L \in \mathcal{L}} L\right] = \sum_{L \in \mathcal{L}} \mu(L)[L]
\]
follows from the Möbius inversion formula; cf. Section 3.7 of [27].

Now, for each $L \in \mathcal{L}$ and $m = n, n + t, \ldots, n + td$ we compute the values of $E_L(m\Delta)$ as in Section 6, compute
\[
\nu(m\Delta) = \sum_{L \in \mathcal{L}} \mu(L)E_L(m\Delta),
\]
and find $\nu_{d-k}(\Delta; n) = e_{d-k}(\Delta, n)$ by interpolation.

8. Possible extensions and further questions

8.1. Computing more general expressions. Let $P \subset \mathbb{R}^d$ be a rational polytope, let $\alpha \geq 0$ be a rational number, and let $u \in \mathbb{R}^d$ be a rational vector. One can show (cf. Section 4.1) that
\[
\frac{1}{\mathbb{Z}^d}\left((n + \alpha)P + u\right) \cap \mathbb{Z}^d = \sum_{i=0}^{d} e_i(P, \alpha, u; n) n^i \quad \text{for all } n \in \mathbb{N},
\]
where $e_i(P, \alpha, u; \cdot) : \mathbb{N} \rightarrow \mathbb{Q}$, $i = 0, \ldots, d$, satisfy
\[
e_i(P, \alpha, u; n + t) = e_i(P, \alpha, u; n) \quad \text{for all } n \in \mathbb{N},
\]
provided $t \in \mathbb{N}$ is a number such that $tP$ is an integer polytope. As long as $k$ is fixed in advance, for given $\alpha$, $u$, $n$, and a rational simplex $\Delta \subset \mathbb{R}^d$, one can compute $e_{d-k}(\Delta, \alpha; u; n)$ in polynomial time. Similarly, Theorem 1.1 and its proof extend to this more general situation in a straightforward way.

8.2. Computing the generating function. Let $P \subset \mathbb{R}^d$ be a rational polytope. Then, for every $0 \leq i \leq d$, the series
\[
\sum_{n=1}^{+\infty} e_i(P; n) t^n
\]
converges to a rational function $f_i(P; t)$ for $|t| < 1$.

It is not clear whether $f_{d-k}(\Delta; t)$ can be efficiently computed as a “closed form expression” in any meaningful sense, although it seems that by adjusting the methods of Sections 5–7, for any given $t$ such that $|t| < 1$ one can compute the value of $f_{d-k}(\Delta; t)$ in polynomial time (again, $k$ is assumed to be fixed in advance).
8.3. Extensions to other classes of polytopes. If \( k \) is fixed in advance, the coefficient \( e_{d-k}(P;n) \) can be computed in polynomial time, if the rational polytope \( P \subset \mathbb{R}^d \) is given by the list of its \( d+c \) vertices or the list of its \( d+c \) inequalities, where \( c \) is a constant fixed in advance. A similar result holds for rational parallelepipeds \( P \), that is, for Minkowski sums of \( d \) rational intervals that do not lie in the same affine hyperplane in \( \mathbb{R}^d \).

8.4. Possible applications to integer programming and integer point counting. If \( P \subset \mathbb{R}^m \) is a rational polytope given by the list of its defining linear inequalities, the problem of testing whether \( P \cap \mathbb{Z}^m = \emptyset \) is a typical problem of integer programming; see [16] and [25]. Moreover, a general construction of “aggregation” (see Section 16.6 of [25] and Section 2.2 of [29]) establishes a bijection between the sets \( P \cap \mathbb{Z}^m \) and \( \Delta \cap \mathbb{Z}^d \) provided \( P \) is defined by \( d+1 \) linear inequalities. Here \( \Delta \subset \mathbb{R}^d \) is a rational simplex whose definition is computable in polynomial time from that of \( P \). It would be interesting to find out whether approximating valuation \( E \) by valuation \( \nu \) of Theorem 1.1 for some \( k \ll d \) and applying the algorithm of this paper to compute \( \nu(\Delta) \) can be of any practical use to solve higher-dimensional integer programs and integer point counting problems. It could complement existing software packages [11] and [10] based on the “short rational functions” calculus.

References


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48109-1043

E-mail address: barvinok@umich.edu