LOWER BOUNDS FOR THE CONDITION NUMBER OF A REAL CONFLUENT VANDERMONDE MATRIX

REN-CANG LI

Abstract. Lower bounds on the condition number $\kappa_p(V_c)$ of a real confluent Vandermonde matrix $V_c$ are established in terms of the dimension $n$, or $n$ and the largest absolute value among all nodes that define the confluent Vandermonde matrix and the interval that contains the nodes. In particular, it is proved that for any modest $k_{\text{max}}$ (the largest multiplicity of distinct nodes), $\kappa_p(V_c)$ behaves no smaller than $O(n((1+\sqrt{2})^n))$, or than $O(n((1+\sqrt{2})^{2n})$ if all nodes are nonnegative. It is not clear whether those bounds are asymptotically sharp for modest $k_{\text{max}}$.

1. Introduction

Given $n$ numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ called nodes, the associated Vandermonde matrix is defined as

$$V \overset{\text{def}}{=} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}.$$

It, for example, arises from polynomial interpolation and others. $V$ is invertible if all nodes $\alpha_i$ are distinct, i.e., $\alpha_i \neq \alpha_j$ for $i \neq j$, but it becomes singular whenever $\alpha_i = \alpha_j$ for some $i \neq j$. A generalization of $V$ for nodes not all of which are distinct is the so-called confluent Vandermonde matrices, e.g.,

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ \alpha_1 & 1 & 0 & \alpha_4 & \alpha_5 & 1 \\ \alpha_1^2 & 2\alpha_1 & 2 & \alpha_4^2 & \alpha_5^2 & 2\alpha_5 \\ \alpha_1^3 & 3\alpha_1^2 & 6\alpha_1 & \alpha_4^3 & \alpha_5^3 & 3\alpha_5^2 \\ \alpha_1^4 & 4\alpha_1^3 & 12\alpha_1^2 & \alpha_4^4 & \alpha_5^4 & 4\alpha_5^3 \\ \alpha_1^5 & 5\alpha_1^4 & 20\alpha_1^3 & 20\alpha_1^2 & \alpha_4^5 & 5\alpha_5^4 \end{pmatrix},$$

where $\alpha_1 = \alpha_2 = \alpha_3$ and $\alpha_5 = \alpha_6$. The second, third, and sixth columns are obtained by “differentiating” the previous column. Confluent Vandermonde matrices

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arise in Hermite interpolation [1], for example. Adopting the formulation in [8], we define the confluent Vandermonde matrix $V_c$ as follows. First

\begin{equation}
\{\alpha_j\}_{j=1}^{n} \text{ are ordered so that equal nodes are contiguous, i.e., } \alpha_i = \alpha_j \quad (i < j) \implies \alpha_i = \alpha_{i+1} = \cdots = \alpha_j.
\end{equation}

Define

\begin{equation}
V_c = (f_1(\alpha_1) \ f_2(\alpha_2) \ \cdots \ f_n(\alpha_n)),
\end{equation}

where the vector function $f_j(t)$ is defined recursively by

\begin{equation}
f_j(t) = \begin{cases}
(1 \ t \ \cdots \ t^{n-1})^T, & \text{if } j = 1 \text{ or } \alpha_j \neq \alpha_{j-1}, \\
\frac{d}{dt} f_{j-1}(t), & \text{otherwise},
\end{cases}
\end{equation}

where ".$^T$" is the transpose of a vector or matrix. As far as defining $V_c$ is concerned, $\alpha_j$ can be real or complex. But in this paper, we shall focus on real $\alpha_j$. In what follows, $\alpha_j$ and $V_c$, as well as

\[ \alpha_{\max} \overset{\text{def}}{=} \max_j |\alpha_j|, \]

are reserved for their assignments here.

(Optimal) condition numbers for real Vandermonde matrices have been systematically studied by Gautschi and his coauthor (see [7] and references therein), and more recently by Tyrtyshnikov [12], Beckermann [2], and Li [10]. In this paper, we shall establish three lower bounds on the condition number $\kappa_p(V_c) \equiv \|V_c\|_p\|V_c^{-1}\|_p$ in terms of $n$, or $n$ and $\alpha_{\max}$ and the interval $[\alpha, \beta]$ that contains all nodes. In particular, we will show that for fixed $k_{\max}$ (the largest multiplicity of distinct nodes), $\kappa_p(V_c)$ behaves no smaller than $\mathcal{O}_n((1 + \sqrt{2})^n)$, where notation $a_n = \mathcal{O}_n(b_n)$ means $c_1 n^{d_1} \leq a_n / b_n \leq c_2 n^{d_2}$ for some constants $c_1$, $c_2$, $d_1$, and $d_2$.

Optimally conditioned confluent Vandermonde matrices can be much worse ill-conditioned than optimally conditioned Vandermonde matrices. One extreme example would be that all nodes are equal $\alpha_1 = \cdots = \alpha_n$ for which $V_c$ is lower triangular, and thus

\[ \kappa_p(V_c) \geq (n-1)! \sim \sqrt{2\pi n^{1/2}} e^{-n} \]

by Stirling’s asymptotic formula [11, Page 18], and it becomes an equality for $\alpha_1 = \cdots = \alpha_n = 0$. While for optimally conditioned Vandermonde matrices, $\kappa_p(V)$ goes to $\infty$ as fast as $(1 + \sqrt{2})^n$ modulo a factor $n^d$ for $|d| \leq 1$ [2, 10].

The rest of this paper is organized as follows. A general lower bound on $\kappa_p(V_c)$ is established in Section 2 but it is not uniform. Uniform bounds for $p = \infty$ are obtained in Section 3 for all real $V_c$ and for $V_c$ with nonnegative nodes. Finally we present our concluding remarks in Section 4.

2. A GENERAL LOWER BOUND

Given $1 \leq p \leq \infty$, the $\ell_p$-norm of vector $u = (\mu_1 \ \mu_2 \ \cdots \ \mu_n)^T$ is defined as

\[ \|u\|_p = \left( \sum_{j=1}^{n} |\mu_j|^p \right)^{1/p}, \]
and \(\|u\|_\infty = \lim_{p \to \infty} \|u\|_p = \max_j |\mu_j|\). The associated \(\ell_p\)-operator norm of the \(m \times n\) matrix \(A\) is defined as

\[
\|A\|_p = \max_{u \neq 0} \frac{\|Au\|_p}{\|u\|_p}
\]

(2.1)

It can be proved that \(\|A\|_p = \|A^T\|_{p'}\), upon noticing

\[
\|A\|_p = \max_{u \neq 0, v \neq 0} \frac{|u^T Av|}{\|u\|_p \|v\|_{p'}}
\]

where \(1/p + 1/p' = 1\) (see also [9]).

Let \([\alpha, \beta]\) be the interval in which all \(\alpha_j\) lie.

\[
T_n(t) = \cos(n \arccos t) \quad \text{for } |t| \leq 1,
\]

(2.2)

\[
\frac{1}{2} \left( t + \sqrt{t^2 - 1} \right)^n + \frac{1}{2} \left( t - \sqrt{t^2 - 1} \right)^n \quad \text{for } |t| \geq 1
\]

(2.3)

is the \(n\)th Chebyshev polynomial of the first kind. Define the \(n\)th translated Chebyshev polynomial \(T_n(x; \omega, \tau) \equiv T_n(x/\omega + \tau)\), where

\[
\omega = \frac{\beta - \alpha}{2} > 0, \quad \tau = \frac{\beta - \alpha}{\beta + \alpha}
\]

Let \(a_{jn} \equiv a_{jn}(\omega, \tau)\) be the coefficient of \(x^j\) in \(T_n(x; \omega, \tau)\), i.e.,

\[
T_n(x; \omega, \tau) = a_{nn}x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.
\]

(2.4)

Define [10]

\[
S_{n,p}(\omega, \tau) = \left( \sum_{j=0}^{n} |a_{jn}|^p \right)^{1/p}.
\]

Now we are ready to state our main theorem for the section.

**Theorem 2.1.** Assume that there are \(\ell\) distinct nodes \(\alpha_j\), and let \(k_{\max}\) be the largest multiplicity of the distinct nodes. Then

\[
\kappa_p(V_\xi) \geq \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \omega^{k-1} \times \max\{\ell/p', \alpha_{\max}^{n-1} / n^{1/p'} \}
\]

(2.5)

**Proof.** Inequality (2.5) is a consequence of Lemmas [2.1] and [2.3] below. \(\square\)

For \(k_{\max} = 1\), i.e., \(\ell = n\) and \(k_1 = \cdots = k_n = 1\) (and thus \(V_\xi = V\)), (2.5) becomes one of the lower bounds for \(\kappa_p(V)\) in [10]. The right-hand side of (2.5) entails the explicit computation of \(S_{n,p}(\omega, \tau)\). It can also be estimated fairly well, too, by

\[
n^{-1/p}S_{n-1,1}(\omega, \tau) \leq S_{n-1,p}(\omega, \tau) \leq S_{n-1,1}(\omega, \tau),
\]

(2.6)

\[
[n/2]^{-1/p}S_{n-1,1}(\omega, 0) \leq S_{n-1,p}(\omega, 0) \leq S_{n-1,1}(\omega, 0),
\]

(2.7)

in connection with the explicit formulas for \(S_{n-1,1}(\omega, \tau)\) for \(\tau = 0\) or \(|\tau| \geq 1\) in [10]. Here \(|\xi|\) is the smallest integer that is larger than \(\xi\). The formulas are

\[
S_{n-1,1}(\omega, 0) = T_{n-1}(t/\omega) \sim \frac{1}{2} \left( \frac{1}{\omega} + \sqrt{1 + \frac{1}{\omega^2}} \right)^{n-1},
\]

(2.8)
where \( \ell = \sqrt{-1} \), and for \( \alpha \geq 0 \) (for which \( \tau \leq -1 \),

\[
S_{n-1,1}(\omega, \tau) = T_{n-1}(|\tau| + 1/\omega) \sim \frac{1}{2} \left[ \left( \frac{1}{\omega} + |\tau| \right) + \sqrt{\left( \frac{1}{\omega} + |\tau| \right)^2 - 1} \right]^{n-1}.
\]

**Lemma 2.1.** Assume that there are \( \ell \) distinct nodes \( \alpha_j \). Then

\[
\|V_c\|_p \geq \max \left\{ \ell^{1/p'}, \alpha_{\text{max}}^{n-1} \right\},
\]

\[
\|V_c\|_p \geq \left( \sum_{j=1}^{n} \alpha_{\text{max}}^{(j-1)p} \right)^{1/p}.
\]

**Proof.** Let \( c_j \) be the \( j \)th column of the \( n \times n \) identity matrix \( I_n \) (or simply \( I \) if \( n \) is clear from the context). Use (2.10), and use \( \|V_c\|_p \geq \|V_c^T e_1\|_{p'} \) and \( \|V_c\|_p \geq \|V_c^T e_n\|_{p'} \) to get (2.10), and use \( \|V_c\|_p \geq \max_j \|V_c^T e_j\|_{p'} \) to get (2.11). \( \square \)

**Lemma 2.2.** For \( 0 \leq k \leq n \),

\[
\left| \frac{d}{dx} T_n(x; \omega, \tau) \right| \leq \frac{n(n-1)\cdots(n-k+1)^2}{\omega^k} \text{ for } x \in [\alpha, \beta].
\]

**Proof.** It follows from \( T_n(x; \omega, \tau) = T_n(x/\omega + \tau) \equiv T_n(t) \) that

\[
\frac{d^k}{dx^k} T_n(x; \omega, \tau) = \frac{1}{\omega^k} T_n^{(k)}(t),
\]

where \( t \equiv t(x) = x/\omega + \tau \). It suffices to show that \( |T_n^{(k)}(t)| \leq |n(n-1)\cdots(n-k+1)|^2 \) for \( t \in [-1,1] \) since \( t(x) \) maps \( x \in [\alpha, \beta] \) to \( t \in [-1,1] \). By Markov’s inequality [5, Page 233],

\[
\max_{t \in [-1,1]} |T_n^{(k)}(t)| \leq \frac{2\max_{t \in [-1,1]} |T_n^{(k-1)}(t)|}{n(n-1)\cdots(n-k+1)} \leq \ldots \leq \frac{\max_{t \in [-1,1]} |T_n(t)|}{n(n-1)\cdots(n-k+1)} = \left( \frac{n(n-1)\cdots(n-k+1)}{\omega^k} \right)^2,
\]

as expected. \( \square \)

**Lemma 2.3.** Under the conditions of Theorem 2.1

\[
\|V_c^{-1}\|_p \geq \min_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \omega^{k-1} \times \frac{S_{n-1,p'}(\omega, \tau)}{\frac{\alpha_{\text{max}}^{(k-1)p}}{n^{1/p'}}}.
\]

**Proof.** For the sake of this proof, let the \( \ell \) distinct nodes have multiplicities \( k_1, k_2, \ldots, k_{\ell} \), respectively, where \( k_1 + k_2 + \cdots + k_{\ell} = n \), and the first \( k_1 \alpha_j \)'s are equal, the next \( k_2 \alpha_j \)'s are equal, and so on. Let \( v \) be the vector of the coefficients of the translated Chebyshev polynomial \( T_{n-1}(x; \omega, \tau) \), i.e., \( v = (a_{0,n-1} a_{1,n-1} \cdots a_{n-1,n-1})^T \). Then

\[
V_c^T v = (T_{n-1}(\alpha_1; \omega, \tau) T_{n-1}(\alpha_1; \omega, \tau) \cdots T_{n-1}(\alpha_1; \omega, \tau) \cdots \cdots)^T,
\]
which yields, by Lemma 2.2 for $1 \leq p' < \infty$

\[
(2.14) \quad \|V^T_e v\|_{p'}^p \leq \sum_{j=1}^\ell \left(1^{p'} + \frac{(n-1)^2}{\omega} \right)^{p'} \\
+ \cdots + \left[\frac{(n-1)(n-2)\cdots(n-k_j+1)^2}{\omega^{k_j-1}}\right]^{p'} \\
\leq \sum_{j=1}^\ell k_j \times \left(\max_{1 \leq k \leq k_j} \left[\frac{(n-1)!}{(n-k)!} \frac{1}{\omega^{k-1}}\right]\right)^{p'} \\
\leq n \times \left(\max_{1 \leq k \leq k_{\max}} \left[\frac{(n-1)!}{(n-k)!} \frac{1}{\omega^{k-1}}\right]\right)^{p'},
\]

which gives

\[
(2.15) \quad \|V^T_e v\|_{p'} \leq n^{1/p'} \times \max_{1 \leq k \leq k_{\max}} \left[\frac{(n-1)!}{(n-k)!} \frac{1}{\omega^{k-1}}\right].
\]

This is proved so far for $1 \leq p' < \infty$, but it can be verified that (2.16) holds for $p' = \infty$, too. Therefore, we have

\[
\|V^{-T}_e\|_{p'} = \max_u \frac{\|u\|_{p'}\|V^T_e u\|_{p'}}{\|V^T_e v\|_{p'}} \geq \frac{\|v\|_{p'}}{\|V^T_e v\|_{p'}} \\
\geq \min_{1 \leq k \leq k_{\max}} \left[\frac{(n-k)!}{(n-1)!} \omega^{k-1}\right]^{1/p'} \times S_{n-1,p'}(\omega, \tau) / n^{1/p'},
\]

as was to be shown. \[\square\]

In general, we may use (2.14), instead of (2.15), in estimating $\|V^{-1}_e\|_p$. Doing so, however, will lead to a more complicated lower bound on $\kappa_p(V_e)$.

**Remark 2.1.** Lemma 2.3 is made possible by Lemma 2.2 which is proved with the help of Markov’s inequality. Another classical inequality for the same purpose is Bernstein’s inequality [5, Page 233], using which we can obtain the following. For $0 \leq k \leq n$, if $\alpha < a \overset{\text{def}}{=} \min_j \alpha_j < b \overset{\text{def}}{=} \max_j \alpha_j < \beta$, then

\[
(2.17) \quad \frac{d}{dx} T_n(x; \omega, \tau) \leq \frac{n(n-1)\cdots(n-k+1)}{\omega^{1 - \left(\frac{\max(\beta-b, a-\alpha)}{\omega}\right)^2}} \quad \text{for } x \in [\alpha, \beta].
\]

This inequality improves (2.12) in the numerator part but has complications in the denominator, and also it requires the interval $[\alpha, \beta]$ to be (slightly) larger than the smallest interval containing all nodes. This can be bad because larger $[\alpha, \beta]$ will weaken the effectiveness of $S_{n,p'}(\omega, \tau)$ in the later bounds on $\kappa_p(V_e)$; for example $S_{n,p'}(\omega, \tau)$ is decreasing in $\omega$ [10].

3. **Two uniform bounds**

We present two theorems here, one for any real $V_e$ and one for $V_e$ with nonnegative nodes. Their proofs will be given later after two lemmas. Again let $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$. 
Theorem 3.1. Under the conditions of Theorem 2.1, if

\[ k_{\text{max}} - 1 \leq \frac{n-1}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-2n+2} \right] \sim \frac{n-1}{\sqrt{2}}, \]

then

\[ \kappa_p(V_c) \geq \left[ \frac{(n-k_{\text{max}})!}{(n-1)!} \right]^2 \frac{1}{2^{k_{\text{max}}-1}} \frac{S_{n-1,1}(1,0)}{n} \sim \frac{1}{2^{k_{\text{max}}-1}} \frac{[1 + \sqrt{2}]^{n-1}}{n}. \]

Theorem 3.2. Under the conditions of Theorem 2.1, if all \( \alpha_i \geq 0 \) and

\[ k_{\text{max}} - 1 \leq \frac{n-1}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-4(n-1)} \right]^{-1} \sim \frac{n-1}{\sqrt{2}}, \]

then

\[ \kappa_p(V_c) \geq \left[ \frac{(n-k_{\text{max}})!}{(n-1)!} \right]^2 \frac{1}{2^{k_{\text{max}}-1}} \frac{S_{n-1,1}(1/2,1)}{n} \sim \frac{1}{2^{k_{\text{max}}-1}} \frac{[1 + \sqrt{2}]^{2(n-1)}}{n}. \]

Lemma 3.1. Let \( j \geq 0 \) and \( m \geq 1 \). \( \rho^j S_{m,1}(\rho,0) \) is decreasing in \( \rho \) for \( 0 \leq \rho \leq 1 \) if

\[ j \leq \frac{m}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-2m} \right] \sim \frac{m}{\sqrt{2}}. \]

Proof. We claim that under inequality (3.3), \( \frac{d}{d\rho} \rho^j S_{m,1}(\rho,0) \leq 0 \) for \( 0 \leq \rho \leq 1 \). To this end, we notice that

\[ \frac{d}{d\rho} \rho^j S_{m,1}(\rho,0) = j \rho^{j-1} S_{m,1}(\rho,0) + \rho^j \frac{d}{d\rho} S_{m,1}(\rho,0). \]

Now for \( 0 \leq \rho \leq 1 \) and by (2.8), we have

\[ S_{m,1}(\rho,0) \leq \frac{1}{2} \left[ \frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}} \right]^m \left[ 1 + \epsilon^{-2m} \right], \]

\[ -\frac{d}{d\rho} S_{m,1}(\rho,0) \geq \frac{m}{2} \left[ \frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}} \right]^{m-1} \left[ 1 - \delta^{-2m} \right] \times \left[ \frac{1}{\rho^2} + \frac{1}{\rho^2 \sqrt{1 + \rho^2}} \right], \]
where $\epsilon = 1 + \sqrt{2}$ and $\delta = 0$ for even $m$, and $\epsilon = 0$ and $\delta = 1 + \sqrt{2}$ for odd $m$. Therefore, for $\rho \leq 1$,

$$\frac{d}{d\rho} \rho^j p_{m,1}(\rho, 0) = \frac{j}{m} + \rho \frac{d}{d\rho} S_{m,1}(\rho, 0) \leq \frac{j}{m} - \frac{\rho}{\rho^2} + \frac{1}{\rho^2} \frac{1 - \delta^{-2m}}{1 + \epsilon^{-2m}}$$

$$= \frac{j}{m} - \frac{1}{\sqrt{1 + \rho^2}} \frac{1 - \delta^{-2m}}{1 + \epsilon^{-2m}} \leq \frac{j}{m} - \frac{1}{\sqrt{2}} \frac{1 - (1 + \sqrt{2})^{-2m}}{1} \leq 0$$

upon using (3.3).

\(\square\)

**Lemma 3.2.** Let $j \geq 0$, $\gamma \geq 1$, and $m \geq 1$. For $j$ satisfying (3.3) and $\rho > 0$,

$$\rho^j \max\{\gamma, \rho^m\} S_{m,1}(\rho, 0) \geq S_{m,1}(1, 0).$$

**Proof.** Let $\Phi_1 = \rho^j \times S_{m,1}(\rho, 0)$ and $\Phi_2 = \rho^j \times S_{m,1}(\rho, 0)$. Then $\max\{\Phi_1, \Phi_2\}$ is $\Phi_1$ for $\rho \leq \gamma^{1/m}$ and $\Phi_2$ for $\rho \geq \gamma^{1/m}$. $\Phi_2$ is increasing in $\rho$ for $\rho > 0$ because $\rho^m S_{m,1}(\rho, 0)$ is a polynomial in $\rho$ with nonnegative coefficients and thus increasing in $\rho$ for $\rho > 0$. So

$$\max\{\Phi_1, \Phi_2\} \geq \Phi_2 \geq S_{m,1}(1, 0) \quad \text{for } \rho \geq 1.$$

For $0 \leq \rho \leq 1$, $\Phi_1$ is decreasing in $\rho$ by Lemma 3.1 and thus

$$\max\{\Phi_1, \Phi_2\} \geq \Phi_1 \geq S_{m,1}(1, 0) \quad \text{for } \rho \leq 1.$$

This completes the proof. \(\square\)

**Proof of Theorem 3.1** Setting $\gamma = \beta = \alpha_{\max}$ in (2.5), we have, upon using (2.7),

$$\kappa_p(V_c) \geq \min \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \frac{\alpha_{\max}^{k-1} \times \max\{\ell^{1/p}, \alpha_{\max}^{-1}\} S_{n-1,1}(\alpha_{\max}, 0)}{n^{1/p} \left[ \frac{n}{2} \right]^{1/p}} \int_{1 \leq k \leq k_{\max}} \Phi,$$

where $\Phi = \alpha_{\max}^{k-1} \times \max\{\ell^{1/p}, \alpha_{\max}^{-1}\} S_{n-1,1}(\alpha_{\max}, 0)$. Apply Lemma 3.2 with $j = k - 1$, $m = n - 1$, $\gamma = \ell^{1/p}$, and $\rho = \alpha_{\max}$ to get $\Phi \geq S_{n-1,1}(1, 0)$, as needed. \(\square\)

**Proof of Theorem 3.2** Setting $0 = \alpha < \beta = \alpha_{\max}$ in (2.5), we have, upon using (2.0),

$$\kappa_p(V_c) \geq \min \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \frac{\alpha_{\max}^{-1} \times \max\{\ell^{1/p}, \alpha_{\max}^{-1}\} S_{n-1,1}(\alpha_{\max}/2, 1)}{n^{1/p} \left[ \frac{n}{2} \right]^{1/p}} \int_{1 \leq k \leq k_{\max}} \Psi,$$
where
\[ \tilde{\Psi} = \alpha_{\text{max}}^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\text{max}}^{n-1}\} S_{n-1,1}(\alpha_{\text{max}}/2, 1). \]
It can be verified by (2.3), (2.8), and (2.9) that
\[ S_{n-1,1}(\alpha_{\text{max}}/2, 1) = S_{2(n-1),1}(\sqrt{\alpha_{\text{max}}}, 0). \]
Therefore
\[ \tilde{\Psi} = (\sqrt{\alpha_{\text{max}}})^{2(k-1)} \times \max\{\ell^{1/p'}, (\sqrt{\alpha_{\text{max}}})^{2(n-1)}\} S_{2(n-1),1}(\sqrt{\alpha_{\text{max}}}, 0) \]
\[ \geq S_{2(n-1),1}(1, 0), \]
on upon using Lemma 3.2 with \( j = 2(k-1), m = 2(n-1), \gamma = \ell^{1/p'}, \) and \( \rho = \sqrt{\alpha_{\text{max}}}. \)

4. CONCLUDING REMARKS

We have obtained three lower bounds on the condition number \( \kappa_p(V_c) \) of a real confluent Vandermonde matrix \( V_c \). Two of them are uniform in the sense that they depend on \( n \), the dimension of \( V_c \) only, while the other one is more general, as is the function of \( n \) and \( \alpha_{\text{max}} \) and the interval \([\alpha, \beta]\) that contains all \( \alpha_j \). These bounds grow exponentially for any fixed \( k_{\text{max}} \), much as expected. While it is not clear in general if (any of) our bounds are asymptotically optimal, in contrast to those for Vandermonde matrices by Beckermann [2] and recently by the author [10], our bounds are unlikely to be asymptotically optimal if \( k_{\text{max}} \) also grows, e.g., linearly in \( n \). This is illustrated by the extreme example \( k_{\text{max}} = n \), as we commented in Section 4.

We have focused on real confluent Vandermonde matrices here. It is conceivable that there would be much better conditioned complex confluent Vandermonde matrices or confluent Vandermonde-like matrices. This is partly an intuition one might get from that although real Vandermonde matrices are very ill-conditioned [7, 2, 10, 12], there exist very well-conditioned complex Vandermonde matrices and Vandermonde-like matrices [3, 11]. We plan to investigate this issue in future work.

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REFERENCES


Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506
E-mail address: rcli@ms.uky.edu
URL: http://www.ms.uky.edu/~rcli