

LOWER BOUNDS FOR THE CONDITION NUMBER OF A REAL CONFLUENT VANDERMONDE MATRIX

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ABSTRACT. Lower bounds on the condition number $\kappa_p(V_c)$ of a real confluent Vandermonde matrix V_c are established in terms of the dimension n , or n and the largest absolute value among all nodes that define the confluent Vandermonde matrix and the interval that contains the nodes. In particular, it is proved that for any modest k_{\max} (the largest multiplicity of distinct nodes), $\kappa_p(V_c)$ behaves no smaller than $\mathcal{O}_n((1 + \sqrt{2})^n)$, or than $\mathcal{O}_n((1 + \sqrt{2})^{2n})$ if all nodes are nonnegative. It is not clear whether those bounds are asymptotically sharp for modest k_{\max} .

1. INTRODUCTION

Given n numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ called *nodes*, the associated *Vandermonde matrix* is defined as

$$(1.1) \quad V \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}.$$

It, for example, arises from polynomial interpolation and others [3]. V is invertible if all nodes α_j are distinct, i.e., $\alpha_i \neq \alpha_j$ for $i \neq j$, but it becomes singular whenever $\alpha_i = \alpha_j$ for some $i \neq j$. A generalization of V for nodes not all of which are distinct is the so-called *confluent Vandermonde matrices*, e.g.,

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ \alpha_1 & 1 & 0 & \alpha_4 & \alpha_5 & 1 \\ \alpha_1^2 & 2\alpha_1 & 2 & \alpha_4^2 & \alpha_5^2 & 2\alpha_5 \\ \alpha_1^3 & 3\alpha_1^2 & 6\alpha_1 & \alpha_4^3 & \alpha_5^3 & 3\alpha_5^2 \\ \alpha_1^4 & 4\alpha_1^3 & 12\alpha_1^2 & \alpha_4^4 & \alpha_5^4 & 4\alpha_5^3 \\ \alpha_1^5 & 5\alpha_1^4 & 20\alpha_1^3 & \alpha_4^5 & \alpha_5^5 & 5\alpha_5^4 \end{pmatrix},$$

where $\alpha_1 = \alpha_2 = \alpha_3$ and $\alpha_5 = \alpha_6$. The second, third, and sixth columns are obtained by “differentiating” the previous column. Confluent Vandermonde matrices

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arise in Hermite interpolation [4], for example. Adopting the formulation in [8], we define the *confluent Vandermonde matrix* V_c as follows. First

$$(1.2) \quad \boxed{\begin{array}{l} \{\alpha_j\}_{j=1}^n \text{ are ordered so that equal nodes are contiguous, i.e.,} \\ \alpha_i = \alpha_j \quad (i < j) \quad \Rightarrow \quad \alpha_i = \alpha_{i+1} = \cdots = \alpha_j. \end{array}}$$

Define

$$(1.3) \quad V_c = (f_1(\alpha_1) \ f_2(\alpha_2) \ \cdots \ f_n(\alpha_n)),$$

where the vector function $f_j(t)$ is defined recursively by

$$(1.4) \quad f_j(t) = \begin{cases} (1 \ t \ \cdots \ t^{n-1})^T, & \text{if } j = 1 \text{ or } \alpha_j \neq \alpha_{j-1}, \\ \frac{d}{dx} f_{j-1}(t), & \text{otherwise,} \end{cases}$$

where “ \cdot^T ” is the transpose of a vector or matrix. As far as defining V_c is concerned, α_j can be real or complex. But in this paper, we shall focus on real α_j . In what follows, α_j and V_c , as well as

$$\alpha_{\max} \stackrel{\text{def}}{=} \max_j |\alpha_j|,$$

are reserved for their assignments here.

(Optimal) condition numbers for real Vandermonde matrices have been systematically studied by Gautschi and his coauthor (see [7] and references therein), and more recently by Tyrtysnikov [12], Beckermann [2], and Li [10]. In this paper, we shall establish three lower bounds on the ℓ_p -condition number $\kappa_p(V_c) \equiv \|V_c\|_p \|V_c^{-1}\|_p$ in terms of n , or n and α_{\max} and the interval $[\alpha, \beta]$ that contains all nodes. In particular, we will show that for fixed k_{\max} (the largest multiplicity of distinct nodes), $\kappa_p(V_c)$ behaves no smaller than $\mathcal{O}_n((1 + \sqrt{2})^n)$, where notation $a_n = \mathcal{O}_n(b_n)$ means $c_1 n^{d_1} \leq a_n/b_n \leq c_2 n^{d_2}$ for some constants c_1, c_2, d_1 , and d_2 .

Optimally conditioned confluent Vandermonde matrices can be much worse ill-conditioned than optimally conditioned Vandermonde matrices. One extreme example would be that all nodes are equal $\alpha_1 = \cdots = \alpha_n$ for which V_c is lower triangular, and thus

$$\kappa_p(V_c) \geq (n-1)! \sim \sqrt{2\pi} n^{n-1/2} e^{-n}$$

by Stirling's asymptotic formula [1, Page 18], and it becomes an equality for $\alpha_1 = \cdots = \alpha_n = 0$. While for optimally conditioned Vandermonde matrices, $\kappa_p(V)$ goes to ∞ as fast as $(1 + \sqrt{2})^n$ modulo a factor n^d for $|d| \leq 1$ [2, 10].

The rest of this paper is organized as follows. A general lower bound on $\kappa_p(V_c)$ is established in Section 2, but it is not uniform. Uniform bounds for $p = \infty$ are obtained in Section 3 for all real V_c and for V_c with nonnegative nodes. Finally we present our concluding remarks in Section 4.

2. A GENERAL LOWER BOUND

Given $1 \leq p \leq \infty$, the ℓ_p -norm of vector $u = (\mu_1 \ \mu_2 \ \cdots \ \mu_n)^T$ is defined as

$$\|u\|_p = \left(\sum_{j=1}^n |\mu_j|^p \right)^{1/p},$$

and $\|u\|_\infty = \lim_{p \rightarrow \infty} \|u\|_p = \max_j |\mu_j|$. The associated ℓ_p -operator norm of the $m \times n$ matrix A is defined as

$$(2.1) \quad \|A\|_p = \max_{u \neq 0} \frac{\|Au\|_p}{\|u\|_p}.$$

It can be proved that $\|A\|_p = \|A^T\|_{p'}$, upon noticing

$$\|A\|_p = \max_{u \neq 0, v \neq 0} \frac{|v^T Au|}{\|v\|_{p'} \|u\|_p},$$

where $1/p + 1/p' = 1$ (see also [9]).

Let $[\alpha, \beta]$ be the interval in which all α_j lie.

$$(2.2) \quad T_n(t) = \cos(n \arccost) \quad \text{for } |t| \leq 1,$$

$$(2.3) \quad = \frac{1}{2} (t + \sqrt{t^2 - 1})^n + \frac{1}{2} (t - \sqrt{t^2 - 1})^n \quad \text{for } |t| \geq 1$$

is the n th Chebyshev polynomial of the first kind. Define the n th translated Chebyshev polynomial $T_n(x; \omega, \tau) \stackrel{\text{def}}{=} T_n(x/\omega + \tau)$, where

$$\omega = \frac{\beta - \alpha}{2} > 0, \quad \tau = -\frac{\beta + \alpha}{\beta - \alpha}.$$

Let $a_{jn} \equiv a_{jn}(\omega, \tau)$ be the coefficient of x^j in $T_n(x; \omega, \tau)$, i.e.,

$$(2.4) \quad T_n(x; \omega, \tau) = a_{nn}x^n + a_{n-1,n}x^{n-1} + \dots + a_{1n}x + a_{0n}.$$

Define [10]

$$S_{n,p}(\omega, \tau) = \left(\sum_{j=0}^n |a_{jn}|^p \right)^{1/p}.$$

Now we are ready to state our main theorem for the section.

Theorem 2.1. *Assume that there are ℓ distinct nodes α_j , and let k_{\max} be the largest multiplicity of the distinct nodes. Then*

$$(2.5) \quad \kappa_p(V_c) \geq \min_{1 \leq k \leq k_{\max}} \left[\frac{(n-k)!}{(n-1)!} \right]^2 \omega^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^{n-1}\} \frac{S_{n-1,p'}(\omega, \tau)}{n^{1/p'}}.$$

Proof. Inequality (2.5) is a consequence of Lemmas 2.1 and 2.3 below. □

For $k_{\max} = 1$, i.e., $\ell = n$ and $k_1 = \dots = k_n = 1$ (and thus $V_c = V$), (2.5) becomes one of the lower bounds for $\kappa_p(V)$ in [10]. The right-hand side of (2.5) entails the explicit computation of $S_{n,p'}(\omega, \tau)$. It can also be estimated fairly well, too, by

$$(2.6) \quad n^{-1/p} S_{n-1,1}(\omega, \tau) \leq S_{n-1,p'}(\omega, \tau) \leq S_{n-1,1}(\omega, \tau),$$

$$(2.7) \quad [n/2]^{-1/p} S_{n-1,1}(\omega, 0) \leq S_{n-1,p'}(\omega, 0) \leq S_{n-1,1}(\omega, 0),$$

in connection with the explicit formulas for $S_{n-1,1}(\omega, \tau)$ for $\tau = 0$ or $|\tau| \geq 1$ in [10]. Here $[\xi]$ is the smallest integer that is larger than ξ . The formulas are

$$(2.8) \quad S_{n-1,1}(\omega, 0) = T_{n-1}(\iota/\omega) \sim \frac{1}{2} \left(\frac{1}{\omega} + \sqrt{1 + \frac{1}{\omega^2}} \right)^{n-1},$$

where $\iota = \sqrt{-1}$, and for $\alpha \geq 0$ (for which $\tau \leq -1$),

$$(2.9) \quad S_{n-1,1}(\omega, \tau) = T_{n-1}(|\tau| + 1/\omega) \sim \frac{1}{2} \left[\left(\frac{1}{\omega} + |\tau| \right) + \sqrt{\left(\frac{1}{\omega} + |\tau| \right)^2 - 1} \right]^{n-1}.$$

Lemma 2.1. *Assume that there are ℓ distinct nodes α_j . Then*

$$(2.10) \quad \|V_c\|_p \geq \max \left\{ \ell^{1/p'}, \alpha_{\max}^{n-1} \right\},$$

$$(2.11) \quad \|V_c\|_p \geq \left(\sum_{j=1}^n \alpha_{\max}^{(j-1)p} \right)^{1/p}.$$

Proof. Let e_j be the j th column of the $n \times n$ identity matrix I_n (or simply I if n is clear from the context). Use $\|V_c\|_p \geq \|V_c^T e_1\|_{p'}$ and $\|V_c\|_p \geq \|V_c^T e_n\|_{p'}$ to get (2.10), and use $\|V_c\|_p \geq \max_j \|V_c^T e_j\|_{p'}$ to get (2.11). \square

Lemma 2.2. *For $0 \leq k \leq n$,*

$$(2.12) \quad \left| \frac{d}{dx^k} T_n(x; \omega, \tau) \right| \leq \frac{[n(n-1) \cdots (n-k+1)]^2}{\omega^k} \quad \text{for } x \in [\alpha, \beta].$$

Proof. It follows from $T_n(x; \omega, \tau) = T_n(x/\omega + \tau) \equiv T_n(t)$ that

$$\frac{d^k}{dx^k} T_n(x; \omega, \tau) = \frac{1}{\omega^k} T_n^{(k)}(t),$$

where $t \equiv t(x) = x/\omega + \tau$. It suffices to show that $|T_n^{(k)}(t)| \leq [n(n-1) \cdots (n-k+1)]^2$ for $t \in [-1, 1]$ since $t(x)$ maps $x \in [\alpha, \beta]$ to $t \in [-1, 1]$. By Markov's inequality [5, Page 233],

$$\begin{aligned} \max_{t \in [-1, 1]} |T_n^{(k)}(t)| &\leq (n-k+1)^2 \max_{t \in [-1, 1]} |T_n^{(k-1)}(t)| \\ &\leq \cdots \\ &\leq [n(n-1) \cdots (n-k+1)]^2 \max_{t \in [-1, 1]} |T_n(t)| \\ &= [n(n-1) \cdots (n-k+1)]^2, \end{aligned}$$

as expected. \square

Lemma 2.3. *Under the conditions of Theorem 2.1,*

$$(2.13) \quad \|V_c^{-1}\|_p \geq \min_{1 \leq k \leq k_{\max}} \left[\frac{(n-k)!}{(n-1)!} \right]^2 \omega^{k-1} \times \frac{S_{n-1,p'}(\omega, \tau)}{n^{1/p'}}.$$

Proof. For the sake of this proof, let the ℓ distinct nodes have multiplicities k_1, k_2, \dots, k_ℓ , respectively, where $k_1 + k_2 + \cdots + k_\ell = n$, and the first k_1 α_j 's are equal, the next k_2 α_j 's are equal, and so on. Let v be the vector of the coefficients of the translated Chebyshev polynomial $T_{n-1}(x; \omega, \tau)$, i.e., $v = (a_{0n-1} \ a_{1n-1} \ \cdots \ a_{n-1n-1})^T$. Then

$$V_c^T v = (T_{n-1}(\alpha_1; \omega, \tau) \ T'_{n-1}(\alpha_1; \omega, \tau) \ \cdots \ T_{n-1}^{(k_1-1)}(\alpha_1; \omega, \tau) \ \cdots \ \cdots)^T,$$

which yields, by Lemma 2.2, for $1 \leq p' < \infty$

$$\begin{aligned}
 (2.14) \quad \|V_c^T v\|_{p'}^{p'} &\leq \sum_{j=1}^{\ell} \left(1^{p'} + \left[\frac{(n-1)^2}{\omega} \right]^{p'} \right. \\
 &\quad \left. + \dots + \left[\frac{[(n-1)(n-2)\cdots(n-k_j+1)]^2}{\omega^{k_j-1}} \right]^{p'} \right) \\
 &\leq \sum_{j=1}^{\ell} k_j \times \left(\max_{1 \leq k \leq k_j} \left[\frac{(n-1)!}{(n-k)!} \right]^2 \frac{1}{\omega^{k-1}} \right)^{p'} \\
 (2.15) \quad &\leq n \times \left(\max_{1 \leq k \leq k_{\max}} \left[\frac{(n-1)!}{(n-k)!} \right]^2 \frac{1}{\omega^{k-1}} \right)^{p'} ,
 \end{aligned}$$

which gives

$$(2.16) \quad \|V_c^T v\|_{p'} \leq n^{1/p'} \times \max_{1 \leq k \leq k_{\max}} \left[\frac{(n-1)!}{(n-k)!} \right]^2 \frac{1}{\omega^{k-1}} .$$

This is proved so far for $1 \leq p' < \infty$, but it can be verified that (2.16) holds for $p' = \infty$, too. Therefore, we have

$$\begin{aligned}
 \|V_c^{-T}\|_{p'} &= \max_u \frac{\|u\|_{p'}}{\|V_c^T u\|_{p'}} \geq \frac{\|v\|_{p'}}{\|V_c^T v\|_{p'}} \\
 &\geq \min_{1 \leq k \leq k_{\max}} \left[\frac{(n-k)!}{(n-1)!} \right]^2 \omega^{k-1} \times \frac{S_{n-1,p'}(\omega, \tau)}{n^{1/p'}} ,
 \end{aligned}$$

as was to be shown. □

In general, we may use (2.14), instead of (2.15), in estimating $\|V_c^{-1}\|_p$. Doing so, however, will lead to a more complicated lower bound on $\kappa_p(V_c)$.

Remark 2.1. Lemma 2.3 is made possible by Lemma 2.2 which is proved with the help of Markov’s inequality. Another classical inequality for the same purpose is Bernstein’s inequality [5, Page 233], using which we can obtain the following. For $0 \leq k \leq n$, if $\alpha < a \stackrel{\text{def}}{=} \min_j \alpha_j < b \stackrel{\text{def}}{=} \max_j \alpha_j < \beta$, then

$$(2.17) \quad \left| \frac{d}{dx^k} T_n(x; \omega, \tau) \right| \leq \frac{n(n-1)\cdots(n-k+1)}{\left[\omega \sqrt{1 - \left(\frac{\max\{\beta-b, a-\alpha\}}{\omega} \right)^2} \right]^k} \quad \text{for } x \in [\alpha, \beta].$$

This inequality improves (2.12) in the numerator part but has complications in the denominator, and also it requires the interval $[\alpha, \beta]$ to be (slightly) larger than the smallest interval containing all nodes. This can be bad because larger $[\alpha, \beta]$ will weaken the effectiveness of $S_{n,p'}(\omega, \tau)$ in the later bounds on $\kappa_p(V_c)$; for example $S_{n,p'}(\omega, \tau)$ is decreasing in ω [10].

3. TWO UNIFORM BOUNDS

We present two theorems here, one for any real V_c and one for V_c with nonnegative nodes. Their proofs will be given later after two lemmas. Again let $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$.

Theorem 3.1. *Under the conditions of Theorem 2.1, if*

$$(3.1) \quad k_{\max} - 1 \leq \frac{n-1}{\sqrt{2}} \left[1 - (1 + \sqrt{2})^{-2n+2} \right] \sim \frac{n-1}{\sqrt{2}},$$

then

$$\begin{aligned} \kappa_p(V_c) &\geq \left[\frac{(n - k_{\max})!}{(n-1)!} \right]^2 \frac{S_{n-1,1}(1, 0)}{n^{1/p'} [n/2]^{1/p}} \\ &\sim \left[\frac{(n - k_{\max})!}{(n-1)!} \right]^2 \frac{[1 + \sqrt{2}]^{n-1}}{n^{1/p'} [n/2]^{1/p}}. \end{aligned}$$

Theorem 3.2. *Under the conditions of Theorem 2.1, if all $\alpha_i \geq 0$ and*

$$(3.2) \quad k_{\max} - 1 \leq \frac{n-1}{\sqrt{2}} \left[1 - (1 + \sqrt{2})^{-4(n-1)} \right]^{-1} \sim \frac{n-1}{\sqrt{2}},$$

then

$$\begin{aligned} \kappa_p(V_c) &\geq \left[\frac{(n - k_{\max})!}{(n-1)!} \right]^2 \frac{1}{2^{k_{\max}-1}} \frac{S_{n-1,1}(1/2, 1)}{n} \\ &\sim \left[\frac{(n - k_{\max})!}{(n-1)!} \right]^2 \frac{1}{2^{k_{\max}-1}} \frac{[1 + \sqrt{2}]^{2(n-1)}}{n}. \end{aligned}$$

Lemma 3.1. *Let $j \geq 0$ and $m \geq 1$. $\rho^j S_{m,1}(\rho, 0)$ is decreasing in ρ for $0 \leq \rho \leq 1$ if*

$$(3.3) \quad j \leq \frac{m}{\sqrt{2}} \left[1 - (1 + \sqrt{2})^{-2m} \right] \sim \frac{m}{\sqrt{2}}.$$

Proof. We claim that under inequality (3.3), $\frac{d}{d\rho} \rho^j S_{m,1}(\rho, 0) \leq 0$ for $0 \leq \rho \leq 1$. To this end, we notice that

$$\frac{d}{d\rho} \rho^j S_{m,1}(\rho, 0) = j \rho^{j-1} S_{m,1}(\rho, 0) + \rho^j \frac{d}{d\rho} S_{m,1}(\rho, 0).$$

Now for $0 \leq \rho \leq 1$ and by (2.8), we have

$$\begin{aligned} S_{m,1}(\rho, 0) &\leq \frac{1}{2} \left[\frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}} \right]^m [1 + \epsilon^{-2m}], \\ -\frac{d}{d\rho} S_{m,1}(\rho, 0) &\geq \frac{m}{2} \left[\frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}} \right]^{m-1} [1 - \delta^{-2m}] \\ &\quad \times \left[\frac{1}{\rho^2} + \frac{1}{\rho^2 \sqrt{1 + \rho^2}} \right], \end{aligned}$$

where $\epsilon = 1 + \sqrt{2}$ and $\delta = 0$ for even m , and $\epsilon = 0$ and $\delta = 1 + \sqrt{2}$ for odd m . Therefore, for $\rho \leq 1$,

$$\begin{aligned} \frac{\frac{d}{d\rho}\rho^j S_{m,1}(\rho, 0)}{m \rho^{j-1} S_{m,1}(\rho, 0)} &= \frac{j}{m} + \frac{\rho \frac{d}{d\rho} S_{m,1}(\rho, 0)}{m S_{m,1}(\rho, 0)} \\ &\leq \frac{j}{m} - \frac{\rho \left[\frac{1}{\rho^2} + \frac{1}{\rho^2 \sqrt{1+\rho^2}} \right]}{\frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}}} \frac{1 - \delta^{-2m}}{1 + \epsilon^{-2m}} \\ &= \frac{j}{m} - \frac{1}{\sqrt{1 + \rho^2}} \frac{1 - \delta^{-2m}}{1 + \epsilon^{-2m}} \\ &\leq \frac{j}{m} - \frac{1}{\sqrt{2}} \frac{1 - \delta^{-2m}}{1 + \epsilon^{-2m}} \\ &\leq \frac{j}{m} - \frac{1}{\sqrt{2}} \left[1 - (1 + \sqrt{2})^{-2m} \right] \\ &\leq 0 \end{aligned}$$

upon using (3.3). □

Lemma 3.2. *Let $j \geq 0$, $\gamma \geq 1$, and $m \geq 1$. For j satisfying (3.3) and $\rho > 0$,*

$$\rho^j \max\{\gamma, \rho^m\} S_{m,1}(\rho, 0) \geq S_{m,1}(1, 0).$$

Proof. Let $\Phi_1 = \rho^j \times \gamma S_{m,1}(\rho, 0)$ and $\Phi_2 = \rho^j \times \rho^m S_{m,1}(\rho, 0)$. Then $\max\{\Phi_1, \Phi_2\}$ is Φ_1 for $\rho \leq \gamma^{1/m}$ and Φ_2 for $\rho \geq \gamma^{1/m}$. Φ_2 is increasing in ρ for $\rho > 0$ because $\rho^m S_{m,1}(\rho, 0)$ is a polynomial in ρ with nonnegative coefficients and thus increasing in ρ for $\rho > 0$. So

$$\max\{\Phi_1, \Phi_2\} \geq \Phi_2 \geq S_{m,1}(1, 0) \quad \text{for } \rho \geq 1.$$

For $0 \leq \rho \leq 1$, Φ_1 is decreasing in ρ by Lemma 3.1, and thus

$$\max\{\Phi_1, \Phi_2\} \geq \Phi_1 \geq S_{m,1}(1, 0) \quad \text{for } \rho \leq 1.$$

This completes the proof. □

Proof of Theorem 3.1. Setting $-\alpha = \beta = \alpha_{\max}$ in (2.5), we have, upon using (2.7),

$$\begin{aligned} \kappa_p(V_c) &\geq \min_{1 \leq k \leq k_{\max}} \left[\frac{(n-k)!}{(n-1)!} \right]^2 \alpha_{\max}^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^{n-1}\} \frac{S_{n-1,1}(\alpha_{\max}, 0)}{n^{1/p'} \lceil n/2 \rceil^{1/p}} \\ (3.4) \quad &\geq \left[\frac{(n-k_{\max})!}{(n-1)!} \right]^2 \frac{1}{n^{1/p'} \lceil n/2 \rceil^{1/p}} \min_{1 \leq k \leq k_{\max}} \tilde{\Phi}, \end{aligned}$$

where $\tilde{\Phi} = \alpha_{\max}^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^{n-1}\} S_{n-1,1}(\alpha_{\max}, 0)$. Apply Lemma 3.2 with $j = k - 1$, $m = n - 1$, $\gamma = \ell^{1/p'}$, and $\rho = \alpha_{\max}$ to get $\tilde{\Phi} \geq S_{n-1,1}(1, 0)$, as needed. □

Proof of Theorem 3.2. Setting $0 = \alpha < \beta = \alpha_{\max}$ in (2.5), we have, upon using (2.6),

$$\begin{aligned} \kappa_p(V_c) &\geq \min_{1 \leq k \leq k_{\max}} \left[\frac{(n-k)!}{(n-1)!} \right]^2 \left[\frac{\alpha_{\max}}{2} \right]^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^{n-1}\} \frac{S_{n-1,1}(\alpha_{\max}/2, 1)}{n} \\ &\geq \left[\frac{(n-k_{\max})!}{(n-1)!} \right]^2 \frac{1}{n 2^{k_{\max}-1}} \min_{1 \leq k \leq k_{\max}} \tilde{\Psi}, \end{aligned}$$

where

$$\tilde{\Psi} = \alpha_{\max}^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^{n-1}\} S_{n-1,1}(\alpha_{\max}/2, 1).$$

It can be verified by (2.3), (2.8), and (2.9) that

$$S_{n-1,1}(\alpha_{\max}/2, 1) = S_{2(n-1),1}(\sqrt{\alpha_{\max}}, 0).$$

Therefore

$$\begin{aligned} \tilde{\Psi} &= (\sqrt{\alpha_{\max}})^{2(k-1)} \times \max\left\{\ell^{1/p'}, (\sqrt{\alpha_{\max}})^{2(n-1)}\right\} S_{2(n-1),1}(\sqrt{\alpha_{\max}}, 0) \\ &\geq S_{2(n-1),1}(1, 0), \end{aligned}$$

upon using Lemma 3.2 with $j = 2(k - 1)$, $m = 2(n - 1)$, $\gamma = \ell^{1/p'}$, and $\rho = \sqrt{\alpha_{\max}}$. \square

4. CONCLUDING REMARKS

We have obtained three lower bounds on the condition number $\kappa_p(V_c)$ of a real confluent Vandermonde matrix V_c . Two of them are uniform in the sense that they depend on n , the dimension of V_c only, while the other one is more general, as is the function of n and α_{\max} and the interval $[\alpha, \beta]$ that contains all α_j . These bounds grow exponentially for any fixed k_{\max} , much as expected. While it is not clear in general if (any of) our bounds are asymptotically optimal, in contrast to those for Vandermonde matrices by Beckermann [2] and recently by the author [10], our bounds are unlikely to be asymptotically optimal if k_{\max} also grows, e.g., linearly in n . This is illustrated by the extreme example $k_{\max} = n$, as we commented in Section 1.

We have focused on real confluent Vandermonde matrices here. It is conceivable that there would be much better conditioned complex confluent Vandermonde matrices or confluent Vandermonde-like matrices. This is partly an intuition one might get from that although real Vandermonde matrices are very ill-conditioned [7, 2, 10, 12], there exist very well-conditioned complex Vandermonde matrices and Vandermonde-like matrices [6, 11]. We plan to investigate this issue in future work.

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