LOWER BOUNDS FOR THE CONDITION NUMBER OF A REAL CONFLUENT VANDERMONDE MATRIX

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Abstract. Lower bounds on the condition number $\kappa_p(V_c)$ of a real confluent Vandermonde matrix $V_c$ are established in terms of the dimension $n$, or $n$ and the largest absolute value among all nodes that define the confluent Vandermonde matrix and the interval that contains the nodes. In particular, it is proved that for any modest $k_{\text{max}}$ (the largest multiplicity of distinct nodes), $\kappa_p(V_c)$ behaves no smaller than $O(n((1+\sqrt{2})^n))$, or than $O(n((1+\sqrt{2})^{2n})$ if all nodes are nonnegative. It is not clear whether those bounds are asymptotically sharp for modest $k_{\text{max}}$.

1. Introduction

Given $n$ numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ called nodes, the associated Vandermonde matrix is defined as

$$V \equiv \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}.$$ 

It, for example, arises from polynomial interpolation and others [3]. $V$ is invertible if all nodes $\alpha_i$ are distinct, i.e., $\alpha_i \neq \alpha_j$ for $i \neq j$, but it becomes singular whenever $\alpha_i = \alpha_j$ for some $i \neq j$. A generalization of $V$ for nodes not all of which are distinct is the so-called confluent Vandermonde matrices, e.g.,

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ \alpha_1 & 1 & 0 & \alpha_4 & \alpha_5 & 1 \\ \alpha_1^2 & 2\alpha_2 & 2 & \alpha_4^2 & \alpha_5^2 & 2\alpha_5 \\ \alpha_1^3 & 3\alpha_2^2 & 3\alpha_1 & \alpha_4^3 & \alpha_5^3 & 3\alpha_5^2 \\ \alpha_1^4 & 4\alpha_2^3 & 12\alpha_1\alpha_2 & \alpha_4^4 & \alpha_5^4 & 4\alpha_5^3 \\ \alpha_1^5 & 5\alpha_2^4 & 20\alpha_1\alpha_2^2 & 10\alpha_1^2 & \alpha_4^5 & 5\alpha_5^4 \end{pmatrix},$$

where $\alpha_1 = \alpha_2 = \alpha_3$ and $\alpha_5 = \alpha_6$. The second, third, and sixth columns are obtained by “differentiating” the previous column. Confluent Vandermonde matrices

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arise in Hermite interpolation \[1\], for example. Adopting the formulation in \[8\], we define the confluent Vandermonde matrix \(V_c\) as follows. First

\[
\{\alpha_j\}_{j=1}^{n} \text{ are ordered so that equal nodes are contiguous, i.e.,} \\
\alpha_i = \alpha_j \quad (i < j) \quad \Rightarrow \quad \alpha_i = \alpha_{i+1} = \cdots = \alpha_j.
\]

Define

\[
V_c = (f_1(\alpha_1) \ f_2(\alpha_2) \ \cdots \ f_n(\alpha_n)),
\]
where the vector function \(f_j(t)\) is defined recursively by

\[
f_j(t) = \begin{cases} 
(1 \ t \ \cdots \ t^{n-1})^T, & \text{if } j = 1 \text{ or } \alpha_j \neq \alpha_{j-1}, \\
\frac{d}{dt} f_{j-1}(t), & \text{otherwise,}
\end{cases}
\]
where "\(\cdot^T\)" is the transpose of a vector or matrix. As far as defining \(V_c\) is concerned, \(\alpha_j\) can be real or complex. But in this paper, we shall focus on real \(\alpha_j\). In what follows, \(\alpha_j\) and \(V_c\), as well as

\[
\alpha_{\max} \overset{\text{def}}{=} \max_j |\alpha_j|,
\]
are reserved for their assignments here.

(Optimal) condition numbers for real Vandermonde matrices have been systematically studied by Gautschi and his coauthor (see \[7\] and references therein), and more recently by Tyrtyshnikov \[12\], Beckermann \[2\], and Li \[10\]. In this paper, we shall establish three lower bounds on the \(\ell_p\)-condition number \(\kappa_p(V_c) \equiv \|V_c\|_p \|V_c^{-1}\|_p\) in terms of \(n\), or \(n\) and \(\alpha_{\max}\) and the interval \([\alpha, \beta]\) that contains all nodes. In particular, we will show that for fixed \(k_{\max}\) (the largest multiplicity of distinct nodes), \(\kappa_p(V_c)\) behaves no smaller than \(O_n((1 + \sqrt{2})^n)\), where notation \(a_n = O_n(b_n)\) means \(c_1 n^{d_1} \leq a_n / b_n \leq c_2 n^{d_2}\) for some constants \(c_1, c_2, d_1, \text{ and } d_2\).

Optimally conditioned confluent Vandermonde matrices can be much worse ill-conditioned than optimally conditioned Vandermonde matrices. One extreme example would be that all nodes are equal \(\alpha_1 = \cdots = \alpha_n\) for which \(V_c\) is lower triangular, and thus

\[
\kappa_p(V_c) \geq (n-1)! \sim \sqrt{2\pi} n^{n-1/2} e^{-n}
\]
by Stirling’s asymptotic formula \[11\] Page 18], and it becomes an equality for \(\alpha_1 = \cdots = \alpha_n = 0\). While for optimally conditioned Vandermonde matrices, \(\kappa_p(V)\) goes to \(\infty\) as fast as \((1 + \sqrt{2})^n\) modulo a factor \(n^d\) for \(|d| \leq 1\) \(\[2\] \[10\].

The rest of this paper is organized as follows. A general lower bound on \(\kappa_p(V_c)\) is established in Section \[2\] but it is not uniform. Uniform bounds for \(p = \infty\) are obtained in Section \[3\] for all real \(V_c\) and for \(V_c\) with nonnegative nodes. Finally we present our concluding remarks in Section \[4\].

2. A GENERAL LOWER BOUND

Given \(1 \leq p \leq \infty\), the \(\ell_p\)-norm of vector \(u = (\mu_1 \ \mu_2 \ \cdots \ \mu_n)^T\) is defined as

\[
\|u\|_p = \left(\sum_{j=1}^{n} |\mu_j|^p \right)^{1/p}.
\]
and \[\|u\|_\infty = \lim_{p \to \infty} \|u\|_p = \max_j |\mu_j|\]. The associated \(\ell_p\)-operator norm of the \(m \times n\) matrix \(A\) is defined as
\[
\|A\|_p = \max_{u \neq 0} \frac{\|Au\|_p}{\|u\|_p},
\]
It can be proved that \(\|A\|_p = \|A^T\|_{p'}\), upon noticing
\[
\|A\|_p = \max_{u \neq 0, v \neq 0} \frac{|u^T A v|}{\|u\|_p \|v\|_{p'}},
\]
where \(1/p + 1/p' = 1\) (see also [3]).

Let \([\alpha, \beta]\) be the interval in which all \(\alpha_j\) lie.

\[
T_m(t) = \cos(n \arccos t) \quad \text{for } |t| \leq 1,
\]
\[
\frac{1}{2} \left( t + \sqrt{t^2 - 1} \right)^n + \frac{1}{2} \left( t - \sqrt{t^2 - 1} \right)^n \quad \text{for } |t| \geq 1
\]
is the \(n\)th Chebyshev polynomial of the first kind. Define the \(n\)th translated Chebyshev polynomial \(T_n(x; \omega, \tau) \defeq T_n(x/\omega + \tau)\), where
\[
\omega = \frac{\beta - \alpha}{2} > 0, \quad \tau = \frac{-\beta + \alpha}{\beta - \alpha}
\]
Let \(a_{jn} \equiv a_{jn}(\omega, \tau)\) be the coefficient of \(x^j\) in \(T_n(x; \omega, \tau)\), i.e.,
\[
T_n(x; \omega, \tau) = a_{nn}x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.\]
Define \([10]\)
\[
S_{n,p}(\omega, \tau) = \left( \sum_{j=0}^{n} |a_{jn}|^p \right)^{1/p}.
\]
Now we are ready to state our main theorem for the section.

**Theorem 2.1.** Assume that there are \(\ell\) distinct nodes \(\alpha_j\), and let \(k_{\max}\) be the largest multiplicity of the distinct nodes. Then
\[
\kappa_p(V_\xi) \geq \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \omega^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^n\} \frac{S_{n-1,p}(\omega, \tau)}{n^{1/p'}}.
\]
**Proof.** Inequality (2.5) is a consequence of Lemmas [2.1] and [2.3] below. \(\square\)

For \(k_{\max} = 1\), i.e., \(\ell = n\) and \(k_1 = \cdots = k_n = 1\) (and thus \(V_\xi = V\)), (2.5) becomes one of the lower bounds for \(\kappa_p(V)\) in [10]. The right-hand side of (2.5) entails the explicit computation of \(S_{n,p}(\omega, \tau)\). It can also be estimated fairly well, too, by
\[
n^{-1/p} S_{n-1,1}(\omega, \tau) \leq S_{n-1,1}(\omega, \tau) \leq S_{n-1,1}(\omega, \tau),
\]
\[
[n/2]^{-1/p} S_{n-1,1}(\omega, 0) \leq S_{n-1,1}(\omega, 0) \leq S_{n-1,1}(\omega, 0),
\]
in connection with the explicit formulas for \(S_{n-1,1}(\omega, \tau)\) for \(\tau = 0\) or \(|\tau| \geq 1\) in [10]. Here \(|\xi|\) is the smallest integer that is larger than \(\xi\). The formulas are
\[
S_{n-1,1}(\omega, 0) = T_{n-1}(t/\omega) \sim \frac{1}{2} \left( \frac{1}{\omega} + \sqrt{1 + \frac{1}{\omega^2}} \right)^{n-1},
\]
where \( \iota = \sqrt{-1} \), and for \( \alpha \geq 0 \) (for which \( \tau \leq -1 \)),

\[
S_{n-1,1}(\omega, \tau) = T_{n-1}(|\tau| + 1/\omega) \sim \frac{1}{2} \left[ \left( \frac{1}{\omega} + |\tau| \right) + \sqrt{\left( \frac{1}{\omega} + |\tau| \right)^2 - 1} \right]^{n-1}.
\]

**Lemma 2.1.** Assume that there are \( \ell \) distinct nodes \( \alpha_j \). Then

\[
\|V_c\|_p \geq \max \left\{ \ell^{1/p'}, \alpha_{\max}^{(n-1)1/p} \right\},
\]

\[
\|V_c\|_p \geq \left( \sum_{j=1}^{n} \alpha_{\max}^{(j-1)1/p} \right)^{1/p}.
\]

**Proof.** Let \( e_j \) be the \( j \)th column of the \( n \times n \) identity matrix \( I_n \) (or simply \( I \) if \( n \) is clear from the context). Use \( \|V_c\|_p \geq \|V_c^T e_1\|_{p'} \) and \( \|V_c\|_p \geq \|V_c^T e_n\|_{p'} \) to get \((2.10)\), and use \( \|V_c\|_p \geq \max_j \|V_c^T e_j\|_{p'} \) to get \((2.11)\). \( \square \)

**Lemma 2.2.** For \( 0 \leq k \leq n \),

\[
\left| \frac{d}{dx^k} T_n(x; \omega, \tau) \right| \leq \frac{n(n-1) \cdots (n-k+1) \tau}{\omega^k} \quad \text{for } x \in [\alpha, \beta].
\]

**Proof.** It follows from \( T_n(x; \omega, \tau) = T_n(x/\omega + \tau) \equiv T_n(t) \) that

\[
\frac{d^k}{dx^k} T_n(x; \omega, \tau) = \frac{1}{\omega^k} T_n^{(k)}(t),
\]

where \( t = t(x) = x/\omega + \tau \). It suffices to show that \( |T_n^{(k)}(t)| \leq \frac{n(n-1) \cdots (n-k+1) \tau}{\omega^k} \)

for \( t \in [-1, 1] \) since \( t(x) \) maps \( x \in [\alpha, \beta] \) to \( t \in [-1, 1] \). By Markov’s inequality [5, Page 233],

\[
\max_{t \in [-1,1]} |T_n^{(k)}(t)| \leq (n-k+1)^2 \max_{t \in [-1,1]} |T_n^{(k-1)}(t)| \leq \cdots \leq \left[ n(n-1) \cdots (n-k+1) \tau \right] \max_{t \in [-1,1]} |T_n(t)| = \left[ n(n-1) \cdots (n-k+1) \tau \right],
\]

as expected. \( \square \)

**Lemma 2.3.** Under the conditions of Theorem 2.1.

\[
\|V_c^{-1}\|_p \geq \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n-k)!}{(n-1)!} \right]^{2} \omega^{k-1} \times \frac{S_{n-1,p}(\omega, \tau)}{n^{1/p'}},
\]

**Proof.** For the sake of this proof, let the \( \ell \) distinct nodes have multiplicities \( k_1, k_2, \ldots, k_{\ell} \), respectively, where \( k_1 + k_2 + \cdots + k_{\ell} = n \), and the first \( k_1 \) \( \alpha_j \)'s are equal, the next \( k_2 \) \( \alpha_j \)'s are equal, and so on. Let \( v \) be the vector of the coefficients of the translated Chebyshev polynomial \( T_{n-1}(x; \omega, \tau) \), i.e., \( v = (a_{0,n-1} \ a_{1,n-1} \cdots \ a_{n-1,n-1})^T \). Then

\[
V_c^T v = (T_{n-1}(\alpha_1; \omega, \tau) \ T_{n-1}^{(k_2-1)}(\alpha_1; \omega, \tau) \cdots \ T_{n-1}^{(k_1-1)}(\alpha_1; \omega, \tau) \cdots)^T,
\]

\[
V_c^T V_c = (T_{n-1}(\alpha_1; \omega, \tau) V_{c,T} V_{c} T_{n-1}(\alpha_1; \omega, \tau) \cdots V_{c,T} V_{c} T_{n-1}^{(k_1-1)}(\alpha_1; \omega, \tau) \cdots)^T.
\]
which yields, by Lemma 2.2 for $1 \leq p' < \infty$

\[
(2.14) \quad \|V_{c}^{-T}v\|_{p'}^{p'} \leq \sum_{j=1}^{\ell} \left( 1^{p'} + \left( \frac{(n-1)^{2}}{\omega} \right)^{p'} + \cdots + \left( \frac{(n-1)(n-2)\cdots(n-k_{j}+1)}{\omega^{k_{j}-1}} \right)^{p'} \right)
\]

\[
\leq \sum_{j=1}^{\ell} k_{j} \times \left( \max_{1 \leq k \leq k_{j}} \left[ \frac{(n-1)!}{(n-k)!} \right] \frac{1}{\omega^{k-1}} \right)^{p'}
\]

\[
(2.15) \quad \leq n \times \left( \max_{1 \leq k \leq k_{\max}} \left[ \frac{(n-1)!}{(n-k)!} \right] \frac{1}{\omega^{k-1}} \right)^{p'},
\]

which gives

\[
(2.16) \quad \|V_{c}^{-T}v\|_{p'} \leq n^{1/p'} \times \max_{1 \leq k \leq k_{\max}} \left[ \frac{(n-1)!}{(n-k)!} \right]^{2} \frac{1}{\omega^{k-1}}.
\]

This is proved so far for $1 \leq p' < \infty$, but it can be verified that (2.16) holds for $p' = \infty$, too. Therefore, we have

\[
\|V_{c}^{-T}\|_{p'} = \max_{u} \frac{\|u\|_{p'}}{\|V_{c}^{-T}u\|_{p'}} \geq \|v\|_{p'} \geq \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n-k)!}{(n-1)!} \omega^{k-1} \right]^{2} \times \frac{S_{n-1,p'}(\omega, \tau)}{n^{1/p'}}
\]

as was to be shown. \(\square\)

In general, we may use (2.14), instead of (2.15), in estimating $\|V_{c}^{-1}\|_{p}$. Doing so, however, will lead to a more complicated lower bound on $\kappa_{p}(V_{c})$.

**Remark 2.1.** Lemma 2.3 is made possible by Lemma 2.2 which is proved with the help of Markov’s inequality. Another classical inequality for the same purpose is Bernstein’s inequality [5, Page 233], using which we can obtain the following. For $0 \leq k \leq n$, if $\alpha < a \overset{\text{def}}{=} \min_{j} \alpha_{j} < b \overset{\text{def}}{=} \max_{j} \alpha_{j} < \beta$, then

\[
(2.17) \quad \left| \frac{d}{dx^{k}} T_{n}(x; \omega, \tau) \right| \leq \frac{n(n-1)\cdots(n-k+1)}{\omega^{\sqrt{1 - \left( \frac{\max(b-a, a-c)}{\omega} \right)^{2}}}^{k}} \quad \text{for } x \in [\alpha, \beta].
\]

This inequality improves (2.12) in the numerator part but has complications in the denominator, and also it requires the interval $[\alpha, \beta]$ to be (slightly) larger than the smallest interval containing all nodes. This can be bad because larger $[\alpha, \beta]$ will weaken the effectiveness of $S_{n,p}(\omega, \tau)$ in the later bounds on $\kappa_{p}(V_{c})$; for example $S_{n,p}(\omega, \tau)$ is decreasing in $\omega$ [10].

3. **Two uniform bounds**

We present two theorems here, one for any real $V_{c}$ and one for $V_{c}$ with nonnegative nodes. Their proofs will be given later after two lemmas. Again let $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$. 


Theorem 3.1. Under the conditions of Theorem 2.1, if

\( k_{\text{max}} - 1 \leq \frac{n-1}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-2n+2} \right] \sim \frac{n-1}{\sqrt{2}}, \)

then

\[ \kappa_p(V_c) \geq \left[ \frac{(n - k_{\text{max}})!}{(n - 1)!} \right]^2 \frac{S_{n-1,1}(1,0)}{n^{1/p'} (n/2)^{1/p}} \sim \left[ \frac{(n - k_{\text{max}})!}{(n - 1)!} \right]^2 \frac{1}{2^{k_{\text{max}}-1}} \frac{S_{n-1,1}(1/2,1)}{n} \]

\[ \geq \left[ \frac{(n - k_{\text{max}})!}{(n - 1)!} \right]^2 \frac{1}{2^{k_{\text{max}}-1}} \frac{[1 + \sqrt{2}]^{2(n-1)}}{n}. \]

Theorem 3.2. Under the conditions of Theorem 2.1, if all \( \alpha_i \geq 0 \) and

\( k_{\text{max}} - 1 \leq \frac{n-1}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-4(n-1)} \right]^{-1} \sim \frac{n-1}{\sqrt{2}}, \)

then

\[ \kappa_p(V_c) \geq \left[ \frac{(n - k_{\text{max}})!}{(n - 1)!} \right]^2 \frac{1}{2^{k_{\text{max}}-1}} \frac{S_{n-1,1}(1/2,1)}{n} \sim \left[ \frac{(n - k_{\text{max}})!}{(n - 1)!} \right]^2 \frac{1}{2^{k_{\text{max}}-1}} \frac{[1 + \sqrt{2}]^{2(n-1)}}{n}. \]

Lemma 3.1. Let \( j \geq 0 \) and \( m \geq 1 \). \( \rho^j S_{m,1}(\rho,0) \) is decreasing in \( \rho \) for \( 0 \leq \rho \leq 1 \) if

\[ j \leq \frac{m}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-2m} \right] \sim \frac{m}{\sqrt{2}}. \]

Proof. We claim that under inequality (3.3), \( \frac{d}{d\rho}\rho^j S_{m,1}(\rho,0) \leq 0 \) for \( 0 \leq \rho \leq 1 \). To this end, we notice that

\[ \frac{d}{d\rho}\rho^j S_{m,1}(\rho,0) = j\rho^{j-1} S_{m,1}(\rho,0) + \rho^j \frac{d}{d\rho} S_{m,1}(\rho,0). \]

Now for \( 0 \leq \rho \leq 1 \) and by (2.8), we have

\[ S_{m,1}(\rho,0) \leq \frac{1}{2} \left[ \frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}} \right]^m \left[ 1 + \epsilon^{-2m} \right], \]

\[ -\frac{d}{d\rho} S_{m,1}(\rho,0) \geq \frac{m}{2} \left[ \frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}} \right]^{m-1} \left[ 1 - \delta^{-2m} \right] \times \left[ \frac{1}{\rho^2} + \frac{1}{\rho^2 \sqrt{1 + \rho^2}} \right], \]
where \( \epsilon = 1 + \sqrt{2} \) and \( \delta = 0 \) for even \( m \), and \( \epsilon = 0 \) and \( \delta = 1 + \sqrt{2} \) for odd \( m \). Therefore, for \( \rho \leq 1 \),

\[
\frac{d}{d\rho} \rho^j S_{m,1}(\rho, 0) = \frac{j}{m} \rho^j S_{m,1}(\rho, 0) \leq \frac{j}{m} \rho \left[ \frac{1}{\rho^2} \right] - \frac{1}{1 + \rho^2} \left[ 1 - \delta^{-2m} \right] \leq \frac{j}{m} \rho \left[ 1 - \frac{1}{1 + \rho^2} \right] \left[ 1 - \delta^{-2m} \right] \leq \frac{j}{m} \rho \left[ 1 - (1 + \sqrt{2})^{-2m} \right] \leq 0
\]

upon using \((3.3)\).

\[\square\]

**Lemma 3.2.** Let \( j \geq 0 \), \( \gamma \geq 1 \), and \( m \geq 1 \). For \( j \) satisfying \((3.3)\) and \( \rho > 0 \),

\[\rho^j \max\{\gamma, \rho^m\} S_{m,1}(\rho, 0) \geq S_{m,1}(1, 0)\]

**Proof.** Let \( \Phi_1 = \rho^j \times \gamma S_{m,1}(\rho, 0) \) and \( \Phi_2 = \rho^j \times \rho^m S_{m,1}(\rho, 0) \). Then \( \max\{\Phi_1, \Phi_2\} \) is \( \Phi_1 \) for \( \rho \leq \gamma^{1/m} \) and \( \Phi_2 \) for \( \rho > \gamma^{1/m} \). \( \Phi_2 \) is increasing in \( \rho \) for \( \rho > 0 \) because \( \rho^m S_{m,1}(\rho, 0) \) is a polynomial in \( \rho \) with nonnegative coefficients and thus increasing in \( \rho \) for \( \rho > 0 \). So

\[\max\{\Phi_1, \Phi_2\} \geq \Phi_2 \geq S_{m,1}(1, 0) \quad \text{for} \quad \rho \geq 1.\]

For \( 0 \leq \rho \leq 1 \), \( \Phi_1 \) is decreasing in \( \rho \) by Lemma \((3.1)\) and thus

\[\max\{\Phi_1, \Phi_2\} \geq \Phi_1 \geq S_{m,1}(1, 0) \quad \text{for} \quad \rho \leq 1.\]

This completes the proof. \[\square\]

**Proof of Theorem \((3.1)\)** Setting \( -\alpha = \beta = \alpha_{\text{max}} \) in \((2.5)\), we have, upon using \((2.7)\),

\[
\kappa_p(V_c) \geq \min_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n - k)!}{(n - 1)!} \right]^2 \alpha_{k_{\text{max}}}^{-1} \times \max\{\ell^{1/p'}, \alpha_{m_{\text{max}}}^{-1}\} \frac{S_{n-1,1}(\alpha_{\text{max}}, 0)}{n^{1/p'} \left[ n^2 \right]^{1/p'}} \Phi_{k_{\text{max}}},
\]

where \( \Phi = \alpha_{k_{\text{max}}}^{-1} \times \max\{\ell^{1/p'}, \alpha_{m_{\text{max}}}^{-1}\} S_{n-1,1}(\alpha_{\text{max}}, 0) \). Apply Lemma \((3.2)\) with \( j = k - 1 \), \( m = n - 1 \), \( \gamma = \ell^{1/p'} \), and \( \rho = \alpha_{\text{max}} \) to get \( \Phi \geq S_{n-1,1}(1, 0) \), as needed. \[\square\]

**Proof of Theorem \((3.2)\)** Setting \( 0 = \alpha < \beta = \alpha_{\text{max}} \) in \((2.5)\), we have, upon using \((2.0)\),

\[
\kappa_p(V_c) \geq \min_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n - k)!}{(n - 1)!} \right]^2 \left[ \frac{\alpha_{\text{max}}}{2} \right]^{k-1} \times \max\{\ell^{1/p'}, \alpha_{m_{\text{max}}}^{-1}\} \frac{S_{n-1,1}(\alpha_{\text{max}}/2, 1)}{n} \Psi_{k_{\text{max}}},
\]

where \( \Psi = \alpha_{k_{\text{max}}}^{-1} \times \max\{\ell^{1/p'}, \alpha_{m_{\text{max}}}^{-1}\} S_{n-1,1}(\alpha_{\text{max}}/2, 1) \). Apply Lemma \((3.2)\) with \( j = k - 1 \), \( m = n - 1 \), \( \gamma = \ell^{1/p'} \), and \( \rho = \alpha_{\text{max}} \) to get \( \Psi \geq S_{n-1,1}(1, 0) \), as needed. \[\square\]
where 
\[ \tilde{\Psi} = \alpha^{k-1}_{\text{max}} \times \max\{\ell^{1/\rho'}, \alpha^{n-1}_{\text{max}}\} S_{n-1,1}(\alpha_{\text{max}}/2, 1). \]
It can be verified by (2.3), (2.8), and (2.9) that 
\[ S_{n-1,1}(\alpha_{\text{max}}/2, 1) = S_{2(n-1),1}(\sqrt{\alpha_{\text{max}}}, 0). \]
Therefore 
\[ \tilde{\Psi} = (\sqrt{\alpha_{\text{max}}})^{2(k-1)} \times \max\left\{\ell^{1/\rho'}, \left(\sqrt{\alpha_{\text{max}}}\right)^{2(n-1)}\right\} S_{2(n-1),1}(\sqrt{\alpha_{\text{max}}}, 0) \geq S_{2(n-1),1}(1, 0), \]
upon using Lemma 3.2 with \( j = 2(k - 1), m = 2(n - 1), \gamma = \ell^{1/\rho'}, \) and \( \rho = \sqrt{\alpha_{\text{max}}}. \)

\[ \square \]

4. CONCLUDING REMARKS

We have obtained three lower bounds on the condition number \( \kappa_{p}(V_{c}) \) of a real confluent Vandermonde matrix \( V_{c} \). Two of them are uniform in the sense that they depend on \( n \), the dimension of \( V_{c} \) only, while the other one is more general, as is the function of \( n \) and \( \alpha_{\text{max}} \) and the interval \( [\alpha, \beta] \) that contains all \( \alpha_{j} \). These bounds grow exponentially for any fixed \( k_{\text{max}} \), much as expected. While it is not clear in general if (any of) our bounds are asymptotically optimal, in contrast to those for Vandermonde matrices by Beckermann [2] and recently by the author [10], our bounds are unlikely to be asymptotically optimal if \( k_{\text{max}} \) also grows, e.g., linearly in \( n \). This is illustrated by the extreme example \( k_{\text{max}} = n \), as we commented in Section 4.

We have focused on real confluent Vandermonde matrices here. It is conceivable that there would be much better conditioned complex confluent Vandermonde matrices or confluent Vandermonde-like matrices. This is partly an intuition one might get from that although real Vandermonde matrices are very ill-conditioned [7, 2, 10, 12], there exist very well-conditioned complex Vandermonde matrices and Vandermonde-like matrices [3, 11]. We plan to investigate this issue in future work.

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REFERENCES


