LOWER BOUNDS FOR THE CONDITION NUMBER OF A REAL CONFLUENT VANDERMONDE MATRIX

REN-CANG LI

Abstract. Lower bounds on the condition number $\kappa_p(V_c)$ of a real confluent Vandermonde matrix $V_c$ are established in terms of the dimension $n$, or $n$ and the largest absolute value among all nodes that define the confluent Vandermonde matrix and the interval that contains the nodes. In particular, it is proved that for any modest $k_{\text{max}}$ (the largest multiplicity of distinct nodes), $\kappa_p(V_c)$ behaves no smaller than $O(n((1+\sqrt{2})^n))$, or than $O(n((1+\sqrt{2})^{2n})$ if all nodes are nonnegative. It is not clear whether those bounds are asymptotically sharp for modest $k_{\text{max}}$.

1. Introduction

Given $n$ numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ called nodes, the associated Vandermonde matrix is defined as

$$ V \overset{\text{def}}{=} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots & \alpha_n^{n-1} \end{pmatrix}. $$

It, for example, arises from polynomial interpolation and others. $V$ is invertible if all nodes $\alpha_i$ are distinct, i.e., $\alpha_i \neq \alpha_j$ for $i \neq j$, but it becomes singular whenever $\alpha_i = \alpha_j$ for some $i \neq j$. A generalization of $V$ for nodes not all of which are distinct is the so-called confluent Vandermonde matrices, e.g.,

$$ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ \alpha_1 & 1 & 0 & \alpha_4 & \alpha_5 & 1 \\ \alpha_1^2 & 2\alpha_1 & 2 & \alpha_4^2 & \alpha_5^2 & 2\alpha_5 \\ \alpha_1^3 & 3\alpha_1^2 & 6\alpha_1 & \alpha_4^3 & \alpha_5^3 & 3\alpha_5^2 \\ \alpha_1^4 & 4\alpha_1^3 & 12\alpha_1^2 & \alpha_4^4 & \alpha_5^4 & 4\alpha_5^3 \\ \alpha_1^5 & 5\alpha_1^4 & 20\alpha_1^3 & \alpha_4^5 & \alpha_5^5 & 5\alpha_5^4 \end{pmatrix}, $$

where $\alpha_1 = \alpha_2 = \alpha_3$ and $\alpha_5 = \alpha_6$. The second, third, and sixth columns are obtained by “differentiating” the previous column. Confluent Vandermonde matrices

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arise in Hermite interpolation [11], for example. Adopting the formulation in [8], we define the confluent Vandermonde matrix $V_c$ as follows. First
\begin{equation}
\{\alpha_j\}_{j=1}^n \text{ are ordered so that equal nodes are contiguous, i.e.,}
\alpha_i = \alpha_j \quad (i < j) \implies \alpha_i = \alpha_{i+1} = \cdots = \alpha_j.
\end{equation}
Define
\begin{equation}
V_c = (f_1(\alpha_1) \ f_2(\alpha_2) \ \cdots \ f_n(\alpha_n)),
\end{equation}
where the vector function $f_j(t)$ is defined recursively by
\begin{equation}
f_j(t) = \begin{cases}
(1 \ t \ \ldots \ t^{n-1})^T, & \text{if } j = 1 \text{ or } \alpha_j \neq \alpha_{j-1},
\frac{d}{dt} f_{j-1}(t), & \text{otherwise},
\end{cases}
\end{equation}
where $\cdot^T$ is the transpose of a vector or matrix. As far as defining $V_c$ is concerned, $\alpha_j$ can be real or complex. But in this paper, we shall focus on real $\alpha_j$. In what follows, $\alpha_j$ and $V_c$, as well as
\begin{equation*}
\alpha_{\max} \overset{\text{def}}{=} \max_j |\alpha_j|,
\end{equation*}
are reserved for their assignments here.

(Optimal) condition numbers for real Vandermonde matrices have been systematically studied by Gautschi and his coauthor (see [7] and references therein), and more recently by Tyrtyshnikov [12], Beckermann [2], and Li [10]. In this paper, we shall establish three lower bounds on the $\ell_p$-condition number $\kappa_p(V_c) \equiv \|V_c\|_p \|V_c^{-1}\|_p$ in terms of $n$, or $n$ and $\alpha_{\max}$ and the interval $[\alpha, \beta]$ that contains all nodes. In particular, we will show that for fixed $k_{\max}$ (the largest multiplicity of distinct nodes), $\kappa_p(V_c)$ behaves no smaller than $O_n ((1 + \sqrt{2})^n)$, where notation $a_n = O_n(b_n)$ means $c_1 n^{d_1} \leq a_n / b_n \leq c_2 n^{d_2}$ for some constants $c_1$, $c_2$, $d_1$, and $d_2$.

Optimally conditioned confluent Vandermonde matrices can be much worse ill-conditioned than optimally conditioned Vandermonde matrices. One extreme example would be that all nodes are equal $\alpha_1 = \cdots = \alpha_n$ for which $V_c$ is lower triangular, and thus
\begin{equation*}
\kappa_p(V_c) \geq (n - 1)! \sim \sqrt{2\pi n^{n-1/2} e^{-n}}
\end{equation*}
by Stirling’s asymptotic formula [11, Page 18], and it becomes an equality for $\alpha_1 = \cdots = \alpha_n = 0$. While for optimally conditioned Vandermonde matrices, $\kappa_p(V)$ goes to $\infty$ as fast as $(1 + \sqrt{2})^n$ modulo a factor $n^d$ for $|d| \leq 1$ [2, 10].

The rest of this paper is organized as follows. A general lower bound on $\kappa_p(V_c)$ is established in Section [2] but it is not uniform. Uniform bounds for $p = \infty$ are obtained in Section [3] for all real $V_c$ and for $V_c$ with nonnegative nodes. Finally we present our concluding remarks in Section [4].

2. A GENERAL LOWER BOUND

Given $1 \leq p \leq \infty$, the $\ell_p$-norm of vector $u = (\mu_1 \ \mu_2 \ \cdots \ \mu_n)^T$ is defined as
\begin{equation*}
\|u\|_p = \left( \sum_{j=1}^{n} |\mu_j|^p \right)^{1/p}.
\end{equation*}
and \( \|u\|_\infty = \lim_{p \to \infty} \|u\|_p = \max_j |\mu_j| \). The associated \( \ell_p \)-operator norm of the \( m \times n \) matrix \( A \) is defined as
\[
\|A\|_p = \max_{u \neq 0} \frac{\|Au\|_p}{\|u\|_p}.
\]
(2.1)

It can be proved that \( \|A\|_p = \|A^T\|_{p'} \), upon noticing
\[
\|A\|_p = \max_{u \neq 0, v \neq 0} \|v\|_{p'} \|u\|_p,
\]
where \( 1/p + 1/p' = 1 \) (see also [9]).

Let \([\alpha, \beta]\) be the interval in which all \( \alpha_j \) lie.
\[
T_n(t) = \cos(n \arccos t) \quad \text{for } |t| \leq 1,
\]
(2.2) \[
\frac{1}{2} \left( t + \sqrt{t^2 - 1} \right)^n + \frac{1}{2} \left( t - \sqrt{t^2 - 1} \right)^n \quad \text{for } |t| \geq 1
\]
is the \( n \)th Chebyshev polynomial of the first kind. Define the \( n \)th translated Chebyshev polynomial \( T_n(x; \omega, \tau) \) \( \overset{\text{def}}{=} T_n(x/\omega + \tau) \), where
\[
\omega = \frac{\beta - \alpha}{2} > 0, \quad \tau = -\frac{\beta + \alpha}{\beta - \alpha}
\]
Let \( a_{jn} \equiv a_{jn}(\omega, \tau) \) be the coefficient of \( x^j \) in \( T_n(x; \omega, \tau) \), i.e.,
\[
T_n(x; \omega, \tau) = a_{nn}x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0.
\]
(2.4) Define \([10]\)
\[
S_{n,p}(\omega, \tau) = \left( \sum_{j=0}^{n} |a_{jn}|^p \right)^{1/p}.
\]

Now we are ready to state our main theorem for the section.

**Theorem 2.1.** Assume that there are \( \ell \) distinct nodes \( \alpha_j \), and let \( k_{\max} \) be the largest multiplicity of the distinct nodes. Then
\[
\kappa_p(V_c) \geq \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \omega^{k-1} \times \max\{\ell/\ell', \alpha_{\max}^{n-1} \} \frac{S_{n-1,p}(\omega, \tau)}{n^{1/\ell'}}.
\]
(2.5)

**Proof.** Inequality (2.5) is a consequence of Lemmas [2.1] and [2.3] below.

For \( k_{\max} = 1 \), i.e., \( \ell = n \) and \( k_1 = \cdots = k_n = 1 \) (and thus \( V_c = V \)), (2.5) becomes one of the lower bounds for \( \kappa_p(V) \) in [10]. The right-hand side of (2.5) entails the explicit computation of \( S_{n,p}(\omega, \tau) \). It can also be estimated fairly well, too, by
\[
n^{-1/\ell} S_{n-1,1}(\omega, \tau) \leq S_{n-1,p}(\omega, \tau) \leq S_{n-1,1}(\omega, \tau),
\]
(2.6) \[
[n/2]^{-1/\ell} S_{n-1,1}(\omega, 0) \leq S_{n-1,p}(\omega, 0) \leq S_{n-1,1}(\omega, 0),
\]
(2.7) in connection with the explicit formulas for \( S_{n-1,1}(\omega, \tau) \) for \( \tau = 0 \) or \( |\tau| \geq 1 \) in [10]. Here \( |\xi| \) is the smallest integer that is larger than \( \xi \). The formulas are
\[
S_{n-1,1}(\omega, 0) = T_{n-1}(t/\omega) \sim \frac{1}{2} \left( \frac{1}{\omega} + \sqrt{1 + \frac{1}{\omega^2}} \right)^{n-1},
\]
(2.8)
where $\iota = \sqrt{-1}$, and for $\alpha \geq 0$ (for which $\tau \leq -1$),

$$S_{n-1,1}(\omega, \tau) = T_{n-1}(|\tau| + 1/\omega) \approx \frac{1}{2} \left[ \frac{1}{\omega} + |\tau| \right] + \sqrt{\left( \frac{1}{\omega} + |\tau| \right)^2 - 1} \right]^{n-1}.$$

**Lemma 2.1.** Assume that there are $\ell$ distinct nodes $\alpha_j$. Then

$$\|V_{c}^\ell\|_p \geq \max \left\{ \|e_j\|_p, \alpha_1^{n-1} \right\},$$

$$\|V_{c}^\ell\|_p \geq \sum_{j=1}^{n} \alpha_{\max}^{(j-1)p}.$$

**Proof.** Let $e_j$ be the $j$th column of the $n \times n$ identity matrix $I_n$ (or simply $I$ if $n$ is clear from the context). Use $\|V_{c}^\ell\|_p \geq \|V_{c}^\ell e_j\|_p$ and $\|V_{c}^\ell\|_p \geq \|V_{c}^\ell e_n\|_p$ to get (2.10), and use $\|V_{c}^\ell\|_p \geq \max_j \|V_{c}^\ell e_j\|_p$ to get (2.11). \qed

**Lemma 2.2.** For $0 \leq k \leq n$,

$$\left| \frac{d}{dx^k} T_n(x; \omega, \tau) \right| \leq \frac{n(n-1) \cdots (n-k+1)^2}{\omega^k} \text{ for } x \in [\alpha, \beta].$$

**Proof.** It follows from $T_n(x; \omega, \tau) = T_n(x/\omega + \tau) \equiv T_n(t)$ that

$$\left| \frac{d}{dx^k} T_n(x; \omega, \tau) \right| = \frac{1}{\omega^k} T_n^{(k)}(t),$$

where $t \equiv t(x) = x/\omega + \tau$. It suffices to show that $|T_n^{(k)}(t)| \leq |n(n-1) \cdots (n-k+1)|^2$ for $t \in [-1,1]$ since $t(x)$ maps $x \in [\alpha, \beta]$ to $t \in [-1,1]$. By Markov’s inequality [5, Page 233],

$$\max_{t \in [-1,1]} |T_n^{(k)}(t)| \leq \frac{(n-k)^2}{\omega^k} \max_{t \in [-1,1]} |T_n^{(k-1)}(t)| \leq \cdots \leq \frac{n(n-1) \cdots (n-k+1)^2}{\omega^k} \max_{t \in [-1,1]} |T_n(t)| = \frac{n(n-1) \cdots (n-k+1)^2}{\omega^k},$$

as expected. \qed

**Lemma 2.3.** Under the conditions of Theorem 2.1.

$$\|V_{c}^{-1}\|_p \geq \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \omega^{k-1} \times S_{n-1,p'}(\omega, \tau)\frac{S_{n-1,p'}(\omega, \tau)}{n^{1/p'}}.$$

**Proof.** For the sake of this proof, let the $\ell$ distinct nodes have multiplicities $k_1, k_2, \ldots, k_{\ell}$, respectively, where $k_1 + k_2 + \cdots + k_{\ell} = n$, and the first $k_1 \alpha_j$’s are equal, the next $k_2 \alpha_j$’s are equal, and so on. Let $v$ be the vector of the coefficients of the translated Chebyshev polynomial $T_{n-1}(x; \omega, \tau)$, i.e., $v = (a_{0,n-1} a_{1,n-1} \cdots a_{n-1,n-1})^T$. Then

$$V_{c}^T v = \left( T_{n-1}(\alpha_1; \omega, \tau) \ T_{n-1}(\alpha_1; \omega, \tau) \cdots T_{n-1}(\alpha_1; \omega, \tau) \cdots \right)^T.$$
which yields, by Lemma 2.2 for $1 \leq p' < \infty$

\[(2.14) \| V_c^T v \|_{p'} \leq \sum_{j=1}^{\ell} \left( 1^{p'} + \left( \frac{(n-1)^2}{\omega} \right)^{p'} \right) + \cdots + \left( \frac{(n-1)(n-2)\cdots(n-k_j+1)^2}{\omega^{k_j-1}} \right)^{p'} \right) \]

\[\leq \sum_{j=1}^{\ell} k_j \times \left( \max_{1 \leq k \leq k_j} \left[ \frac{(n-1)!}{(n-k)!} \right]^2 \frac{1}{\omega^{k-1}} \right)^{p'}. \]

(2.15) \[\leq n \times \left( \max_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n-1)!}{(n-k)!} \right]^2 \frac{1}{\omega^{k-1}} \right)^{p'}, \]

which gives

\[(2.16) \| V_c^T v \|_{p'} \leq n^{1/p'} \times \max_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n-1)!}{(n-k)!} \right]^2 \frac{1}{\omega^{k-1}}. \]

This is proved so far for $1 \leq p' < \infty$, but it can be verified that (2.16) holds for $p' = \infty$, too. Therefore, we have

\[\| V_c^{-T} \|_{p'} = \max_{u} \frac{\| u \|_{p'}}{\| V_c^T u \|_{p'}} \geq \frac{\| v \|_{p'}}{\| V_c^T v \|_{p'}} \geq \min_{1 \leq k \leq k_{\text{max}}} \left[ \frac{(n-1)!}{(n-k)!} \right]^2 \omega^{k-1} \times \frac{S_{n-1, p'}(\omega, \tau)}{n^{1/p'}}, \]

as was to be shown. \[\square\]

In general, we may use (2.14), instead of (2.15), in estimating $\| V_c^{-1} \|_p$. Doing so, however, will lead to a more complicated lower bound on $\kappa_p(V_c)$.

**Remark 2.1.** Lemma 2.3 is made possible by Lemma 2.2 which is proved with the help of Markov’s inequality. Another classical inequality for the same purpose is Bernstein’s inequality [5, Page 233], using which we can obtain the following. For $0 \leq k \leq n$, if $\alpha < a \overset{\text{def}}{=} \min_j \alpha_j < b \overset{\text{def}}{=} \max_j \alpha_j < \beta$, then

\[(2.17) \frac{d}{dx^k} T_n(x; \omega, \tau) \leq \frac{n(n-1)\cdots(n-k+1)}{\omega \sqrt{1 - \left( \frac{\max(b-a, a-\omega)}{\omega} \right)^2}} \text{ for } x \in [\alpha, \beta]. \]

This inequality improves (2.12) in the numerator part but has complications in the denominator, and also it requires the interval $[\alpha, \beta]$ to be (slightly) larger than the smallest interval containing all nodes. This can be bad because larger $[\alpha, \beta]$ will weaken the effectiveness of $S_{n, p'}(\omega, \tau)$ in the later bounds on $\kappa_p(V_c)$; for example $S_{n, p'}(\omega, \tau)$ is decreasing in $\omega$ [10].

3. **Two uniform bounds**

We present two theorems here, one for any real $V_c$ and one for $V_c$ with nonnegative nodes. Their proofs will be given later after two lemmas. Again let $1 \leq p \leq \infty$ and $1/p + 1/p' = 1$. 
Theorem 3.1. Under the conditions of Theorem 2.1, if

\[
(3.1) \quad k_{\max} - 1 \leq \frac{n - 1}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-2n+2} \right] \sim \frac{n - 1}{\sqrt{2}},
\]

then

\[
\kappa_p(V_e) \geq \left[ \frac{(n - k_{\max})!}{(n-1)!} \right]^2 \frac{S_{n-1,1}(1,0)}{n^{1/p} \sqrt{n/2}^{1/p} / n} \sim \left[ \frac{(n - k_{\max})!}{(n-1)!} \right]^2 \frac{1 + \sqrt{2}}{n^{1/p} \sqrt{n/2}^{1/p}}.
\]

Theorem 3.2. Under the conditions of Theorem 2.1, if all \(\alpha_i \geq 0\) and

\[
(3.2) \quad k_{\max} - 1 \leq \frac{n - 1}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-4(n-1)} \right]^{-1} \sim \frac{n - 1}{\sqrt{2}},
\]

then

\[
\kappa_p(V_e) \geq \left[ \frac{(n - k_{\max})!}{(n-1)!} \right]^2 \frac{1}{2^{k_{\max}-1} n} \frac{S_{n-1,1}(1/2,1)}{n} \sim \left[ \frac{(n - k_{\max})!}{(n-1)!} \right]^2 \frac{1}{2^{k_{\max}-1} n} \frac{1 + \sqrt{2}}{n}. \]

Lemma 3.1. Let \(j \geq 0\) and \(m \geq 1\). \(\rho^j S_{m,1}(\rho,0)\) is decreasing in \(\rho\) for \(0 \leq \rho \leq 1\) if

\[
(3.3) \quad j \leq \frac{m}{\sqrt{2}} \left[ 1 - (1 + \sqrt{2})^{-2m} \right] \sim \frac{m}{\sqrt{2}}
\]

Proof. We claim that under inequality (3.3), \(d/d\rho \rho^j S_{m,1}(\rho,0) \leq 0\) for \(0 \leq \rho \leq 1\). To this end, we notice that

\[
\frac{d}{d\rho} \rho^j S_{m,1}(\rho,0) = j \rho^{j-1} S_{m,1}(\rho,0) + \rho^j \frac{d}{d\rho} S_{m,1}(\rho,0).
\]

Now for \(0 \leq \rho \leq 1\) and by (2.8), we have

\[
S_{m,1}(\rho,0) \quad \leq \quad \frac{1}{2} \left[ \frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}} \right]^m \left[ 1 + \epsilon^{-2m} \right],
\]

\[
-\frac{d}{d\rho} S_{m,1}(\rho,0) \quad \geq \quad \frac{m}{2} \left[ \frac{1}{\rho} + \sqrt{1 + \frac{1}{\rho^2}} \right]^{m-1} \left[ 1 - \delta^{-2m} \right]
\times \left[ \frac{1}{\rho^2} + \frac{1}{\rho^2 \sqrt{1 + \rho^2}} \right],
\]

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where $\epsilon = 1 + \sqrt{2}$ and $\delta = 0$ for even $m$, and $\epsilon = 0$ and $\delta = 1 + \sqrt{2}$ for odd $m$. Therefore, for $\rho \leq 1$,
\[
\frac{d}{d\rho} \rho^j S_{m,1}(\rho,0) = j \rho^j S_{m,1}(\rho,0) \leq \frac{\rho^j S_{m,1}(\rho,0)}{m S_{m,1}(\rho,0)} = \frac{j}{m} \rho^j \left( \frac{1}{\rho^j} + \frac{1}{\rho^j \sqrt{1 + \frac{1}{\rho^j}}} \right) \frac{1}{1 - \delta^{-2m}} \leq \frac{j}{m} \left( 1 - \delta^{-2m} \right) \leq \frac{j}{m} 1 - \epsilon^{-2m} \leq 0
\]
upon using (3.3).

**Lemma 3.2.** Let $j \geq 0$, $\gamma \geq 1$, and $m \geq 1$. For $j$ satisfying $j \leq m$, and $\rho > 0$,
\[\rho^j \max\{\gamma, \rho^m\} S_{m,1}(\rho,0) \geq S_{m,1}(1,0).\]

**Proof.** Let $\Phi_1 = \rho^j \times \gamma S_{m,1}(\rho,0)$ and $\Phi_2 = \rho^j \times \rho^m S_{m,1}(\rho,0)$. Then $\max\{\Phi_1, \Phi_2\}$ is $\Phi_1$ for $\rho \leq \gamma^{1/m}$ and $\Phi_2$ for $\rho > \gamma^{1/m}$. $\Phi_2$ is increasing in $\rho$ for $\rho > 0$ because $\rho^m S_{m,1}(\rho,0)$ is a polynomial in $\rho$ with nonnegative coefficients and thus increasing in $\rho$ for $\rho > 0$. So
\[\max\{\Phi_1, \Phi_2\} \geq \Phi_2 \geq S_{m,1}(1,0) \text{ for } \rho \geq 1.\]

For $0 \leq \rho \leq 1$, $\Phi_1$ is decreasing in $\rho$ by Lemma 3.1 and thus
\[\max\{\Phi_1, \Phi_2\} \geq \Phi_1 \geq S_{m,1}(1,0) \text{ for } \rho \leq 1.\]
This completes the proof. \hfill \square

**Proof of Theorem 3.1** Setting $-\alpha = \beta = \alpha_{\max}$ in (2.5), we have, upon using (2.7),
\[
\kappa_p(V_c) \geq \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \alpha_{\max}^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^{n-1}\} \frac{S_{n-1,1}(\alpha_{\max},0)}{n^{1/p'} [n/2] \ell^{1/p'}} \geq \left[ \frac{(n-k_{\max})!}{(n-1)!} \right]^2 \frac{1}{n^{1/p'} [n/2] \ell^{1/p'}} \min_{1 \leq k \leq k_{\max}} \psi,
\]
where $\psi = \alpha_{\max}^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^{n-1}\} S_{n-1,1}(\alpha_{\max},0)$. Apply Lemma 3.2 with $j = k - 1, m = n - 1$, $\gamma = \ell^{1/p'}$, and $\rho = \alpha_{\max}$ to get $\psi \geq S_{n-1,1}(1,0)$, as needed. \hfill \square

**Proof of Theorem 3.2** Setting $0 = \alpha < \beta = \alpha_{\max}$ in (2.5), we have, upon using (2.0),
\[
\kappa_p(V_c) \geq \min_{1 \leq k \leq k_{\max}} \left[ \frac{(n-k)!}{(n-1)!} \right]^2 \left[ \frac{\alpha_{\max}^{n-1}}{2} \right]^{k-1} \times \max\{\ell^{1/p'}, \alpha_{\max}^{n-1}\} \frac{S_{n-1,1}(\alpha_{\max}/2,1)}{n} \geq \left[ \frac{(n-k_{\max})!}{(n-1)!} \right]^2 \frac{1}{n^{2k_{\max}-1}} \min_{1 \leq k \leq k_{\max}} \psi,
where\[\tilde{\Psi} = \alpha_{\text{max}}^{k-1} \times \max \{\ell^{1/p'}, \alpha_{\text{max}}^{n-1} \} S_{n-1,1}(\alpha_{\text{max}}/2, 1),\]

It can be verified by (2.3), (2.8), and (2.9) that\[S_{n-1,1}(\alpha_{\text{max}}/2, 1) = S_{2(n-1),1}(\sqrt{\alpha_{\text{max}}}, 0).\]

Therefore\[\tilde{\Psi} = (\sqrt{\alpha_{\text{max}}})^{2(k-1)} \times \max \left\{\ell^{1/p'}, (\sqrt{\alpha_{\text{max}}})^{2(n-1)}\right\} S_{2(n-1),1}(\sqrt{\alpha_{\text{max}}}, 0) \geq S_{2(n-1),1}(1, 0),\]

upon using Lemma 3.2 with \(j = 2(k - 1), m = 2(n - 1), \gamma = \ell^{1/p'},\) and \(\rho = \sqrt{\alpha_{\text{max}}}\). □

4. CONCLUDING REMARKS

We have obtained three lower bounds on the condition number \(\kappa_p(V_c)\) of a real confluent Vandermonde matrix \(V_c\). Two of them are uniform in the sense that they depend on \(n\), the dimension of \(V_c\) only, while the other one is more general, as is the function of \(n\) and \(\alpha_{\text{max}}\) and the interval \([\alpha, \beta]\) that contains all \(\alpha_j\). These bounds grow exponentially for any fixed \(k_{\text{max}},\) much as expected. While it is not clear in general if (any of) our bounds are asymptotically optimal, in contrast to those for Vandermonde matrices by Beckermann [2] and recently by the author [10], our bounds are unlikely to be asymptotically optimal if \(k_{\text{max}}\) also grows, e.g., linearly in \(n\). This is illustrated by the extreme example \(k_{\text{max}} = n,\) as we commented in Section 1.

We have focused on real confluent Vandermonde matrices here. It is conceivable that there would be much better conditioned complex confluent Vandermonde matrices or confluent Vandermonde-like matrices. This is partly an intuition one might get from that although real Vandermonde matrices are very ill-conditioned [7, 2, 10, 12], there exist very well-conditioned complex Vandermonde matrices and Vandermonde-like matrices [6, 11]. We plan to investigate this issue in future work.

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REFERENCES


Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506
E-mail address: rcli@ms.uky.edu
URL: http://www.ms.uky.edu/~rcli