SHEPARD–BERNOULLI OPERATORS

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ABSTRACT. We introduce the Shepard–Bernoulli operator as a combination of the Shepard operator with a new univariate interpolation operator: the generalized Taylor polynomial. Some properties and the rate of convergence of the new combined operator are studied and compared with those given for classical combined Shepard operators. An application to the interpolation of discrete solutions of initial value problems is given.

1. THE PROBLEM

Let $X = \{x_1, \ldots, x_N\}$ be a set of N distinct points of $\mathbb{R}^s, s \in \mathbb{N}$, and let f be a function defined on a domain D containing X. The classical Shepard operators (first introduced in [24] in the particular case $s = 2$) are defined by

$$(1.1) \qquad S_{N,\mu}[f](x) = \sum_{i=1}^{N} A_{\mu,i}(x) f(x_i), \quad \mu > 0,$$

where the weight functions $A_{\mu,i}(x)$ in barycentric form are

$$(1.2) \qquad A_{\mu,i}(x) = \frac{|x - x_i|^{-\mu}}{\sum\limits_{k=1}^{N} |x - x_k|^{-\mu}}$$

and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^s$. The interpolation operator $S_{N,\mu}[f]$ is stable, in the sense that

$$\min_i f(x_i) \le S_{N,\mu}[f](x) \le \max_i f(x_i),$$

but for $\mu > 1$ the interpolating function $S_{N,\mu}[f](x)$ has flat spots in the neighborhood of all data points. Also, the degree of exactness of the operator $S_{N,\mu}[\cdot]$ is 0, in the sense that if it is restricted to the polynomial space $P^m := \{p : \deg(p) \le m\}$, then $S_{N,\mu}[\cdot]|_{P^m} = Id_{P^m}$ (the identity function on P^m) only for $m = 0$. These drawbacks can be avoided by replacing each value $f(x_i)$ in (1.1) with an interpolation operator in x_i, applied to f, having a certain degree of exactness $m > 0$. More precisely, if for each $i = 1, \ldots, N$ $P[\cdot, x_i]$ denotes such an interpolation operator in x_i, then the related combined Shepard operator is

$$(1.3) \qquad S_{N,\mu}P[f](x) = \sum_{i=1}^{N} A_{\mu,i}(x) P[f, x_i](x).$$

Shepard himself proposed such a generalization (in the bivariate case) by using for each $i = 1, \ldots, N$ instead of $f(x_i)$ the Taylor polynomial of f of degree 1 at x_i, when the gradient of f at each node x_i is also given. In this case the combined operator has degree of exactness 1 and if $\mu > 1$ it also interpolates the first partial derivatives at the interpolation nodes [24].

Several interpolation operators have been introduced and studied to increase the degree of exactness of the Shepard operator: Taylor [7, 8, 14, 15, 24], Lagrange [8, 9], Hermite [5, 8], Birkhoff [6, 8], least square approximations [20, 21, 22, 25], and splines [8]. The present paper is devoted to a further combination of the Shepard operator with a new interpolation operator, the generalized Taylor polynomial, recently proposed by F. Costabile [10]. In section 2 we recall some known properties of the generalized Taylor polynomial and give new results on the error of approximation that will be used later in the paper. In section 3 we define the univariate Shepard–Bernoulli operator and study its rate of convergence when applied to functions in the class C^m in an interval containing the nodes. This operator has degree of exactness equal to m, while the Shepard–Taylor, Shepard–Lagrange, and Shepard–Hermite operators all have degree of exactness equal to $m - 1$. The considered operators all have the same rate of convergence; numerical examples, given in section 4, demonstrate the accuracy of the proposed combination in some special situations. Finally, in section 5 we apply the combined Shepard operators to the problem of interpolating the discrete solutions of initial value problems for ordinary differential equations. We interpolate the exact solution of such a problem by a rational (real analytic) function with an order of approximation that is comparable with that of the classical Hermite cubic spline interpolation. In our opinion this kind of approximation is desirable, in particular if the exact solution is analytic.

2. Some remarks about the generalized Taylor polynomial

The generalized Taylor polynomial is an expansion in Bernoulli polynomials, i.e., in the polynomials defined recursively by means of the following relations [18]:

$$(2.1) \qquad \begin{cases} B_0(x) = 1, \\ B_n'(x) = nB_{n-1}(x), & n \geq 1, \\ \int\limits_0^1 B_n(x)\, dx = 0, & n \geq 1. \end{cases}$$

For functions in the class $C^m([a,b])$, $a, b \in \mathbb{R}$, $a < b$, this expansion is realized by the equation

$$(2.2) \qquad f(x) = P_m[f, a, b](x) + R_m[f, a, b](x), \quad x \in [a, b],$$

where the polynomial approximant is defined by

$$(2.3) \quad P_m[f, a, b](x) = f(a) + \sum_{k=1}^{m} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!} h^{k-1} \left(f^{(k-1)}(b) - f^{(k-1)}(a)\right)$$

and the remainder term is
$$(2.4)$$

$$R_m[f, a, b](x) = \frac{h^{m-1}}{m!} \int\limits_a^b f^{(m)}(t) \left(B_m\left(\frac{b-t}{h}\right) - B_m\left(\frac{(x-t) - [x-t]}{h}\right)\right) dt,$$

where we have set $B_k = B_k(0)$ and we have denoted by $[\cdot]$ the integer part of the argument and $h = b - a$. The name chosen for the polynomial approximant $P_m[f, a, b]$ is derived from a nice property of this operator: its limit when $h \to 0$ is the Taylor polynomial $T_m[f, a](x)$ of degree m of f about a:

$$\lim_{h \to 0} P_m[f, a, b](x) = T_m[f, a](x).$$

Moreover, the polynomial $P_m[f, a, b]$ satisfies the following interpolation conditions:

(2.5) $$P_m[f, a, b](a) = f(a), \quad P_m[f, a, b](b) = f(b),$$

and
(2.6)
$$\frac{d^k}{dx^k} P_m[f, a, b](b) - \frac{d^k}{dx^k} P_m[f, a, b](a) = f^{(k)}(b) - f^{(k)}(a), \quad k = 1, \ldots, m - 1.$$

Another notable property of (2.3) is that, for each fixed $m \geq 1$, this approximant is exact for polynomials of degree not greater than m, since

$$\int_a^b \left(B_m \left(\frac{b - t}{h} \right) - B_m \left(\frac{(x - t) - [x - t]}{h} \right) \right) dt = 0$$

for each $x \in [a, b]$.

Clearly a polynomial can be defined by means of equation (2.3) also in the case when $a > b$; a property of $P_m[f, a, b](x)$ not stated in [10] is that this polynomial does not depend on the order of the points a, b.

Theorem 2.1. *For each $a, b \in \mathbb{R}, \ a \neq b$,*

$$P_m[f, b, a](x) = P_m[f, a, b](x), \quad \textit{for each } m \geq 1, \ x \in \mathbb{R}.$$

Proof. We will use the well-known symmetry of the Bernoulli polynomials [1]

$$B_n(1 - x) = (-1)^n B_n(x), \ n \geq 0, \ x \in \mathbb{R},$$

and the property of the Bernoulli numbers [1]

$$B_n = (-1)^n B_n \quad \text{for each } n \geq 2.$$

By a straightforward calculation we have

$$P_m\left[f,b,a\right](x)$$

$$= f\left(b\right) + \sum_{k=1}^{m} \frac{B_k\left(\frac{x-b}{a-b}\right) - B_k}{k!} \left(a-b\right)^{k-1}\left(f^{(k-1)}\left(a\right) - f^{(k-1)}\left(b\right)\right)$$

$$= f\left(b\right) + \sum_{k=1}^{m} \frac{B_k\left(1-\frac{x-a}{b-a}\right) - B_k}{k!} \left(-1\right)^k \left(b-a\right)^{k-1}\left(f^{(k-1)}\left(b\right) - f^{(k-1)}\left(a\right)\right)$$

$$= f\left(b\right) - \left(B_1\left(1-\frac{x-a}{b-a}\right) - B_1\right)\left(f\left(b\right) - f\left(a\right)\right)$$

$$+ \sum_{k=2}^{m} \frac{B_k\left(1-\frac{x-a}{b-a}\right) - B_k}{k!} \left(-1\right)^k \left(b-a\right)^{k-1}\left(f^{(k-1)}\left(b\right) - f^{(k-1)}\left(a\right)\right)$$

$$= f\left(b\right) - \left(1 - \frac{x-a}{b-a}\right)\left(f\left(b\right) - f\left(a\right)\right)$$

$$+ \sum_{k=2}^{m} \frac{\left(-1\right)^k B_k\left(\frac{x-a}{b-a}\right) - \left(-1\right)^k B_k}{k!} \left(-1\right)^k \left(b-a\right)^{k-1}\left(f^{(k-1)}\left(b\right) - f^{(k-1)}\left(a\right)\right)$$

$$= P_m\left[f,a,b\right](x). \qquad\qquad \square$$

With regard to the successive derivatives of the polynomial $P_m\left[f,a,b\right](x)$ we can prove the following result.

Theorem 2.2. *Let $f \in C^m\left([a,b]\right)$ and suppose that the derivative $f^{(m+1)}$ exists and is bounded in (a,b); set $h = b - a$. Then for each $j = 1, \ldots, m$ we have*

$$\left. \frac{d^j}{dx^j} P_m\left[f,a,b\right](x)\right|_{x=a} = f^{(j)}\left(a\right) + O\left(h^{m-j+1}\right).$$

Proof. By differentiating j times the polynomial $P_m\left[f,a,b\right](x)$ we have

$$\frac{d^j}{dx^j} P_m\left[f,a,b\right](x) = \sum_{k=j}^{m} \frac{B_{k-j}\left(\frac{x-a}{h}\right)}{(k-j)!} h^{k-j-1}\left(f^{(k-1)}\left(b\right) - f^{(k-1)}\left(a\right)\right),$$

so that

$$\left. \frac{d^j}{dx^j} P_m\left[f,a,b\right](x)\right|_{x=a} = \sum_{k=j}^{m} \frac{B_{k-j}}{(k-j)!} h^{k-j-1}\left(f^{(k-1)}\left(b\right) - f^{(k-1)}\left(a\right)\right)$$

$$= \sum_{k=0}^{m-j} \frac{B_k}{k!} h^{k-1}\left(f^{(k+j-1)}\left(b\right) - f^{(k+j-1)}\left(a\right)\right).$$

If $f \in C^m\left([a,b]\right)$ and the derivative $f^{(m+1)}$ exists and is bounded in (a,b), then by the well-known Taylor expansion with Lagrange remainder we have, for each $k = 0, 1, \ldots, m-j$,

$$h^{k-1}\left(f^{(k+j-1)}\left(b\right) - f^{(k+j-1)}\left(a\right)\right)$$

$$= \sum_{i=k+j}^{m} \frac{f^{(i)}\left(a\right)}{(i-k-j+1)!} h^{i-j} + \frac{f^{(m+1)}\left(\xi_k\right)}{(m-k-j+2)!} h^{m-j+1},$$

for some $\xi_k \in (a, b)$. Consequently

$$
\frac{d^j}{dx^j} P_m [f, a, b] (x) \Big|_{x=a}
$$

$$
= \sum_{k=0}^{m-j} \frac{B_k}{k!} \left(\sum_{i=k+j}^{m} \frac{f^{(i)}(a)}{(i-k-j+1)!} h^{i-j} + \frac{f^{(m+1)}(\xi_k)}{(m-k-j+2)!} h^{m-j+1} \right)
$$

$$
= \sum_{k=0}^{m-j} \sum_{i=k+j}^{m} \frac{B_k}{k!} \frac{f^{(i)}(a)}{(i-k-j+1)!} h^{i-j} + \sum_{k=0}^{m-j} \frac{B_k}{k!} \frac{f^{(m+1)}(\xi_k)}{(m-k-j+2)!} h^{m-j+1}
$$

$$
= \sum_{i=j}^{m} \left(\sum_{k=0}^{i-j} \frac{B_k}{k!(i-j+1-k)!} \right) f^{(i)}(a) h^{i-j} + \sum_{k=0}^{m-j} \frac{B_k}{k!} \frac{f^{(m+1)}(\xi_k)}{(m-k-j+2)!} h^{m-j+1}.
$$

Since [18, p. 233]

$$
\sum_{k=0}^{p} \frac{B_k}{k!(p+1-k)!} = \left\{ \begin{array}{l} 1, \text{ if } p = 0, \\ 0, \text{ if } p \geq 1, \end{array} \right.
$$

we find

$$
\frac{d^j}{dx^j} P_m [f, a, b] (x) \Big|_{x=a} = f^{(j)}(a) + \sum_{k=0}^{m-j} \frac{B_k}{k!} \frac{f^{(m+1)}(\xi_k)}{(m-k-j+2)!} h^{m-j+1}
$$

$$
= f^{(j)}(a) + O\left(h^{m-j+1} \right). \qquad \square
$$

Note that the polynomial $P_m[f, a, b]$ can be extended in a natural way to the whole real line. In order to obtain bounds for the remainder (2.4) even in points outside the interval $[a, b]$, we consider the operator

$$
f \rightarrow P_m[f, a, b]
$$

as acting on the space $C^m([c, d])$ with $c < a$ and $b < d$. Peano's kernel theorem [13, p. 70] provides an integral expression for the remainder (2.4), as indicated in the following theorem.

Theorem 2.3. *Let* $f \in C^m([c, d])$ *and* $x \in [c, d]$. *Then for the remainder*

(2.7) $$R_m[f, a, b](x) = f(x) - P_m[f, a, b](x)$$

the following integral representations hold:

(2.8) $$R_m[f, a, b](x) = \left\{ \begin{array}{ll} \frac{1}{(m-1)!} \int_x^b f^{(m)}(t) K_{a,b}(x, t)\, dt, & c \leq x \leq a, \\[2mm] \frac{1}{(m-1)!} \int_a^b f^{(m)}(t) K_{a,b}(x, t)\, dt, & a \leq x \leq b, \\[2mm] \frac{1}{(m-1)!} \int_a^x f^{(m)}(t) K_{a,b}(x, t)\, dt, & b \leq x \leq d, \end{array} \right.$$

where

$$
K_{a,b}(x, t) = (x - t)_+^{m-1} - (a - t)_+^{m-1}
$$

(2.9) $$- \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} \left[(b-t)_+^{m-k} - (a-t)_+^{m-k} \right] h^{k-1} \frac{B_k\left(\frac{x-a}{h} \right) - B_k}{k!}$$

and $(\cdot)_+^k$ *denotes the positive part of the* kth *power of the argument, i.e.,*

$$
(s)_+^k = \max\left\{ s^k, 0 \right\}.
$$

Proof. First note that in the expansion (2.3) there are evaluations of derivatives of f up to the order $m-1$ in points of $[c,d]$; on the other hand the exactness of the approximant (2.3) on the polynomial space P^m implies the exactness of this operator on the space P^{m-1}. By applying Peano's kernel theorem, we then have

$$(2.10) \qquad R_m\left[f,a,b\right](x) = \frac{1}{(m-1)!}\int_c^d f^{(m)}(t)\, K_{a,b}(x,t)\, dt,$$

where the kernel function (2.9) is obtained by applying the linear functional $f \to R_m\left[f,a,b\right](x)$ to $(x-t)_+^{m-1}$ considered as a function of x. If $x \in [c,a]$, then by the additivity of the integral we have

(2.11)
$$R_m\left[f,a,b\right](x)$$
$$= \frac{1}{(m-1)!}\int_c^x f^{(m)}(t)\, K_{a,b}(x,t)\, dt + \frac{1}{(m-1)!}\int_x^a f^{(m)}(t)\, K_{a,b}(x,t)\, dt$$
$$+ \frac{1}{(m-1)!}\int_a^b f^{(m)}(t)\, K_{a,b}(x,t)\, dt + \frac{1}{(m-1)!}\int_b^d f^{(m)}(t)\, K_{a,b}(x,t)\, dt.$$

Note that if $c < t < x$, then

$$K_{a,b}(x,t) = (x-t)^{m-1} - (a-t)^{m-1}$$
$$- \sum_{k=1}^m \frac{(m-1)!}{(m-k)!}\left[(b-t)^{m-k} - (a-t)^{m-k}\right] h^{k-1}\frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!} = 0,$$

because $(x-t)^{m-1}$ is a polynomial in x of degree $m-1$ so that it must coincide with its generalized Taylor expansion (2.3). By definition of the positive part, the kernel (2.9) is also zero in the interval $b < t < d$, so we have the first case of (2.8).

The remaining expressions may be obtained by analogous arguments. $\qquad\square$

The previous theorem allows us to obtain the desired bounds.

Theorem 2.4. *Let $f \in C^m\left([c,d]\right)$ and $x \in [c,d]$. Then for the remainder (2.7) we have*

$$(2.12) \qquad \left|R_m\left[f,a,b\right](x)\right| \leq \begin{cases} C\left(m\right)\left\|f^{(m)}\right\|(b-x)^m, & c < x < a, \\ C\left(m\right)\left\|f^{(m)}\right\|(b-a)^m, & a < x < b, \\ C\left(m\right)\left\|f^{(m)}\right\|(x-a)^m, & b < x < d, \end{cases}$$

where $\|\cdot\|$ denotes the sup-norm on $[c,d]$ and

$$(2.13) \qquad C\left(m\right) = \frac{1}{m!}\left(1 + \sum_{k=1}^m\sum_{l=1}^k\binom{m}{k}\binom{k}{l}|B_{k-l}|\right), \qquad m = 1,2,\dots.$$

Proof. Let us assume that $c < x < a$. In this case we have from (2.8) that

$$R_m\left[f,a,b\right](x)$$
$$= \frac{1}{(m-1)!}\int_x^a f^{(m)}(t)\, K_{a,b}(x,t)\, dt + \frac{1}{(m-1)!}\int_a^b f^{(m)}(t)\, K_{a,b}(x,t)\, dt.$$

If $x < t < a$, then

$K_{a,b}(x,t)$

$$= -(a-t)^{m-1} - \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} \left[(b-t)^{m-k} - (a-t)^{m-k} \right] h^{k-1} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!}$$

so that

$$\int_x^a K_{a,b}(x,t) f^{(m)}(t)\, dt$$

$$= -\int_x^a (a-t)^{m-1} f^{(m)}(t)\, dt - \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} h^{k-1} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!}$$

$$\times \int_x^a \left((b-t)^{m-k} - (a-t)^{m-k} \right) f^{(m)}(t)\, dt.$$

Note that the integrands are of type $g(t) f^{(m)}(t)$ with a $g(t)$ that does not change sign in $[x,a]$. In view of this last property, by the first mean value theorem for integrals [13, Theorem 1.4.2] we have

$$\int_x^a K_{a,b}(x,t) f^{(m)}(t)\, dt$$

$$= -f^{(m)}(\xi) \int_x^a (a-t)^{m-1}\, dt - \sum_{k=1}^{m} \frac{(m-1)!}{(m-k)!} h^{k-1} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!}$$

$$\times f^{(m)}(\xi_k) \int_x^a \left((b-t)^{m-k} - (a-t)^{m-k} \right) dt$$

for some $\xi, \xi_k \in [c,b]$, $k = 1, \ldots, m$, so that we find after some calculations

$$\int_x^a K_{a,b}(x,t) f^{(m)}(t)\, dt$$

$$= -f^{(m)}(\xi) \frac{(a-x)^m}{m} - h^m \sum_{k=1}^{m} \frac{(m-1)!}{(m-k+1)!} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!}$$

(2.14)
$$\times f^{(m)}(\xi_k) \left(-1 + \sum_{j=0}^{m-k} \left(\frac{b-x}{h} \right)^{m-k-j} \left(\frac{a-x}{h} \right)^{j} \right).$$

If $a < t < b$, then

$$K_{a,b}(x,t) = -\sum_{k=1}^{m} \frac{(m-1)!}{(m-i)!} (b-t)^{m-k} h^{k-1} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!}$$

and

$$\int_a^b K_{a,b}(x,t) f^{(m)}(t)\, dt$$

$$= -\sum_{k=1}^{m} \frac{(m-1)!}{(m-i)!} h^{k-1} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!} \int_a^b (b-t)^{m-k} f^{(m)}(t)\, dt,$$

and by applying the first mean value theorem for integrals we find after some calculations that

$$(2.15) \quad \int_a^b K_{a,b}(x,t) f^{(m)}(t)\, dt = -h^m \sum_{k=1}^m f^{(m)}(\eta_k) \frac{(m-1)!}{(m-k+1)!} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!}$$

for some $\eta_k \in [c, b]$, $k = 1, \ldots, m$. Substituting into (2.11) the left-hand sides of (2.14), (2.15) with their respective right-hand sides, we finally obtain the following expression for the error:

$$R_m[f, a, b](x)$$
$$= -f^{(m)}(\xi) \frac{(a-x)^m}{m!} - \frac{h^m}{m!} \sum_{k=1}^m f^{(m)}(\xi_k) \frac{m!}{(m-k+1)!} \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!}$$
$$\times \left(-1 + \sum_{j=0}^{m-k} \left(\frac{b-x}{h}\right)^{m-k-j} \left(\frac{a-x}{h}\right)^j\right) - \frac{h^m}{m!} \sum_{k=1}^m f^{(m)}(\eta_k) \frac{m!}{(m-k+1)!}$$
$$\times \frac{B_k\left(\frac{x-a}{h}\right) - B_k}{k!}$$

In order to establish the bound (2.12) we make use of the well-known identities

$$(2.16) \qquad B_k(x) - B_k = \sum_{l=1}^k \binom{k}{l} B_{k-l} x^l, \quad k = 1, 2, \ldots,$$

from which it results after some calculations

$$|R_m[f, a, b](x)|$$
$$\leq \frac{\|f^{(m)}\|}{m!} \left((a-x)^m + h^m \sum_{k=1}^m \sum_{l=1}^k \binom{m}{k}\binom{k}{l} |B_{k-l}| \left(\frac{b-x}{h}\right)^{m-(k-l)}\right)$$
$$\leq h^m \frac{\|f^{(m)}\|}{m!} \left(1 + \sum_{k=1}^m \sum_{l=1}^k \binom{m}{k}\binom{k}{l} |B_{k-l}|\right) \left(\frac{b-x}{h}\right)^m.$$

The third expression (2.12) can be proved in an analogous manner.   □

Since the degree of exactness of the operator $P_m[\cdot, a, b]$ is equal to m, we can prove the following result in an analogous manner.

Theorem 2.5. *Let $f \in C^{m+1}([c, d])$ and $x \in [c, d]$. Then for the remainder (2.7) the inequalities*

$$|R_m[f, a, b](x)| \leq \begin{cases} C(m+1) \|f^{(m+1)}\| (b-x)^{m+1}, & c < x < a, \\ C(m+1) \|f^{(m+1)}\| (b-a)^{m+1}, & a < x < b, \\ C(m+1) \|f^{(m+1)}\| (x-a)^{m+1}, & b < x < d, \end{cases}$$

hold with $C(m)$ defined in (2.13).

The first values of $C(m)$ for initial values of m are given in Table 1.

For large values of m the constant $C(m)$ tends quickly to 0; in fact the following upper bound holds.

TABLE 1. First values of $C(m)$

m	1	2	3	4	5	6	7	8	9
$C(m)$	2.000	2.500	2.166	1.375	0.677	0.272	0.092	0.027	0.007

Theorem 2.6. *For each $m = 1, 2, \ldots$ the constant $C(m)$ defined in (2.13) satisfies the inequality*

$$C(m) \leq \frac{1}{m!} + 2\left(\frac{m}{2}\right)^{1-\frac{m}{2}} + \left(\frac{m-1}{2}\right)^{1-\frac{m-1}{2}}$$

$$+ \frac{2\pi^2}{3}\left[\frac{m-1}{2}\right]\frac{\left(\frac{m-2\left[\frac{m-31}{2}\right]}{2}\right)^{1-\frac{m-2\left[\frac{m-31}{2}\right]}{2}}}{(2\pi)^{2\left[\frac{m-31}{2}\right]}}.$$

Proof. Let us consider the sums involved in (2.13). By inverting their order and substituting $k - l$ with s as a dummy index we have

$$\sum_{k=1}^{m}\sum_{l=1}^{k}\binom{m}{k}\binom{k}{l}|B_{k-l}| = \sum_{l=1}^{m}\sum_{k=l}^{m}\binom{m}{k}\binom{k}{l}|B_{k-l}|$$

$$= \sum_{l=1}^{m}\sum_{s=0}^{m-l}\binom{m}{l+s}\binom{l+s}{l}|B_s|$$

$$= \sum_{s=0}^{m-1}\sum_{l=1}^{m-s}\binom{m}{l+s}\binom{l+s}{l}|B_s|.$$

Since by [1]

$$B_0 = 1, \;\; B_1 = -\frac{1}{2}, \;\; B_{2s+1} = 0, \;\; s = 1, 2, \ldots,$$

(2.13) becomes

$$C(m) = \frac{1}{m!} + \sum_{l=1}^{m}\frac{1}{m!}\binom{m}{l} + \sum_{l=1}^{m-1}\frac{1}{2m!}\binom{m}{l+1}l$$

$$+ \frac{1}{m!}\sum_{s=1}^{\left[\frac{m-1}{2}\right]}\sum_{l=1}^{m-2s}\binom{m}{l+2s}\binom{l+2s}{l}|B_{2s}|.$$

Now set

$$C_{m,s} := \frac{1}{m!}\sum_{l=1}^{m-2s}\binom{m}{l+2s}\binom{l+2s}{l}|B_{2s}|.$$

By the formula [16]

$$B_{2s} = \frac{(-1)^{s-1}2(2s)!}{(2\pi)^{2s}}\zeta(2s), \;\; s = 1, 2, \ldots,$$

where, as usual, ζ denotes the Riemann zeta function, and by the inequality [16]

$$n! > n^{\frac{n}{2}}, \;\; n = 1, 2, \ldots,$$

we obtain after some calculations

$$C_{m,s} \le 2 \sum_{l=1}^{m-2s} \frac{1}{(m-2s-l)^{\frac{m-2s-l}{2}} l^{\frac{l}{2}}} \times \frac{\zeta(2s)}{(2\pi)^{2s}}.$$

For each pair of fixed positive integers m, s, s.t. $m - 2s > 0$, the function

$$l \mapsto (m-2s-l)^{\frac{m-2s-l}{2}} l^{\frac{l}{2}}, \quad 0 < l < m - 2s,$$

attains its minimum value $\left(\frac{m-2s}{2}\right)^{\frac{m-2s}{2}}$ in $l = \frac{m-2s}{2}$, so that we get

$$C_{m,s} \le 2 \sum_{l=1}^{m-2s} \left(\frac{m-2s}{2}\right)^{-\frac{m-2s}{2}} \frac{\zeta(2s)}{(2\pi)^{2s}},$$

and since [16]

$$\zeta(2s) \le \zeta(2) = \frac{\pi^2}{6}, \ s = 1, 2, \ldots,$$

we finally obtain

$$C_{m,s} \le 4 \left(\frac{m-2s}{2}\right)^{1-\frac{m-2s}{2}} \frac{\frac{\pi^2}{6}}{(2\pi)^{2s}}.$$

We can prove in an analogous manner that

$$\sum_{l=1}^{m} \frac{1}{m!} \binom{m}{l} \le 2 \left(\frac{m}{2}\right)^{1-\frac{m}{2}}$$

and

$$\sum_{l=1}^{m-1} \frac{1}{2m!} \binom{m}{l+1} l \le \left(\frac{m-1}{2}\right)^{1-\frac{m-1}{2}}.$$

A straightforward calculation furthermore shows that

$$\left(\frac{m-2s}{2}\right)^{1-\frac{m-2s}{2}} \frac{1}{(2\pi)^{2s}} \le \frac{\left(\frac{m-2\left[\frac{m-31}{2}\right]}{2}\right)^{1-\frac{m-2\left[\frac{m-31}{2}\right]}{2}}}{(2\pi)^{2\left[\frac{m-31}{2}\right]}},$$

$$m = 1, 2, \ldots; \ s = 1, 2, \ldots, \left[\frac{m-1}{2}\right],$$

from which it easily follows that

$$\sum_{s=1}^{\left[\frac{m-1}{2}\right]} 4 \left(\frac{m-2s}{2}\right)^{1-\frac{m-2s}{2}} \frac{\frac{\pi^2}{6}}{(2\pi)^{2s}} \le \frac{2\pi^2}{3} \left[\frac{m-1}{2}\right] \frac{\left(\frac{m-2\left[\frac{m-31}{2}\right]}{2}\right)^{1-\frac{m-2\left[\frac{m-31}{2}\right]}{2}}}{(2\pi)^{2\left[\frac{m-31}{2}\right]}}$$

and then the assertion.   □

3. The Shepard–Bernoulli operator:
definition, error estimation, and rate of convergence

Let $x_1 < x_2 < \cdots < x_{N-1} < x_N$ be fixed points of an interval $I \subset \mathbb{R}$; we set $X = \{x_1, \ldots, x_N\}$ and $x_{N+1} = x_{N-1}$.

Definition 3.1. For each fixed $\mu > 0$ and $m = 1, 2, \ldots$ the Shepard–Bernoulli univariate operator is defined by

$$(3.1) \qquad S_{B_m}[f](x) = \sum_{i=1}^{N} A_{\mu,i}(x) P_m[f, x_i, x_{i+1}](x), \ x \in I,$$

where $P_m[f, x_i, x_{i+1}](x)$ is the natural extension of the polynomial approximant defined in (2.3) over the whole interval I.

The following statements can be checked without any difficulty.

Theorem 3.2. *The operator $S_{B_m}[\cdot]$ is an interpolation operator in x_i, $i = 1, \ldots, N$.*

Proof. The assertion follows from the relations (2.5) in view of the well-known property

$$(3.2) \qquad A_{\mu,i}(x_k) = \delta_{ik}, \quad i, k = 1, \ldots, N. \qquad \square$$

Theorem 3.3. *The degree of exactness of the operator $S_{B_m}[\cdot]$ is m, i.e., $S_{B_m}[p] = p$ for each univariate polynomial $p \in P^m$.*

Proof. The assertion follows from the well-known property

$$\sum_{i=1}^{N} A_{\mu,i}(x) \equiv 1$$

since the degree of exactness of $P_m[\cdot, x_i, x_{i+1}]$ is m for $i = 1, \ldots, N$. $\qquad \square$

As for the continuity class of the Shepard operator, and consequently the continuity class of the Shepard–Bernoulli operators, there is the following result [4].

Theorem 3.4. *If $P[\cdot, x_i]$, $i = 1, \ldots, N$, are polynomial interpolation operators in x_i, then the continuity class of the operator (1.3) depends upon μ and, for $\mu > 0$, is as follows:*

    i) *if μ is an even integer, then $S_{N,\mu}P[\cdot] \in C^{\infty}$;*
    ii) *if μ is an odd integer, then $S_{N,\mu}P[\cdot] \in C^{\mu-1}$;*
    iii) *if μ is not an integer, then $S_{N,\mu}P[\cdot] \in C^{[\mu]}$;*

here $[\mu]$ is the largest integer $< \mu$.

Remark 3.5. We emphasize that if μ is an even integer, then $S_{N,\mu}P[\cdot] \in C^{\omega}$ (the class of real analytic functions); in fact [8]

$$A_{\mu,i}(x) = \frac{\prod_{\substack{j=1 \\ j \neq i}}^{N} |x - x_j|^{\mu}}{\sum_{k=1}^{N} \prod_{\substack{j=1 \\ j \neq k}}^{N} |x - x_j|^{\mu}}.$$

Theorem 3.6. *For each j s.t. $1 \leq j < \mu$ we have*

$$\left. \frac{d^j}{dx^j} S_{B_m}[f](x) \right|_{x=x_k} = \left. \frac{d^j}{dx^j} P_m[f, x_k, x_{k+1}](x) \right|_{x=x_k}, \quad k = 1, .., N.$$

Proof. It is known that for each j s.t. $1 \leq j < \mu$ [8]

$$(3.3) \qquad \frac{d^j}{dx^j} A_{\mu,i}(x) \bigg|_{x=x_k} = 0, \quad i,k = 1,\ldots,N.$$

The assertion of the theorem then follows by applying the Leibniz rule and by using relations (3.2), (3.3):

$$\frac{d^j}{dx^j} S_{B_m}[f](x) \bigg|_{x=x_k}$$

$$= \sum_{i=1}^{N} \frac{d^j}{dx^j} (A_{\mu,i}(x) P_m[f,x_i,x_{i+1}](x)) \bigg|_{x=x_k}$$

$$= \sum_{i=1}^{N} \sum_{l=0}^{j} \binom{j}{l} \frac{d^{j-l}}{dx^{j-l}} A_{\mu,i}(x) \bigg|_{x=x_k} \frac{d^l}{dx^l} P_m[f,x_i,x_{i+1}](x) \bigg|_{x=x_k}$$

$$= \frac{d^j}{dx^j} P_m[f,x_k,x_{k+1}](x) \bigg|_{x=x_k}. \qquad \square$$

In order to study the rate of convergence of the Shepard–Bernoulli operator we follow the ideas of Farwig's paper [15], later pursued in [8]. We introduce the following notations:

$$I_\rho(x) = [x-\rho, x+\rho], \ \rho > 0,$$
$$r = \inf\{\rho > 0 : \forall x \in I, \ I_\rho(x) \cap X \neq \varnothing\},$$

and

$$M = \sup_{x \in I} \#(I_r(x) \cap X),$$

where $\#(\cdot)$ denotes the cardinality function, so M is the maximum number of points from X contained in an interval $I_r(x)$. For the Shepard–Bernoulli univariate operator we then have the following error estimates.

Theorem 3.7. *If $f \in C^m(I)$, then*

$$\|S_{B_m}[f] - f\| \leq CM \left\|f^{(m)}\right\| \varepsilon_\mu^{m-1}(r),$$

where $\|\cdot\|$ denotes the sup norm in I,

$$(3.4) \qquad \varepsilon_\mu^{m-1}(r) = \begin{cases} |\log r|^{-1}, & \mu = 1, \\ r^{\mu-1}, & 1 < \mu < m+1, \\ r^{\mu-1}|\log r|, & \mu = m+1, \\ r^m, & \mu > m+1, \end{cases}$$

C is a positive constant independent of x and X, and r is given above.

Proof. Let $a,b \in I$ be fixed, $a < b$; for each $x \in I$ we set

$$(3.5) \qquad d[a,b](x) = \begin{cases} b-x, & x < a, \\ b-a, & a \leq x \leq b, \\ x-a, & b < x, \end{cases}$$

i.e., $d[a,b](x)$ is the distance of x from the interval $[a,b]$, plus $b-a$; we set also

$$d^m[a,b](x) = (d[a,b](x))^m.$$

In view of (2.12) the following inequality holds :

$$|S_{B_m}[f](x) - f(x)| \le \left| \sum_{i=1}^{N} A_i(x) P_m[f, x_i, x_{i+1}](x) - f(x) \right|$$

$$\le \sum_{i=1}^{N} A_i(x) |P_m[f, x_i, x_{i+1}](x) - f(x)|$$

$$\le C(m) \left\| f^{(m)} \right\|_\infty s_\mu^m(x),$$

where

$$(3.6) \qquad s_\mu^m(x) = \frac{\sum\limits_{i=1}^{N} |x - x_i|^{-\mu} d^m[x_i, x_{i+1}](x)}{\sum\limits_{k=1}^{N} |x - x_k|^{-\mu}}.$$

We want to show that

$$s_\mu^m(x) \le CM\varepsilon_\mu^{m-1}(r),$$

where C is a constant independent of x and X. Denoting by $\operatorname{diam}(I)$ the diameter of the interval I, we set

$$n = \left[\frac{\operatorname{diam}(I)}{2r} \right] + 1,$$

$$Q_\rho(u) = (u - \rho, u + \rho], \quad u \in I, \ \rho > 0,$$

and

$$T_j = Q_r(x - 2rj) \cup Q_r(x + 2rj), \quad j = 0, 1, \ldots, n.$$

Clearly the set $\bigcup_{j=-n}^{n} Q_r(x + 2rj)$ is a covering of I with half open intervals; consequently, for each $i \in \{1, \ldots, N\}$ there exists a $j \in \{0, \ldots, n\}$ such that $x_i \in T_j$. Then the following inequalities hold:

$$(3.7) \qquad (2j - 1)r \le |x - x_i| \le (2j + 1)r,$$

and by (3.5)

$$(3.8) \qquad d[x_i, x_{i+1}](x) \le (2(j+1) + 1)r.$$

On the other hand, by definition of M it follows that

$$1 \le \#(X \cap T_j) \le M$$

for each $j = 0, 1, \ldots, n$. Let us denote by x_d the node closest to x; since

$$\frac{|x - x_d|^{-\mu}}{\sum\limits_{k=1}^{N} |x - x_k|^{-\mu}} \le 1$$

we get, by applying (3.7), (3.8)

$$s_\mu^m (x) \leq \sum_{x_i \in T_0} \frac{|x - x_i|^{-\mu} \, d^m \, [x_i, x_{i+1}] \, (x)}{\sum\limits_{k=1}^{N} |x - x_k|^{-\mu}} + \sum_{j=1}^{n} \sum_{x_i \in T_j} \frac{|x - x_i|^{-\mu} \, d^m \, [x_i, x_{i+1}] \, (x)}{\sum\limits_{k=1}^{N} |x - x_k|^{-\mu}}$$

$$\leq \sum_{x_i \in T_0} d^m \, [x_i, x_{i+1}] \, (x) + |x - x_d|^{\mu} \sum_{j=1}^{n} \sum_{x_i \in T_j} |x - x_i|^{-\mu} \, d^m \, [x_i, x_{i+1}] \, (x)$$

$$\leq M \, (3r)^m + 2Mr^{\mu} \sum_{j=1}^{n} ((2j-1) \, r)^{-\mu} \, ((2j+3) \, r)^m$$

$$\leq M5^m r^m \left(1 + 2 \sum_{j=1}^{n} j^{m-\mu} \right),$$

where the last inequality follows from

$$\begin{cases} 2j - 1 \geq j, & j = 1, 2, \ldots, \\ 2j + 3 \leq 5j, & j = 1, 2, \ldots. \end{cases}$$

We have two cases:

Case 1 ($\mu > 1$).

Subcase 1a. If $1 < \mu < m + 1$, then

$$r^m \left(1 + 2 \sum_{j=1}^{n} j^{m-\mu} \right) = O \left(r^{\mu-1} \right).$$

Subcase 1b. If $\mu = m + 1$, then

$$\sum_{j=1}^{n} j^{m-\mu} = \log n \geq |\log r| \, .$$

Subcase 1c. If $\mu > m + 1$, then $\sum_{j=1}^{n} j^{m-\mu}$ is bounded.

Case 2 ($\mu = 1$). The function $s_1^m (x)$ defined in (3.6),

$$s_1^m (x) = \frac{\sum\limits_{i=1}^{N} |x - x_i|^{-1} \, d^m \, [x_i, x_{i+1}] \, (x)}{\sum\limits_{k=1}^{N} |x - x_k|^{-1}}$$

has its denominator greater than or equal to

$$\sum_{x_k \in T_0} |x - x_k|^{-1} + \sum_{j=1}^{n} \sum_{x_k \in T_j} |x - x_k|^{-1} \geq \sum_{x_k \in T_0} |x - x_k|^{-1} + \frac{C}{r} \, |\log r|$$

and its numerator less than or equal to

$$\sum_{x_i \in T_0} |x - x_i|^{-1} \, d^m \, [x_i, x_{i+1}] \, (x) + \sum_{j=1}^{n} \sum_{x_i \in T_j} |x - x_i|^{-1} \, d^m \, [x_i, x_{i+1}] \, (x).$$

By applying the inequality

$$\frac{\sum a_i}{\sum b_i} \leq \sum \frac{a_i}{b_i}, \quad a_i \geq 0, \ b_i > 0,$$

we get

$$s_1^m(x) \leq \frac{\sum\limits_{x_i \in T_0} |x - x_i|^{-1} d^m [x_i, x_{i+1}] (x) + \sum\limits_{j=1}^{n} \sum\limits_{x_i \in T_j} |x - x_i|^{-1} d^m [x_i, x_{i+1}] (x)}{\sum\limits_{x_k \in T_0} |x - x_k|^{-1} + \frac{C}{r} |\log r|}$$

$$\leq \sum\limits_{x_i \in T_0} d^m [x_i, x_{i+1}] (x) + \frac{C_1 r}{|\log r|} \sum\limits_{j=1}^{n} \sum\limits_{x_i \in T_j} |x - x_i|^{-1} d^m [x_i, x_{i+1}] (x)$$

$$\leq M (3r)^m + \frac{C_1}{|\log r|} \sum\limits_{j=1}^{n} \sum\limits_{x_i \in T_j} (2j - 1)^{-1} (2j + 3)^m r^m$$

$$\leq M 5^m r^m \left(1 + \frac{C_2}{|\log r|} \sum\limits_{j=1}^{n} j^{m-1} \right)$$

$$\leq M 5^m r^m \left(1 + \frac{C_2}{|\log r|} O \left(r^{-m} \right) \right)$$

$$= O \left(|\log r|^{-1} \right). \qquad \square$$

Remark 3.8. In the papers [15], [8] estimates similar to those of Theorem 3.7 have been proven for the Shepard–Taylor, Shepard–Lagrange, and Shepard–Hermite combinations, thus the rate of convergence is equal for all considered operators. Nevertheless we underline the following fact: for fixed $m \geq 1$ and $f \in C^m (I)$, the Shepard–Bernoulli operator has degree of exactness m, while the degree of exactness of Shepard–Taylor, Shepard–Lagrange, and Shepard–Hermite combinations is $m - 1$.

In an analogous manner we can prove the following result:

Theorem 3.9. *If $f \in C^{m+1} (I)$, then*

$$\|S_{B_m} [f] - f\|_I \leq CM \left\| f^{(m+1)} \right\| \varepsilon_\mu^m (r)$$

where

(3.9) $$\varepsilon_\mu^m (r) = \begin{cases} |\log r|^{-1}, & \mu = 1, \\ r^{\mu-1}, & 1 < \mu < m + 2, \\ r^{\mu-1} |\log r|, & \mu = m + 2, \\ r^{m+1}, & \mu > m + 2, \end{cases}$$

and C is a positive constant independent of x and X.

4. NUMERICAL TESTS

Let us investigate the accuracy of the combined Shepard–Bernoulli approximation operator when applied to the following set of functions on the interval

$I = [0, 1]$:

$$\text{Saddle} \quad f_1 = \frac{1.25}{6 + 6\,(3x - 1)^2},$$

$$\text{Sphere} \quad f_2 = \frac{\sqrt{64 - 81(x - .5)^2}}{9} - .5,$$

$$\text{Cliff} \quad f_3 = \frac{\tanh(-9x + 1)}{2} + .5,$$

$$\text{Gentle} \quad f_4 = \frac{\exp(-\frac{81}{16}(x - .5)^2)}{3},$$

$$\text{Steep} \quad f_5 = \frac{\exp(-\frac{81}{4}(x - .5)^2)}{3},$$

$$\text{Exponential} \quad f_6 = .75 \exp\left(-\frac{(9x - 2)^2}{4}\right) + .75 \exp\left(-\frac{(9x + 1)^2}{49}\right)$$

$$+ .5 \exp\left(-\frac{(9x - 7)^2}{4}\right) + .2 \exp\left(-(9x - 4)^2\right).$$

These functions result from adapting to the univariate case test functions generally used in the multivariate interpolation of large sets of scattered data [19]. For each function f_i we will compare the numerical results obtained by applying the approximation $S_{B_m}[f]$ with those resulting from applying the following combined Shepard approximation operators known in literature: the Shepard–Lagrange univariate operator $S_{L_m}[f]$ and the Shepard–Taylor univariate operator $S_{T_m}[f]$ defined as follows [8, 15, 25]:

$$S_{L_m}[f](x) = \sum_{i=1}^{N} A_i(x) L_m[f, x_i](x),$$

with

$$L_m[f, x_i](x) = \sum_{j=0}^{m} \prod_{\substack{k=0 \\ k \neq j}}^{m} \frac{x - x_{i+k}}{x_{i+j} - x_{i+k}} f(x_{i+j}), \quad x_{N+j} = x_{N-m+j-1}, \ j = 1, \ldots, m,$$

and

$$S_{T_m}[f](x) = \sum_{i=1}^{N} A_i(x) T_m[f, x_i](x)$$

with

$$T_m[f, x_i](x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_i)}{j!} (x - x_i)^j.$$

In the following section we will also consider a special case of the Shepard–Hermite univariate operator

$$S_{H_{2m-1}}[f](x) = \sum_{i=1}^{N} A_i(x) H_{2m-1}[f, x_i](x),$$

obtained from the Shepard operator in combination with the two-point Taylor interpolating polynomial [2], a special case of the Hermite interpolating polynomials:

$$H_{2m-1}[f, x_i](x)$$
$$= \sum_{j=0}^{m-1} \sum_{k=0}^{m-j-1} \binom{m+k-1}{k} \left[\frac{(x-x_i)^j}{j!} \left(\frac{x-x_{i+1}}{x_i-x_{i+1}} \right)^m \left(\frac{x-x_i}{x_{i+1}-x_i} \right)^k f^{(j)}(x_i) \right.$$
$$\left. + \frac{(x-x_{i+1})^j}{j!} \left(\frac{x-x_i}{x_{i+1}-x_i} \right)^m \left(\frac{x-x_{i+1}}{x_i-x_{i+1}} \right)^k f^{(j)}(x_{i+1}) \right].$$

Tables 2–7 show mean and maximum absolute interpolation errors, computed for different values of the parameters μ and m. Taking into account the number of degrees of freedom of each operator according to [3], we used uniform grids of 33 points in the interval $[0, 1]$ for S_{L_0} and S_{T_0}; at the same time we used grids of 17 points for S_{B_1}, S_{L_1}, and S_{T_1}, grids of 11 points for S_{B_2}, S_{L_2}, and S_{T_2}, and finally grids of 8 points for S_{B_3}. In order to have as accurate an estimation of the error as possible, we computed the resulting function approximations at the points

TABLE 2. Saddle

(μ, m)	$S_{L_{m-1}}[f_1](x)$		$S_{T_{m-1}}[f_1](x)$		$S_{B_m}[f_1](x)$	
	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}
$(2,1)$	0.001328	0.003901	0.001328	0.003901	0.001050	0.004954
$(2,2)$	0.001050	0.004954	0.001067	0.003575	0.001062	0.004715
$(2,3)$	0.001248	0.007232	0.001043	0.003988	0.001490	0.005153
$(3,1)$	0.001246	0.002785	0.001246	0.002785	0.000476	0.003314
$(3,2)$	0.000476	0.003314	0.000321	0.002282	0.000333	0.002302
$(3,3)$	0.000718	0.004230	0.000151	0.001188	0.000206	0.001096
$(4,1)$	0.001437	0.003308	0.001437	0.003308	0.000457	0.003233
$(4,2)$	0.000457	0.003233	0.000244	0.001837	0.000259	0.001908
$(4,3)$	0.000699	0.004160	0.000052	0.000503	0.000136	0.001460

TABLE 3. Sphere

(μ, m)	$S_{L_{m-1}}[f_2](x)$		$S_{T_{m-1}}[f_2](x)$		$S_{B_m}[f_2](x)$	
	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}
$(2,1)$	0.001330	0.006394	0.001330	0.006394	0.002145	0.005623
$(2,2)$	0.002145	0.005623	0.002263	0.005076	0.000312	0.000842
$(2,3)$	0.000260	0.000758	0.000371	0.000945	0.000586	0.002344
$(3,1)$	0.001408	0.004570	0.001408	0.004570	0.000583	0.001620
$(3,2)$	0.000583	0.001620	0.000480	0.001313	0.000058	0.000247
$(3,3)$	0.000106	0.000421	0.000034	0.000133	0.000079	0.000323
$(4,1)$	0.001628	0.005677	0.001628	0.005677	0.000510	0.001447
$(4,2)$	0.000510	0.001447	0.000287	0.000958	0.000039	0.000255
$(4,3)$	0.000096	0.000452	0.000007	0.000024	0.000025	0.000113

TABLE 4. Cliff

(μ, m)	$S_{L_{m-1}}[f_3](x)$		$S_{T_{m-1}}[f_3](x)$		$S_{B_m}[f_3](x)$	
	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}
$(2,1)$	0.004773	0.039174	0.004773	0.039174	0.006604	0.038815
$(2,2)$	0.006604	0.038815	0.005827	0.019985	0.004710	0.031367
$(2,3)$	0.005537	0.046979	0.007020	0.025692	0.013455	0.062821
$(3,1)$	0.004080	0.026022	0.004080	0.026022	0.002522	0.021627
$(3,2)$	0.002522	0.021627	0.001727	0.015187	0.002466	0.027527
$(3,3)$	0.003962	0.035485	0.001144	0.011410	0.002138	0.016732
$(4,1)$	0.004661	0.030332	0.004661	0.030332	0.002405	0.021752
$(4,2)$	0.002405	0.021752	0.001347	0.013669	0.002170	0.034048
$(4,3)$	0.004146	0.040890	0.000663	0.008714	0.001542	0.024101

TABLE 5. Gentle

(μ, m)	$S_{L_{m-1}}[f_4](x)$		$S_{T_{m-1}}[f_4](x)$		$S_{B_m}[f_4](x)$	
	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}
$(2,1)$	0.002243	0.006798	0.002243	0.006798	0.002590	0.007116
$(2,2)$	0.002590	0.007116	0.002512	0.006030	0.001897	0.005956
$(2,3)$	0.001895	0.006919	0.001898	0.005585	0.001138	0.006015
$(3,1)$	0.002184	0.004667	0.002184	0.004667	0.000681	0.003277
$(3,2)$	0.000681	0.003277	0.000490	0.002431	0.000378	0.001727
$(3,3)$	0.000783	0.003035	0.000187	0.001007	0.000175	0.000940
$(4,1)$	0.002525	0.005541	0.002525	0.005541	0.000618	0.002978
$(4,2)$	0.000618	0.002978	0.000333	0.001689	0.000270	0.001163
$(4,3)$	0.000709	0.002656	0.000045	0.000249	0.000089	0.000575

TABLE 6. Steep

(μ, m)	$S_{L_{m-1}}[f_5](x)$		$S_{T_{m-1}}[f_5](x)$		$S_{B_m}[f_5](x)$	
	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}
$(2,1)$	0.003744	0.012533	0.003744	0.012533	0.002358	0.012532
$(2,2)$	0.002358	0.012532	0.001790	0.007709	0.002950	0.015868
$(2,3)$	0.004414	0.027049	0.002697	0.011764	0.004950	0.019728
$(3,1)$	0.003060	0.008709	0.003060	0.008709	0.001930	0.011016
$(3,2)$	0.001930	0.011016	0.001218	0.006998	0.001501	0.009079
$(3,3)$	0.003487	0.017791	0.000586	0.003912	0.000909	0.005278
$(4,1)$	0.003527	0.010438	0.003527	0.010438	0.001945	0.011413
$(4,2)$	0.001945	0.011413	0.001000	0.005903	0.001323	0.008184
$(4,3)$	0.003478	0.018516	0.0002566	0.001789	0.000815	0.006381

$i/101$, $i = 1, \ldots, 100$. Note that the approximation data corresponding to the classical Shepard operator coincide with those obtained by applying the operators

TABLE 7. Exponential

(μ, m)	$S_{L_{m-1}}[f_6](x)$		$S_{T_{m-1}}[f_6](x)$		$S_{B_m}[f_6](x)$	
	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}	ε_{mean}	ε_{max}
$(2,1)$	0.007930	0.025868	0.007930	0.025868	0.007669	0.034957
$(2,2)$	0.007669	0.034957	0.006299	0.022608	0.005271	0.025436
$(2,3)$	0.012072	0.045807	0.003968	0.019925	0.025296	0.067861
$(3,1)$	0.007543	0.018120	0.007543	0.018120	0.005122	0.021099
$(3,2)$	0.005122	0.021099	0.002905	0.015777	0.004379	0.024620
$(3,3)$	0.011187	0.048933	0.001660	0.010758	0.003523	0.020488
$(4,1)$	0.008484	0.020677	0.008484	0.020677	0.005026	0.022762
$(4,2)$	0.005026	0.022762	0.002402	0.013701	0.004233	0.024080
$(4,3)$	0.011221	0.051697	0.001198	0.008318	0.003020	0.018326

S_{L_0} or S_{T_0}; furthermore $S_{B_1} = S_{L_1}$. Numerical results show that the accuracy of the Shepard–Bernoulli interpolation is comparable with the accuracy of Shepard–Lagrange and Shepard–Taylor interpolation. To verify that the obtained results do not depend on the particular grid, we computed numerical approximations using sets of random nodes of interpolation in the interval $[0, 1]$; the results show that the approximations are comparable in those cases, too.

5. AN APPLICATION OF THE COMBINED SHEPARD OPERATORS

After solving initial value problems

$$(5.1) \qquad \begin{cases} y'(x) = \varphi(x, y(x)), & (x, y) \in I \times \mathbb{R}, \\ y(x_0) = y_0, & x_0 \in I, \end{cases}$$

by means of a discrete method, we often need to know the solution on a set of points that differs from the grid. The nonnodal approximations are generally obtained by interpolation of the discrete data, and different features can be requested from the interpolants according to the problem. When the exact solution is infinitely many times differentiable (analytic), then the combined Shepard operators can be used to interpolate the discrete data. In fact, combinations of these operators with discrete solvers of ODEs will provide, for special values of μ, *infinitely differentiable (analytic)* approximations of the solution of the problem (5.1) on the whole interval I. An algorithm for constructing these interpolants can be arranged as follows. The discrete solver produces an approximation $\tilde{y}_i$ of the exact solution $y(x_i)$, with a certain tolerance, at each node x_i, $i = 1, \ldots, N$, in the interval I of interest. Since a step control algorithm is enclosed in the ODE solver, the distribution of the nodes depends on the function φ. When φ is smooth enough then $y \in C^1(I)$ (at least) and the value $\tilde{y}'_i = \varphi(x_i, \tilde{y}_i)$ approximates the value $y'(x_i)$, $i = 1, \ldots, N$, with the same precision as that of $\tilde{y}_i$ with respect to $y(x_i)$. The proposed interpolants are then obtained by substituting the exact values into the definition of the combined Shepard operators with their respective approximations.

Tables 8–11 show the accuracy of the combined Shepard operators $S_{T_1}[\cdot]$, $S_{B_2}[\cdot]$, and $S_{H_3}[\cdot]$, when applied to interpolating the discrete solutions of the first four nonstiff DETEST problems A1–A4 [17]; a comparison of this accuracy to that of the classical cubic Hermite spline [3], here denoted by C_{H_3}, is possible. The

discrete approximations are computed by means of the MATLAB® ode45 solver in the interval $[0, 20]$ with various settings of the absolute and relative tolerances $\epsilon = \{AbsTol, RelTol\}$. If we denote by $h_\epsilon = \max_{1 \leq n \leq N_\epsilon - 1} \left| x^\epsilon_{n+1} - x^\epsilon_n \right|$ the size of the grid $0 = x^\epsilon_1 < x^\epsilon_2 < \cdots < x^\epsilon_{N_\epsilon} = 20$ of the nodal approximations $\tilde{y} = \{\tilde{y}_i\}$, then for $\mu = 6$, $S_{T_1}[\tilde{y}](x)$, $S_{B_2}[\tilde{y}](x)$, and $S_{H_3}[\tilde{y}](x)$ are rational, infinitely differentiable (analytic) functions, and, according to Theorem 3.9 and [8, Theorem 4],

$$\|S_{T_1}[y] - y\| \leq C_1 \left\|y^{(2)}\right\|_I h_\epsilon^2,$$
$$\|S_{B_2}[y] - y\| \leq C_2 \left\|y^{(3)}\right\|_I h_\epsilon^3,$$
$$\|S_{H_3}[y] - y\| \leq C_3 \left\|y^{(4)}\right\|_I h_\epsilon^4.$$

Similarly, higher order approximations can be obtained by using the Shepard operator in combination with the osculatory interpolation polynomial that involves more than two points (for example three) [2, 3] or the operators $S_{T_{m-1}}[\cdot]$, $S_{B_m}[\cdot]$, and $S_{H_{2m-1}}[\cdot]$, $m \geq 3$; but, in the last three cases, symbolic partial differentiations of the function defining the IVP problem are required. On the other hand it is well known [3] that C_{H_3} is differentiable only once and

$$\|C_{H_3}[y] - y\| \leq C_4 \left\|y^{(4)}\right\|_I h_\epsilon^4.$$

Problem A1: $\dot{y} = -y$, $y(0) = 1$. Solution: $y(x) = e^{-x}$.

With the tolerances $\epsilon_1 = \{1.0e - 06, 1.0e - 03\}$ the MATLAB® ode45 solver returns the discrete solution $\tilde{y}$ on a grid of 77 points and size $h_\epsilon = 5.0e - 1$; by changing the error tolerances to $\epsilon_2 = \{1.0e - 08, 1.0e - 05\}$, ode45 returns the discrete solution on a grid of 161 points and $h_\epsilon = 4.2e - 1$; finally by setting $\epsilon_3 = \{1.0e - 10, 1.0e - 07\}$ ode45 returns the discrete solution on a grid of 373 points and $h_\epsilon = 2.4e - 1$; the maxima of the global errors computed on the output grid for $\tilde{y}$ and on the grid of 1001 points $i/1001$, $i = 1, \ldots, 1000$, with $S_{T_1}, S_{B_2}, S_{H_3}$, and C_{H_3} are presented in the Table 8.

TABLE 8. Accuracy for the A1 DETEST problem

	$\tilde{y}$	S_{T_1}	S_{B_2}	S_{H_3}	C_{H_3}
ϵ_1	$2.93e - 04$	$4.43e - 03$	$7.23e - 04$	$2.96e - 04$	$2.95e - 04$
ϵ_2	$2.49e - 06$	$8.32e - 04$	$2.56e - 05$	$2.50e - 06$	$2.50e - 06$
ϵ_3	$2.27e - 08$	$1.20e - 04$	$2.08e - 06$	$2.69e - 08$	$2.27e - 08$

Problem A2: $\dot{y} = \frac{-y^3}{2}$, $y(0) = 1$. Solution: $y(x) = \frac{1}{\sqrt{1+x}}$.

In this case and with the error tolerances $\epsilon_1 = \{1.0e - 06, 1.0e - 03\}$, the MATLAB® ode45 solver returns the discrete solution $\tilde{y}$ on a grid of 49 points and $h_\epsilon = 5.0e - 1$; by changing the error tolerances to $\epsilon_2 = \{1.0e - 08, 1.0e - 05\}$, ode45 returns the discrete solution on a grid of 65 points and $h_\epsilon = 5.0e - 1$; finally by setting $\epsilon_3 = \{1.0e - 10, 1.0e - 07\}$, ode45 returns the discrete solution on a grid of 117 points and $h_\epsilon = 4.8e - 1$; the maxima of the global errors computed on the output grid for $\tilde{y}$ and on the grid of 1001 points for $S_{T_1}, S_{B_2}, S_{H_3}$, and C_{H_3} are in the Table 9.

Problem A3: $\dot{y} = y \cos x$, $y(0) = 1$. Solution: $y(x) = e^{\sin x}$.

In this case and with the error tolerances $\epsilon_1 = \{1.0e - 06, 1.0e - 03\}$, the MATLAB® ode45 solver returns the discrete solution $\tilde{y}$ on a grid of 93 points

TABLE 9. Accuracy for the A2 DETEST problem

	$\tilde{y}$	S_{T_1}	S_{B_2}	S_{H_3}	C_{H_3}
ϵ_1	$3.65e-04$	$1.61e-03$	$3.67e-04$	$3.70e-04$	$3.65e-04$
ϵ_2	$2.85e-06$	$4.01e-04$	$2.85e-05$	$3.06e-06$	$2.85e-06$
ϵ_3	$4.79e-08$	$7.29e-05$	$2.95e-06$	$9.06e-08$	$4.91e-08$

and $h_\epsilon = 3.1e-1$; by changing the error tolerances to $\epsilon_2 = \{1.0e-08, 1.0e-05\}$, ode45 returns the discrete solution on a grid of 197 points and $h_\epsilon = 1.5e-1$; finally, by setting $\epsilon_3 = \{1.0e-10, 1.0e-07\}$, ode45 returns the discrete solution on a grid of 465 points and $h_\epsilon = 6.3e-2$; the maxima of the global errors computed on the output grid for $\tilde{y}$ and on the grid of 1001 points for $S_{T_1}, S_{B_2}, S_{H_3}$, and C_{H_3} are in the Table 10.

TABLE 10. Accuracy for the A3 DETEST problem

	$\tilde{y}$	S_{T_1}	S_{B_2}	S_{H_3}	C_{H_3}
ϵ_1	$3.76e-03$	$2.54e-02$	$7.43e-03$	$3.79e-03$	$3.79e-03$
ϵ_2	$7.49e-05$	$4.55e-03$	$6.73e-04$	$7.46e-05$	$7.46e-05$
ϵ_3	$7.08e-07$	$7.65e-04$	$3.22e-05$	$8.72e-07$	$7.07e-07$

Problem A4: $\dot{y} = \frac{y}{4}\left(1 - \frac{y}{20}\right)$, $y(0) = 1$. Solution: $y(x) = \frac{20}{1+19e^{-x/4}}$.

In this case and with the error tolerances $\epsilon_1 = \{1.0e-06, 1.0e-03\}$, the MATLAB$^\circledR$ ode45 solver returns the discrete solution $\tilde{y}$ on a grid of 45 points and $h_\epsilon = 5.0e-1$; by changing the error tolerances to $\epsilon_2 = \{1.0e-08, 1.0e-05\}$, ode45 returns the discrete solution on a grid of 49 points and $h_\epsilon = 5.0e-1$; finally, by setting $\epsilon_3 = \{1.0e-10, 1.0e-07\}$, ode45 returns the discrete solution on a grid of 93 points and $h_\epsilon = 3.6e-1$; the maxima of the global errors computed on the output grid for $\tilde{y}$ and on the grid of 1001 points for $S_{T_1}, S_{B_2}, S_{H_3}$, and C_{H_3} are in the Table 11.

TABLE 11. Accuracy for the A4 DETEST problem

	$\tilde{y}$	S_{T_1}	S_{B_2}	S_{H_3}	C_{H_3}
ϵ_1	$4.64e-05$	$3.82e-03$	$3.57e-04$	$4.64e-05$	$4.64e-05$
ϵ_2	$3.07e-05$	$3.81e-03$	$3.74e-04$	$3.07e-05$	$3.07e-05$
ϵ_3	$2.05e-06$	$1.75e-03$	$1.26e-04$	$2.17e-06$	$2.05e-06$

Remark 5.1. The combined Shepard operators are global (i.e., their values at each point x are affected by *all* the data); however, the global character of these interpolants can be avoided by substituting the weight functions $A_{\mu,i}(x)$ with basis functions $W_{\mu,i}(x)$ of an opportune class of differentiability $C_0^s(\mathbb{R})$, $0 \le s \le \infty$, with small compact support and which may depend on the local distribution of data points [4, 23, 24, 26].

6. CONCLUSIONS

The univariate Shepard–Bernoulli operator can be extended to the multivariate case by using the expansions studied in [11, 12].

Acknowledgments

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